GLOBAL CONTINUATION OF PERIODIC SOLUTIONS FOR RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS ON MANIFOLDS

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Abstract. We consider T-periodic parametrized retarded functional differential equations, with infinite delay, on (possibly) noncompact manifolds. Using a topological approach, based on the notions of degree of a tangent vector field and of the fixed point index, we prove a global continuation result for T-periodic solutions of such equations.

Our main theorem is a generalization to the case of retarded equations of a global continuation result obtained by the last two authors for ordinary differential equations on manifolds. As corollaries we obtain a Rabinowitz type global bifurcation result and a continuation principle of Mawhin type.

Dedicated to our friend and outstanding mathematician Jean Mawhin

1. Introduction

In this paper we prove a global continuation result for periodic solutions of the following *retarded functional differential equation* (*RFDE* for short) on a manifold, depending on a parameter $\lambda \geq 0$:

$$x'(t) = \lambda f(t, x_t). \tag{1.1}$$

Let us present the setting of the problem. Consider a boundaryless smooth m-dimensional manifold $M \subseteq \mathbb{R}^k$ and, given any $p \in M$, let $T_pM \subseteq \mathbb{R}^k$ stand for the tangent space of M at p. Denote by $\widetilde{M} := BU((-\infty, 0], M)$ the set of bounded and uniformly continuous maps from $(-\infty, 0]$ into M, and observe that this is a metric space, as a subset of the Banach space $\widetilde{\mathbb{R}}^k := BU((-\infty, 0], \mathbb{R}^k)$ with the usual supremum norm. Given $T > 0$, let $f\colon \mathbb{R} \times \widetilde{M} \to \mathbb{R}^k$ be a continuous function verifying the following conditions:

1. $f(t, \varphi) = f(t + T, \varphi), \ \forall\, (t, \varphi) \in \mathbb{R} \times \widetilde{M}$;
2. $f(t, \varphi) \in T_{\varphi(0)}M, \ \forall\, (t, \varphi) \in \mathbb{R} \times \widetilde{M}$;
3. f is locally Lipschitz in the second variable.

A *solution* of (1.1) is a function x with values in the *ambient manifold* M, defined on an open real interval J with $\inf J = -\infty$, bounded and uniformly continuous on any closed half-line $(-\infty, b] \subset J$, such that the equality $x'(t) = \lambda f(t, x_t)$ is eventually verified. We use here the standard notation in functional equations: whenever it makes sense, $x_t \in \widetilde{M}$ denotes the function $\theta \mapsto x(t + \theta)$.

To proceed with the exposition of our problem, we need some further notation. Given $p \in M$, p^- denotes the constant p-valued function defined on $\mathbb{R}$, or on any convenient subinterval of $\mathbb{R}$. The actual domain of p^- will be clear from the context. Moreover, given any $A \subseteq M$, A^- stands for the set $\{p^- : p \in A\}$. All the functions

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of A^- will be considered defined on the same interval, suggested by the context. By $C_T(M)$ we mean the set of all continuous T-periodic maps $x\colon \mathbb{R} \to M$. This set, which contains M^-, is a metric subspace of the Banach space $C_T(\mathbb{R}^k)$ with the standard supremum norm. We call $(\lambda, x) \in [0,+\infty) \times C_T(M)$ a *T-periodic pair* of the equation (1.1) if $x\colon \mathbb{R} \to M$ is a solution of (1.1) corresponding to λ. Among these pairs we distinguish the *trivial* ones; that is, the elements of the set $\{0\} \times M^-$, which can be isometrically identified with M. Notice that any T-periodic pair of the type $(0, x)$ is trivial, since the function x turns out to be necessarily constant. An element $p \in M$ will be called a *bifurcation point* of (1.1) if any neighborhood of $(0, p^-)$ in $[0,+\infty) \times C_T(M)$ contains nontrivial T-periodic pairs. Roughly speaking, $p \in M$ is a bifurcation point if any of its neighborhoods in M contains T-periodic orbits corresponding to arbitrarily small values of $\lambda > 0$.

The main outcome of this paper, Theorem 3.3 below, is a *global continuation result* for T-periodic solutions of the equation (1.1). That is, given an open subset Ω of $[0,+\infty) \times C_T(M)$, it is a result which provides sufficient conditions for the existence of a *global bifurcating branch in* Ω, meaning a *connected subset of* Ω *of nontrivial T-periodic pairs whose closure in* Ω *is noncompact and intersects the set of trivial T-periodic pairs*. The proof of Theorem 3.3 is based on a relation, obtained in a technical result, Lemma 3.8 below, between the degree (in an open subset of M) of the tangent vector field

$$w(p) = \frac{1}{T}\int_0^T f(t, p^-)\, dt, \quad p \in M,$$

and the fixed point index of a sort of Poincaré T-translation operator acting inside the Banach space $C([-T, 0], \mathbb{R}^k)$.

The prelude of our approach can be found in some papers of the last two authors (see for instance [9]), where the notions of degree of a tangent vector field and of fixed point index of a suitable Poincaré T-translation operator are related in order to get continuation results for ODEs on differentiable manifolds.

Theorem 3.3 extends and unifies two results recently obtained by the authors in [1] and [2]. In [1] the ambient manifold M is not necessarily compact, but our investigation regards delay differential equations with finite time lag. On the other hand, in [2] we consider RFDEs with infinite delay; nevertheless in this case M is compact and the map f is defined on $\mathbb{R} \times C((-\infty, 0], M)$ with a topology which is too weak, making the continuity assumption on f a too heavy condition.

We point out that, in order to obtain our continuation result for RFDEs with infinite delay without assuming the compactness of the ambient manifold M, we had to tackle strong technical difficulties. Therefore, we were forced to undertake a thorough preliminary investigation on the general properties of RFDEs with infinite delay on (possibly) noncompact manifolds. This was the purpose of our recent paper [3].

