Mel'nikov methods and homoclinic orbits in discontinuous systems

Alessandro Calamai and Matteo Franca *

Abstract

We consider a discontinuous system exhibiting a, possibly non-smooth, homoclinic trajectory. We assume that the critical point lies on the discontinuity level. We study the persistence of such a trajectory when the system is subject to a smooth non-autonomous perturbation. We use a Mel'nikov type approach and we introduce conditions which enable us to reformulate the problem in the setting of smooth systems so that we can follow the outline of the classical theory.

Keywords. Homoclinic solutions, discontinuous systems, Mel'nikov integrals.

AMS Subject Classification. 34C23, 34C37, 37G20

1 Introduction

In this paper we are interested in the persistence of a homoclinic trajectory of a system, either continuous or discontinuous, subject to a smooth perturbation. Let $\vec{f}$ be a smooth function and assume that the system $\dot{\vec{x}} = \vec{f}(\vec{x})$ admits a trajectory $\vec{\gamma}(t)$ homoclinic to the origin (which is assumed to be a fixed point). It is well known that the perturbed system

$$\dot{\vec{x}} = \vec{f}(\vec{x}) + \varepsilon \vec{g}(t, \vec{x}, \varepsilon) \tag{1.1}$$

where $\vec{g}$ is a smooth function and ε is a small positive constant, admits a homoclinic trajectory $\vec{x}(t, \varepsilon)$ close to $\vec{\gamma}(t)$, if a generic Mel'nikov condition is satisfied, see [9, 5, 18, 10] and Theorem 2.3 below. Furthermore when $\vec{g}$ is periodic in t such a condition is sufficient for the appearance of a chaotic pattern for the perturbed system (1.1), see e.g. [18]. In this framework we quote also [15] where the authors overview all codimension 1 bifurcations for a planar autonomous discontinuous (but piecewise smooth) vector field.

*Dipartimento di Ingegneria Industriale e Scienze Matematiche, Università Politecnica delle Marche, Via Brecce Bianche 1, 60131 Ancona - Italy. Partially supported by G.N.A.M.P.A. - INdAM (Italy) and MURST (Italy)

Recently several papers extended this well developed theory to the case of discontinuous equations, see [13, 14] for the 2-dimensional non-autonomous case. Battelli and Fečkan in [3] consider the case of higher dimensional discontinuous systems: they assume that $\vec{f}$ is discontinuous on a smooth surface Ω^0 and that the flow is transversal on Ω^0 (so there is no sliding). Moreover they assume that there is a homoclinic trajectory $\vec{\gamma}(t)$ which crosses Ω^0 transversally twice, thus $\vec{\gamma}(t)$ is in fact obtained gluing together (in a Lipschitz way) two smooth functions, $\vec{\gamma}^-(t)$ and $\vec{\gamma}^+(t)$. In this context they are able to produce a Mel'nikov condition sufficient to guarantee the persistence of the homoclinic trajectory in the perturbed system.

More precisely, let G be a C^r function, $r \geq 2$, on $\Omega \subset \mathbb{R}^n$, and let $\Omega^\pm = \{\vec{x} \in \Omega \mid \pm G(\vec{x}) > 0\}$, $\Omega^0 := \{\vec{x} \in \Omega \mid G(\vec{x}) = 0\}$; here and in the sequel we use the shorthand notation $\pm$ to represent both the $+$ and $-$ equations and functions. Let $\vec{f}^\pm \in C^r_b(\Omega^\pm, \mathbb{R}^n)$ and consider the equation

$$\dot{\vec{x}} = \vec{f}^\pm(\vec{x}) + \varepsilon \vec{g}(t, \vec{x}, \varepsilon), \quad x \in \Omega^\pm, \tag{1.2}$$

where $\vec{g} \in C^r_b(\mathbb{R} \times \Omega \times \mathbb{R}, \mathbb{R}^n)$ and $\varepsilon \in \mathbb{R}$ is a small parameter. In [3] it is discussed the case where (1.2) admits a continuous but not smooth homoclinic trajectory $\vec{\gamma}(t)$ for $\varepsilon = 0$, such that $\vec{\gamma}(t) \in \Omega^-$ for $|t| > T$, $\vec{\gamma}(t) \in \Omega^+$ for $|t| < T$ and which crosses transversally the hypersurface Ω^0 at $|t| = T$ for a certain $T > 0$. The authors prove the persistence of the homoclinic trajectory for $|\varepsilon| > 0$, assuming that the critical point lies in the interior of Ω^-, i.e. $G(\vec{0}) < 0$. The main difficulty in [3] is to take care of the jump discontinuities of the projector maps of the exponential dichothomy of the variational system. Furthermore finding an explicit expression of the bounded solution of the adjoint variational system leads to rather cumbersome computations.

The main purpose of this paper is to generalize the results of [3] to the case where the fixed point of both the systems $\dot{\vec{x}} = \vec{f}^\pm(\vec{x})$ is the origin, and lies on the discontinuity level Ω^0. The key point, that makes the extension not trivial, is that we need to locate trajectories when they are exponentially small, so we cannot simply rely on a continuity argument, but we need to give precise asymptotic estimates.

With the same notation as above, here we assume that $G(\vec{0}) = 0$ and that for $\varepsilon = 0$ equation (1.2) admits a continuous (not necessarily C^1) trajectory $\vec{\gamma}(t)$ homoclinic to the origin, which in fact consists of two solutions:

$$\vec{\gamma}(t) = \begin{cases} \vec{\gamma}^-(t) & \text{if } t \leq 0 \\ \vec{\gamma}^+(t) & \text{if } t \geq 0 \end{cases}$$

where $\vec{\gamma}^\pm(t) \in \Omega^\pm$ for $t \neq 0$, and $\vec{\gamma}^-(0) = \vec{\gamma}^+(0) = \vec{\gamma}(0) \in \Omega^0$ (with $\vec{\gamma}(0) \neq 0$, see figure 1).

We prove via Mel'nikov theory the existence of a solution $\vec{x}_b(t, \varepsilon)$, $|\varepsilon| > 0$ small, which is homoclinic to the origin and close to $\vec{\gamma}(t)$, and we look for conditions on $\vec{f}^\pm$ and $\vec{g}$ which guarantee that $\vec{x}_b(t, \varepsilon) \in \Omega^-$ for $t < 0$ and $\vec{x}_b(t, \varepsilon) \in \Omega^+$ for $t > 0$ (see Theorem 3.7 below).

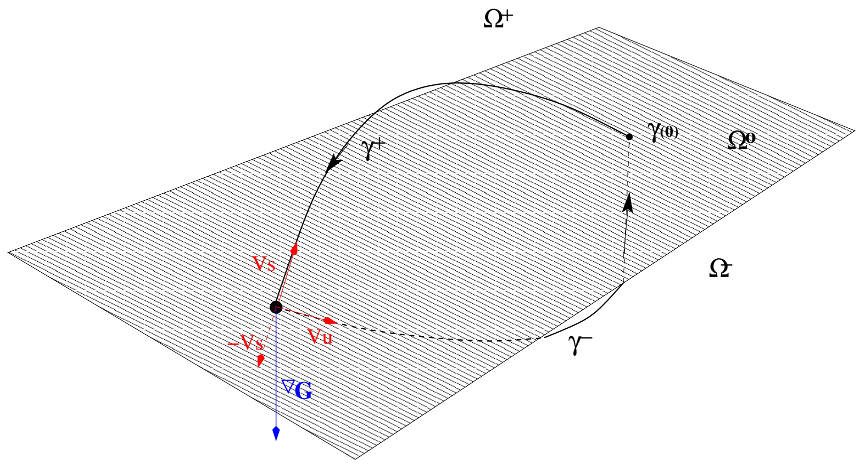

Figure 1: A sketch of the piecewise smooth homoclinic trajectory $\vec{\gamma}(t)$ of the unperturbed problem (1.2).

As a first step we consider a simpler case. We assume that, for $\varepsilon = 0$, equation (1.2) has a smooth trajectory $\vec{\gamma}(t)$ homoclinic to the origin and which lies entirely, say, on Ω^+, and we look for conditions to get $\vec{x}_b(t, \varepsilon) \in \Omega^+$ for any $t \in \mathbb{R}$. Now, a crucial remark is that this problem can be equivalently stated in the continuous setting. As a consequence of this observation, we start our investigation in the following way. We focus on the continuous case, that is we consider equation (1.1), and we assume that the system $\dot{\vec{x}} = f(\vec{x})$ admits a trajectory $\vec{\gamma}(t)$ homoclinic to the origin (which still lies on Ω^0) and such that $\vec{\gamma}(t) \in \Omega^+$ for any $t \in \mathbb{R}$. Then we just have to guarantee that the perturbed homoclinic $\vec{x}_b(t, \varepsilon)$, obtained via classical Mel'nikov theory (see Theorem 2.3 below, see also [18, 3]), lies in Ω^+. We present this first result in Theorem 2.9 below and the application to the discontinuous case in Corollary 3.1. This result may be applied to study a pendulum subject to small spatial dependent forcing and small dry friction. With our tool one may show the existence of a homoclinic trajectory in which there is no inversion of the motion.

As pointed out, we are able to develop all the proofs in a simpler context, which involves the least of technicalities and already introduces all the main ideas of this paper. In fact, we just need to follow the outline of the proof of the known results and to modify a lemma in order to locate stable and unstable trajectories. Then in Theorem 3.7 we consider the motivating case of equation (1.2) where $\vec{\gamma}^{\pm}(t) \in \Omega^{\pm}$.

The main goal of this paper concerns the comprehension of non-smooth and discontinuous differential equations. These problems occur typically in mechanical systems with dry friction or with impacts, and have received a great interest for their relevance in applications. They turn out to be useful also in control theory, electronics, economics and biology (see [6, 7, 14, 15] for more details). On the other hand, as a corollary of our approach we are able to locate homoclinic trajectories of perturbed problems, even in the continuous case, and this fact proves to be useful also in different settings. In the last section, devoted to applications, we give a result concerning radial solutions of an elliptic equation: we show that our method can be used to guarantee positivity of the solutions found, which is one of the main issues in that context.

The plan of this paper is the following. In Section 2.1 we fix the notation and we give a list of the assumptions that are used in the sequel. Moreover we collect some known results in Mel'nikov theory concerning the continuous setting. Then, in Section 2.2, we assume $\vec{\gamma}(t) \in \Omega^+$ for any $t \in \mathbb{R}$ and we give conditions in order that the perturbed homoclinic $\vec{x}_b(t,\varepsilon)$ lies in Ω^+.

In Section 3 we consider the discontinuous setting where $\vec{\gamma}(t) \in \Omega^+$ for $t > 0$, $\vec{\gamma}(t) \in \Omega^-$ for $t < 0$ and $\vec{\gamma}$ crosses transversally Ω^0 at $t = 0$. We still assume that $\vec{\gamma}(t)$ is asymptotic to the origin which lies on Ω^0. The generalization to this case in fact needs just little effort. Then we show that we can analyze the case where $\vec{\gamma}$ crosses Ω^0 twice (or more) and is asymptotic to the origin, which lies on Ω^0, see figures 1 and 3. Here we put together the ideas of Section 2 with the argument developed in [3].

In Section 4 we construct, for illustrative purposes, some examples where all the computation can be carried on analytically. In the Appendix we discuss a result related to the roughness of exponential dichotomy, which we did not find in literature, even if it seems to be known by experts.

Finally we note that a related problem is studied in [1], where the authors consider the case where a finite part of the homoclinic solution of the unperturbed problem remains on a discontinuity level, i.e. sliding is allowed. We stress that in this paper we always give conditions which ensure that sliding phenomena do not appear.

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2 The continuous case

2.1 Preliminary results and notation

In this section we collect some known results in Mel'nikov theory and we fix the notation.

Throughout the paper we will use the following notation. We denote scalars by small letters, e.g. a, vectors in $\mathbb{R}^n$ with an arrow, e.g. $\vec{a}$, and $n \times n$ matrices by bold letters, e.g. $\mathbf{A}$. By $\vec{a}^*$ and $\mathbf{A}^*$ we mean the transpose of the vector $\vec{a}$ and of the matrix $\mathbf{A}$ respectively, so that $\vec{a}^*\vec{b}$ denotes the scalar product of the vectors $\vec{a}$, $\vec{b}$. We will denote by $||\cdot||$ the Euclidean norm in $\mathbb{R}^n$ and in the space of $n \times n$ matrices. We will use the shorthand notation $\boldsymbol{f_x} = \frac{\boldsymbol{\partial f}}{\boldsymbol{\partial x}}$ unless this may cause confusion.

As in the Introduction, let G be a C^r function on the open set $\Omega \subset \mathbb{R}^n$, $r \geq 2$, such that $G(\vec{0}) = 0$ and let $\Omega^{\pm} = \{\vec{x} \in \Omega \mid \pm G(\vec{x}) > 0\}$, $\Omega^0 := \{\vec{x} \in \Omega \mid$

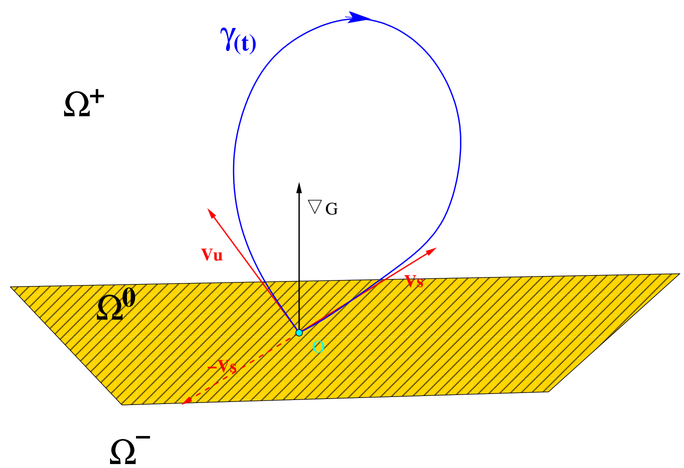

Figure 2: A sketch of the homoclinic trajectory $\vec{\gamma}(t)$ of the unperturbed problem (2.1). Here we assume that $\vec{\gamma}(t) \in \Omega^+$ and is smooth for any $t \in \mathbb{R}$.

$G(\vec{x}) = 0\}$. We consider equation

$$\dot{\vec{x}} = \vec{f}(\vec{x}) + \varepsilon \vec{g}(t, \vec{x}, \varepsilon), \tag{2.1}$$

where $\vec{f} \in C_b^r(\Omega, \mathbb{R}^n)$, $\vec{g} \in C_b^r(\mathbb{R} \times \Omega \times \mathbb{R}, \mathbb{R}^n)$ and $\varepsilon \in \mathbb{R}$ is a small parameter. We suppose that the unperturbed equation $\dot{\vec{x}} = \vec{f}(\vec{x})$ admits a trajectory $\vec{\gamma}(t)$ homoclinic to the origin (which is assumed to be a fixed point) and such that $\vec{\gamma}(t) \in \Omega^+$ for any $t \in \mathbb{R}$.

First of all we observe that, since the map $\vec{g}$ is bounded, the critical point bifurcates into a bounded solution $\vec{x}_0(t; \varepsilon)$ such that

$$\|\vec{x}_0(\cdot; \varepsilon)\|_\infty := \sup\{\|\vec{x}_0(t; \varepsilon)\| \mid t \in \mathbb{R}\} < K|\varepsilon|,$$

for some $K > 0$. Then, via Mel'nikov theory, one can prove the existence of a solution $\vec{x}_b(t; \varepsilon)$ which is homoclinic to $\vec{x}_0(t; \varepsilon)$, see Theorem 2.3 below. Our next step (see Section 2.2 below) is to give sufficient conditions in order to guarantee that the homoclinic solution verifies $\vec{x}_b(t, \varepsilon) \in \Omega^+$ for any $t \in \mathbb{R}$.

Along the paper we will consider the following assumptions. We list them here for reader's convenience.

F0 $\boldsymbol{f_x}(\vec{0})$ has no eigenvalues with real part equal to zero.

We denote by λ_u and λ_s the eigenvalues of $\boldsymbol{f_x}(\vec{0})$ respectively with smallest positive real part and with largest negative real part.

F1 λ_u and λ_s are real and simple.

We denote by $\vec{v}_u$ and $\vec{v}_s$ the normalized eigenvectors corresponding to λ_u and λ_s. We assume w.l.o.g. that $[\vec{\nabla} G(\vec{0})]^* \vec{v}_s$ and $[\vec{\nabla} G(\vec{0})]^* \vec{v}_u$ are both nonnegative and we consider the following hypotheses:

F2 $\vec{v}_u$ and $\vec{v}_s$ are not orthogonal to $\vec{\nabla}G(\vec{0})$.

F3 The limits $\lim_{t\to-\infty} \frac{\dot{\vec{\gamma}}(t)}{\|\dot{\vec{\gamma}}(t)\|}$ and $\lim_{t\to+\infty} \frac{\dot{\vec{\gamma}}(t)}{\|\dot{\vec{\gamma}}(t)\|}$ exist and coincide with $\vec{v}_u$ and $-\vec{v}_s$ respectively.

We stress that conditions $\mathbf{F0}, \mathbf{F1}, \mathbf{F2}, \mathbf{F3}$ are generic. We know that $\vec{v}_u$ and $\vec{v}_s$ belong respectively to the unstable space and the stable space of the linearized system $\dot{\vec{x}} = \boldsymbol{f_x}(\vec{0})\vec{x}$. Condition **F1** ensures that a "generic" trajectory $\vec{\phi}(t)$ of $\dot{\vec{x}} = \vec{f}(\vec{x})$, converging to the origin as $|t| \to \infty$, has a definite direction, hence the limits $\lim_{t\to\pm\infty} \vec{\phi}(t)/\|\vec{\phi}(t)\|$ exist. Condition **F3** guarantees that $\vec{\gamma}(t)$ is one of the "generic" trajectories just described.

Remark 2.1. We point out that the results of this paper are still valid weakening condition **F1** by assuming that λ_u and λ_s are real and semisimple. In this case we still suppose that the limits $\lim_{t\to-\infty} \frac{\dot{\vec{\gamma}}(t)}{\|\dot{\vec{\gamma}}(t)\|}$ and $\lim_{t\to+\infty} \frac{\dot{\vec{\gamma}}(t)}{\|\dot{\vec{\gamma}}(t)\|}$ exist, and we define $\vec{v}_u$ and $-\vec{v}_s$ as in **F3**. Then, we observe that $\vec{v}_u$ and $\vec{v}_s$ are still normalized eigenvectors corresponding to λ_u and λ_s respectively, and we assume that condition **F2** holds.

Concerning the perturbation term $\vec{g}$ we will assume boundedness and condition **G0** below, i.e., superlinearity in $\vec{x}$. We stress that the basic assumption of boundedness of the map $\vec{g}$ is needed to guarantee the persistence of a bounded solution bifurcating from the origin, for ε small enough.

G0 $\vec{g}(t; \vec{0}, \varepsilon) = 0$ and $\boldsymbol{g_x}(t; \vec{0}, \varepsilon) = 0$ for any $t \in \mathbb{R}$ and any $\varepsilon \in \mathbb{R}$.

Condition **F0** implies that the autonomous system $\dot{\vec{x}} = \boldsymbol{f_x}(\vec{0})\vec{x}$ admits an exponential dichotomy on the whole of $\mathbb{R}$, with projection $\boldsymbol{P^0}$ and constant $k^0 \geq 1$. Hence, if $\boldsymbol{X^0}(t) = \exp[t\boldsymbol{f_x}(\vec{0})]$ is the fundamental matrix of this system, we have

$$
\begin{aligned}
&\|\boldsymbol{P^0}\boldsymbol{X^0}(t-s)\| \leq k^0 e^{\lambda_s(t-s)} && \text{if } s \leq t \\
&\|(\mathbb{I} - \boldsymbol{P^0})\boldsymbol{X^0}(t-s)\| \leq k^0 e^{\lambda_u(t-s)} && \text{if } t \leq s.
\end{aligned}
\tag{2.2}
$$

Because of *roughness* of exponential dichotomies (see [8, 18, 19] and the Appendix) the linear system $\dot{\vec{x}} = \boldsymbol{f_x}(\vec{\gamma}(t))\vec{x}$ admits an exponential dichotomy on $(-\infty, 0]$ and $[0, \infty)$ respectively, that is, there exist projections $\boldsymbol{P^{\pm}} : \mathbb{R}^n \to \mathbb{R}^n$ and a positive number $k \geq 1$ such that the following hold:

$$
\begin{aligned}
&\|\boldsymbol{X}(t)\boldsymbol{P^-}\boldsymbol{X^{-1}}(s)\| \leq k\, \mathrm{e}^{\lambda_s(t-s)} && \text{if } s \leq t \leq 0 \\
&\|\boldsymbol{X}(t)(\mathbb{I} - \boldsymbol{P^-})\boldsymbol{X^{-1}}(s)\| \leq k\, \mathrm{e}^{\lambda_u(t-s)} && \text{if } t \leq s \leq 0 \\
&\|\boldsymbol{X}(t)\boldsymbol{P^+}\boldsymbol{X^{-1}}(s)\| \leq k\, \mathrm{e}^{\lambda_s(t-s)} && \text{if } 0 \leq s \leq t \\
&\|\boldsymbol{X}(t)(\mathbb{I} - \boldsymbol{P^+})\boldsymbol{X^{-1}}(s)\| \leq k\, \mathrm{e}^{\lambda_u(t-s)} && \text{if } 0 \leq t \leq s
\end{aligned}
\tag{2.3}
$$

where $\boldsymbol{X}(t)$ is the fundamental matrix of the linear system $\dot{\vec{x}} = \boldsymbol{f_x}(\vec{\gamma}(t))\vec{x}$ such that $\boldsymbol{X}(0) = \mathbb{I}$.

Remark 2.2. We stress the fact that the exponents λ_u, λ_s in the exponential dichotomies (2.3) are the same as in (2.2). This will be a crucial point in the proof of Theorem 2.9. In the Appendix we give a proof of this roughness result.

Next, setting $\boldsymbol{P}^{\pm}(t) = \boldsymbol{X}(\pm t)\boldsymbol{P}^{\pm}\boldsymbol{X}^{-1}(\pm t)$ we have (see [18])

$$\lim_{t\to+\infty} \|\boldsymbol{P}^{\pm}(t) - \boldsymbol{P}^{\mathbf{0}}\| = 0\,.$$

To apply Mel'nikov theory we need a further non-degeneracy condition:

F4 Bounded solutions of the linear system $\dot{\vec{x}} = \boldsymbol{f_x}(\vec{\gamma}(t))\vec{x}$ have the form $c\dot{\vec{\gamma}}(t)$ where c is an arbitrary constant.

We stress that $c\dot{\vec{\gamma}}(t)$, $c \in \mathbb{R}$, is always a bounded solution of the above system. Condition **F4** requires that all the bounded solutions have this form. Notice that **F4** is equivalent to the assumption $\mathcal{N}\boldsymbol{P}^{-} \cap \mathcal{R}\boldsymbol{P}^{+} = \mathrm{span}[\dot{\vec{\gamma}}(0)]$, and this condition is always satisfied in dimension $n \le 3$. Moreover, such a condition is equivalent to the following:

F4* The adjoint variational system $\dot{\vec{x}} = -\boldsymbol{f_x}^*(\vec{\gamma}(t))\vec{x}$ admits a bounded solution $\vec{\psi}(t)$ which is unique up to multiplication by a constant.

We review first briefly the standard case: we formulate a Mel'nikov condition that guarantees the existence of the perturbed homoclinic $\vec{x}_b(t,\varepsilon)$. Assume that conditions **F0**, **F4** hold. Define the Mel'nikov function

$$M(\alpha) = \int_{-\infty}^{+\infty} \vec{\psi}^*(t)\vec{g}(t+\alpha, \vec{\gamma}(t), 0)dt \tag{2.4}$$

and consider its derivative

$$M'(\alpha) = \int_{-\infty}^{+\infty} \vec{\psi}^*(t)\frac{\partial \vec{g}}{\partial t}(t+\alpha, \vec{\gamma}(t), 0)dt.$$

The next theorem guarantees the existence of the perturbed homoclinic.

Theorem 2.3 ([18],[3])**.** *Assume that conditions* **F0**, **F4** *hold, and that there is* α_0 *such that* $M(\alpha_0) = 0$ *and* $M'(\alpha_0) \neq 0$*. Then there exists* $\varepsilon_0 > 0$ *such that for any* $0 < |\varepsilon| < \varepsilon_0$ *system (2.1) admits a unique* C^{r-1} *solution* $\vec{x}_b(t;\varepsilon)$ *bounded on* $\mathbb{R}$*, and there is a* C^{r-1} *function* $\alpha(\varepsilon)$*, with* $\alpha(0) = \alpha_0$*, with the following property:*

$$\sup_{t\in\mathbb{R}} ||\vec{x}_b(t+\alpha(\varepsilon);\varepsilon) - \vec{\gamma}(t)|| \to 0 \quad \textit{as } \varepsilon \to 0. \tag{2.5}$$

We stress that Theorem 2.3 does not allow to locate $\vec{x}_b(t;\varepsilon)$. In order to ensure that $\vec{x}_b(t;\varepsilon) \in \Omega^+$ we need to require $\vec{\gamma}(t) \in \Omega^+$ for any $t \in \mathbb{R}$ and some further assumptions on $\vec{f}$ and $\vec{g}$: this will be the content of Theorem 2.9. For a further discussion on the regularity of $\vec{x}_b$ with respect to the parameter ε, see Remark 3.5.

Even if Theorem 2.3 is known, we sketch the proof for convenience since this gives us the outline of the proof of our results (in particular Theorem 3.7 below). We refer the interested reader to [18] or [3] for more details.

The starting point is the following lemma (see e.g. Lemma 2.5 in [3]).

Lemma 2.4. *For any $\alpha \in \mathbb{R}$ there are $\varepsilon_0 > 0$ and $\eta_0 > 0$, such that for any $\vec{\eta}^- \in \mathcal{N}\boldsymbol{P}^-$ and $\vec{\eta}^+ \in \mathcal{R}\boldsymbol{P}^+$, $\|\vec{\eta}^+\| + \|\vec{\eta}^-\| < \eta_0$, $0 \le |\varepsilon| < \varepsilon_0$, there are unique C^r-solutions $\vec{x}^-(t, \vec{\eta}^-, \alpha, \varepsilon)$ defined for $t \le \alpha$, and $\vec{x}^+(t, \vec{\eta}^+, \alpha, \varepsilon)$ defined for $t \ge \alpha$ of (2.1) satisfying*

$$\begin{array}{l}(\mathbb{I} - \boldsymbol{P}^-)[\vec{x}^-(\alpha, \vec{\eta}^-, \alpha, \varepsilon) - \vec{\gamma}(0)] = \vec{\eta}^- \\ \boldsymbol{P}^+[\vec{x}^+(\alpha, \vec{\eta}^+, \alpha, \varepsilon) - \vec{\gamma}(0)] = \vec{\eta}^+\end{array}$$

and

$$\begin{array}{l}\sup_{t \le 0} \|\vec{x}^-(t+\alpha, \vec{\eta}^-, \alpha, \varepsilon) - \vec{\gamma}(t)\| \le c[\|\vec{\eta}^-\| + |\varepsilon|] \\ \sup_{t \ge 0} \|\vec{x}^+(t+\alpha, \vec{\eta}^+, \alpha, \varepsilon) - \vec{\gamma}(t)\| \le c[\|\vec{\eta}^+\| + |\varepsilon|]\end{array}$$

for a constant $c > 0$.

Let us sketch the proof of Lemma 2.4. First we set $\vec{z}(t) = \vec{x}^-(t+\alpha) - \vec{\gamma}(t)$; then $\vec{z}(t)$ solves

$$\dot{\vec{z}} = \boldsymbol{f_x}(\vec{\gamma}(t))\vec{z} + \vec{H}_0(t; \vec{z}, \alpha, \varepsilon)\,, \tag{2.6}$$

where

$$\vec{H}_0(t; \vec{z}, \alpha, \varepsilon) = \vec{f}(\vec{z} + \vec{\gamma}(t)) - \vec{f}(\vec{\gamma}(t)) - \boldsymbol{f_x}(\vec{\gamma}(t))\vec{z} + \varepsilon\vec{g}(t+\alpha, \vec{z} + \vec{\gamma}(t), \varepsilon)\,. \tag{2.7}$$

Observe that $\|\vec{H}_0(t; \vec{z}, \alpha, \varepsilon)\| = O(\varepsilon) + o(\|\vec{z}\|)$ as $\varepsilon \to 0$ and $\|\vec{z}\| \to 0$. The existence of $\vec{x}^-(t, \vec{\eta}^-, \alpha, \varepsilon)$ is now obtained rewriting (2.6) as the fixed point equation

$$\vec{z} = \mathcal{T}(\vec{z}),$$

where the operator $\mathcal{T}$, acting on the Banach space of bounded continuous functions $C^0_b((-\infty, 0], \mathbb{R}^n)$ with the standard supremum norm, is defined by

$$\begin{aligned}\mathcal{T}(\vec{z})(t) = \quad & \mathbf{X}(t)\vec{\eta}^- + \int_{-\infty}^{t} \mathbf{X}(t)\boldsymbol{P}^-\boldsymbol{X}^{-1}(s)\vec{H}_0(s; z(s), \alpha, \varepsilon)ds - \\ - \quad & \int_t^0 \mathbf{X}(t)[\mathbb{I} - \boldsymbol{P}^-]\boldsymbol{X}^{-1}(s)\vec{H}_0(s; z(s), \alpha, \varepsilon)ds\,,\end{aligned} \tag{2.8}$$

with $\vec{\eta}^- \in \mathcal{N}\boldsymbol{P}^-$. Using exponential dichotomy and the estimates on $\vec{H}_0$ it is not difficult to see that for $\|\vec{\eta}^-\|$, $|\varepsilon|$ small enough, the map $\mathcal{T}$ is a contraction and maps a small ball centered at the origin in itself. So we can apply the Banach fixed point theorem, and prove the existence and uniqueness of $\vec{x}^-(t, \vec{\eta}^-, \alpha, \varepsilon)$. Analogously we can show the existence and uniqueness of $\vec{x}^+(t, \vec{\eta}^+, \alpha, \varepsilon)$ and conclude the proof of Lemma 2.4.

Remark 2.5. Differentiating the fixed point equation $\vec{z} = \mathcal{T}(\vec{z})$, it is possible to compute the following derivatives (compare also [18] and [3]):

$$\frac{\partial \vec{x}^-}{\partial \varepsilon}(\alpha; \vec{0}, \alpha, 0) = \int_{-\infty}^{0} \boldsymbol{P}^- \boldsymbol{X}^{-1}(s) \vec{g}(s+\alpha, \vec{\gamma}(s), 0) ds$$
$$\frac{\partial \vec{x}^+}{\partial \varepsilon}(\alpha; \vec{0}, \alpha, 0) = -\int_{0}^{+\infty} [\mathbb{I} - \boldsymbol{P}^+] \boldsymbol{X}^{-1}(s) \vec{g}(s+\alpha, \vec{\gamma}(s), 0) ds$$

and

$$\frac{\partial \vec{\boldsymbol{x}}^-}{\partial \boldsymbol{\eta}^-}(t+\alpha; \vec{0}, \alpha, 0) = \boldsymbol{X}(t)|_{\mathcal{N}\boldsymbol{P}^-}$$
$$\frac{\partial \vec{\boldsymbol{x}}^+}{\partial \boldsymbol{\eta}^+}(t+\alpha; \vec{0}, \alpha, 0) = \boldsymbol{X}(t)|_{\mathcal{R}\boldsymbol{P}^-} .$$

The existence of the homoclinic trajectory $\vec{x}_b(t;\varepsilon)$, now amounts to the existence of a solution for the equation

$$\vec{F}(\vec{\eta}^-, \vec{\eta}^+, \alpha, \varepsilon) := \vec{x}^+(\alpha; \vec{\eta}^+, \alpha, \varepsilon) - \vec{x}^-(\alpha; \vec{\eta}^-, \alpha, \varepsilon) = \vec{0}. \tag{2.9}$$

To see that such a solution exists we apply the implicit function theorem but we need first to make a Lijapunov-Schmidt reduction. First observe that, from the uniqueness and t-invariance of the solutions of $\dot{\vec{x}} = \vec{f}(\vec{x})$ it follows that for any $\alpha, \beta \in \mathbb{R}$ we have

$$\vec{x}^-(t+\alpha, (\mathbb{I} - \boldsymbol{P}^-)[\vec{\gamma}(\beta) - \vec{\gamma}(0)], \alpha, 0) = \vec{\gamma}(t+\beta) \text{ for } t \le 0\,,$$

and

$$\vec{x}^+(t+\alpha, \boldsymbol{P}^+[\vec{\gamma}(\beta) - \vec{\gamma}(0)], \alpha, 0) = \vec{\gamma}(t+\beta) \text{ for } t \ge 0\,.$$

Let us denote by M_0 the following smooth curve in $\mathcal{N}\boldsymbol{P}^- \times \mathcal{R}\boldsymbol{P}^+$:

$$M_0 := \left\{ \left((\mathbb{I} - \boldsymbol{P}^-)[\vec{\gamma}(\beta) - \vec{\gamma}(0)], \boldsymbol{P}^+[\vec{\gamma}(\beta) - \vec{\gamma}(0)]\right) \mid \beta \in \mathbb{R} \right\} .$$

We have

$$\vec{F}(\vec{\eta}^-, \vec{\eta}^+, \alpha, 0) \equiv \vec{0} \quad \text{for any } \alpha \in \mathbb{R} \text{ and } (\vec{\eta}^-, \vec{\eta}^+) \in M_0. \tag{2.10}$$

Roughly speaking this means that when $\varepsilon = 0$ there is a whole "matching curve" M_0. We will see that the situation is quite similar for $\varepsilon \neq 0$ small.

Observe that $(\vec{0}, \vec{0}) \in M_0$, and the tangent space of M_0 at $(\vec{0}, \vec{0})$ is spanned by $(\dot{\vec{\gamma}}(0), \dot{\vec{\gamma}}(0))$; in fact

$$\boldsymbol{P}^+[\vec{\gamma}(\beta) - \vec{\gamma}(0)] = \beta \boldsymbol{P}^+[\dot{\vec{\gamma}}(0)] + o(\beta) = \beta \dot{\vec{\gamma}}(0) + o(\beta), \quad \beta \to 0,$$

and similarly for $(\mathbb{I} - \boldsymbol{P}^-)[\vec{\gamma}(\beta) - \vec{\gamma}(0)]$, since $\dot{\vec{\gamma}}(0) \in \mathcal{R}\boldsymbol{P}^+ \cap \mathcal{N}\boldsymbol{P}^-$.

Now, assumption **F4** implies that $\mathcal{N}\boldsymbol{P}^-(t) \cap \mathcal{R}\boldsymbol{P}^+(t) = \text{span}[\dot{\vec{\gamma}}(t)]$ for all $t \in \mathbb{R}$. Let us denote by $V^- := \mathcal{N}\boldsymbol{P}^- \cap [\dot{\vec{\gamma}}(0)]^\perp$, and $V^+ := \mathcal{R}\boldsymbol{P}^+ \cap [\dot{\vec{\gamma}}(0)]^\perp$. It follows that $\mathcal{N}\boldsymbol{P^-} + \mathcal{R}\boldsymbol{P^+} = V^+ \bigoplus V^- \bigoplus \mathrm{span}[\dot{\vec{\gamma}}(0)]$. We write $\vec{\eta}^-$ and $\vec{\eta}^+$ as follows: $\vec{\eta}^\pm := \vec{\eta}^\pm_\perp + \mu^\pm\dot{\vec{\gamma}}(0)$, where $\vec{\eta}^-_\perp \in V^-$ and $\vec{\eta}^+_\perp \in V^+$, respectively.

By **F4**, the adjoint variational system

$$\dot{\vec{x}} = -\boldsymbol{f_x}^*(\vec{\gamma}(t))\vec{x}$$

admits a unique solution $\vec{\psi}(t)$ which is bounded in the whole of $\mathbb{R}$, up to multiplication by a constant. Moreover $\mathrm{span}[\vec{\psi}(t)] = [\mathcal{N}\boldsymbol{P^-}(t)]^\perp \cap [\mathcal{R}\boldsymbol{P^+}(t)]^\perp$ for all $t \in \mathbb{R}$. Therefore $\mathbb{R}^n = V^+ \bigoplus V^- \bigoplus \mathrm{span}[\dot{\vec{\gamma}}(0)] \bigoplus \mathrm{span}[\vec{\psi}(0)]$. Denote by $\mathbf{\Pi} : \mathbb{R}^n \to \mathbb{R}^n$ the orthogonal projection such that $\mathcal{R}\mathbf{\Pi} = V^- \bigoplus V^+$ and $\mathcal{N}\mathbf{\Pi} = \mathrm{span}[\vec{\psi}(0)] \bigoplus \mathrm{span}[\dot{\vec{\gamma}}(0)]$.

According to the above settings, we write equation (2.9) as

$$\vec{F}(\vec{\eta}^-_\perp + \mu^-\dot{\vec{\gamma}}(0), \vec{\eta}^+_\perp + \mu^+\dot{\vec{\gamma}}(0), \alpha, \varepsilon) = \vec{0},$$

and we decompose it as

$$\begin{cases} \vec{F}_0(\vec{\eta}^-_\perp + \mu^-\dot{\vec{\gamma}}(0), \vec{\eta}^+_\perp + \mu^+\dot{\vec{\gamma}}(0), \alpha, \varepsilon) = \vec{0}, \\ F_1(\vec{\eta}^-_\perp + \mu^-\dot{\vec{\gamma}}(0), \vec{\eta}^+_\perp + \mu^+\dot{\vec{\gamma}}(0), \alpha, \varepsilon) = 0, \\ F_2(\vec{\eta}^-_\perp + \mu^-\dot{\vec{\gamma}}(0), \vec{\eta}^+_\perp + \mu^+\dot{\vec{\gamma}}(0), \alpha, \varepsilon) = 0, \end{cases}$$

where

$$\vec{F}_0(\vec{\eta}^-_\perp + \mu^-\dot{\vec{\gamma}}(0), \vec{\eta}^+_\perp + \mu^+\dot{\vec{\gamma}}(0), \alpha, \varepsilon) := \mathbf{\Pi}\vec{F}(\vec{\eta}^-_\perp + \mu^-\dot{\vec{\gamma}}(0), \vec{\eta}^+_\perp + \mu^+\dot{\vec{\gamma}}(0), \alpha, \varepsilon),$$

$$F_1(\vec{\eta}^-_\perp + \mu^-\dot{\vec{\gamma}}(0), \vec{\eta}^+_\perp + \mu^+\dot{\vec{\gamma}}(0), \alpha, \varepsilon) := [\dot{\vec{\gamma}}(0)]^*\vec{F}(\vec{\eta}^-_\perp + \mu^-\dot{\vec{\gamma}}(0), \vec{\eta}^+_\perp + \mu^+\dot{\vec{\gamma}}(0), \alpha, \varepsilon),$$

$$F_2(\vec{\eta}^-_\perp + \mu^-\dot{\vec{\gamma}}(0), \vec{\eta}^+_\perp + \mu^+\dot{\vec{\gamma}}(0), \alpha, \varepsilon) := [\vec{\psi}(0)]^*\vec{F}(\vec{\eta}^-_\perp + \mu^-\dot{\vec{\gamma}}(0), \vec{\eta}^+_\perp + \mu^+\dot{\vec{\gamma}}(0), \alpha, \varepsilon).$$

Notice that

$$\vec{F}(\vec{0}, \vec{0}, \alpha, 0) = \vec{0}, \qquad \text{for any } \alpha \in \mathbb{R}.$$

Now, from Remark 2.5 it follows that, for any $\alpha \in \mathbb{R}$,

$$\frac{\boldsymbol{\partial \vec{F}_0}}{\boldsymbol{\partial \vec{\eta}^-_\perp}}(\vec{0}, \vec{0}, \alpha, 0) = -\mathbb{I}_{V^-}, \qquad \frac{\boldsymbol{\partial \vec{F}_0}}{\boldsymbol{\partial \vec{\eta}^+_\perp}}(\vec{0}, \vec{0}, \alpha, 0) = \mathbb{I}_{V^+}$$

and

$$\frac{\partial F_1}{\partial \mu^\pm}(\vec{0}, \vec{0}, \alpha, 0) = \pm\|\dot{\gamma}(0)\|^2.$$

So we can apply the implicit function theorem and we find unique smooth functions $\vec{\eta}^-_\perp = \vec{\eta}^-_\perp(\mu^-, \alpha, \varepsilon)$, $\vec{\eta}^+_\perp = \vec{\eta}^+_\perp(\mu^-, \alpha, \varepsilon)$ and $\mu^+ = \mu^+(\mu^-, \alpha, \varepsilon)$ such that: $\vec{\eta}^-_\perp(0, \alpha, 0) = \vec{0}$, $\vec{\eta}^+_\perp(0, \alpha, 0) = \vec{0}$,

$$\vec{F}_0(\vec{\eta}^-_\perp(\mu^-, \alpha, \varepsilon) + \mu^-\dot{\vec{\gamma}}(0), \vec{\eta}^+_\perp(\mu^-, \alpha, \varepsilon) + \mu^+(\mu^-, \alpha, \varepsilon)\dot{\vec{\gamma}}(0), \alpha, \varepsilon) = \vec{0},$$

$$F_1(\vec{\eta}^-_\perp(\mu^-, \alpha, \varepsilon) + \mu^-\dot{\vec{\gamma}}(0), \vec{\eta}^+_\perp(\mu^-, \alpha, \varepsilon) + \mu^+(\mu^-, \alpha, \varepsilon)\dot{\vec{\gamma}}(0), \alpha, \varepsilon) = 0.$$

Thus

$$\tilde{F}_2(\mu^-,\alpha,\varepsilon) := F_2(\vec{\eta}_\perp^-(\mu^-,\alpha,\varepsilon)+\mu^-\dot{\vec{\gamma}}(0), \vec{\eta}_\perp^+(\mu^-,\alpha,\varepsilon)+\mu^+(\mu^-,\alpha,\varepsilon)\dot{\vec{\gamma}}(0),\alpha,\varepsilon)$$

becomes the new bifurcation function for our problem. We want to apply again the implicit function theorem in order to find $\alpha=\alpha(\mu^-,\varepsilon)$ such that $\tilde{F}_2(\mu^-,\alpha(\mu^-,\varepsilon),\varepsilon)=0$, for μ^-,ε small.