In our opinion the existence of a global bifurcating branch ensured by Theorem 3.3 should hold also without the assumption that f is locally Lipschitz in the second variable. However, we are not able to prove or disprove this conjecture, because of some difficulties arising in this case. One is that the uniqueness of the initial value problem for the equation (1.1) is not ensured and, consequently, a Poincaré T-translation operator is not defined as a single valued map. A classical tool to overcome this obstacle, and usually applied in analogous problems, consists in considering a sequence of C^1 maps approximating f. In our situation, however, because of the peculiar domain of f, we do not know how to realize this approach, and this is another difficulty.

We conclude the paper with some consequences of Theorem 3.3. One is a Rabinowitz type global bifurcation result [31], obtained by assuming that the degree of the above tangent vector field w is nonzero on an open subset of M. Another corollary is deduced when M is compact: we get an existence result already proved in [4], and we extend an analogous one obtained in [2] in which the continuity assumption on f is too heavy. A third interesting case occurs when the degree of w is nonzero on a relatively compact open subset of M and suitable *a priori* bounds hold for the T-periodic orbits of the equation (1.1): in this case we obtain a continuation principle *à la* Mawhin [22, 23].

The different and related cases of RFDEs with finite delay in Euclidean spaces have been investigated by many authors. For general reference we suggest the monograph by Hale and Verduyn Lunel [17]. We refer also to the works of Gaines and Mawhin [12], Nussbaum [26, 27] and Mallet-Paret, Nussbaum and Paraskevopoulos [21]. For RFDEs with infinite delay in Euclidean spaces we recommend the article of Hale and Kato [16], the book by Hino, Murakami and Naito [18], and the more recent paper of Oliva and Rocha [30]. For RFDEs with finite delay on manifolds we suggest the papers of Oliva [28, 29]. Finally, for RFDEs with infinite delay on manifolds we cite [3].

2. Preliminaries

2.1. **Fixed point index.** We recall that a metrizable space $\mathcal{X}$ is an *absolute neighborhood retract* (ANR) if, whenever it is homeomorphically embedded as a closed subset C of a metric space $\mathcal{Y}$, there exist an open neighborhood V of C in $\mathcal{Y}$ and a retraction $r\colon V \to C$ (see e.g. [5, 14]). Polyhedra and differentiable manifolds are examples of ANRs. Let us also recall that a continuous map between topological spaces is called *locally compact* if each point in its domain has a neighborhood whose image is contained in a compact set.

Let $\mathcal{X}$ be a metric ANR and consider a locally compact (continuous) $\mathcal{X}$-valued map k defined on a subset $\mathcal{D}(k)$ of $\mathcal{X}$. Given an open subset U of $\mathcal{X}$ contained in $\mathcal{D}(k)$, if the set of fixed points of k in U is compact, the pair (k,U) is called *admissible*. We point out that such a condition is clearly satisfied if $\overline{U} \subseteq \mathcal{D}(k)$, $\overline{k(U)}$ is compact and $k(p) \neq p$ for all p in the boundary of U. To any admissible pair (k,U) one can associate an integer $\mathrm{ind}_{\mathcal{X}}(k,U)$ – the *fixed point index* of k in U – which satisfies properties analogous to those of the classical Leray–Schauder degree [20]. The reader can see for instance [6, 13, 25, 27] for a comprehensive presentation of the index theory for ANRs. As regards the connection with the homology theory we refer to standard algebraic topology textbooks (e.g. [7, 32]).

We summarize below the main properties of the fixed point index.

- (Existence) *If* $\mathrm{ind}_{\mathcal{X}}(k,U) \neq 0$*, then* k *admits at least one fixed point in* U*.*
- (Normalization) *If* $\mathcal{X}$ *is compact, then* $\mathrm{ind}_{\mathcal{X}}(k,\mathcal{X}) = \Lambda(k)$*, where* $\Lambda(k)$ *denotes the Lefschetz number of* k*.*
- (Additivity) *Given two disjoint open subsets* U_1*,* U_2 *of* U*, if any fixed point of* k *in* U *is contained in* $U_1 \cup U_2$*, then* $\mathrm{ind}_{\mathcal{X}}(k,U) = \mathrm{ind}_{\mathcal{X}}(k,U_1) + \mathrm{ind}_{\mathcal{X}}(k,U_2)$*.*
- (Excision) *Given an open subset* U_1 *of* U*, if* k *has no fixed points in* $U \backslash U_1$*, then* $\mathrm{ind}_{\mathcal{X}}(k,U) = \mathrm{ind}_{\mathcal{X}}(k,U_1)$*.*
- (Commutativity) *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be metric ANRs. Suppose that* U *and* V *are open subsets of* $\mathcal{X}$ *and* $\mathcal{Y}$ *respectively and that* $k\colon U \to \mathcal{Y}$ *and* $h\colon V \to \mathcal{X}$ *are locally compact maps. Assume that the set of fixed points of either* hk

in $k^{-1}(V)$ *or* kh *in* $h^{-1}(U)$ *is compact. Then the other set is compact as well and* $\mathrm{ind}_{\mathcal{X}}(hk, k^{-1}(V)) = \mathrm{ind}_{\mathcal{Y}}(kh, h^{-1}(U))$.

- (Generalized homotopy invariance) *Let* I *be a compact real interval and* W *an open subset of* $\mathcal{X} \times I$. *For any* $\lambda \in I$, *denote* $W_\lambda = \{x \in \mathcal{X} : (x, \lambda) \in W\}$. *Let* $H \colon W \to \mathcal{X}$ *be a locally compact map such that the set* $\{(x, \lambda) \in W : H(x, \lambda) = x\}$ *is compact. Then* $\mathrm{ind}_{\mathcal{X}}(H(\cdot, \lambda), W_\lambda)$ *is independent of* λ.