We claim that $\tilde{F}_2(\mu^-,\alpha,0)=0$ for any $\alpha\in\mathbb{R}$ and μ^- small. Indeed, recall that, for any $\alpha\in\mathbb{R}$ fixed and $(\vec{\eta}^-,\vec{\eta}^+)\in M_0$, we have $\vec{F}(\vec{\eta}^-,\vec{\eta}^+,\alpha,0)=\vec{0}$. On the other hand, for μ^- small, the pair

$$\begin{cases} \vec{\eta}^- = \vec{\eta}_\perp^-(\mu^-,\alpha,0)+\mu^-\dot{\vec{\gamma}}(0) \\ \vec{\eta}^+ = \vec{\eta}_\perp^+(\mu^-,\alpha,0)+\mu^+(\mu^-,\alpha,0)\dot{\vec{\gamma}}(0) \end{cases}$$

which describes a smooth curve N_0 in $\mathcal{N}\boldsymbol{P}^-\times\mathcal{R}\boldsymbol{P}^+$, solves the system

$$\begin{cases} \vec{F}_0(\vec{\eta}^-,\vec{\eta}^+,\alpha,0)=\vec{0} \\ F_1(\vec{\eta}^-,\vec{\eta}^+,\alpha,0)=0. \end{cases}$$

Now, by (2.10), such a system is solved by $(\vec{\eta}^-,\vec{\eta}^+)\in M_0$ as well. Moreover, both M_0 and N_0 are smooth curves in $\mathcal{N}\boldsymbol{P}^-\times\mathcal{R}\boldsymbol{P}^+$ and have a point in common (the origin), so they must coincide because of uniqueness. Consequently, we have $F_2(\vec{\eta}^-,\vec{\eta}^+,\alpha,0)=0$ for any $\alpha\in\mathbb{R}$ and $(\vec{\eta}^-,\vec{\eta}^+)\in N_0$, and this proves the claim.

So we define the smooth function

$$\Delta(\mu^-,\alpha,\varepsilon)=\begin{cases} \frac{\tilde{F}_2(\mu^-,\alpha,\varepsilon)}{\varepsilon} & \text{if } \varepsilon\neq 0 \\ \frac{\partial\tilde{F}_2}{\partial\varepsilon}(\mu^-,\alpha,0) & \text{if } \varepsilon=0. \end{cases}$$

In view of Remark 2.5, since $\vec{\psi}^*(t)=\vec{\psi}^*\boldsymbol{X}^{-1}(t)$, $\vec{\psi}^*(\mathbb{I}-\boldsymbol{P}^-)=0$, and $\vec{\psi}^*\boldsymbol{P}^+=0$ we find

$$-\Delta(\mu^-,\alpha,0)=M(\alpha)=\int_{-\infty}^{+\infty}\vec{\psi}^*(t)\vec{g}(t+\alpha,\vec{\gamma}(t),0)dt$$

$$-\frac{\partial\Delta}{\partial\alpha}(\mu^-,\alpha,0)=M'(\alpha)=\int_{-\infty}^{+\infty}\vec{\psi}^*(t)\frac{\partial\vec{g}}{\partial t}(t+\alpha,\vec{\gamma}(t),0)dt$$

(notice in fact that the above integrals do not depend on μ^-).

Assume now, as in Theorem 2.3, that α_0 is a simple zero for $\Delta(\mu^-,\alpha,0)$. Then we can apply the implicit function theorem again to find a C^{r-1} function $\alpha=\alpha(\mu^-,\varepsilon)$ such that $\alpha(\mu^-,0)=\alpha_0$ and $\Delta(\mu^-,\alpha(\mu^-,\varepsilon),\varepsilon)=0$ for any (μ^-,ε) small enough.

So we get

$$\vec{F}(\vec{\eta}_\perp^-(\mu^-,\varepsilon)+\mu^-\dot{\vec{\gamma}}(0),\vec{\eta}_\perp^+(\mu^-,\varepsilon)+\mu^+(\mu^-,\varepsilon)\dot{\vec{\gamma}}(0),\alpha(\mu^-,\varepsilon),\varepsilon)=\vec{0},$$

for any (μ^-,ε) sufficiently small. That is,

$$\vec{F}(\vec{\eta}^-(\mu^-,\varepsilon),\vec{\eta}^+(\mu^-,\varepsilon),\alpha(\mu^-,\varepsilon),\varepsilon)=\vec{0}\,,$$

for (μ^-,ε) small. This allows to define the homoclinic trajectory $x_b(t) = \vec{x}_b(t;\mu^-,\varepsilon)$ by

$$x_b(t) = \begin{cases} \vec{x}^-(t+\alpha(\mu^-,\varepsilon), \vec{\eta}^-(\mu^-,\varepsilon), \alpha(\mu^-,\varepsilon), \varepsilon) & \text{if } t \geq 0 \\ \vec{x}^+(t+\alpha(\mu^-,\varepsilon), \vec{\eta}^+(\mu^-,\varepsilon), \alpha(\mu^-,\varepsilon), \varepsilon) & \text{if } t \leq 0. \end{cases}$$

Finally, let us discuss briefly the dependence on μ^- of such a homoclinic. When $\varepsilon = 0$, arguing as above, we find that for μ^- small the pair

$$\begin{cases} \vec{\eta}^- = \vec{\eta}_\perp^-(\mu^-,0) + \mu^-\dot{\vec{\gamma}}(0) \\ \vec{\eta}^+ = \vec{\eta}_\perp^+(\mu^-,0) + \mu^+(\mu^-,0)\dot{\vec{\gamma}}(0) \end{cases}$$

describes a smooth curve in $\mathcal{N}\boldsymbol{P}^- \times \mathcal{R}\boldsymbol{P}^+$ which solves $\vec{F}(\vec{\eta}^-, \vec{\eta}^+, \alpha_0, 0) = \vec{0}$. Thus, again this curve must coincide with M_0. Hence, for any β small enough there exists $\mu^- = \mu^-(\beta)$ such that

$$\vec{\eta}_\perp^-(\mu^-(\beta),0) + \mu^-\dot{\vec{\gamma}}(0) = (\mathbb{I} - \boldsymbol{P}^-)[\vec{\gamma}(\beta) - \vec{\gamma}(0)].$$

Observe that this equality allows to estimate $\mu^-(\beta)$ for β small. In fact we get $\frac{d\mu}{d\beta}(0) = 1$, that is, $\mu^- = \beta + o(\beta)$ for $\beta \to 0$.

Analogously, for $\varepsilon \neq 0$ small, the pair

$$\begin{cases} \vec{\eta}^- = \vec{\eta}_\perp^-(\mu^-,\varepsilon) + \mu^-\dot{\vec{\gamma}}(0) \\ \vec{\eta}^+ = \vec{\eta}_\perp^+(\mu^-,\varepsilon) + \mu^+(\mu^-,\varepsilon)\dot{\vec{\gamma}}(0) \end{cases}$$

describes, for for μ^- small, a smooth curve M_ε in $\mathcal{N}\boldsymbol{P}^- \times \mathcal{R}\boldsymbol{P}^+$, defined in a neighborhood of $(\vec{0},\vec{0})$ and close to M_0, which again is a sort of "matching curve". Roughly speaking, we see that the t translation term α is constant along M_0 but it changes as μ^- varies along M_ε.

In fact we have obtained a whole one parameter family of homoclinic trajectories, satisfying

$$\sup_{t\in\mathbb{R}} \|\vec{x}_b(t+\alpha(\mu^-,\varepsilon);\mu^-,\varepsilon) - \gamma(t)\| \leq C(|\mu^-| + |\varepsilon|)$$

for a certain $C > 0$. Then, if we set $\alpha(0,\varepsilon) = \alpha(\varepsilon)$ and $\vec{x}_b(t,\varepsilon) = \vec{x}_b(t,0,\varepsilon)$ the estimate (2.5) follows and Theorem 2.3 is proved. We stress that $\vec{x}_b(t+\tau,\varepsilon)$ is a homoclinic solution of (2.1) for any $\tau \in \mathbb{R}$. Thus, exploiting the uniqueness of M_ε we see that the function $\tau \to \vec{x}_b(\tau,\varepsilon)$ gives a further parameterization of M_ε. Therefore if $\vec{w}^+ = \boldsymbol{P}^+[\vec{x}_b(\tau+\alpha(\varepsilon),\varepsilon) - \gamma(0)]$ and $\vec{w}^- = (\mathbb{I} - \boldsymbol{P}^-)[\vec{x}_b(\tau + \alpha(\varepsilon),\varepsilon) - \gamma(0)]$, then there is μ^- such that $\tau + \alpha(\varepsilon) = \alpha(\mu^-,\varepsilon)$, $\vec{\eta}^-(\mu^-,\varepsilon) = \vec{w}^-$ and $\vec{\eta}^+(\mu^-,\varepsilon) = \vec{w}^+$.

2.2 The result in the continuous case

In this section we give sufficient conditions in order to guarantee that the homoclinic solution $\vec{x}_b(t,\varepsilon)$, whose existence is given by Theorem 2.3, lies in Ω^+

for any $t \in \mathbb{R}$. We assume that $\vec{\gamma}(t) \in \Omega^+$ for any $t \in \mathbb{R}$. Thus, when $|t|$ belongs to a bounded interval we can use a continuity argument and we easily get that $\vec{x}_b(t;\varepsilon) \in \Omega^+$; but we need further hypotheses and a better insight to achieve the same conclusion as $|t| \to +\infty$. More precisely we will assume that condition **G0** holds; the idea is to state a result analogous to Lemma 2.4 and to prove it using the Banach fixed point theorem in a suitable space of exponentially bounded functions.

For this purpose we introduce the Banach spaces

$$X^- = \{\vec{z} \in C((-\infty,0],\mathbb{R}^n) \mid \sup_{t\le 0}[\|\vec{z}(t)\|e^{-\lambda_u t}] < \infty\}$$
$$X^+ = \{\vec{z} \in C([0,+\infty),\mathbb{R}^n) \mid \sup_{t\ge 0}[\|\vec{z}(t)\|e^{-\lambda_s t}] < \infty\}$$

endowed respectively with the norms

$$\|\vec{z}\|_{X^-} = \sup_{t\le 0}[\|\vec{z}(t)\|e^{-\lambda_u t}] \quad \text{and} \quad \|\vec{z}\|_{X^+} = \sup_{t\ge 0}[\|\vec{z}(t)\|e^{-\lambda_s t}].$$

Observe that $\vec{\gamma}(t)$ and $\dot{\vec{\gamma}}(t)$ belong to both X^- and X^+, moreover $\vec{\gamma}(t)$ assumes the direction of v_u as $t \to -\infty$ and of v_s as $t \to +\infty$, while by **F3** $\dot{\vec{\gamma}}(t)$ assumes the direction of v_u as $t \to -\infty$ and of $-v_s$ as $t \to +\infty$. Consequently, for any $\delta > 0$, there are positive constants C, C_u, C_s and $T = T(\delta)$ such that

$$\begin{aligned} &\max\{\|\vec{\gamma}\|_{X^-}, \|\vec{\gamma}\|_{X^+}, \|\dot{\vec{\gamma}}\|_{X^-}, \|\dot{\vec{\gamma}}\|_{X^+}\} \le C \\ &\|\vec{\gamma}(t)e^{-\lambda_u t} - C_u \vec{v}_u\| \le \delta \quad \text{for } t < -T \\ &\|\vec{\gamma}(t)e^{-\lambda_s t} - C_s \vec{v}_s\| \le \delta \quad \text{for } t > T \end{aligned} \tag{2.11}$$

Let us set, for any $\rho < 2C$,

$$\Delta_f(\rho) := \sup\{\|\boldsymbol{f_x}(\vec{x_1}) - \boldsymbol{f_x}(\vec{x_2})\| \mid \|\vec{x_1}\|, \|\vec{x_2}\| \le \rho\}$$
$$\Delta_g(\rho) := \sup\{\|\boldsymbol{g_x}(t,\vec{x_1},\varepsilon) - \boldsymbol{g_x}(t,\vec{x_2},\varepsilon)\| \mid \|\vec{x_1}\|, \|\vec{x_2}\| \le \rho, t \in \mathbb{R}, |\varepsilon| < \varepsilon_0\}.$$

Since $\vec{f}$ and $\vec{g}$ are C^r, there are $M_0, N_0 > 0$ such that $\Delta_f(\rho) \le M_0\rho$, and $\Delta_g(\rho) \le N_0\rho$.

We point out that it is possible to prove our results under weaker regularity assumptions (see Remark 2.7 below). However, for the sake of simplicity we always assume that $\vec{f}$ and $\vec{g}$ are of class C^r_b, $r \ge 2$.

To get our result the first step is to prove the following lemma analogous to Lemma 2.4.

Lemma 2.6. *Assume* **G0** *and* **F0**, **F1**, **F3**. *Then for any* $\alpha \in \mathbb{R}$ *there are positive constants* η_0, ε_0, $c(\eta_0,\varepsilon_0)$ *such that, for any* $\vec{\eta}^- \in \mathcal{N}\boldsymbol{P}^-$, $\vec{\eta}^+ \in \mathcal{R}\boldsymbol{P}^+$, $\|\vec{\eta}^-\| + \|\vec{\eta}^+\| \le \eta_0$, *and* $0 < |\varepsilon| < \varepsilon_0$, *there are* C^r*-solutions* $\vec{\tilde{x}}(t,\vec{\eta}^-,\alpha,\varepsilon)$ *defined for* $t \le \alpha$, *and* $\vec{\tilde{x}}(t,\vec{\eta}^+,\alpha,\varepsilon)$ *defined for* $t \ge \alpha$ *of (2.1) satisfying*

$$(\mathbb{I} - \boldsymbol{P}^-)[\vec{\tilde{x}}^-(\alpha,\vec{\eta}^-,\alpha,\varepsilon) - \vec{\gamma}(0)] = \vec{\eta}^-$$
$$\boldsymbol{P}^+[\vec{\tilde{x}}^+(\alpha,\vec{\eta}^+,\alpha,\varepsilon) - \vec{\gamma}(0)] = \vec{\eta}^+$$

and

$$\begin{array}{l}\sup_{t\le 0}\|\vec{\tilde{x}}(t+\alpha;\vec{\eta}^-,\alpha,\varepsilon)-\vec{\gamma}(t)\|e^{-\lambda_u t}\le c[\|\vec{\eta}^-\|+|\varepsilon|]\\ \sup_{t\ge 0}\|\vec{\tilde{x}}(t+\alpha;\vec{\eta}^+,\alpha,\varepsilon)-\vec{\gamma}(t)\|e^{-\lambda_s t}\le c[\|\vec{\eta}^+\|+|\varepsilon|]\end{array} \tag{2.12}$$

Moreover these are the only exponentially bounded solution of (2.1), i.e. if $\vec{x}^-(t;\varepsilon)$ *and* $\vec{x}^+(t;\varepsilon)$ *solve (2.1) and are exponentially bounded respectively for* $t\le\alpha$ *and for* $t\ge\alpha$*, then* $\vec{x}^-(t;\varepsilon)=\vec{\tilde{x}}(t,\vec{\eta}^-,\alpha,\varepsilon)$ *and* $\vec{x}^+(t;\varepsilon)=\vec{\tilde{x}}(t,\vec{\eta}^+,\alpha,\varepsilon)$ *where* $(\mathbb{I}-\boldsymbol{P}^-)[\vec{x}^-(\alpha,\varepsilon)-\vec{\gamma}(0)]=\vec{\eta}^-$ *and* $\boldsymbol{P}^+[\vec{x}^+(\alpha,\varepsilon)-\vec{\gamma}(0)]=\vec{\eta}^+$*.*

Proof. We just prove the lemma for $\vec{\tilde{x}}(t,\vec{\eta}^-,\alpha,\varepsilon)$, i.e. for $\alpha\in\mathbb{R}$ fixed and $t\le\alpha$, the case $\vec{\tilde{x}}(t,\vec{\eta}^+,\alpha,\varepsilon)$ being analogous.

Let $\vec{z}(t)=\tilde{x}(t+\alpha)-\vec{\gamma}(t)$, and let $\tilde{\mathcal{T}}$ be the restriction to the space X^- of the operator $\mathcal{T}$ defined in (2.8). We want to apply the Banach fixed point theorem to the operator $\tilde{\mathcal{T}}$.

Let ρ be fixed with $0<\rho<C$. Let us prove that $\tilde{\mathcal{T}}$ is a contraction on the ball centered at the origin of radius ρ, uniformly in $\alpha,\varepsilon,\vec{\eta}^-$ whenever $0<|\varepsilon|<\varepsilon_0$ and $\|\vec{\eta}^-\|\le\eta_0$ with:

$$4k[\lambda_u]^{-1}(M_0\rho+4N_0\varepsilon_0C^2)<1,\qquad 4k[\lambda_u]^{-1}(\eta_0+4N_0\varepsilon_0C^2)<\rho.$$

First notice that, given $\vec{z}\in X^-$ with $\|\vec{z}\|_{X^-}\le\rho$, for any $t\le 0$ we have

$$\begin{array}{l}\|\vec{g}(t+\alpha,\vec{z}(t)+\vec{\gamma}(t),\varepsilon)\|\le\\ \le\displaystyle\int_0^1\left\|\boldsymbol{g_x}(t+\alpha,\theta(\vec{z}(t)+\vec{\gamma}(t)),\varepsilon)[\vec{z}(t)+\vec{\gamma}(t)]\right\|d\theta\le\\ \le N_0e^{2\lambda_u t}\|\vec{\gamma}(t)+\vec{z}(t)\|^2_{X^-}\le 2N_0(C^2+\rho\|\vec{z}\|_{X^-})e^{2\lambda_u t}.\end{array} \tag{2.13}$$

Thus, it is easy to check that

$$\begin{array}{l}\|\vec{H}_0(t;\vec{z}(t),\alpha,\varepsilon)\|\le e^{2\lambda_u t}\big(M_0\rho\|\vec{z}\|_{X^-}+4N_0C^2|\varepsilon|\big)\qquad\text{and}\\ \|\vec{H}_0(t;\vec{z}_2(t),\alpha,\varepsilon)-\vec{H}_0(t;\vec{z}_1(t),\alpha,\varepsilon)\|\le e^{2\lambda_u t}\big(M_0\rho+4N_0C^2|\varepsilon|\big)\|\vec{z}_2-\vec{z}_1\|_{X^-}.\end{array}$$

Using these estimates together with (2.3) on (2.8) we find

$$\begin{array}{l}\|\tilde{\mathcal{T}}(\vec{z})(t)\|\le ke^{\lambda_u t}\left[\|\vec{\eta}^-\|+\big(M_0\rho\|\vec{z}\|_{X^-}+4N_0C^2|\varepsilon|\big)\Big(\dfrac{e^{\lambda_u t}}{2\lambda_u+|\lambda_s|}+\dfrac{1}{\lambda_u}\Big)\right]\\ \|\tilde{\mathcal{T}}(\vec{z}_2)(t)-\tilde{\mathcal{T}}(\vec{z}_1)(t)\|\le\\ \le ke^{\lambda_u t}\big(M_0\rho+4N_0C^2|\varepsilon|\big)\Big(\dfrac{e^{\lambda_u t}}{2\lambda_u+|\lambda_s|}+\dfrac{1}{\lambda_u}\Big)\|\vec{z}_2-\vec{z}_1\|_{X^-}.\end{array}$$

Thus,

$$\begin{array}{l}\|\tilde{\mathcal{T}}(\vec{z})\|_{X^-}\le\dfrac{2k}{\lambda_u}\left[\eta_0+4N_0C^2\varepsilon_0+M_0\rho\|\vec{z}\|_{X^-}\right]\\ \|\tilde{\mathcal{T}}(\vec{z}_2)-\tilde{\mathcal{T}}(\vec{z}_1)\|_{X^-}\le\dfrac{2k}{\lambda_u}\left[M_0\rho+4N_0C^2\varepsilon_0\right]\|\vec{z}_2-\vec{z}_1\|_{X^-}.\end{array}$$

Therefore $\tilde{\mathcal{T}}$ maps the ball of X^- centered at the origin with radius ρ in itself, and it is a contraction with factor of contraction $1/2$. By the Banach fixed point theorem, $\tilde{\mathcal{T}}$ has a unique fixed point $\tilde{z} = \tilde{z}(\cdot, \vec{\eta}^-, \alpha, \varepsilon)$ with $\|\tilde{z}\|_{X^-} \leq \rho$.

To prove the estimate

$$\|\tilde{z}(\cdot, \vec{\eta}^-, \alpha, \varepsilon)\|_{X^-} \leq c(\|\vec{\eta}^-\| + |\varepsilon|)$$

observe that, since $\tilde{z} = \tilde{\mathcal{T}}(\tilde{z})$, we have

$$\begin{aligned}\|\tilde{z}\|_{X^-} = \|\tilde{\mathcal{T}}(\tilde{z})\|_{X^-} &\leq \frac{2k}{\lambda_u}\left(\|\vec{\eta}^-\| + 4N_0C^2|\varepsilon|\right) + \frac{2k}{\lambda_u}M_0\rho\|\tilde{z}\|_{X^-} \leq \\ &\leq \frac{2k}{\lambda_u}\left(\|\vec{\eta}^-\| + 4N_0C^2|\varepsilon|\right) + \frac{1}{2}\|\tilde{z}\|_{X^-}.\end{aligned}$$

Hence

$$\|\tilde{z}\|_{X^-} \leq \frac{4k}{\lambda_u}\left(\|\vec{\eta}^-\| + 4N_0C^2|\varepsilon|\right).$$

This completes the proof. ■

Remark 2.7. As already pointed out, Lemma 2.6 as well as Theorem 2.9 below still hold under weaker regularity assumptions. More precisely, to prove Lemma 2.6 it is sufficient to assume that $\vec{f}$ and $\vec{g}$ are C^1, and that their derivatives with respect to the $\vec{x}$ variable are Hölder continuous, uniformly for any $t \in \mathbb{R}$ and any $\varepsilon \in \mathbb{R}$. For instance, if we assume Hölder continuity with exponent $0 < \varsigma < 1$, we get that there are constants M_1, N_1 such that $\Delta_f(\rho) \leq M_1\rho^\varsigma$, and $\Delta_g(\rho) \leq N_1\rho^\varsigma$ for any $\rho < 2C$. Therefore, formula (2.13) above is replaced by the estimate

$$\|\vec{g}(t + \alpha, \vec{z}(t) + \vec{\gamma}(t), \varepsilon)\| \leq N_1(C + \rho)^{(1+\varsigma)}e^{\lambda_u(1+\varsigma)t}$$

and the proof of Lemma 2.6 can be modified accordingly.

Since the fixed point of $\tilde{\mathcal{T}}$ and $\mathcal{T}$ is unique, we have the following:

Remark 2.8. Each solution $\vec{x}(t, \vec{\eta}_\pm, \alpha, \varepsilon)$, constructed through Lemma 2.4, satisfies (2.12) so it is in fact exponentially bounded.

In view of the above remark, we get that $\vec{x}_b(t + \alpha; \varepsilon) - \vec{\gamma}(t)$ is exponentially bounded both in the past and in the future. Moreover we recall that in order to construct the homoclinic trajectory $\vec{x}_b(t; \varepsilon)$ through Theorem 2.3, the solutions $\vec{z}(t, \vec{\eta}^-, \alpha, \varepsilon)$ and $\vec{z}(t, \vec{\eta}^+, \alpha, \varepsilon)$ are selected in such a way that $\|\vec{\eta}^-\| + \|\vec{\eta}^+\| = O(\varepsilon)$ as $\varepsilon \to 0$. Hence we can find a constant $c > 0$ such that $\|\vec{x}_b(t+\alpha; \varepsilon) - \vec{\gamma}(t)\|_{X^-} + \|\vec{x}_b(t + \alpha; \varepsilon) - \vec{\gamma}(t)\|_{X^+} \leq c|\varepsilon|$.

So, from (2.11), as $t \to -\infty$ we find the following

$$\begin{aligned}G(\vec{x}_b(t + \alpha; \varepsilon)) &= [\vec{\nabla}G(\vec{0})]^*[\vec{\gamma}(t) + \vec{z}(t)] + o(e^{\lambda_u t}) \geq \\ &\geq \left\{C_u[\vec{\nabla}G(\vec{0})]^*\vec{v}_u - c\|\vec{\nabla}G(\vec{0})\|\delta\right\}e^{\lambda_u t} + o(e^{\lambda_u t}),\end{aligned}$$

and analogously as $t \to +\infty$ we get

$$
\begin{aligned}
G(\vec{x}_b(t+\alpha;\varepsilon)) &= [\vec{\nabla}G(\vec{0})]^*[\vec{\gamma}(t)+\vec{z}(t)] + o(e^{\lambda_s t}) \geq \\
&\geq \left\{C_s[\vec{\nabla}G(\vec{0})]^*\vec{v}_s - c\|\vec{\nabla}G(\vec{0})\|\delta\right\}e^{\lambda_s t} + o(e^{\lambda_s t}).
\end{aligned}
$$

Thus, possibly choosing a larger T in (2.11), from **F2** we find $G(\vec{x}_b(t;\varepsilon)) > 0$ for $|t| > T$. Furthermore $G(\vec{\gamma}(t)) > 0$ for any $t \in \mathbb{R}$, so using a continuity argument we easily get that $G(\vec{x}_b(t;\varepsilon))$ is positive for $|t| \leq T$, too. In this way we have proved the following.

Theorem 2.9. *Assume that* **F0**, **F1**, **F2**, **F3**, **F4** *and* **G0** *are satisfied and that there is $\alpha_0 \in \mathbb{R}$ such that the function $M(\alpha)$ defined in (2.4) has a simple zero at α_0. Assume further that $\vec{\gamma}(t) \in \Omega^+$ for any $t \in \mathbb{R}$. Then, there is $\varepsilon_0 > 0$ such that for any $0 < |\varepsilon| < \varepsilon_0$ the homoclinic solution $\vec{x}_b(t;\varepsilon)$ as in the assertion of Theorem 2.3 belongs to Ω^+ for any $t \in \mathbb{R}$.*

3 The discontinuous case

In this section we consider the discontinuous equation

$$
\dot{\vec{x}} = \vec{f}^{\pm}(\vec{x}) + \varepsilon\vec{g}(t,\vec{x},\varepsilon), \quad x \in \Omega^{\pm}, \tag{3.1}
$$

where, as above, $\Omega^{\pm} = \{\vec{x} \in \Omega \mid \pm G(\vec{x}) > 0\}$ are open subsets of $\mathbb{R}^n$, the maps $\vec{f}^{\pm} \in C_b^r(\Omega^{\pm},\mathbb{R}^n)$, $\vec{g} \in C_b^r(\mathbb{R}\times\Omega\times\mathbb{R},\mathbb{R}^n)$ are given, and $\varepsilon \in \mathbb{R}$ is a small parameter.

By a *solution* of (3.1) we mean a continuous, piecewise C^1 function $\vec{x}$ that satisfies equation (3.1) on $\Omega^{\pm}$. That is, $\vec{x}$ verifies

$$
\dot{\vec{x}}(t) = \vec{f}^{-}(\vec{x}(t)) + \varepsilon\vec{g}(t,\vec{x}(t),\varepsilon), \quad \text{whenever } \vec{x}(t) \in \Omega^{-}, \tag{3.1$-$}
$$

$$
\dot{\vec{x}}(t) = \vec{f}^{+}(\vec{x}(t)) + \varepsilon\vec{g}(t,\vec{x}(t),\varepsilon), \quad \text{whenever } \vec{x}(t) \in \Omega^{+}. \tag{3.1$+$}
$$

Moreover, if for some t_0 we have that $\vec{x}(t_0)$ belongs to $\Omega^0 = \{\vec{x} \in \Omega \mid G(\vec{x}) = 0\}$, then for t in some left neighborhood of t_0, say $(t_0-\tau, t_0)$ with $\tau > 0$, we should have either $\vec{x}(t) \in \Omega^-$ or $\vec{x}(t) \in \Omega^+$. In the first case the left derivative of $\vec{x}(t)$ at $t = t_0$ has to satisfy $\dot{\vec{x}}(t_0^-) = \vec{f}^-(\vec{x}(t_0)) + \varepsilon\vec{g}(t_0,\vec{x}(t_0),\varepsilon)$; while in the second case, $\dot{\vec{x}}(t_0^-) = \vec{f}^+(\vec{x}(t_0)) + \varepsilon\vec{g}(t_0,\vec{x}(t_0),\varepsilon)$. A similar meaning it is assumed when the right derivative $\dot{\vec{x}}(t_0^+)$ is concerned. We stress that, in this paper, we do not consider solutions of equation (3.1) that belong to Ω^0 for t in some nontrivial interval.

Assume first that for $\varepsilon = 0$ equation (3.1) admits a homoclinic trajectory $\gamma(t)$ such that $\gamma(t) \in \Omega^+$ for any $t \in \mathbb{R}$. In this case we obtain the following immediate consequence of Theorem 2.9.

Corollary 3.1. *Assume that for $\varepsilon = 0$, equation (3.1) admits a homoclinic trajectory $\gamma(t)$ such that $\gamma(t) \in \Omega^+$ for any $t \in \mathbb{R}$. Assume that the maps $\vec{f}^+$ and $\vec{g}$ verify the hypotheses of Theorem 2.9. Then there is $\varepsilon_0 > 0$ such that for any $0 < |\varepsilon| < \varepsilon_0$, equation (3.1) admits a homoclinic solution $\vec{x}_b(t;\varepsilon)$ with $\vec{x}_b(t;\varepsilon) \in \Omega^+$ for any $t \in \mathbb{R}$.*

Now we turn to consider the main object of this section, i.e., we assume that for $\varepsilon = 0$ equation (3.1) admits a homoclinic trajectory $\vec{\gamma}(t)$, which in fact consists of two solutions:

$$\vec{\gamma}(t) = \begin{cases} \vec{\gamma}^-(t) & \text{if } t \le 0 \\ \vec{\gamma}^+(t) & \text{if } t \ge 0\,, \end{cases}$$

such that $\vec{\gamma}(t) \in \Omega^-$ for $t < 0$, $\vec{\gamma}(0) \in \Omega^0$ and $\vec{\gamma}(t) \in \Omega^+$ for $t > 0$, so $\vec{\gamma}(t)$ is continuous but not necessarily C^1. To get our result we need to translate for this setting the hypotheses **F0–F4**.

F0′ $\boldsymbol{f_x^-}(\vec{0})$ and $\boldsymbol{f_x^+}(\vec{0})$ have no eigenvalues with real part equal to zero.

In analogy to the previous section we denote by λ_u^- and λ_s^+ respectively the eigenvalue of $\boldsymbol{f_x^-}(\vec{0})$ with smallest positive real part, and the eigenvalue of $\boldsymbol{f_x^+}(\vec{0})$ with largest negative real part.

F1′ λ_u^- and λ_s^+ are real and simple.

We denote by $\vec{v}_u^-$ and $\vec{v}_s^+$ the corresponding normalized eigenvectors, such that $[\vec{\nabla}G(\vec{0})]^*\vec{v}_s^+ \le 0 \le [\vec{\nabla}G(\vec{0})]^*\vec{v}_u^-$.

As in Remark 2.1 we observe that our results remain valid if λ_u^- and λ_s^+ are real and semisimple, modifying the definition of $\vec{v}_u^-$ and $\vec{v}_s^+$ accordingly.

F2′ $\vec{v}_u^-$ and $\vec{v}_s^+$ are not orthogonal to $\vec{\nabla}G(\vec{0})$.

F3′ The limits $\lim_{t\to-\infty} \frac{\dot{\vec{\gamma}}^-(t)}{\|\dot{\vec{\gamma}}^-(t)\|} = \vec{v}_u^-$ and $\lim_{t\to+\infty} \frac{\dot{\vec{\gamma}}^+(t)}{\|\dot{\vec{\gamma}}^+(t)\|} = -\vec{v}_s^+$.

As we already pointed out, from the roughness of exponential dichotomies we get that the linear systems $\dot{\vec{x}} = \boldsymbol{f_x^-}(\vec{\gamma}^-(t))\vec{x}$ and $\dot{\vec{x}} = \boldsymbol{f_x^+}(\vec{\gamma}^+(t))\vec{x}$ have an exponential dichotomy on $(-\infty, 0]$ and $[0,\infty)$ respectively. Hence, there are projections $\boldsymbol{P}^{\pm} : \mathbb{R}^n \to \mathbb{R}^n$ and positive numbers $k \ge 1$ such that

$$\begin{array}{ll}
\|\boldsymbol{X}^-(t)\boldsymbol{P}^-[\boldsymbol{X}^-(s)]^{-1}\| \le k\,\mathrm{e}^{\lambda_s^-(t-s)} & \text{if } s \le t \le 0 \\
\|\boldsymbol{X}^-(t)(\mathbb{I}-\boldsymbol{P}^-)[\boldsymbol{X}^-(s)]^{-1}\| \le k\,\mathrm{e}^{\lambda_u^-(t-s)} & \text{if } t \le s \le 0 \\
\|\boldsymbol{X}^+(t)\boldsymbol{P}^+[\boldsymbol{X}^+(s)]^{-1}\| \le k\,\mathrm{e}^{\lambda_s^+(t-s)} & \text{if } 0 \le s \le t \\
\|\boldsymbol{X}^+(t)(\mathbb{I}-\boldsymbol{P}^+)[\boldsymbol{X}^+(s)]^{-1}\| \le k\,\mathrm{e}^{\lambda_u^+(t-s)} & \text{if } 0 \le t \le s
\end{array}$$

where $\boldsymbol{X}^{\pm}(t)$ are the fundamental matrices of the linear systems

$$\dot{\vec{x}} = \boldsymbol{f_x^{\pm}}(\vec{\gamma}^{\pm}(t))\vec{x}$$

respectively, such that $\boldsymbol{X}^+(0) = \boldsymbol{X}^-(0) = \mathbb{I}$.

Next, setting $\boldsymbol{P}^{\pm}(t) = \boldsymbol{X}^{\pm}(t)\boldsymbol{P}^{\pm}[\boldsymbol{X}^{\pm}(t)]^{-1}$ we have (see [18, 19])

$$\lim_{t\to\pm\infty} \|\boldsymbol{P}^{\pm}(t) - \boldsymbol{P}_{\boldsymbol{0}}^{\pm}\| = 0\,,$$

where $\boldsymbol{P}_{\boldsymbol{0}}^{\pm}$ denote the projections on the stable manifold of the discontinuous autonomous system $\dot{\vec{x}} = \boldsymbol{f}_{\boldsymbol{x}}^{\pm}(\vec{0})\vec{x}$.

Now, following [3], we translate for this context condition **F4**, and we introduce the transversality condition **F5′**:

F4′ $\dim([\vec{\nabla}G(\vec{\gamma}(0))]^{\perp} \cap \mathcal{N}\boldsymbol{P}^{-} \cap \mathcal{R}\boldsymbol{P}^{+}) = n-2$

F5′ $[\vec{\nabla}G(\vec{\gamma}(0))]^{*}\vec{f}^{-}(\vec{\gamma}(0)) > 0$ and $[\vec{\nabla}G(\vec{\gamma}(0))]^{*}\vec{f}^{+}(\vec{\gamma}(0)) > 0$.

We stress that, **F4′** is a generic condition since $\dim([\vec{\nabla}G(\vec{\gamma}(0))]^{\perp} \cap \mathcal{N}\boldsymbol{P}^{-} \cap \mathcal{R}\boldsymbol{P}^{+}) \leq n-2$. Moreover we observe that, if **F5′** holds and $n \leq 3$ then **F4′** is automatically satisfied as in the continuous case. Roughly speaking, condition **F4′** amounts to the uniqueness (up to multiplicative constants) of bounded solutions for the adjoint variational system. The geometrical meaning of **F4′** is discussed in more detail in Section 3 of [3].

The first step in the proof of our main result is to state in this context a lemma analogous to Lemmas 2.4 and 2.6 above.

Step 1. *We find solutions $\vec{x}^{-}, \vec{x}^{+}$ such that $\vec{x}^{-}(t+\alpha, \vec{\eta}^{-}, \alpha, \varepsilon) \in \Omega^{-}$ and is orbitally close to $\vec{\gamma}^{-}$ for $t \leq 0$ and $\vec{x}^{+}(t+\alpha, \vec{\eta}^{+}, \alpha, \varepsilon) \in \Omega^{+}$ and is close to $\vec{\gamma}^{+}$ for $t \geq 0$.*

In this step we use the same ideas of Section 2 to locate trajectories. We stress that this construction is not present in [3].

We sketch the argument just for $t \leq 0$, the case $t \geq 0$ being analogous. First of all we set $\vec{z}(t) = \vec{x}^{-}(t+\alpha) - \vec{\gamma}(t)$; then for $t \leq 0$, $\vec{z}(t) = \vec{z}(t, \vec{\eta}^{-}, \alpha, \varepsilon)$ solves

$$\dot{\vec{z}} = \boldsymbol{f}_{\boldsymbol{x}}^{-}(\vec{\gamma}(t))\vec{z} + \vec{H}_0^{-}(t; \vec{z}, \alpha, \varepsilon)\,, \tag{3.2}$$

where

$$\vec{H}_0^{-}(t; \vec{z}, \alpha, \varepsilon) = \vec{f}^{-}(\vec{z} + \vec{\gamma}(t)) - \vec{f}^{-}(\vec{\gamma}(t)) - \boldsymbol{f}_{\boldsymbol{x}}^{-}(\vec{\gamma}(t))\vec{z} + \varepsilon\vec{g}(t+\alpha, \vec{z} + \vec{\gamma}(t), \varepsilon)\,,$$

and the existence of $\vec{x}^{-}(\cdot, \vec{\eta}^{-}, \alpha, \varepsilon)$ is obtained rewriting (3.2) as a fixed point equation, as in Section 2.

Remark 3.2. Let us note that, due to the term $\vec{f}^{-}(\vec{z} + \vec{\gamma}(t))$, the right hand side of (3.2) is defined only when $\vec{z}(t) + \vec{\gamma}(t) \in \Omega^{-}$. In the following we will actually see that the solutions of (3.2) we consider have this property. However, for clarity, we observe the following. Let $B(\vec{x}, r)$ denote the ball centered at $\vec{x}$ of radius r. The set $\Omega^{-} \cup B(\vec{\gamma}(0), \rho_1) \cup B(\vec{0}, \rho_1)$ is an open neighborhood of the compact set $\{\vec{0}\} \cup \{\vec{\gamma}(t) \mid t \leq 0\}$. Hence $\rho > 0$ exists such that

$$U_\rho := B(\vec{0}, \rho) \cup \bigcup_{t\leq 0} B(\vec{\gamma}(t), \rho) \subset \left[\Omega^{-} \cup B(\vec{\gamma}(0), \rho_1) \cup B(\vec{0}, \rho_1)\right].$$

If ρ_1 is small enough, $\vec{f}^-$ can be extended to a C^r function on $\Omega^- \cup B(\vec{\gamma}(0), \rho_1) \cup B(\vec{0}, \rho_1)$ and hence on U_ρ. In this way the right hand side of equation (3.2) is defined for $\|\vec{z}\| < \rho$. Therefore we look for solutions of (3.2) such that $\|\vec{z}(t)\| < \rho$ for all $t \leq 0$. In other words, we can reformulate the fixed point equation and look for solutions $\vec{z} = \vec{z}(\cdot, \vec{\eta}^-, \alpha, \varepsilon))$ such that $\|\vec{z}(t, \vec{\eta}^-, \alpha, \varepsilon)\| < \rho$ for all $t \leq 0$. Later we show that $\vec{z}(t, \vec{\eta}^-, \alpha, \varepsilon) + \vec{\gamma}^-(t) \in \Omega^-$ for $t < 0$ so that $\vec{x}(\cdot, \vec{\eta}^-, \alpha, \varepsilon)$ is a solution of (3.1–), no matter of the extension of $\vec{f}$ we take.

In view of Remark 3.2, from now on we refer to (3.1–) to mean the same equation in which the function $\vec{f}^-$ is replaced by any continuous function $\vec{f}_0^-$ which extends $\vec{f}^-$ on U_ρ, and analogously for equation (3.1+). As pointed out, our argument does not depend on the extensions $\vec{f}_0^-$ and $\vec{f}_0^+$ we choose.