2.2. **Degree of a vector field.** Let us recall some basic notions on degree theory for tangent vector fields on differentiable manifolds. Let $v \colon M \to \mathbb{R}^k$ be a continuous (autonomous) tangent vector field on a smooth manifold M, and let U be an open subset of M. We say that the pair (v, U) is *admissible* (or, equivalently, that v is admissible in U) if $v^{-1}(0) \cap U$ is compact. In this case one can assign to the pair (v, U) an integer, $\deg(v, U)$, called the *degree* (or *Euler characteristic*, or *rotation*) of the tangent vector field v in U which, roughly speaking, counts algebraically the number of zeros of v in U (for general references see e.g. [15, 19, 24, 33]). Notice that the condition for $v^{-1}(0) \cap U$ to be compact is clearly satisfied if U is a relatively compact open subset of M and $v(p) \neq 0$ for all p in the boundary of U.

As a consequence of the Poincaré–Hopf theorem, when M is compact, $\deg(v, M)$ equals $\chi(M)$, the Euler–Poincaré characteristic of M.

In the particular case when U is an open subset of $\mathbb{R}^k$, $\deg(v, U)$ is just the classical Brouwer degree of v in U when the map v is regarded as a vector field; namely, the degree $\deg(v, U, 0)$ of v in U with *target value* $0 \in \mathbb{R}^k$. All the standard properties of the Brouwer degree in the flat case, such as homotopy invariance, excision, additivity, existence, still hold in the more general context of differentiable manifolds. To see this, one can use an equivalent definition of degree of a tangent vector field based on the fixed point index theory as presented in [9] and [10].

Let us stress that, actually, in [9] and [10] the definition of degree of a tangent vector field on M is given in terms of the fixed point index of a Poincaré-type translation operator associated to a suitable ODE on M. Such a definition provides a formula that will play a central role in Lemma 3.8 below, and this will be a crucial step in the proof of our main result.

We point out that no orientability of M is required for $\deg(v, U)$ to be defined. This highlights the fact that the extension of the Brouwer degree for tangent vector fields in the non-flat case does not coincide with the one regarding maps between oriented manifolds with a given target value (as illustrated, for example, in [19, 24]). This dichotomy of the notion of degree in the non- flat situation is not evident in $\mathbb{R}^k$: it is masked by the fact that an equation of the type $f(x) = y$ can be written as $f(x) - y = 0$. Anyhow, in the context of RFDEs (ODEs included), it is the degree of a vector field that plays a significative role.

It is known that, if (v, U) is admissible, then

$$\deg(v, U) = (-1)^m \deg(-v, U), \tag{2.1}$$

where m denotes the dimension of M. Moreover, if v has an isolated zero p and U is an isolating (open) neighborhood of p, then $\deg(v, U)$ is called the *index* of v at p. The excision property ensures that this is a well-defined integer.

2.3. **Retarded functional differential equations.** Given an arbitrary subset A of $\mathbb{R}^k$, we denote by $BU((-\infty, 0], A)$ the set of bounded and uniformly continuous maps from $(-\infty, 0]$ into A. For brevity, we will use the notation

$$\widetilde{A} := BU((-\infty, 0], A).$$

Notice that $\widetilde{\mathbb{R}}^k$ is a Banach space, being closed in the space $BC((-\infty,0],\mathbb{R}^k)$ of the bounded and continuous functions from $(-\infty,0]$ into $\mathbb{R}^k$ (endowed with the standard supremum norm).

Throughout the paper, the norm in $\mathbb{R}^k$ will be denoted by $|\cdot|$ and the norm in the infinite dimensional space $\widetilde{\mathbb{R}}^k$ by $\|\cdot\|$. Thus, the distance between two elements ϕ and ψ of $\widetilde{A}$ will be denoted $\|\phi-\psi\|$, even when $\phi-\psi$ does not belong to $\widetilde{A}$. We observe that $\widetilde{A}$, as a metric space, is complete if and only if A is closed in $\mathbb{R}^k$.

Let M be a boundaryless smooth manifold in $\mathbb{R}^k$. A continuous map

$$g\colon \mathbb{R}\times\widetilde{M}\to\mathbb{R}^k$$

is said to be a *retarded functional tangent vector field over M* if $g(t,\varphi)\in T_{\varphi(0)}M$ for all $(t,\varphi)\in\mathbb{R}\times\widetilde{M}$. In the sequel, any map with this property will be briefly called a *functional field (over M)*.

Let us consider a *retarded functional differential equation* (*RFDE*) of the type

$$x'(t)=g(t,x_t), \tag{2.2}$$

where $g\colon\mathbb{R}\times\widetilde{M}\to\mathbb{R}^k$ is a functional field over M. Here, as usual and whenever it makes sense, given $t\in\mathbb{R}$, by $x_t\in\widetilde{M}$ we mean the function $\theta\mapsto x(t+\theta)$.

A *solution* of (2.2) is a function $x\colon J\to M$, defined on an open real interval J with $\inf J=-\infty$, bounded and uniformly continuous on any closed half-line $(-\infty,b]\subset J$, and which verifies eventually the equality $x'(t)=g(t,x_t)$. That is, $x\colon J\to M$ is a solution of (2.2) if $x_t\in\widetilde{M}$ for all $t\in J$ and there exists $\tau\in J$ such that x is C^1 on the interval $(\tau,\sup J)$ and $x'(t)=g(t,x_t)$ for all $t\in(\tau,\sup J)$. Observe that the derivative of a solution x may not exist at $t=\tau$. However, the right derivative $D_+x(\tau)$ of x at τ always exists and is equal to $g(\tau,x_\tau)$. Also, notice that $t\mapsto x_t$ is a continuous curve in $\widetilde{M}$, since x is uniformly continuous on any closed half-line $(-\infty,b]$ of J.