As in Section 2, we assume that condition **G0** holds. We consider the spaces X^+ and X^- and the functions $\vec{z}^\pm(t) = \vec{x}^\pm(t+\alpha) - \vec{\gamma}(t)$, where $\vec{x}^\pm(t) = \vec{x}^\pm(t, \vec{\eta}^\pm, \alpha, \varepsilon)$ is a solution of the perturbed problem (3.1±) while $\vec{\gamma}(t)$ is the non-smooth homoclinic solution of the unperturbed problem. We observe that $\vec{\gamma}^\pm(t)$ and $\dot{\vec{\gamma}}^\pm(t)$ belong to $X^\pm$. Then, repeating the argument of Lemma 2.6, we get the following.

Lemma 3.3. *Assume that* **G0** *and* $\mathbf{F0'}, \mathbf{F1'}, \mathbf{F3'}$ *hold. Then for any* $\alpha \in \mathbb{R}$ *there are positive constants* η_0, ε_0, c *such that, for any* $\vec{\eta}^- \in \mathcal{N}\boldsymbol{P}^-$, $\vec{\eta}^+ \in \mathcal{R}\boldsymbol{P}^+$ *with* $\|\vec{\eta}^-\| + \|\vec{\eta}^+\| \leq \eta_0$, *and* $0 < |\varepsilon| < \varepsilon_0$, *there are unique* C^r*-solutions* $\vec{x}^-(t, \vec{\eta}^-, \alpha, \varepsilon)$ *of (3.1–) defined for* $t \leq \alpha$, *and* $\vec{x}^+(t, \vec{\eta}^+, \alpha, \varepsilon)$ *defined for* $t \geq \alpha$ *of (3.1+) satisfying*

$$\begin{aligned} \sup_{t\leq 0} \|\vec{x}^-(t+\alpha; \vec{\eta}^-, \alpha, \varepsilon) - \vec{\gamma}^+(t)\| e^{-\lambda_u^- t} &\leq c[\|\vec{\eta}^-\| + |\varepsilon|] \\ \sup_{t\geq 0} \|\vec{x}^+(t+\alpha; \vec{\eta}^+, \alpha, \varepsilon) - \vec{\gamma}^-(t)\| e^{-\lambda_s^+ t} &\leq c[\|\vec{\eta}^+\| + |\varepsilon|]\,. \end{aligned}$$

Step 2. *We select the solutions* $\vec{x}^\pm(t, \vec{\eta}^\pm, \alpha, \varepsilon)$ *of Step 1 in such a way that* $\vec{x}^\pm(\alpha, \vec{\eta}^\pm, \alpha, \varepsilon) \in \Omega^0$, $\vec{x}^-(t, \vec{\eta}^-, \alpha, \varepsilon) \in \Omega^-$ *for* $t < \alpha$ *and* $\vec{x}^+(t, \vec{\eta}^+, \alpha, \varepsilon) \in \Omega^+$ *for* $t > \alpha$.

For this purpose it is enough to choose $\vec{\eta}^-$ and $\vec{\eta}^+$ in such a way that $\vec{x}^\pm(\alpha, \vec{\eta}^\pm, \alpha, \varepsilon) \in \Omega^0$, i.e., $G(\vec{x}^\pm(\alpha, \vec{\eta}^\pm, \alpha, \varepsilon)) = 0$. Then from condition $\mathbf{F5'}$ we find that the flow of (3.1) on $\vec{x}^\pm(\alpha, \vec{\eta}^\pm, \alpha, \varepsilon)$ is transversal to Ω^0 if ε is small enough, hence, from a continuity argument and the results of Step 1, we easily get that $\vec{x}^-(t, \vec{\eta}^+, \alpha, \varepsilon) \in \Omega^-$ for $t < \alpha$ and $\vec{x}^+(t, \vec{\eta}^+, \alpha, \varepsilon) \in \Omega^+$ for $t > \alpha$.

In order to apply the implicit function theorem we consider the functions $G(\vec{x}^\pm(\alpha, \vec{\eta}^\pm, \alpha, \varepsilon))$, observing that $G(\vec{x}^\pm(\alpha, \vec{0}, \alpha, 0)) = G(\vec{\gamma}(0)) = 0$. We denote by $\mathcal{S}^- = \mathcal{N}\boldsymbol{P}^- \cap [\vec{\nabla} G(\vec{\gamma}(0))]^\perp$ and by $\mathcal{S}^+ = \mathcal{R}\boldsymbol{P}^+ \cap [\vec{\nabla} G(\vec{\gamma}(0))]^\perp$, and we write $\vec{\eta}^-$ and $\vec{\eta}^+$ as $\vec{\eta}^\pm := \vec{\eta}_\perp^\pm + \mu^\pm \dot{\vec{\gamma}}^\pm(0)$ with $\vec{\eta}_\perp^- \in \mathcal{S}^-$ and $\vec{\eta}_\perp^+ \in \mathcal{S}^+$. Then, we compute the derivatives of $G(\vec{x}^\pm(\alpha, \vec{\eta}^\pm, \alpha, \varepsilon))$ in the directions $\dot{\vec{\gamma}}^\pm(0)$ at $\varepsilon = 0$ and $\vec{\eta}^\pm = \vec{0}$. For example, in the case of $G(\vec{x}^-(t, \vec{\eta}^-, \alpha, \varepsilon))$ we get

$$\begin{aligned} \frac{\partial}{\partial \vec{\eta}^-} G(\vec{x}^-(\alpha, \vec{\eta}^-, \alpha, \varepsilon))\lfloor_{(\vec{\eta}^-, \varepsilon) = (\vec{0}, 0)} &= \vec{\nabla} G(\vec{x}^-(\alpha, \vec{0}, \alpha, 0)) \frac{\partial \vec{x}^-}{\partial \vec{\eta}^-}(\alpha, \vec{0}, \alpha, 0) = \\ &= \vec{\nabla} G(\vec{\gamma}(0))(\mathbb{I} - \boldsymbol{P}^-)\,. \end{aligned}$$

Thus,

$$\left[\frac{\partial}{\partial\vec{\eta}^-}G(\vec{x}^-(\alpha,\vec{\eta}^-,\alpha,\varepsilon))\lfloor_{(\vec{\eta}^-,\varepsilon)=(\vec{0},0)}\right]^*\dot{\vec{\gamma}}^-(0)=[\vec{\nabla}G(\vec{\gamma}^-(0))]^*\vec{f}^-(\vec{\gamma}(0))>0\,.$$

Hence using the implicit function theorem (compare also Lemma 2.6 in [3]) we get the following.

Lemma 3.4. *Assume that* $\mathbf{G0}$ *and* $\mathbf{F0}', \mathbf{F1}', \mathbf{F3}', \mathbf{F5}'$ *hold. Then, for any fixed* $\alpha \in \mathbb{R}$, $\vec{\eta}_\perp^- \in \mathcal{S}^-, \vec{\eta}_\perp^+ \in \mathcal{S}^+$, ε *sufficiently small, there are unique* C^r *functions* $\mu^-(\vec{\eta}_\perp^-,\alpha,\varepsilon)$ *and* $\mu^+(\vec{\eta}_\perp^+,\alpha,\varepsilon)$ *such that*

$$\begin{aligned}G(\vec{x}^-(\alpha,\vec{\eta}_\perp^-+\mu^-(\vec{\eta}_\perp^-,\alpha,\varepsilon)\dot{\vec{\gamma}}^-(0),\alpha,\varepsilon))=0\\ G(\vec{x}^+(\alpha,\vec{\eta}_\perp^++\mu^+(\vec{\eta}_\perp^+,\alpha,\varepsilon)\dot{\vec{\gamma}}^+(0),\alpha,\varepsilon))=0\end{aligned} \tag{3.3}$$

along with a constant $c>0$ *such that*

$$\left|\mu^-(\vec{\eta}_\perp^-,\alpha,\varepsilon)\right|\le c[\|\vec{\eta}_\perp^-\|+|\varepsilon|]\,,\quad \left|\mu^+(\vec{\eta}_\perp^+,\alpha,\varepsilon)\right|\le c[\|\vec{\eta}_\perp^+\|+|\varepsilon|]\,.$$

Step 3. *We match together the solutions found at Step 2.*

The existence of the homoclinic trajectory for (3.1) amounts to the existence of a solution for the equation

$$\begin{aligned}\vec{x}^+(\alpha,\vec{\eta}_\perp^++\mu^+(\vec{\eta}_\perp^+,\alpha,\varepsilon)\dot{\vec{\gamma}}^+(0),\alpha,\varepsilon)-\\ -\vec{x}^-(\alpha,\vec{\eta}_\perp^-+\mu^-(\vec{\eta}_\perp^-,\alpha,\varepsilon)\dot{\vec{\gamma}}^-(0),\alpha,\varepsilon)=\vec{0}\,.\end{aligned} \tag{3.4}$$

We consider the orthogonal projection $\mathbf{\Theta}:\mathbb{R}^n\to\mathbb{R}^n$ such that $\mathcal{R}\mathbf{\Theta}=[\vec{\nabla}G(\vec{\gamma}(0))]^\perp$ and $\mathcal{N}\mathbf{\Theta}=\mathrm{span}[\vec{\nabla}G(\vec{\gamma}(0))]$; we set

$$\begin{aligned}\vec{F}(\vec{\eta}_\perp^-,\vec{\eta}_\perp^+,\alpha,\varepsilon):=\mathbf{\Theta}[\vec{x}^+(\alpha,\vec{\eta}_\perp^++\mu^+(\vec{\eta}_\perp^+,\alpha,\varepsilon)\dot{\vec{\gamma}}^+(0),\alpha,\varepsilon)-\\ -\vec{x}^-(\alpha,\vec{\eta}_\perp^-+\mu^-(\vec{\eta}_\perp^-,\alpha,\varepsilon)\dot{\vec{\gamma}}^-(0),\alpha,\varepsilon)].\end{aligned}$$

Since Ω^0 is a graph on $\mathcal{R}\mathbf{\Theta}$ it follows (see also Lemma 2.10 of [3]) that solving (3.4) is equivalent to solve

$$\vec{F}(\vec{\eta}_\perp^-,\vec{\eta}_\perp^+,\alpha,\varepsilon)=\vec{0}. \tag{3.5}$$

From $\mathbf{F4}'$ we know that there is $\vec{\psi}\in[\vec{\nabla}G(\vec{\gamma}(0)]^\perp$ such that $\vec{\psi}\in\left[\mathcal{S}^-\bigoplus\mathcal{S}^+\right]^\perp$, that is $\mathcal{R}\mathbf{\Theta}=\mathrm{span}[\vec{\psi}]\bigoplus\mathcal{S}^-\bigoplus\mathcal{S}^+$. We can assume w.l.o.g. that $\|\vec{\psi}\|=1$. We recall that the existence of a vector $\vec{\psi}$ such that $\mathrm{span}[\vec{\psi}]\bigoplus\left[\mathcal{S}^-+\mathcal{S}^+\right]\subset\mathcal{R}\mathbf{\Theta}$ is always ensured. However $\mathbf{F4}'$ requires that $\mathcal{S}^-\cap\mathcal{S}^+=\emptyset$ and consequently $\mathrm{span}[\vec{\psi}]\bigoplus\mathcal{S}^-\bigoplus\mathcal{S}^+=\mathcal{R}\mathbf{\Theta}$ (as in [3], in the case of multiple crossings of the discontinuity levels, $\mathbf{F4}'$ has to be replaced by more complicated transversality conditions, see e.g. $\mathbf{F4}''$ below). Then we apply a Lijapunov-Schmidt reduction method. So we introduce the projection $\mathbf{\Pi}:\mathcal{R}\mathbf{\Theta}\to\mathcal{R}\mathbf{\Theta}$ such that $\mathcal{R}\mathbf{\Pi}=$

$\mathcal{S}^- \bigoplus \mathcal{S}^+$ and $\mathcal{N}\mathbf{\Pi} = \text{span}[\vec{\psi}]$, and we rewrite equation (3.5) as the following system

$$\begin{aligned} \vec{F}_1(\vec{\eta}_\perp^-, \vec{\eta}_\perp^+, \alpha, \varepsilon) &:= \mathbf{\Pi}[\vec{F}(\vec{\eta}_\perp^-, \vec{\eta}_\perp^+, \alpha, \varepsilon)] = \vec{0}\,, \\ F_2(\vec{\eta}_\perp^-, \vec{\eta}_\perp^+, \alpha, \varepsilon) &:= \vec{\psi}^*[\vec{F}(\vec{\eta}_\perp^-, \vec{\eta}_\perp^+, \alpha, \varepsilon)] = 0. \end{aligned} \tag{3.6}$$

The next step will be to solve implicitly the first of equations (3.6). For this purpose, observe first that $\vec{F}_1(\vec{0}, \vec{0}, \alpha, 0) = \vec{0}$. Then, as in (**??**) we find, for any $\alpha \in \mathbb{R}$, $\vec{\eta}^- \in \mathcal{N}\boldsymbol{P}^-$, $\vec{\eta}^+ \in \mathcal{R}\boldsymbol{P}^+$,

$$\tfrac{\partial \vec{x}^-}{\partial \vec{\eta}^-}(\alpha, \vec{\eta}^-, \alpha, 0) = \mathbb{I} - \boldsymbol{P}^- \text{ and } \tfrac{\partial \vec{x}^+}{\partial \vec{\eta}^+}(\alpha, \vec{\eta}^+, \alpha, 0) = \boldsymbol{P}^+. \tag{3.7}$$

Thus, computing the derivative of the first of equations (3.3) with respect to $\vec{\eta}_\perp^-$ at $\varepsilon = 0$ and $\vec{\eta}_\perp^- = \vec{0}$ we get, for any $\vec{v} \in \mathcal{S}^-$,

$$[\vec{\nabla} G(\vec{\gamma}(0))]^* (\mathbb{I} - \boldsymbol{P}^-) \left(\vec{v} + \left[\left[\frac{\partial \mu^-}{\partial \vec{\eta}_\perp^-}(\vec{0}, \alpha, 0) \right]^* \vec{v} \right] \dot{\vec{\gamma}}^-(0) \right) = 0\,.$$

Recalling that $\mathcal{S}^- \subset [\vec{\nabla} G(\vec{\gamma}(0))]^\perp$, since $(\mathbb{I} - \boldsymbol{P}^-)\vec{v} = \vec{v}$ and $(\mathbb{I} - \boldsymbol{P}^-)\dot{\vec{\gamma}}^-(0) = \dot{\vec{\gamma}}^-(0)$, we find

$$\left[\frac{\partial \mu^-}{\partial \vec{\eta}_\perp^-}(\vec{0}, \alpha, 0) \right]^* \vec{v} = - \frac{[\vec{\nabla} G(\vec{\gamma}(0))]^* \vec{v}}{[\vec{\nabla} G(\vec{\gamma}(0))]^* \dot{\vec{\gamma}}^-(0)} = 0$$

whenever $\vec{v} \in \mathcal{S}^-$. Similarly differentiating the second of equations (3.3) with respect to $\vec{\eta}_\perp^+$, and using the fact that $\boldsymbol{P}^+ \vec{w} = \vec{w}$ and $[\vec{\nabla} G(\vec{\gamma}(0))]^* \vec{w} = 0$ whenever $\vec{w} \in \mathcal{S}^+$ we find

$$\frac{\partial \mu^-}{\partial \vec{\eta}_\perp^-}(\vec{0}, \alpha, 0) = \vec{0} \qquad \text{and} \qquad \frac{\partial \mu^+}{\partial \vec{\eta}_\perp^+}(\vec{0}, \alpha, 0) = \vec{0}\,.$$

Consequently,

$$\tfrac{\partial \vec{F}_1}{\partial \vec{\eta}_\perp^-}(\vec{0}, \vec{0}, \alpha, 0)\vec{v} = -\mathbf{\Pi}\mathbf{\Theta}(\mathbb{I} - \boldsymbol{P}^-)\vec{v} = -\vec{v} \quad \text{for any } \vec{v} \in \mathcal{S}^-\,.$$

Hence $\tfrac{\partial \vec{F}_1}{\partial \vec{\eta}_\perp^-}(\vec{0}, \vec{0}, \alpha, 0) = -\mathbb{I}_{\mathcal{S}^-}$, and analogously $\tfrac{\partial \vec{F}_1}{\partial \vec{\eta}_\perp^+}(\vec{0}, \vec{0}, \alpha, 0) = \mathbb{I}_{\mathcal{S}^+}$. So for any fixed $\alpha \in \mathbb{R}$ and ε small enough, we can apply the implicit function theorem and prove the existence of smooth functions $\vec{\eta}_\perp^- = \vec{\eta}_\perp^-(\alpha, \varepsilon)$ and $\vec{\eta}_\perp^+ = \vec{\eta}_\perp^+(\alpha, \varepsilon)$ such that $\vec{\eta}_\perp^-(\alpha, 0) = \vec{0}$, $\vec{\eta}_\perp^+(\alpha, 0) = \vec{0}$ and $\vec{F}_1(\vec{\eta}_\perp^-(\alpha, \varepsilon), \vec{\eta}_\perp^+(\alpha, \varepsilon), \alpha, \varepsilon) = \vec{0}$, identically. Now we have to look for the zeroes of the following bifurcation function:

$$\tilde{F}_2(\alpha, \varepsilon) = F_2(\vec{\eta}_\perp^-(\alpha, \varepsilon), \vec{\eta}_\perp^+(\alpha, \varepsilon), \alpha, \varepsilon)\,.$$

Before going on, it is convenient to compute also the partial derivatives of the functions $\mu^\pm$ with respect to ε.

Arguing as in Remark 2.5 we find

$$\begin{aligned} \frac{\partial \vec{x}^-}{\partial \varepsilon}(\alpha;\vec{0},\alpha,0) &= \int_{-\infty}^{0} \boldsymbol{P}^-(\boldsymbol{X}^-(s))^{-1}\vec{g}(s+\alpha,\vec{\gamma}^-(s),0)ds \\ \frac{\partial \vec{x}^+}{\partial \varepsilon}(\alpha;\vec{0},\alpha,0) &= -\int_{0}^{+\infty} (\mathbb{I}-\boldsymbol{P}^+)(\boldsymbol{X}^+(s))^{-1}\vec{g}(s+\alpha,\vec{\gamma}^+(s),0)ds \end{aligned} \tag{3.8}$$

Differentiating the first of equations (3.3) with respect to ε we find

$$[\vec{\nabla}G(\vec{\gamma}(0))]^* \left[\frac{\boldsymbol{\partial x^-}}{\boldsymbol{\partial \vec{\eta}^-}}(\alpha;\vec{0},\alpha,0)\frac{\partial \mu^-}{\partial \varepsilon}(\vec{0},\alpha,0)\dot{\vec{\gamma}}^-(0) + \frac{\partial \vec{x}^-}{\partial \varepsilon}(\alpha;\vec{0},\alpha,0)\right] = 0 \tag{3.9}$$

So using the fact that $\frac{\boldsymbol{\partial \vec{x}^-}}{\boldsymbol{\partial \vec{\eta}^-}}(\alpha;\vec{0},\alpha,0)\dot{\vec{\gamma}}^-(0) = (\mathbb{I}-\boldsymbol{P}^-)\dot{\vec{\gamma}}^-(0) = \dot{\vec{\gamma}}^-(0)$ we get

$$\frac{\partial \mu^+}{\partial \varepsilon}(\vec{0},\alpha,0) = \frac{[\vec{\nabla}G(\vec{\gamma}(0))]^* \int_0^{+\infty} (\mathbb{I}-\boldsymbol{P}^+)(\boldsymbol{X}^+(s))^{-1}\vec{g}(s+\alpha,\vec{\gamma}^+(s),0)ds}{[\vec{\nabla}G(\vec{\gamma}(0))]^*\dot{\vec{\gamma}}^+(0)}\,, \tag{3.10}$$

In an analogous way, differentiating with respect to ε the second equation in (3.3), and using the fact that $\frac{\boldsymbol{\partial \vec{x}^+}}{\boldsymbol{\partial \vec{\eta}^+}}(\alpha;\vec{0},\alpha,0)\dot{\vec{\gamma}}^+(0) = \boldsymbol{P}^+\dot{\vec{\gamma}}^+(0) = \dot{\vec{\gamma}}^+(0)$ we obtain

$$\frac{\partial \mu^-}{\partial \varepsilon}(\vec{0},\alpha,0) = -\frac{[\vec{\nabla}G(\vec{\gamma}(0))]^* \int_{-\infty}^{0} \boldsymbol{P}^-(\boldsymbol{X}^-(s))^{-1}\vec{g}(s+\alpha,\vec{\gamma}^-(s),0)ds}{[\vec{\nabla}G(\vec{\gamma}(0))]^*\dot{\vec{\gamma}}^-(0)} \tag{3.11}$$

Now, we go back to the bifurcation function $\tilde{F}_2(\alpha,\varepsilon)$. We observe that $\tilde{F}_2(\alpha,0)=0$ for any $\alpha\in\mathbb{R}$, so we consider the function:

$$\Delta(\alpha,\varepsilon) = \begin{cases} \frac{\tilde{F}_2(\alpha,\varepsilon)}{\varepsilon} & \text{if } \varepsilon \neq 0 \\ \frac{\partial \tilde{F}_2(\alpha,0)}{\partial \varepsilon} & \text{if } \varepsilon = 0. \end{cases}$$

Our final step is to obtain a standard form for the Mel'nilov function.

Remark 3.5. We stress that if systems (3.1–) and (3.1+) are C^r in the $\vec{x}$ variable and C^s in the ε variable, $r\geq 1$ and $s\geq 2$, then the functions $\vec{\eta}_{\perp}^-(\alpha,\varepsilon)$ and $\vec{\eta}_{\perp}^+(\alpha,\varepsilon)$ are C^r in the α variable and C^s in the ε variable. On the other hand, Δ inherits the regularity of $\tilde{F}_2$ in the α variable, so it is C^r in α and it is C^{s-1} in ε. If we find an α_0 such that $\Delta(\alpha_0,0) = \frac{\partial \tilde{F}_2(\alpha_0,0)}{\partial \varepsilon} = 0$ but $\frac{\partial \Delta}{\partial \alpha}(\alpha_0,0) = \frac{\partial^2 \tilde{F}_2(\alpha_0,0)}{\partial \alpha \partial \varepsilon} \neq 0$, we can apply the implicit function theorem and we find a (unique) C^r function $\alpha(\varepsilon)$ such that $\alpha(0)=\alpha_0$ and $\tilde{F}_2(\alpha(\varepsilon),\varepsilon)=0$ whenever ε is small enough, if $r\geq s-1$.

Remark 3.6. Till now α has been a fixed parameter. More precisely we have fixed $\alpha\in\mathbb{R}$ and then we have found ε_0 and η_0 (depending on α) for which we have applied the implicit function theorem. In fact ε_0 and η_0 are uniformly positive as long as $|\alpha|$ is finite but they tend to 0 as $|\alpha|\to+\infty$. From now on we let α vary; however we work with values of α bounded, say $\alpha\in[\alpha_0-1,\alpha_0+1]$, so that ε_0 and η_0 are uniformly positive and may be regarded as positive constants (depending on α_0 which will be fixed later).

In order to compute the derivative of $\tilde{F}_2$ with respect to ε we introduce the following maps. We define $\boldsymbol{R}^+, \boldsymbol{R}^- : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\boldsymbol{R}^+ \vec{w} = \vec{w} - \frac{[\vec{\nabla} G(\vec{\gamma}(0))]^* \vec{w}}{[\vec{\nabla} G(\vec{\gamma}(0))]^* \dot{\vec{\gamma}}^+(0)} \dot{\vec{\gamma}}^+(0), \quad \boldsymbol{R}^- \vec{w} = \vec{w} - \frac{[\vec{\nabla} G(\vec{\gamma}(0))]^* \vec{w}}{[\vec{\nabla} G(\vec{\gamma}(0))]^* \dot{\vec{\gamma}}^-(0)} \dot{\vec{\gamma}}^-(0).$$

Observe that $\boldsymbol{R}^+$, and $\boldsymbol{R}^-$ are projections with $\mathcal{R}\boldsymbol{R}^\pm = \mathcal{R}\boldsymbol{\Theta}$ and $\mathcal{N}\boldsymbol{R}^\pm = \text{span}[\dot{\vec{\gamma}}^\pm(0)]$.

Let us set, for brevity, $\mu^\pm(\alpha, \varepsilon) := \mu^\pm(\vec{\eta}_\perp^\pm(\alpha, \varepsilon), \alpha, \varepsilon)$ and consider

$$\tilde{F}_2(\alpha, \varepsilon) = \vec{\psi}^* \boldsymbol{\Theta}[\vec{x}^+(\alpha; \vec{\eta}_\perp^+(\alpha, \varepsilon) + \mu^+(\alpha, \varepsilon)\dot{\vec{\gamma}}^+(0), \alpha, \varepsilon) - \\ - \vec{x}^-(\alpha; \vec{\eta}_\perp^-(\alpha, \varepsilon) + \mu^-(\alpha, \varepsilon)\dot{\vec{\gamma}}^-(0), \alpha, \varepsilon)].$$

From (3.8), we see that (3.10) and (3.11) can be rewritten using the projections $\boldsymbol{R}^\pm$ in the following way:

$$\frac{\partial \mu^\pm}{\partial \varepsilon}(\alpha, \varepsilon)\lfloor_{\varepsilon=0} \dot{\vec{\gamma}}^\pm(0) = (\boldsymbol{R}^\pm - \mathbb{I}) \frac{\partial \vec{x}^\pm}{\partial \varepsilon}(\alpha; \vec{0}, \alpha, 0). \tag{3.12}$$

Furthermore, for any (α, ε) we have $\vec{\eta}_\perp^+(\alpha, \varepsilon) \in \mathcal{R}\boldsymbol{P}^+$, $\vec{\eta}_\perp^-(\alpha, \varepsilon) \in \mathcal{N}\boldsymbol{P}^-$, and $\boldsymbol{\Theta}\vec{\eta}_\perp^\pm(\alpha, \varepsilon) = \vec{\eta}_\perp^\pm(\alpha, \varepsilon) \in \mathcal{S}^\pm \subset [\vec{\psi}]^\perp$. Thus

$$\vec{\psi}^* \boldsymbol{\Theta} \boldsymbol{P}^+ \vec{\eta}_\perp^+(\alpha, \varepsilon) = 0 \quad \text{and} \quad \vec{\psi}^* \boldsymbol{\Theta}(\mathbb{I} - \boldsymbol{P}^-) \vec{\eta}_\perp^-(\alpha, \varepsilon) = 0,$$

identically. In particular, we get

$$\vec{\psi}^* \big[\boldsymbol{\Theta} \boldsymbol{P}^+ \frac{\partial \vec{\eta}_\perp^+}{\partial \varepsilon}(\alpha, 0)\big] = 0 \quad \text{and} \quad \vec{\psi}^* \big[\boldsymbol{\Theta}(\mathbb{I} - \boldsymbol{P}^-) \frac{\partial \vec{\eta}_\perp^-}{\partial \varepsilon}(\alpha, 0)\big] = 0. \tag{3.13}$$

Now, from (3.7) we get

$$\frac{\partial}{\partial \varepsilon} \vec{\psi}^* \Big[\boldsymbol{\Theta} \vec{x}^+(\alpha; \vec{\eta}_\perp^+(\alpha, \varepsilon) + \mu^+(\alpha, \varepsilon)\dot{\vec{\gamma}}^+(0), \alpha, \varepsilon)\Big]\lfloor_{\varepsilon=0} = \\ \vec{\psi}^* \Big[\boldsymbol{\Theta}\Big(\frac{\partial \vec{x}^+}{\partial \vec{\eta}^+}(\alpha; \vec{0}, \alpha, 0) \frac{\partial \vec{\eta}^+}{\partial \varepsilon}(\alpha, 0) + \frac{\partial \vec{x}^+}{\partial \varepsilon}(\alpha; \vec{0}, \alpha, 0)\Big)\Big] = \\ \vec{\psi}^* \Big[\boldsymbol{\Theta}\Big(\boldsymbol{P}^+ \frac{\partial \vec{\eta}^+}{\partial \varepsilon}(\alpha, 0) + \frac{\partial \vec{x}^+}{\partial \varepsilon}(\alpha; \vec{0}, \alpha, 0)\Big)\Big].$$

On the other hand, (3.12) and (3.13) imply

$$\vec{\psi}^* \Big[\boldsymbol{\Theta} \boldsymbol{P}^+ \frac{\partial \vec{\eta}^+}{\partial \varepsilon}(\alpha, 0)\Big] = \vec{\psi}^* \Big[\boldsymbol{\Theta} \frac{\partial \mu^+}{\partial \varepsilon}(\alpha, 0)\dot{\vec{\gamma}}^+(0)\Big] = \\ = \vec{\psi}^* \boldsymbol{\Theta}\Big[(\boldsymbol{R}^+ - \mathbb{I}) \frac{\partial \vec{x}^+}{\partial \varepsilon}(\alpha; \vec{0}, \alpha, 0)\Big].$$

Consequently,

$$\frac{\partial}{\partial \varepsilon} \vec{\psi}^* \Big[\boldsymbol{\Theta} \vec{x}^+(\alpha; \vec{\eta}_\perp^+(\alpha, \varepsilon) + \mu^+(\alpha, \varepsilon)\dot{\vec{\gamma}}^+(0), \alpha, \varepsilon)\Big]\lfloor_{\varepsilon=0} = \\ \vec{\psi}^* \Big[\boldsymbol{\Theta} \boldsymbol{R}^+ \Big(\frac{\partial \vec{x}^+}{\partial \varepsilon}(\alpha; \vec{0}, \alpha, 0)\Big)\Big] = \vec{\psi}^* \Big[\boldsymbol{R}^+ \Big(\frac{\partial \vec{x}^+}{\partial \varepsilon}(\alpha; \vec{0}, \alpha, 0)\Big)\Big]. \tag{3.14}$$

In an analogous way we get

$$\frac{\partial}{\partial \varepsilon}\vec{\psi}^*\Big[\mathbf{\Theta}\vec{x}^-(\alpha;\vec{\eta}_\perp^-(\alpha,\varepsilon)+\mu^-(\alpha,\varepsilon)\dot{\vec{\gamma}}^-(0),\alpha,\varepsilon)\Big]\lfloor_{\varepsilon=0}= \\ \vec{\psi}^*\Big[\mathbf{\Theta}\boldsymbol{R}^-\Big(\frac{\partial\vec{x}^-}{\partial\varepsilon}(\alpha;\vec{0},\alpha,0)\Big)\Big]=\vec{\psi}^*\Big[\boldsymbol{R}^-\Big(\frac{\partial\vec{x}^-}{\partial\varepsilon}(\alpha;\vec{0},\alpha,0)\Big)\Big]. \tag{3.15}$$

Therefore, using (3.8) as well as (3.14) and (3.15) we find

$$\begin{aligned}\frac{\partial\tilde{F}_2(\alpha,0)}{\partial\varepsilon}=&\vec{\psi}^*\left[\boldsymbol{R}^+\frac{\partial\vec{x}^+}{\partial\varepsilon}(\alpha;\vec{0},\alpha,0)-\boldsymbol{R}^-\frac{\partial\vec{x}^-}{\partial\varepsilon}(\alpha;\vec{0},\alpha,0)\right]=\\ &=-\int_0^{+\infty}\vec{\psi}^*\big[\boldsymbol{R}^+[\boldsymbol{X}^+(s)]^{-1}\vec{g}(s+\alpha,\vec{\gamma}^+(s),0)\big]ds+\\ &\quad-\int_{-\infty}^0\vec{\psi}^*\big[\boldsymbol{R}^-[\boldsymbol{X}^-(s)]^{-1}\vec{g}(s+\alpha,\vec{\gamma}^-(s),0)\big]ds\,.\end{aligned} \tag{3.16}$$

Here we have used the fact that by construction

$$\vec{\psi}^*\boldsymbol{R}^+\boldsymbol{P}^+=\vec{0}\quad\text{and}\quad\vec{\psi}^*\boldsymbol{R}^-(\mathbb{I}-\boldsymbol{P}^-)=\vec{0}\,.$$

Hence, the Mel'nilov function $M(\alpha)=-\frac{\partial\tilde{F}_2}{\partial\varepsilon}(\alpha,0)$ and its derivative $M'(\alpha)=-\frac{\partial^2\tilde{F}_2}{\partial\alpha\partial\varepsilon}(\alpha,0)$ take the standard form:

$$M(\alpha)=\int_{-\infty}^{+\infty}\vec{\psi}^*(t)\vec{g}(t+\alpha,\vec{\gamma}(t),0)dt\,,\quad M'(\alpha)=\int_{-\infty}^{+\infty}\vec{\psi}^*(t)\frac{\partial\vec{g}}{\partial t}(t+\alpha,\vec{\gamma}(t),0)dt$$

if we set

$$\vec{\psi}(t)=\begin{cases}[[\boldsymbol{X}^-(t)]^{-1}]^*[\boldsymbol{R}^-]^*\vec{\psi} & \text{if } t<0\\ [[\boldsymbol{X}^+(t)]^{-1}]^*[\boldsymbol{R}^+]^*\vec{\psi} & \text{if } t\geq 0.\end{cases}$$

Summing up we have the following.

Theorem 3.7. *Assume that* $\mathbf{F0}',\mathbf{F1}',\mathbf{F2}',\mathbf{F3}',\mathbf{F4}',\mathbf{F5}'$ *and* $\mathbf{G0}$ *hold, and that there is* α_0 *such that* $M(\alpha_0)=0$ *and* $M'(\alpha_0)\neq 0$. *Then there exists* $\varepsilon_0>0$ *such that for any* $0<|\varepsilon|<\varepsilon_0$ *system (3.1) admits a unique continuous, piecewise* C^{r-1} *solution* $\vec{x}_b(t;\varepsilon)$ *bounded on* $\mathbb{R}$ *and homoclinic to the origin, and there is a* C^{r-1} *function* $\alpha(\varepsilon)$, *with* $\alpha(0)=\alpha_0$ *with the following property:*

$$\sup_{t\in\mathbb{R}}\|\vec{x}_b(t+\alpha(\varepsilon);\varepsilon)-\vec{\gamma}(t)\|\to 0\quad \text{as } \varepsilon\to 0. \tag{3.17}$$

3.1 The case of multiple crossings

We stress that our existence results can be further generalized to the case in which the unperturbed homoclinic crosses two or more discontinuity levels. Obviously computations become longer and a bit cumbersome, so we do not reproduce all of them here: the results can be obtained combining the ideas of [3] with the procedure explained in this paper, in particular Step 1 of the proof

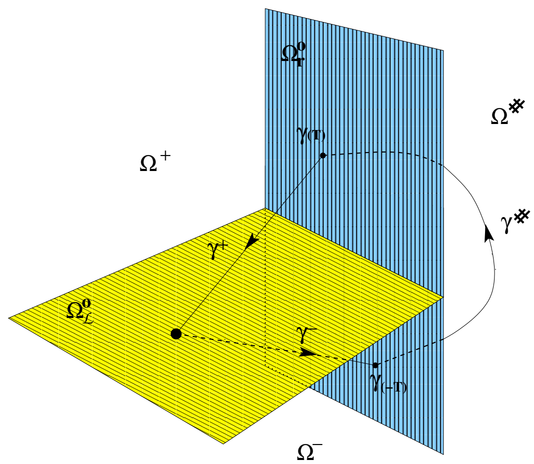

Figure 3: A sketch of the piecewise smooth homoclinic trajectory $\vec{\gamma}(t)$ defined in (3.19) of the unperturbed problem (1.1): we assume that f is as in (3.19) and $\dot{\vec{\gamma}}(t)$ has two jump discontinuities.

of Theorem 3.7, i.e. Lemma 2.6. Here we give a very brief sketch of the construction (in the case of two discontinuity surfaces) leaving the details to the interested reader.

Let $G_1, G_2 : \mathbb{R}^n \to \mathbb{R}$ be smooth functions and denote by

$$\begin{aligned} &\Omega^- := \{\vec{x} \in \mathbb{R}^n \,|\, G_1(\vec{x}) < 0 \;\&\; G_2(\vec{x}) < 0\} \quad \Omega^\# := \{\vec{x} \in \mathbb{R}^n \,|\, G_1(\vec{x}) > 0\} \\ &\Omega^+ := \{\vec{x} \in \mathbb{R}^n \,|\, G_1(\vec{x}) < 0 \;\&\; G_2(\vec{x}) > 0\} \\ &\Omega^0_l := \{\vec{x} \in \mathbb{R}^n \,|\, G_1(\vec{x}) < 0 \;\&\; G_2(\vec{x}) = 0\} \quad \Omega^0_r := \{\vec{x} \in \mathbb{R}^n \,|\, G_1(\vec{x}) = 0\} \end{aligned} \tag{3.18}$$

see figure 3, and by

$$\vec{f}(\vec{x}) := \begin{cases} \vec{f}^-(\vec{x}) & \text{if } \vec{x} \in \Omega^- \\ \vec{f}^\#(\vec{x}) & \text{if } \vec{x} \in \Omega^\# \\ \vec{f}^+(\vec{x}) & \text{if } \vec{x} \in \Omega^+ \end{cases}, \qquad \vec{\gamma}(t) := \begin{cases} \vec{\gamma}^-(t) \in \Omega^- & \text{if } t \le -T \\ \vec{\gamma}^\#(t) \in \Omega^\# & \text{if } -T \le t \le T \\ \vec{\gamma}^+(t) \in \Omega^+ & \text{if } t \ge T \end{cases} \tag{3.19}$$

where $\vec{f}^-, \vec{f}^\#, \vec{f}^+$ are C^r functions with $r \ge 1$, $\vec{\gamma}^\pm(t)$ is a solution of $\dot{\vec{x}} = \vec{f}^\pm(\vec{x})$, and $\vec{\gamma}^\#(t)$ is a solution of $\dot{\vec{x}} = \vec{f}^\#(\vec{x})$.

We assume that $\vec{0} \in \Omega^0_l$, and that $\vec{f}^-(\vec{0}) = \vec{f}^+(\vec{0}) = \vec{0}$. It follows that $\vec{\gamma}(t)$ is a homoclinic trajectory for $\dot{\vec{x}} = \vec{f}(\vec{x})$.

We consider Hypotheses **F0′**, **F1′**, **F2′**, **F3′**, and we adapt to this setting the transversality hypothesis **F5′** as follows

F5″ $[\vec{f}^-(\vec{\gamma}(-T))]^* \vec{\nabla} G_1(\vec{\gamma}(-T)) > 0$, $[\vec{f}^\#(\vec{\gamma}(-T))]^* \vec{\nabla} G_1(\vec{\gamma}(-T)) > 0$ and $[\vec{f}^\#(\vec{\gamma}(T))]^* \vec{\nabla} G_1(\vec{\gamma}(T)) < 0$, $[\vec{f}^+(\vec{\gamma}(T))]^* \vec{\nabla} G_1(\vec{\gamma}(T)) < 0$.

Hypothesis $\mathbf{F4'}$ needs more discussion (we sum up [3]). We denote by $\boldsymbol{X}^{-}$ the fundamental matrix of the variational system $\dot{\vec{x}} = \boldsymbol{f}_{\boldsymbol{x}}^{-}(\vec{\gamma}(t))\vec{x}$, defined for $t \leq -T$ and such that $\boldsymbol{X}^{-}(-T) = \mathbb{I}$; analogously we denote by $\boldsymbol{X}^{+}$ the fundamental matrix of $\dot{\vec{x}} = \boldsymbol{f}_{\boldsymbol{x}}^{+}(\vec{\gamma}(t))\vec{x}$, defined for $t > T$ and such that $\boldsymbol{X}^{+}(T) = \mathbb{I}$, and by $\boldsymbol{X}^{\#}$ the fundamental matrix of $\dot{\vec{x}} = \boldsymbol{f}_{\boldsymbol{x}}^{\#}(\vec{\gamma}(t))\vec{x}$, defined for $-T \leq t \leq T$ and such that $\boldsymbol{X}^{\#}(-T) = \mathbb{I}$. From Hypotheses $\mathbf{F0'}, \mathbf{F1'}, \mathbf{F2'}, \mathbf{F3'}$ we know that the variational systems admit an exponential dichotomy for $t \leq -T$ and $t \geq T$, with projections $\boldsymbol{P}^{-}$, $\boldsymbol{P}^{+}$ and constant k. We denote by $\mathcal{S}^{-} := \mathcal{N}\boldsymbol{P}^{-} \cap [\vec{\nabla} G_1(\vec{\gamma}(-T))]^{\perp}$ and by $V^{+} := \mathcal{R}\boldsymbol{P}^{+} \cap [\vec{\nabla} G_1(\vec{\gamma}(T))]^{\perp}$. From $\mathbf{F5''}$ it follows that $\dim(\mathcal{S}^{-}) + \dim(V^{+}) = n - 1$. Let us introduce the projections $\boldsymbol{R}^{\pm}$ from $\mathbb{R}^n$ onto $[\vec{\nabla} G_1(\vec{\gamma}(\pm T))]^{\perp}$ such that $\mathcal{N}\boldsymbol{R}^{\pm} = \mathrm{span}(\dot{\vec{\gamma}}^{\pm}(\pm T))$, the projection $\boldsymbol{R}^{\#}$ onto $[\vec{\nabla} G_1(\vec{\gamma}(T))]^{\perp}$ such that $\mathcal{N}\boldsymbol{R}^{\#} = \mathrm{span}(\dot{\vec{\gamma}}^{\#}(T))$, i.e.:

$$\boldsymbol{R}^{\pm}\vec{v} = \vec{v} - \frac{[\vec{\nabla} G_1(\vec{\gamma}(\pm T))]^{*}\vec{v}}{[\vec{\nabla} G_1(\vec{\gamma}(\pm T))]^{*}\dot{\vec{\gamma}}^{\pm}(\pm T)}\dot{\vec{\gamma}}^{\pm}(\pm T)\,,$$

and

$$\boldsymbol{R}^{\#}\vec{v} = \vec{v} - \frac{[\vec{\nabla} G_1(\vec{\gamma}(T))]^{*}\vec{v}}{[\vec{\nabla} G_1(\vec{\gamma}(T))]^{*}\dot{\vec{\gamma}}^{\#}(T)}\dot{\vec{\gamma}}^{\#}(T)$$

We denote by $V^{-} := (\boldsymbol{R}^{\#}\,\boldsymbol{X}^{\#}(T)\mathcal{S}^{-}) \cap [\vec{\nabla} G_1(\vec{\gamma}(T))]^{\perp}$: since $\boldsymbol{X}^{\#}(T)$ is invertible it follows that $V^{-} + V^{+}$ is a subset of $[\vec{\nabla} G_1(\vec{\gamma}(T))]^{\perp}$ having codimension at least 1, i.e. $\dim(V^{-} + V^{+}) \leq n - 2$. We require that $\dim(V^{-} \cap V^{+}) = 0$, so they cross transversally, or equivalently

$\mathbf{F4''}$ $\dim(V^{-} \bigoplus V^{+}) = n - 2$.