A solution of (2.2) is said to be *maximal* if it is not a proper restriction of another solution. As in the case of ODEs, Zorn's lemma implies that any solution is the restriction of a maximal solution.

Given $\eta\in\widetilde{M}$, let us associate to equation (2.2) the initial value problem

$$\begin{cases} x'(t)=g(t,x_t),\\ x_0=\eta\,. \end{cases} \tag{2.3}$$

A *solution* of (2.3) is a solution $x\colon J\to M$ of (2.2) such that $\sup J>0$, $x'(t)=g(t,x_t)$ for $t>0$, and $x_0=\eta$.

The continuous dependence of the solutions on initial data is stated in Theorem 2.1 below and is a staightforward consequence of Theorem 4.4 of [3].

Theorem 2.1. *Let M be a boundaryless smooth manifold and $g\colon\mathbb{R}\times\widetilde{M}\to\mathbb{R}^k$ a functional field. Assume, for any $\eta\in\widetilde{M}$, the uniqueness of the maximal solution of problem* (2.3)*. Then, given $T>0$, the set*

$$\mathcal{D}=\big\{\eta\in\widetilde{M}:\ \textit{the maximal solution of (2.3) is defined up to } T\big\}$$

is open and the map $\eta\in\mathcal{D}\mapsto x_T^\eta\in\widetilde{M}$, where $x^\eta(\cdot)$ is the unique maximal solution of problem (2.3)*, is continuous.*

More generally, we will need to consider initial value problems depending on a parameter, such as the equation (1.1) with the initial condition $x_0=\eta$. For these problems the continuous dependence is ensured by the following consequence of Theorem 2.1.

Corollary 2.2 (continuous dependence)**.** *Let M be a boundaryless smooth manifold and $h\colon \mathbb{R}^s \times \mathbb{R} \times \widetilde{M} \to \mathbb{R}^k$ a parametrized functional field. For any $\alpha \in \mathbb{R}^s$ and $\eta \in \widetilde{M}$, assume the uniqueness of the maximal solution of the problem*

$$\begin{cases} x'(t) = h(\alpha, t, x_t), \\ x_0 = \eta\,. \end{cases} \tag{2.4}$$

Then, given $T > 0$, the set

$$\mathcal{D}' = \big\{(\alpha,\eta) \in \mathbb{R}^s \times \widetilde{M} :\ \text{the maximal solution of } (2.4) \text{ is defined up to } T\big\}$$

is open and the map $(\alpha,\eta) \in \mathcal{D}' \mapsto x_T^{(\alpha,\eta)} \in \widetilde{M}$, where $x^{(\alpha,\eta)}(\cdot)$ is the unique maximal solution of problem (2.4), is continuous.

Proof. Apply Theorem 2.1 to the problem

$$\begin{cases} \big(\beta'(t), x'(t)\big) = \big(0, h(\beta(t), t, x_t)\big), \\ \big(\beta(0), x_0\big) = \big(\alpha, \eta\big), \end{cases}$$

that can be regarded as an initial value problem of a RFDE on the ambient manifold $\mathbb{R}^s \times M \subseteq \mathbb{R}^{s+k}$. □

In Theorem 2.1 and in Corollary 2.2 above the hypothesis of the uniqueness of the maximal solution of problems (2.3) and (2.4) is essential in order to make their statements meaningful. Sufficient conditions for the uniqueness are presented in Remark 2.3 below.

Remark 2.3. A functional field $g\colon \mathbb{R} \times \widetilde{M} \to \mathbb{R}^k$ is said to be *compactly Lipschitz* (for short, *c-Lipschitz*) if, given any compact subset Q of $\mathbb{R} \times \widetilde{M}$, there exists $L \geq 0$ such that

$$|g(t,\varphi) - g(t,\psi)| \leq L\|\varphi - \psi\|$$

for all $(t,\varphi), (t,\psi) \in Q$. Moreover, we will say that g is *locally c-Lipschitz* if for any $(\tau,\eta) \in \mathbb{R} \times \widetilde{M}$ there exists an open neighborhood of (τ,η) in which g is c-Lipschitz. In spite of the fact that a locally Lipschitz map is not necessarily (globally) Lipschitz, one could actually show that if g is locally c-Lipschitz, then it is also (globally) c-Lipschitz. As a consequence, if g is locally Lipschitz in the second variable, then it is c-Lipschitz as well. In [3] we proved that, if g is a c-Lipschitz functional field, then problem (2.3) has a unique maximal solution for any $\eta \in \widetilde{M}$. For a characterisation of compact subsets of $\widetilde{M}$ see e.g. [8, Part 1, IV.6.5].

We close this section with the following lemma whose elementary proof is given for the sake of completeness.

Lemma 2.4. *Let $F\colon \mathcal{X} \to \mathcal{Y}$ be a continuous map between metric spaces and let $\{\gamma_n\}$ be a sequence of continuous functions from a compact interval $[a,b]$ (or, more generally, from a compact space) into $\mathcal{X}$. If $\{\gamma_n(s)\}$ converges to $\gamma(s)$ uniformly for $s \in [a,b]$, then also $F(\gamma_n(s)) \to F(\gamma(s))$ uniformly for $s \in [a,b]$.*

Proof. Notice that, if K is a compact subset of $\mathcal{X}$, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $x \in \mathcal{X}$, $k \in K$, $\mathrm{dist}_{\mathcal{X}}(x,k) < \delta$ imply $\mathrm{dist}_{\mathcal{Y}}(F(x),F(k)) < \varepsilon$. Now, our assertion follows immediately by taking the compact K to be the image of the limit function $\gamma\colon [a,b] \to \mathcal{X}$. □