Hence, up to a multiplicative constant, there is a unique vector $\vec{\psi} \in [\vec{\nabla} G_1(\vec{\gamma}(T))]^{\perp}$ such that $\vec{\psi} \in [V^{-} \bigoplus V^{+}]^{\perp}$; so we have $[V^{-} \bigoplus V^{+} \bigoplus \mathrm{span}(\vec{\psi})] = [\vec{\nabla} G_1(\vec{\gamma}(T))]^{\perp}$. We can assume w.l.o.g. that $\|\vec{\psi}\| = 1$, and we can define the analogous of the bounded solution of the adjoint variational system for this context, i.e.,

$$\vec{\psi}(t) = \begin{cases} \left[[\boldsymbol{X}^{-}(t)]^{*}\right]^{-1}[\boldsymbol{R}^{-}]^{*}[\boldsymbol{X}^{\#}(T)]^{*}[\boldsymbol{R}^{\#}]^{*}\vec{\psi} & \text{if } t < -T \\ \left[[\boldsymbol{X}^{\#}(t)]^{*}\right]^{-1}[\boldsymbol{X}^{\#}(T)]^{*}[\boldsymbol{R}^{\#}]^{*}\vec{\psi} & \text{if } -T < t < T \\ \left[[\boldsymbol{X}^{+}(t)]^{*}\right]^{-1}[\boldsymbol{R}^{+}]^{*}\vec{\psi} & \text{if } t > T. \end{cases} \tag{3.20}$$

Finally we set

$$M(\alpha) = \int_{-\infty}^{+\infty} \vec{\psi}^{*}(t)\vec{g}(t+\alpha, \vec{\gamma}(t), 0)dt\,, \quad M'(\alpha) = \int_{-\infty}^{+\infty} \vec{\psi}^{*}(t)\frac{\partial \vec{g}}{\partial t}(t+\alpha, \vec{\gamma}(t), 0)dt$$

Now we are ready to state the generalization of Theorem 3.7 to this context.

Theorem 3.8. *Consider the perturbed system*

$$\dot{\vec{x}} = \vec{f}(\vec{x}) + \varepsilon\vec{g}(t, \vec{x}, \varepsilon)$$

with $\vec{f}$ as in (3.19), and assume that, for $\varepsilon = 0$, it admits a possibly not smooth homoclinic solution $\vec{\gamma}(t)$ as in (3.19). Assume that $\mathbf{F0}', \mathbf{F1}', \mathbf{F2}', \mathbf{F3}', \mathbf{F4}'', \mathbf{F5}''$ and $\mathbf{G0}$ hold, and that there is α_0 such that $M(\alpha_0) = 0$ and $M'(\alpha_0) \neq 0$. Then there exists $\varepsilon_0 > 0$ such that for any $0 < |\varepsilon| < \varepsilon_0$ such a system admits a unique continuous, piecewise C^{r-1} solution $\vec{x}_b(t; \varepsilon)$ bounded on $\mathbb{R}$ and homoclinic to the origin, and there is a C^{r-1} function $\alpha(\varepsilon)$, with $\alpha(0) = \alpha_0$ satisfying:

$$\sup_{t\in\mathbb{R}} \|\vec{x}_b(t + \alpha(\varepsilon); \varepsilon) - \vec{\gamma}(t)\| \to 0 \quad \textit{as } \varepsilon \to 0. \tag{3.21}$$

4 Examples

This section is devoted to the application of our results to some examples. Let us first consider the following Hamiltonian system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2a & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -qbx_1|x_1|^{q-2} \end{pmatrix} \tag{4.1}$$

where $a > 0$, $b > 0$ and $q > 2$. It is easy to check that system (4.1) admits a trajectory $\vec{\gamma}(t) = (\gamma_1(t), \gamma_2(t))$ homoclinic to the origin, given by

$$\gamma_1(t) = A(\mathrm{e}^{Ct} + \mathrm{e}^{-Ct})^{-\sqrt{2a}/C}, \qquad \gamma_2(t) = \dot{\gamma}_1(t), \tag{4.2}$$

where $C = \sqrt{2a}\frac{q-2}{2}$ and $A = (\frac{4a}{b})^{1/(q-2)}$. Observe that, due to the translation invariance in t of (4.1), such a system admits in fact a one parameter family of homoclinic trajectories, given by $\vec{\gamma}^\dagger(t, \alpha) = \vec{\gamma}(t - \alpha)$. All these solutions have the same graph, which is contained in the zero level set of the Hamiltonian

$$H(x_1, x_2) := \frac{|x_2|^2}{2} - a|x_1|^2 + b|x_1|^q .$$

Notice that $\vec{\gamma}(0) = ((a/b)^{1/(q-2)}, 0)$ and the point $\vec{\gamma}^\dagger(0, \alpha)$ moves along the level set $H(\vec{x}) = 0$. Furthermore we have also the symmetric family of homoclinic trajectories given by $-\vec{\gamma}^\dagger(0, \alpha)$, but we will not consider it in these examples.

We start from a continuous equation, so our purpose is to localize the homoclinic trajectory of the perturbed system. Let us consider the following perturbed equation:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2a & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -(1 + \varepsilon\phi(t))x_1|x_1|^{q-2} \end{pmatrix} \tag{4.3}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a smooth bounded function.

Let us point out (see also [11]) that the existence of trajectories $\vec{w}(t) = (w_1(t), w_2(t))$ of (4.3), homoclinic to the origin and satisfying $w_1(t) > 0$ for any $t \in \mathbb{R}$, is equivalent to the existence of positive radial ground states $u(\vec{y})$ with fast decay for the equation

$$\Delta u + [1 + \varepsilon\phi \ln(|\vec{y}|)]u(\vec{y})|u(\vec{y})|^{q-2} = 0, \qquad \vec{y} \in \mathbb{R}^N. \tag{4.4}$$

In fact a solution $\vec{x}(t) = (x_1(t), x_2(t))$ of (4.3) can be obtained from a radial solution $u(r)$ of (4.4) setting $x_1(t) = u(e^t)e^{\sqrt{2a}\,t}$. By a *ground state* we mean a solution u of (4.4) which is well defined and positive and tends to 0 as $|\vec{y}| \to \infty$, and we say that it has *fast decay* if $u(\vec{y})|\vec{y}|^{2-N}$ has positive finite limit as $|\vec{y}| \to \infty$. When $a = (N-2)^2/8$ and $q = 2N/(N-2)$, equation (4.4) and system (4.3) are known as *scalar curvature equation* and the existence of ground states amounts to the existence of conformal metrics having $1 + \varepsilon\phi \ln(|\vec{y}|)$ as scalar curvature. Equation (4.4) finds application in many different areas, such as quantum mechanics, and usually it is required that u is positive, or equivalently for (4.3) $x_1 > 0$, in order to have physically meaningful solutions.

Note that for $\varepsilon = 0$ system (4.3) reduces to (4.1) with $b = 1/q$. Consequently, it admits a one parameter family of homoclinic trajectories $\vec{\gamma}^\dagger(t, \alpha)$ as above.

It is not difficult to prove that $\mathbf{F0}, \mathbf{F1}, \mathbf{F2}, \mathbf{F3}$ and $\mathbf{G0}$ hold. We recall that condition $\mathbf{F4}$ (as well as $\mathbf{F4'}$) is automatically satisfied if $n \le 3$ and that the Mel'nikov integral takes a simpler standard form if $n = 2$. More precisely, if $\vec{\gamma}(t)$ is the homoclinic trajectory of the unperturbed problem (2.1) we have

$$M(\alpha) = \int_{-\infty}^{+\infty} \mathrm{e}^{\int_0^t \mathrm{tr}\boldsymbol{f_x}(\vec{\gamma}(s-\alpha))ds} \vec{f}(\vec{\gamma}(t-\alpha)) \wedge \vec{g}(\vec{\gamma}(t-\alpha))dt\,. \tag{4.5}$$

Here $\wedge$ stands for the cross product in $\mathbb{R}^2$, that is, if $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, then $\vec{a} \wedge \vec{b} = a_1 b_2 - a_2 b_1$. Then, following Lemma 6.1 in [11] we find

$$\begin{aligned} M(\alpha) &= -\int_{-\infty}^{+\infty} \phi(t)\gamma_2(t-\alpha)[\gamma_1(t-\alpha)]^{q-1}dt = \frac{1}{q}\int_{-\infty}^{+\infty} \dot{\phi}(t)[\gamma_1(t-\alpha)]^q dt \\ M'(\alpha) &= \frac{1}{q}\int_{-\infty}^{+\infty} \ddot{\phi}(t)[\gamma_1(t-\alpha)]^q dt\,. \end{aligned}$$

Thus, if there exists α_0 such that $M(\alpha_0) = 0$ and $M'(\alpha_0) \neq 0$, we can apply Theorem 2.9 and we find that, for $\varepsilon > 0$ sufficiently small, system (4.3) admits a homoclinic solution $\vec{x}(t)$ such that $x_1(t) > 0$ for any $t \in \mathbb{R}$. So, the corresponding solution $u(r)$ is positive for any $r > 0$ and it is a ground state with fast decay.

We stress that the positivity of the solution $u(r)$ of (4.4) was already proved in [11] via standard transversality and geometrical techniques. Let us point out that the 2-dimensional examples in this section are given for illustrative purposes. However, in our opinion the main advantage of our method (in the continuous case) lies in the possibility to be used in higher dimensions.

The next example shows an application of Theorem 3.7. Let us consider the discontinuous system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2\tilde{a}(x_2) & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -q\tilde{a}(x_2)x_1|x_1|^{q-2} \end{pmatrix} \tag{4.6}$$

where

$$\tilde{a}(x_2) = \begin{cases} 1 & \text{if } x_2 \ge 0 \\ 4 & \text{if } x_2 < 0 \end{cases}$$

It is easy to check that system (4.6) still admits a homoclinic trajectory $\vec{\tilde{\gamma}}(t) = (\tilde{\gamma}_1(t), \tilde{\gamma}_2(t))$ which is not C^1 (even if its graph is smooth), given by

$$\tilde{\gamma}_1(t) = A(\mathrm{e}^{Ct} + \mathrm{e}^{-Ct})^{-\sqrt{2a}/C}\,, \qquad \tilde{\gamma}_2(t) = \dot{\tilde{\gamma}}_1(t)\,, \tag{4.7}$$

where $A = 4^{1/(q-2)}$ is again a constant while

$$C = \tilde{C}(t) = \begin{cases} \frac{\sqrt{2}}{2}(q-2) & \text{if } t \geq 0 \\ \sqrt{2}(q-2) & \text{if } t < 0 \end{cases}$$

is a discontinuous function. The graph of $\vec{\tilde{\gamma}}(t)$ is contained in the set $\{\vec{x} \in \mathbb{R}^2 \mid \tilde{H}(\vec{x}) = 0\,,\ x_1 > 0\}$, where $\tilde{H}(\vec{x}) = \frac{|x_2|^2}{2} - \tilde{a}(x_2)(|x_1|^2 - |x_1|^q)$.
We perturb (4.6) by adding to the discontinuous right hand side the term $\varepsilon\vec{g}(t, \vec{x}, \varepsilon)$, where

$$\vec{g}(t, \vec{x}, \varepsilon) = (0, \phi(t)x_1|x_1|^{q-2})^*.$$

Again ϕ is smooth and bounded. It is not difficult to prove that the perturbed system verifies $\mathbf{F0'}, \mathbf{F1'}, \mathbf{F2'}, \mathbf{F3'}, \mathbf{F5'}, \mathbf{G0}$; then $\mathbf{F4'}$ follows from $\mathbf{F5'}$ and the fact that the system is 2-dimensional. Moreover the formula for the Mel'nikov function $M(\alpha)$ is again given by (4.5):

$$M(\alpha) = \frac{1}{q}\int_{-\infty}^{+\infty} \dot{\phi}(t)[\tilde{\gamma}_1(t-\alpha)]^q dt\,, \quad M'(\alpha) = \frac{1}{q}\int_{-\infty}^{+\infty} \ddot{\phi}(t)[\tilde{\gamma}_1(t-\alpha)]^q dt$$

where we have used the fact that $\vec{\tilde{\gamma}}$ is still continuous. So if α_0 is a simple zero for $M(\alpha)$ we can apply Theorem 3.7 to conclude the existence of a piecewise smooth homoclinic trajectory $\vec{x}_b(t, \varepsilon)$ for $\varepsilon > 0$ sufficiently small, and of a function $\alpha(\varepsilon)$ satisfying (3.17).

We conclude this section with a 3-dimensional discontinuous example. We stress that this is just an illustrative example. We would like also to point out that that through piecewise linear systems it is possible to obtain homoclinic trajectories which are explicitly computable, so that our perturbation arguments can be easily applied.

We consider the following piecewise linear system

$$\begin{cases} \dot{x} = a^{\pm}x \\ \dot{y} = -b^{\pm}y \\ \dot{z} = -(b^{\pm}+c^{\pm})z \end{cases} \text{for } (x,y,z) \in \Omega^{\pm}; \quad \begin{cases} \dot{x} = -z \\ \dot{y} = x+y+z-2 \\ \dot{z} = 1-y \end{cases} \text{for } (x,y,z) \in \Omega^{\#}, \tag{4.8}$$

where $a^{\pm}, b^{\pm}, c^{\pm}$ are positive constants

$$\Omega^{\#} = \{(x,y,z) \mid G_1(x,y,z) > 0\}\,, \quad \Omega^{\pm} = \{(x,y,z) \mid G_1(x,y,z) < 0,\ \pm G_2(x,y,z) > 0\},$$
$$\Omega^0_r = \{(x,y,z) \mid G_1(x,y,z) = 0\}\,, \quad \Omega^0_l = \{(x,y,z) \mid G_1(x,y,z) < 0,\ G_2(x,y,z) = 0\}\,.$$

and $G_1(x,y,z) = x+y+z-2$, $G_2(x,y,z) = -x+y$, see figure 3. It is easy to check that system (4.8) admits a piecewise smooth homoclinic trajectory $\vec{\gamma}(t)$

$$\vec{\gamma}(t) = \begin{cases} \vec{\gamma}^-(t) = (2\exp[a^-(t+\pi/2)], 0, 0) & \text{for } t \leq -\pi/2 \\ \vec{\gamma}^0(t) = (1-\sin(t), 1+\sin(t), \cos(t)) & \text{for } |t| \leq \pi/2 \\ \vec{\gamma}^+(t) = (0, 2\exp[-b^+(t-\pi/2)], 0) & \text{for } t \geq \pi/2 \end{cases}$$

Here and in the following we will adopt the same notation as in Section 3. It is straightforward to check that $\mathbf{F0}', \mathbf{F1}', \mathbf{F2}', \mathbf{F3}', \mathbf{F5}''$ are satisfied; moreover $\mathbf{F4}''$ follows from $\mathbf{F5}''$ and the fact that (4.8) has dimension less than 4.

It is easy to check that $\mathcal{R}\boldsymbol{P}^{\pm} = \mathrm{span}\{(0,1,0);(0,0,1)\}$ and $\mathcal{N}\boldsymbol{P}^{\pm} = \mathrm{span}\{(1,0,0)\}$, so that $\mathcal{S}^{-} = \mathcal{N}\boldsymbol{P}^{-} \cap [\vec{\nabla} G_1(\vec{\gamma}(-\pi/2))]^{\perp} = \{\vec{0}\}$. Therefore $V^{-} = \boldsymbol{R}^{\#}\, \boldsymbol{X}^{\#}(\pi/2)\mathcal{S}^{-} = \{\vec{0}\}$, $\mathcal{S}^{+} := (\mathcal{R}\boldsymbol{P}^{+}) \cap [\vec{\nabla} G_1(\vec{\gamma}(\pi/2))]^{\perp} = \mathrm{span}[(0,1,-1)]$, and we can choose $\vec{\psi} = \frac{1}{\sqrt{6}}(2,-1,-1)$, so that $\vec{\psi} \perp \mathcal{S}^{+}$ and $\vec{\psi} \bigoplus \mathcal{S}^{+} = [\vec{\nabla} G_1(\vec{\gamma}(\pi/2))]^{\perp}$. From a straightforward computation we find $\boldsymbol{R}^{+}(x,y,z) = (x, -x-z, z)$, $\boldsymbol{R}^{\#}(x,y,z) = (x,y,-x-y)$, $\boldsymbol{R}^{-}(x,y,z) = (-y-z, y, z)$. Further,

$$\boldsymbol{X}^{\#}(t-\frac{\pi}{2}) = \begin{pmatrix} \frac{\mathrm{e}^{t}+\cos(t)-\sin(t)}{2} & \frac{\mathrm{e}^{t}-\cos(t)-\sin(t)}{2} & -\sin(t) \\ \frac{\mathrm{e}^{t}-\cos(t)+\sin(t)}{2} & \frac{\mathrm{e}^{t}+\cos(t)+\sin(t)}{2} & \sin(t) \\ \frac{-\mathrm{e}^{t}+\cos(t)+\sin(t)}{2} & \frac{-\mathrm{e}^{t}+\cos(t)-\sin(t)}{2} & \cos(t) \end{pmatrix}$$

$$[[\boldsymbol{X}^{\#}(t-\frac{\pi}{2})]^{-1}]^{*} = \begin{pmatrix} \frac{\sin(t)+\cos(t)+\mathrm{e}^{-t}}{2} & \frac{-\sin(t)-\cos(t)+\mathrm{e}^{-t}}{2} & \frac{-\sin(t)+\cos(t)-\mathrm{e}^{-t}}{2} \\ \frac{\sin(t)-\cos(t)+\mathrm{e}^{-t}}{2} & \frac{-\sin(t)+\cos(t)+\mathrm{e}^{-t}}{2} & \frac{\sin(t)+\cos(t)-\mathrm{e}^{-t}}{2} \\ \sin(t) & -\sin(t) & \cos(t) \end{pmatrix}$$

where $\boldsymbol{X}^{\#}(t)$ is the fundamental matrix of (4.8) for $(x,y,z) \in \Omega^{\#}$ such that $\boldsymbol{X}^{\#}(-\pi/2) = \mathbb{I}$. Moreover we easily find

$$\left\{[\boldsymbol{X}^{\pm}(t)]^{-1}\right\}^{*} = \begin{pmatrix} \exp\left[-a^{\pm}(t \mp \frac{\pi}{2})\right] & 0 & 0 \\ 0 & \exp\left[b^{\pm}(t \mp \frac{\pi}{2})\right] & 0 \\ 0 & 0 & \exp\left[(b^{\pm}+c^{\pm})(t \mp \frac{\pi}{2})\right] \end{pmatrix}$$

From a straightforward computation we find $(\boldsymbol{R}^{+})^{*}\vec{\psi} = \frac{1}{\sqrt{6}}(3,0,0) = (\boldsymbol{R}^{\#})^{*}\vec{\psi}$, and

$$\psi(\tfrac{\pi}{2}^{-}) = [\boldsymbol{X}^{\#}(\pi/2)]^{*}(\boldsymbol{R}^{\#})^{*}\vec{\psi} = \tfrac{3}{2\sqrt{6}}(\mathrm{e}^{\pi}-1, \mathrm{e}^{\pi}+1, 0)$$
$$\psi(-\tfrac{\pi}{2}^{-}) = (\boldsymbol{R}^{-})^{*}[\boldsymbol{X}^{\#}(\pi/2)]^{*}(\boldsymbol{R}^{\#})^{*}\vec{\psi} = \tfrac{3}{2\sqrt{6}}(0, 2, -\mathrm{e}^{\pi}+1)$$

Therefore, using (3.20) we find

$$\frac{2\sqrt{6}}{3}\vec{\psi}(t) = \begin{cases} (0, 2\,\mathrm{e}^{b^{-}(t+\frac{\pi}{2})}, (-\mathrm{e}^{\pi}+1)\,\mathrm{e}^{(b^{-}+c^{-})(t+\frac{\pi}{2})}) & \text{if } t < -\frac{\pi}{2} \\ (\mathrm{e}^{-t+\pi/2}+\sin(t)-\cos(t),\ \mathrm{e}^{-t+\pi/2}-\sin(t)-\cos(t), -2\cos(t)) & \text{if } -\frac{\pi}{2} < t < \frac{\pi}{2} \\ (2\,\mathrm{e}^{-a^{+}(t-\frac{\pi}{2})}, 0, 0) & \text{if } t > \frac{\pi}{2} \end{cases}$$

Let us choose $\vec{g}(t,\vec{x},\varepsilon)$ such that $g_1(t,\vec{x},\varepsilon) = g_3(t,\vec{x},\varepsilon) = -g_2(t,\vec{x},\varepsilon) = \sin(t)[x_1^2+x_2^2+x_3^2] + O(\varepsilon)$; then we find

$$M(\alpha) = A\sin(\alpha) + B\cos(\alpha)$$
$$M'(\alpha) = A\cos(\alpha) - B\sin(\alpha)$$

where A, B are given by

$$A = A^- + A^0 + A^+ , \quad B = B^- + B^0 + B^+$$
$$A^- = \frac{2\sqrt{6}}{(2a^- + b^-)^2 + 1} + \frac{\sqrt{6}(\mathrm{e}^{\pi} - 1)}{(2a^- + b^- + c^-)^2 + 1}$$
$$A^0 = -\frac{13\sqrt{6}\pi}{16}$$
$$A^+ = -\frac{2\sqrt{6}}{(a^+ + 2b^+)^2 + 1}$$
$$B^- = \frac{2\sqrt{6}(2a^- + b^-)}{(2a^- + b^-)^2 + 1} + \frac{\sqrt{6}(\mathrm{e}^{\pi} - 1)(2a^- + b^- + c^-)}{(2a^- + b^- + c^-)^2 + 1}$$
$$B^0 = \frac{15\sqrt{6}\pi}{16}$$
$$B^+ = \frac{2\sqrt{6}(a^+ + 2b^+)}{(a^+ + 2b^+)^2 + 1}$$

Now, whenever $A^2 + B^2 \neq 0$ there are two values of the parameter α in $(-\pi, \pi]$, say α_0^1 and α_0^2, such that $M(\alpha_0^i + 2k\pi) = 0$ and $M'(\alpha^i + 2k\pi) \neq 0$, $i = 1, 2$, for any $k \in \mathbb{Z}$. Moreover by construction we have that $\mathbf{F0'}, \mathbf{F1'}, \mathbf{F2'}, \mathbf{F3'}, \mathbf{F4''}, \mathbf{F5''}$ and **G0** hold; thus we can apply Theorem 3.8 and we get the existence of ε_0 such that for $i = 1, 2$ and any $0 < \varepsilon < \varepsilon_0$ the piecewise linear system (4.8) admits a unique continuous, piecewise smooth solution $\vec{x}_b^i(t; k, \varepsilon)$, bounded on $\mathbb{R}$ and homoclinic to the origin, and there are smooth function $\alpha_k^i(\varepsilon)$, with $\alpha_k^i = \alpha_0^i + 2k\pi$, $i = 1, 2$, satisfying

$$\sup_{t \in \mathbb{R}} |\vec{x}_b^i(t; \varepsilon) - \vec{\gamma}(t - \alpha_k^i(\varepsilon))| \to 0 \quad \text{as } \varepsilon \to 0.$$

A Appendix

In this appendix we give the explicit proof of a roughness result in exponential dichotomies, which is probably known by the experts but for which we are not able to give a precise reference.

Let $t \mapsto \boldsymbol{A}(t)$ be a piecewise continuous $n \times n$ matrix valued function defined on $\mathbb{R}$. We recall that the linear differential equation

$$\dot{\vec{x}} = \boldsymbol{A}(t)\vec{x} \tag{A.1}$$

is said to have an *exponential dichotomy* on an interval J if there are projections $\boldsymbol{P}$ and positive constants k, α_1, α_2 such that

$$\begin{array}{ll} \|\mathbf{X}(t)\boldsymbol{P}\mathbf{X}^{-1}(s)\| \leq k\,\mathrm{e}^{-\alpha_1(t-s)} & \text{for } s, t \in J \text{ with } s \leq t \\ \|\mathbf{X}(t)(\mathbb{I} - \mathbf{P})\mathbf{X}^{-1}(s)\| \leq k\,\mathrm{e}^{-\alpha_2(s-t)} & \text{for } s, t \in J \text{ with } t \leq s \end{array} \tag{A.2}$$

where $\mathbf{X}(t)$ is the fundamental matrix of equation (A.1).

Assume that system (A.1) has an exponential dichotomy on J with constant k and exponents α_1, α_2, and let β_1, β_2 be positive constants such that $\beta_1 < \alpha_1$ and $\beta_2 < \alpha_2$. The well known *roughness property* of exponential dichotomies (see e.g. [19, p. 133]) implies that if $t \mapsto \boldsymbol{B}(t)$ is a piecewise continuous $n \times n$ matrix valued function such that $\|\boldsymbol{B}(\cdot)\|_\infty < \delta$, then for δ sufficiently small the perturbed system

$$\dot{\vec{x}} = [\boldsymbol{A}(t) + \boldsymbol{B}(t)]\vec{x} \tag{A.3}$$

has an exponential dichotomy on J with projection $\boldsymbol{Q}$, constant k' and exponents β_1, β_2. We want to show that if we further assume that the function $\boldsymbol{B}$ is L^1 then equation (A.3) has an exponential dichotomy on J with the same exponents α_1, α_2.

More precisely we will prove the following result.

Proposition A.1. Assume that system (A.1) has an exponential dichotomy on $J = [0, +\infty)$ with projection $\boldsymbol{P}$, constant k and exponents α_1, α_2. Let $\boldsymbol{B} \in L^1[0, +\infty)$ be a piecewise continuous $n \times n$ matrix valued function, and fix $\bar{T} > 0$ such that $\int_{\bar{T}}^{\infty} \|\boldsymbol{B}(\tau)\| \, d\tau < \frac{1}{k}$. Then there is δ_0 such that if $\|\boldsymbol{B}(\cdot)\|_\infty < \delta < \delta_0$ the perturbed system (A.3) has an exponential dichotomy on $[\bar{T}, +\infty)$ (and, consequently, on J) with projection $\boldsymbol{Q}$, constant k' and the same exponents α_1, α_2.

First of all we prove the following Gronwall-like lemma.

Lemma A.2. *Let $\phi : [\bar{T}, +\infty) \to \mathbb{R}$ be a bounded, continuous function such that, for $t \geq \bar{T}$,*

$$\phi(t) \leq k_1 \, \mathrm{e}^{-\alpha_1 t} + k_2 \int_{\bar{T}}^{t} \mathrm{e}^{-\alpha_1 (t-\tau)} b(\tau) \phi(\tau) \, d\tau + k_2 \int_{t}^{\infty} \mathrm{e}^{-\alpha_2 (\tau - t)} b(\tau) \phi(\tau) \, d\tau,$$

where k_1, k_2, α_1, α_2 are positive constants, $b \in L^1[0, +\infty)$ is a nonnegative piecewise continuous function, and $\bar{T} > 0$ is such that $\int_{\bar{T}}^{\infty} b(\tau) \, d\tau < \frac{1}{4k_2}$. Then, there is $c > 0$ such that

$$\phi(t) \leq c k_1 \, \mathrm{e}^{-\alpha_1 t}, \qquad t \geq \bar{T}.$$

Proof. Consider the corresponding integral equation

$$\psi(t) = k_1 \, \mathrm{e}^{-\alpha_1 t} + k_2 \int_{\bar{T}}^{t} \mathrm{e}^{-\alpha_1 (t-\tau)} b(\tau) \psi(\tau) \, d\tau + k_2 \int_{t}^{\infty} \mathrm{e}^{-\alpha_2 (\tau - t)} b(\tau) \psi(\tau) \, d\tau, \tag{A.4}$$

for $t \geq \bar{T}$. Then, any bounded continuous solution ψ is twice differentiable and is in fact a solution of the differential equation

$$\psi'' + (\alpha_1 - \alpha_2)\psi' - [\alpha_1 \alpha_2 - k_2(\alpha_1 + \alpha_2) b(t)]\psi(t) = 0 \, .$$

We have

$$\psi(t) = c_1 \, \mathrm{e}^{-\alpha_1 t} + c_2 \, \mathrm{e}^{\alpha_2 t} + k_2 \int_{\bar{T}}^{t} \left(\mathrm{e}^{-\alpha_1 (t-\tau)} - \mathrm{e}^{\alpha_2 (t-\tau)} \right) b(\tau) \psi(\tau) \, d\tau, \qquad t \geq \bar{T},$$

Since we are assuming that ψ is bounded we have

$$c_2 = k_2 \int_{\bar{T}}^{\infty} \mathrm{e}^{-\alpha_2 \tau} b(\tau)\psi(\tau)\, d\tau,$$

thus

$$\psi(t) = \left(k_2 \int_t^{\infty} \mathrm{e}^{-\alpha_2 \tau} b(\tau)\psi(\tau)\, d\tau \right) \mathrm{e}^{\alpha_2 t} + \\ + \left(c_1 + k_2 \int_{\bar{T}}^{t} \mathrm{e}^{\alpha_1 \tau} b(\tau)\psi(\tau)\, d\tau \right) \mathrm{e}^{-\alpha_1 t}, \qquad t \geq \bar{T},$$

Let us show that $\psi(t) = O(\mathrm{e}^{-\alpha_1 t})$ as $t \to +\infty$. For this purpose we define the Banach space

$$X_1 = \left\{ \psi \in C([\bar{T}, +\infty), \mathbb{R}) \mid \sup_{t \geq \bar{T}} [|\psi(t)| e^{\alpha_1 t}] < \infty \right\}$$

endowed with the norm $|\psi|_{\alpha_1} = \sup_{t \geq \bar{T}} [|\psi(t)| e^{\alpha_1 t}]$.

We define a linear operator $\mathcal{T}$ acting on X_1 as follows

$$\mathcal{T}(\psi)(t) = \left(k_2 \int_t^{\infty} \mathrm{e}^{-\alpha_2 \tau} b(\tau)\psi(\tau)\, d\tau \right) \mathrm{e}^{\alpha_2 t} + \\ + \left(c_1 + k_2 \int_{\bar{T}}^{t} \mathrm{e}^{\alpha_1 \tau} b(\tau)\psi(\tau)\, d\tau \right) \mathrm{e}^{-\alpha_1 t}, \qquad t \geq \bar{T},$$

Let us show that $\mathcal{T}$ is a contraction. We have

$$\|\mathcal{T}(\psi)(t)\| \leq \left(c_1 + 2k_2 \|b\|_{L^1([\bar{T}, +\infty))} \|\psi\|_{\alpha_1} \right) \mathrm{e}^{-\alpha_1 t}, \qquad t \geq \bar{T},$$

therefore the operator $\mathcal{T}$ is well defined. Moreover if $\psi_1, \psi_2 \in X_1$ we have

$$\|\mathcal{T}(\psi_2)(t) - \mathcal{T}(\psi_1)(t)\| \leq 2k_2 \|b\|_{L^1([\bar{T}, +\infty))} \|\psi_2 - \psi_1\|_{\alpha_1} \mathrm{e}^{-\alpha_1 t}, \qquad t \geq \bar{T}.$$

Since $2k_2 \int_{\bar{T}}^{\infty} b(\tau)\, d\tau < \frac{1}{2}$ by assumption, $\mathcal{T}$ is a contraction, hence has a unique fixed point $\bar{\psi} \in X_1$. It follows that

$$\bar{\psi}(t) \leq c k_1 \mathrm{e}^{-\alpha_1 t}, \qquad t \geq \bar{T}$$

for some constant c.

Moreover for any constant $L > \dfrac{k_1}{k_2(1 - \|b\|_{L^1([\bar{T}, +\infty))})}$ we have

$$L \geq k_1 \mathrm{e}^{-\alpha_1 t} + k_2 \int_{\bar{T}}^{t} \mathrm{e}^{-\alpha_1 (t-\tau)} b(\tau) L\, d\tau + k_2 \int_t^{\infty} \mathrm{e}^{-\alpha_2 (\tau - t)} b(\tau) L\, d\tau,$$

for any $t \geq \bar{T}$. So if we choose $L \geq \sup_{t \geq \bar{T}} \phi(t)$ by the upper lower solution method we find a solution $\tilde{\psi}(t)$ of (A.4) such that $\phi(t) \leq \tilde{\psi}(t) \leq L$ for any $t \geq \bar{T}$. Since $\bar{\psi}(t)$ is uniquely determined we find $\tilde{\psi}(t) = \bar{\psi}(t)$ and the result follows. ■

With a similar argument we prove the following lemma.

Lemma A.3. *Let k_1, k_2, α_1, α_2, b and $\bar{T}$ be as in Lemma A.2.*

1) Let $\phi : [\bar{T}, +\infty) \to \mathbb{R}$ be a bounded, continuous function such that, for all $t \geq s \geq \bar{T}$,

$$\phi(t) \leq k_1 \,\mathrm{e}^{-\alpha_1(t-s)} + k_2 \int_s^t \mathrm{e}^{-\alpha_1(t-\tau)} b(\tau)\phi(\tau)\, d\tau + k_2 \int_t^\infty \mathrm{e}^{-\alpha_2(\tau-t)} b(\tau)\phi(\tau)\, d\tau,$$

then, there is $c > 0$ such that

$$\phi(t) \leq ck_1 \,\mathrm{e}^{-\alpha_1(t-s)}, \qquad t \geq s \geq \bar{T}.$$

2) Let $\phi : [\bar{T}, +\infty) \to \mathbb{R}$ be a bounded, continuous function such that, for all $s \geq t \geq \bar{T}$,

$$\phi(t) \leq k_1 \,\mathrm{e}^{-\alpha_2(s-t)} + k_2 \int_{\bar{T}}^t \mathrm{e}^{-\alpha_1(t-\tau)} b(\tau)\phi(\tau)\, d\tau + k_2 \int_t^s \mathrm{e}^{-\alpha_2(\tau-t)} b(\tau)\phi(\tau)\, d\tau,$$

then, there is $c' > 0$ such that

$$\phi(t) \leq c'k_1 \,\mathrm{e}^{-\alpha_2(s-t)}, \qquad s \geq t \geq \bar{T}.$$

We now give the

Proof of Proposition A.1. Assume that $\delta_0 \leq \dfrac{\alpha}{4k}$.

We define a linear operator $\mathcal{T}$, acting on the Banach space of bounded continuous functions $C_b^0([\bar{T}, +\infty), \mathbb{R}^n)$ with the standard supremum norm $\| \cdot \|_\infty$. Let $\xi \in \mathbb{R}^n$ be fixed and let

$$\begin{aligned}\mathcal{T}(\vec{u})(t) =& \mathbf{X}(t)\boldsymbol{P}\vec{\xi} + \int_{\bar{T}}^t \mathbf{X}(t)\boldsymbol{P}\mathbf{X}^{-1}(\tau)\boldsymbol{B}(\tau)\vec{u}(\tau)\, d\tau - \\ &- \int_t^\infty \mathbf{X}(t)[\mathbb{I} - \mathbf{P}]\mathbf{X}^{-1}(\tau)\boldsymbol{B}(\tau)\vec{u}(\tau)\, d\tau\,, \quad t \geq \bar{T}.\end{aligned}$$

Then,

$$\begin{aligned}\|\mathcal{T}(\vec{u})(t)\| &\leq k\,\mathrm{e}^{-\alpha_1 t}\|\vec{\xi}\| + \left(\int_{\bar{T}}^t k\,\mathrm{e}^{-\alpha_1(t-\tau)}\delta\, d\tau + \int_t^\infty k\,\mathrm{e}^{-\alpha_2(\tau-t)}\delta\, d\tau\right)\|\vec{u}\|_\infty \\ &\leq k_1\|\vec{\xi}\| + \frac{2k}{\alpha}\delta\|\vec{u}\|_\infty\,, \quad t \geq \bar{T}.\end{aligned}$$

Moreover,

$$\|\mathcal{T}(\vec{u}_2)(t) - \mathcal{T}(\vec{u}_1)(t)\| \leq \frac{2k}{\alpha}\delta\|\vec{u}_2 - \vec{u}_1\|_\infty\,, \quad t \geq \bar{T}.$$

Since $\dfrac{2k}{\alpha}\delta < 1$ it follows that $\mathcal{T}$ is a contraction and hence has a unique fixed point.

Such a fixed point $\vec{\hat{u}}$ verifies, for $t \geq s \geq \bar{T}$,

$$\begin{aligned}\vec{\hat{u}}(t) =&\mathbf{X}(t)\boldsymbol{P}\mathbf{X}^{-1}(s)\vec{\hat{u}}(s) + \int_{\bar{T}}^{t} \mathbf{X}(t)\boldsymbol{P}\mathbf{X}^{-1}(\tau)\boldsymbol{B}(\tau)\vec{\hat{u}}(\tau)\, d\tau - \\ &- \int_{t}^{\infty} \mathbf{X}(t)[\mathbb{I} - \mathbf{P}]\mathbf{X}^{-1}(\tau)\boldsymbol{B}(\tau)\vec{\hat{u}}(\tau)\, d\tau\,. \end{aligned} \tag{A.5}$$

Then we have, for any $t \geq s \geq \bar{T}$,

$$\begin{aligned}\|\vec{\hat{u}}(t)\| \leq k\,\mathrm{e}^{-\alpha_1 t}\|\vec{\hat{u}}(s)\| + k\int_{\bar{T}}^{t} \mathrm{e}^{-\alpha_1(t-\tau)}\|\boldsymbol{B}(\tau)\|\|\vec{\hat{u}}(\tau)\|\, d\tau + \\ + k\int_{t}^{\infty} \mathrm{e}^{-\alpha_2(\tau-t)}\|\boldsymbol{B}(\tau)\|\|\vec{\hat{u}}(\tau)\|\, d\tau\end{aligned}$$

We apply Lemma A.3 with $\|\vec{u}(t)\|$ in place of $\phi(t)$ and $\|\boldsymbol{B}(t)\|$ in place of $b(t)$. Hence there is $c > 0$ such that

$$\|\vec{u}(t)\| \leq ck\,\mathrm{e}^{-\alpha_1(t-s)}\,\|\vec{u}(s)\|, \qquad t \geq s \geq \bar{T}.$$

Analogously, one can prove that there is $c' > 0$ such that

$$\|\vec{u}(t)\| \leq c'k\,\mathrm{e}^{-\alpha_2(s-t)}\,\|\vec{u}(s)\|, \qquad s \geq t \geq \bar{T}.$$

Following an argument similar to [19, Lemma 7.4] we obtain the estimates of the exponential dichotomy on $[\bar{T}, +\infty)$ and, consequently, on $[0, +\infty)$. ■

We recall that when $\boldsymbol{A}(t) \equiv \boldsymbol{A}$ is a constant function, we have that (A.1) admits exponential dichotomy in the whole of $\mathbb{R}$ if and only if $\boldsymbol{A}$ has no eigenvalues with real part equal to 0. Let us denote by λ_u and λ_s the eigenvalues of $\boldsymbol{A}$ respectively with smallest positive real part and with largest negative real part. If λ_u and λ_s are real and simple then the exponents of the dichotomy are exactly λ_u and λ_s and the constant is 1. Similarly if λ_u and λ_s are semisimple (as we assume in this paper), that is, $\lambda_u = a + ib$ and $\lambda_s = -c + id$ with $a, c > 0$, then the exponents of the dichotomy are again exactly $\mathrm{Re}(\lambda_u) = a$ and $\mathrm{Re}(\lambda_s) = -c$. On the other hand, if λ_u and λ_s have algebraic multiplicity larger than geometric multiplicity, then (A.1) admits exponential dichotomy in the whole of $\mathbb{R}$ but with exponents λ^+ and λ^- where $0 > \lambda^- > -\mathrm{Re}(\lambda_s)$ and $0 < \lambda^+ < \mathrm{Re}(\lambda_u)$.

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