3. Branches of periodic solutions

Let M be a boundaryless smooth m-dimensional manifold in $\mathbb{R}^k$. Given $T > 0$, let

$$\widehat{M} := C([-T, 0], M)$$

denote the metric subspace of $C([-T, 0], \mathbb{R}^k)$ of the M-valued continuous functions on $[-T, 0]$ and set

$$\widehat{M}_* := \big\{\psi \in \widehat{M} : \psi(-T) = \psi(0)\big\}.$$

Moreover, denote by $C_T(\mathbb{R}^k)$ the Banach space of the continuous T-periodic maps $x\colon \mathbb{R} \to \mathbb{R}^k$ (with the standard supremum norm) and by $C_T(M)$ the metric subspace of $C_T(\mathbb{R}^k)$ of the M-valued maps. Observe that, since M is locally compact, then $\widehat{M}$ and $C_T(M)$ (but not $\widetilde{M}$) are locally complete. Moreover, they are complete if and only if M is closed.

Let $f\colon \mathbb{R} \times \widetilde{M} \to \mathbb{R}^k$ be a functional field over M. Given $T > 0$, assume that f is T-periodic in the first variable. Consider the following RFDE depending on a parameter $\lambda \geq 0$:

$$x'(t) = \lambda f(t, x_t). \tag{3.1}$$

As in the introduction, we call $(\lambda, x) \in [0, +\infty) \times C_T(M)$ a *T-periodic pair* (of (3.1)) if the function $x\colon \mathbb{R} \to M$ is a (T-periodic) solution of (3.1) corresponding to λ. Let us denote by X the set of all T-periodic pairs of (3.1). Lemma 3.1 below states some properties of X that will be used in the sequel.

Lemma 3.1. *The set X is closed in $[0, +\infty) \times C_T(M)$ and locally compact.*

Proof. Let $\big\{(\lambda^n, x^n)\big\}$ be a sequence of T-periodic pairs of (3.1) converging to (λ^0, x^0) in $[0, +\infty) \times C_T(M)$. Because of Lemma 2.4, $f(t, x_t^n)$ converges uniformly to $f(t, x_t^0)$ for $t \in \mathbb{R}$. Thus, $(x^n)'(t) = \lambda^n f(t, x_t^n) \to \lambda^0 f(t, x_t^0)$ uniformly and, therefore, $(x^0)'(t) = \lambda^0 f(t, x_t^0)$, that is (λ^0, x^0) belongs to X. This proves that X is closed in $[0, +\infty) \times C_T(M)$.

Now, as observed above, $C_T(M)$ is locally complete. Consequently X is locally complete as well, as a closed subset of a locally complete space. Moreover, by using Ascoli's theorem, we get that it is actually a locally compact space. □

We recall that, given $p \in M$, with the notation p^- we mean the constant p-valued function defined on some real interval that will be clear from the context. Moreover, a T-periodic pair of the type $(0, p^-)$ is said to be *trivial*, and an element $p \in M$ is a *bifurcation point* of the equation (3.1) if any neighborhood of $(0, p^-)$ in $[0, +\infty) \times C_T(M)$ contains a nontrivial T-periodic pair (i.e. a T-periodic pair (λ, x) with $\lambda > 0$). In some sense, p is a bifurcation point if, for $\lambda > 0$ sufficiently small, there are T-periodic orbits of (3.1) arbitrarily close to p.

In the sequel, we are interested in the existence of branches of nontrivial T-periodic pairs that, roughly speaking, emanate from a trivial pair $(0, p^-)$, with p a bifurcation point of (3.1). To this end, we introduce the mean value tangent vector field $w : M \to \mathbb{R}^k$ given by

$$w(p) = \frac{1}{T} \int_0^T f(t, p^-)\, dt. \tag{3.2}$$

Throughout the paper w will play a crucial role in obtaining our continuation results for (3.1). First, in Theorem 3.2 below, we provide a necessary condition for $p \in M$ to be a bifurcation point.

Theorem 3.2. *Let $x \in C_T(M)$ be such that $(0, x)$ is an accumulation point of nontrivial T-periodic pairs of (3.1). Then, there exists $p \in M$ such that $x(t) = p$, for any $t \in \mathbb{R}$, and $w(p) = 0$. Thus, any bifurcation point of (3.1) is a zero of w.*

Proof. By assumption there exists a sequence $\{(\lambda^n, x^n)\}$ of T-periodic pairs of (3.1) such that $\lambda^n > 0$, $\lambda^n \to 0$, and $x^n(t) \to x(t)$ uniformly on $\mathbb{R}$. As proved in Lemma 3.1, the set X of the T-periodic pairs is closed in $[0,+\infty) \times C_T(M)$. Thus, the pair $(0,x)$ belongs to X and, consequently, the function x must be constant, say $x = p^-$ for some $p \in M$. Clearly, the point p is a bifurcation point of (3.1).

Now, given $n \in \mathbb{N}$, recalling that $x^n(T) = x^n(0)$ and that $\lambda^n \neq 0$, we get

$$\int_0^T f(t, x^n_t)\, dt = 0.$$

Observe that the sequence of curves $t \mapsto (t, x^n_t) \in \mathbb{R} \times \widetilde{M}$ converges uniformly to $t \mapsto (t, p^-)$ for $t \in [0,T]$. Hence, because of Lemma 2.4, $f(t, x^n_t) \to f(t, p^-)$ uniformly for $t \in [0,T]$ and the assertion follows passing to the limit in the above integral. □

Let now Ω be an open subset of $[0,+\infty) \times C_T(M)$. Our main result (Theorem 3.3 below) provides a sufficient condition for the existence of a bifurcation point p in M with $(0,p^-) \in \Omega$. More precisely, we give conditions which ensure the existence of a connected subset of Ω of nontrivial T-periodic pairs of equation (3.1) (a *global bifurcating branch* for short), whose closure in Ω is noncompact and intersects the set of trivial T-periodic pairs contained in Ω.

Theorem 3.3. *Let $M \subseteq \mathbb{R}^k$ be a boundaryless smooth manifold, $f\colon \mathbb{R} \times \widetilde{M} \to \mathbb{R}^k$ a functional field on M, T-periodic in the first variable and locally Lipschitz in the second one, and $w\colon M \to \mathbb{R}^k$ the autonomous tangent vector field*

$$w(p) = \frac{1}{T}\int_0^T f(t, p^-)\, dt\,.$$

Let Ω be an open subset of $[0,+\infty) \times C_T(M)$ and let $j\colon M \to [0,+\infty) \times C_T(M)$ be the map $p \mapsto (0,p^-)$. Assume that $\deg(w, j^{-1}(\Omega))$ is defined and nonzero. Then, there exists a connected subset of Ω of nontrivial T-periodic pairs of equation (3.1) *whose closure in Ω is noncompact and intersects $\{0\} \times C_T(M)$ in a (nonempty) subset of $\{(0,p^-) \in \Omega : w(p) = 0\}$.*

Remark 3.4 (On the meaning of global bifurcating branch)**.** In addition to the hypotheses of Theorem 3.3, assume that f sends bounded subsets of $\mathbb{R} \times \widetilde{M}$ into bounded subsets of $\mathbb{R}^k$, and that M is closed in $\mathbb{R}^k$ (or, more generally, that the closure $\overline{\Omega}$ of Ω in $[0,+\infty) \times C_T(M)$ is complete).

Then a connected subset Γ of Ω as in Theorem 3.3 is either unbounded or, if bounded, its closure $\overline{\Gamma}$ in $\overline{\Omega}$ reaches the boundary $\partial\Omega$ of Ω.

To see this, assume that $\overline{\Gamma}$ is bounded. Then, being $f(\overline{\Gamma})$ bounded, because of Ascoli's theorem, Γ is actually totally bounded. Thus, $\overline{\Gamma}$ is compact, being totally bounded and, additionally, complete since $\overline{\Gamma}$ is contained in $\overline{\Omega}$. On the other hand, according to Theorem 3.3, the closure $\overline{\Gamma}_\Omega$ of Γ in Ω is noncompact. Consequently, the set $\overline{\Gamma} \setminus \overline{\Gamma}_\Omega$ is nonempty, and this means that $\overline{\Gamma}$ reaches the boundary of Ω.

The proof of Theorem 3.3 requires some preliminary steps. In the first one we define a parametrized Poincaré-type T-translation operator whose fixed points are the restrictions to the interval $[-T,0]$ of the T-periodic solutions of (3.1). For this purpose, we need to introduce a suitable backward extension of the elements of $\widehat{M}$. The properties of such an extension are contained in Lemma 3.5 below, obtained in [11]. In what follows, by a T-periodic map on an interval J we mean the restriction to J of a T-periodic map defined on $\mathbb{R}$.

Lemma 3.5. *There exist an open neighborhood U of $\widehat{M}_*$ in $\widehat{M}$ and a continuous map from U to $\widetilde{M}$, $\psi \mapsto \widetilde{\psi}$, with the following properties:*

1) $\widetilde{\psi}$ *is an extension of* ψ*;*
2. $\widetilde{\psi}$ *is* T*-periodic on* $(-\infty, -T]$*;*
3. $\widetilde{\psi}$ *is* T*-periodic on* $(-\infty, 0]$*, whenever* $\psi \in \widehat{M}_*$*.*

Let now U be an open subset of $\widehat{M}$ as in the previous lemma and let f be as in Theorem 3.3. Given $\lambda \geq 0$ and $\psi \in U$, consider the initial value problem

$$\begin{cases} x'(t) = \lambda f(t, x_t), \\ x_0 = \widetilde{\psi}, \end{cases} \tag{3.3}$$

where $\widetilde{\psi}$ is the extension of ψ as in Lemma 3.5.

Let

$$D = \big\{(\lambda, \psi) \in [0, +\infty) \times U : \text{ the maximal solution of (3.3) is defined up to } T\big\}.$$

The set D is nonempty since it contains $\{0\} \times U$ (notice that, for $\lambda = 0$, the solution of problem (3.3) is constant for $t > 0$). Moreover, it follows by Corollary 2.2 that D is open in $[0, +\infty) \times \widehat{M}$.

Given $(\lambda, \psi) \in D$, denote by $x^{(\lambda, \widetilde{\psi})}$ the maximal solution of problem (3.3) and define

$$P : D \to \widehat{M}$$

by

$$P(\lambda, \psi)(\theta) = x^{(\lambda, \widetilde{\psi})}(\theta + T), \quad \theta \in [-T, 0].$$

Observe that $P(\lambda, \psi)$ is the restriction of $x_T^{(\lambda, \widetilde{\psi})} \in \widetilde{M}$ to the interval $[-T, 0]$.

The following lemmas regard crucial properties of the operator P. The proof of the first one is standard and will be omitted.

Lemma 3.6. *The fixed points of* $P(\lambda, \cdot)$ *correspond to the* T*-periodic solutions of the equation* (3.1) *in the following sense:* ψ *is a fixed point of* $P(\lambda, \cdot)$ *if and only if it is the restriction to* $[-T, 0]$ *of a* T*-periodic solution.*

Lemma 3.7. *The operator* P *is continuous and locally compact.*

Proof. The continuity of P follows immediately from the continuous dependence on data stated in Corollary 2.2 and by the continuity of the map $\psi \mapsto \widetilde{\psi}$ of Lemma 3.5 and of the map that associates to any $\varphi \in \widetilde{M}$ its restriction to the interval $[-T, 0]$.

Let us prove that P is locally compact. Take $(\lambda^0, \psi^0) \in D$ and denote for simplicity by x^0 the maximal solution $x^{(\lambda^0, \widetilde{\psi^0})}$ of (3.3) corresponding to $(\lambda^0, \widetilde{\psi^0})$. Clearly, x^0 is defined at least up to T and $P(\lambda^0, \psi^0)(\theta) = x^0(\theta + T)$ for any $\theta \in [-T, 0]$. Set

$$K = \{(t, x_t^0) \in \mathbb{R} \times \widetilde{M} : t \in [0, T]\}.$$

Observe that K is compact, being the image of $[0, T]$ under the (continuous) curve $t \mapsto (t, x_t^0)$. Let O be an open neighborhood of K in $\mathbb{R} \times \widetilde{M}$ and $c > 0$ such that $|f(t, \varphi)| \leq c$ for all $(t, \varphi) \in \overline{O}$. Let us show that there exists an open neighborhood W of (λ^0, ψ^0) in D such that if $(\lambda, \psi) \in W$, then $(t, x_t^{(\lambda, \widetilde{\psi})}) \in O$ for $t \in [0, T]$, where $x^{(\lambda, \widetilde{\psi})}$ is the maximal solution of (3.3) corresponding to $(\lambda, \widetilde{\psi})$. By contradiction, for any $n \in \mathbb{N}$ suppose there exist $(\lambda^n, \psi^n) \in D$ and $t^n \in [0, T]$ such that $(\lambda^n, \psi^n) \to (\lambda^0, \psi^0)$ and $(t^n, x_{t^n}^n) \notin O$, where x^n denotes the maximal solution $x^{(\lambda^n, \widetilde{\psi^n})}$ of (3.3) corresponding to $(\lambda^n, \widetilde{\psi^n})$. We may assume $t^n \to \tau \in [0, T]$. Now, from the fact that in $\widetilde{M}$ the convergence is uniform we get the equicontinuity of the sequence $\{x_T^n\}$. This easily implies that $(t^n, x_{t^n}^n) \to (\tau, x_\tau^0)$. A contradiction, since O is open and (τ, x_τ^0) belongs to $K \subseteq O$. Thus, the existence of the required W is proved. Consequently, for any $(\lambda,\psi) \in W$, the maximal solution $x^{(\lambda,\widetilde{\psi})}$ of (3.3) corresponding to $(\lambda,\widetilde{\psi})$ is such that $|(x^{(\lambda,\widetilde{\psi})})'(t)| = |\lambda f(t,x_t^{(\lambda,\widetilde{\psi})})| \leq |\lambda| c$ for all $t \in [0,T]$.

Therefore, by Ascoli's theorem and taking into account the local completeness of $\widehat{M}$, we get that P maps W into a compact subset of $\widehat{M}$. This proves that P is locally compact. □

The following result establishes the relationship between the fixed point index of the Poincaré-type operator $P(\lambda,\cdot)$ and the degree of the mean value vector field w. It will be crucial in the proof of Lemma 3.10.

Lemma 3.8. *Let $\mathcal{V}$ be an open subset of $\widehat{M}$ such that $\mathcal{V} \cap \{p^- \in \widehat{M} : w(p) = 0\}$ is compact and let $\varepsilon > 0$ be such that*

a) $[0,\varepsilon] \times \overline{\mathcal{V}}$ *is contained in the domain D of P;*
b) $P([0,\varepsilon] \times \mathcal{V})$ *is relatively compact;*
c) $P(\lambda,\psi) \neq \psi$ *for $0 < \lambda \leq \varepsilon$ and ψ in the boundary $\partial\mathcal{V}$ of $\mathcal{V}$.*

Consider the open set $V = \{p \in M : p^- \in \mathcal{V}\}$. Then, $\deg(-w,V)$ *is well defined and*

$$\operatorname{ind}_{\widehat{M}}(P(\lambda,\cdot),\mathcal{V}) = \deg(-w,V), \quad 0 < \lambda \leq \varepsilon.$$

Proof. Let U be an open subset of $\widehat{M}$ as in Lemma 3.5. Given $\lambda \geq 0$, $\mu \in [0,1]$ and $\psi \in U$, consider the initial value problem

$$\begin{cases} x'(t) = \lambda\big((1-\mu)f(t,x_t) + \mu w(x(t))\big), \\ x_0 = \widetilde{\psi}, \end{cases} \tag{3.4}$$

where $\widetilde{\psi}$ is associated to ψ as in Lemma 3.5. Since f is locally Lipschitz in the second variable, then it is easy to see that w is locally Lipschitz as well. Hence, for any $\lambda \in [0,+\infty)$ and $\mu \in [0,1]$, the uniqueness of the solution of problem (3.4) is ensured (recall Remark 2.3). Denote by $x^{(\lambda,\widetilde{\psi},\mu)}$ the maximal solution of problem (3.4), and put

$$E = \big\{(\lambda,\psi,\mu) \in [0,+\infty) \times U \times [0,1] : x^{(\lambda,\widetilde{\psi},\mu)} \text{ is defined up to } T\big\}$$

and

$$D' = \big\{(\lambda,\psi) \in [0,+\infty) \times U : (\lambda,\psi,\mu) \in E \text{ for all } \mu \in [0,1]\big\}.$$

Corollary 2.2 implies that E is open in $[0,+\infty) \times U \times [0,1]$. Therefore D' is open in $[0,+\infty) \times \widehat{M}$, because of the compactness of $[0,1]$. Moreover, observe that the slice D'_0 of D' at $\lambda = 0$ coincides with U and that D' is contained into the domain D of the operator P defined above. Define $H\colon D' \times [0,1] \to \widehat{M}$ by

$$H(\lambda,\psi,\mu)(\theta) = x^{(\lambda,\widetilde{\psi},\mu)}(\theta + T), \quad \theta \in [-T,0].$$

Clearly, $H(\cdot,\cdot,0)$ coincides with P on D', while $H(\cdot,\cdot,1)$ is the (infinite dimensional) operator associated to the undelayed problem

$$\begin{cases} x'(t) = \lambda w(x(t)), \\ x_0 = \widetilde{\psi}. \end{cases}$$

As in Lemmas 3.6 and 3.7, one can show that the fixed points of $H(\lambda,\cdot,\mu)$ correspond to the T-periodic solutions of the equation

$$x'(t) = \lambda\big((1-\mu)f(t,x_t) + \mu w(x(t))\big),$$

and that H is continuous and locally compact.

The assertion now will follow by proving some intermediate results on the homotopy H. These results will be carried out in several steps. In what follows set

$$Z = \{p \in M : w(p) = 0\}$$

and, according to our notation,

$$Z^- = \{p^- \in \widehat{M} : p \in Z\}.$$

Step 1. There exist $\sigma > 0$ and an open subset $\mathcal{V}'$ of $\widehat{M}$, containing $\mathcal{V} \cap Z^-$, with $\overline{\mathcal{V}'} \subseteq \mathcal{V}$, and such that

a') $[0,\sigma] \times \overline{\mathcal{V}'} \subseteq D'$ *(i.e. for $0 \le \lambda \le \sigma$, $H(\lambda,\cdot,\cdot)$ is defined in $\overline{\mathcal{V}'} \times [0,1]$);*

b') $H([0,\sigma] \times \mathcal{V}' \times [0,1])$ *is relatively compact.*

To prove Step 1, observe that $\{0\} \times (\mathcal{V} \cap Z^-) \times [0,1]$ is compact and contained in $D' \times [0,1]$, which is open in $[0,+\infty) \times \widehat{M} \times [0,1]$, and recall that H is locally compact.

Step 2. For small values of $\lambda > 0$, $H(\lambda,\psi,\mu) \neq \psi$ for any $\psi \in \partial\mathcal{V}'$ and $\mu \in [0,1]$.

By contradiction, suppose there exists a sequence $\{(\lambda^n,\psi^n,\mu^n)\}$ in $D' \times [0,1]$ such that $\lambda^n > 0$, $\lambda^n \to 0$, $\psi^n \in \partial\mathcal{V}'$ and $H(\lambda^n,\psi^n,\mu^n) = \psi^n$. Without loss of generality, taking into account b'), we may assume that $\psi^n \to \psi^0$ and also that $\mu^n \to \mu^0$. Denote by x^n the T-periodic solution $x^{(\lambda^n,\widetilde{\psi^n},\mu^n)}$ of (3.4) corresponding to $(\lambda^n,\widetilde{\psi^n},\mu^n)$. Since ψ^n is the restriction of x^n to $[-T,0]$, then $\{x^n(t)\}$ converges uniformly on $\mathbb{R}$ to $x^0(t)$, where x^0 is the solution of (3.4) corresponding to the fixed point ψ^0 of $H(0,\cdot,\mu^0)$. Therefore, there exists $p \in M$ such that $x^0(t) = p$ for any $t \in \mathbb{R}$ and, as in the proof of Theorem 3.2, we can show that $w(p) = 0$. Thus, $\psi^0 = p^-$ belongs to $\partial\mathcal{V}' \cap Z^-$, contradicting the choice of $\mathcal{V}'$. This proves Step 2.

Step 3. For small values of $\lambda > 0$, $H(\lambda,\psi,0) \neq \psi$ for any $\psi \in \overline{\mathcal{V}}\backslash\mathcal{V}'$.

The proof is analogous to that of Step 2, noting that $H(\lambda,\psi,0) = P(\lambda,\psi)$ for $(\lambda,\psi) \in D'$ and taking into account assumption b) and the fact that $\overline{\mathcal{V}}\backslash\mathcal{V}'$ is closed in $\widehat{M}$.

Step 4. Let $k\colon \mathcal{V}' \to M$ be defined by $k(\psi) = \psi(0)$ and consider the open set $V' = \{p \in M : p^- \in \mathcal{V}'\}$. Then, there exists $\sigma' \in (0,\sigma]$ such that $H(\lambda,\psi,1) \neq \psi$ for any $(\lambda,\psi) \in (0,\sigma'] \times (\overline{\mathcal{V}'}\backslash k^{-1}(V'))$.

By contradiction, suppose there exists a sequence $\{(\lambda^n,\psi^n)\}$ in D' such that $\lambda^n > 0$, $\lambda^n \to 0$, $\psi^n \in \overline{\mathcal{V}'}\backslash k^{-1}(V')$ and $H(\lambda^n,\psi^n,1) = \psi^n$. Without loss of generality, taking into account b'), we may assume that $\psi^n \to \psi^0$. Therefore, by the continuity of H, we get $H(0,\psi^0,1) = \psi^0$, so that ψ^0 is a constant function of $\overline{\mathcal{V}'}\backslash k^{-1}(V')$. This is impossible, since any constant function of $\mathcal{V}'$ is contained in $k^{-1}(V')$.

Step 5. Let V' and σ' be as in Step 4 and let $Q\colon [0,\sigma'] \times \overline{V'} \to M$ be the T-translation operator $Q(\lambda,p) = x^{(\lambda,p^-,1)}(T)$, where $x^{(\lambda,p^-,1)}$ is the maximal solution of the undelayed problem

$$\begin{cases} x'(t) = \lambda w(x(t)), \\ x_0 = p^-. \end{cases}$$

Then, for small values of λ, $\operatorname{ind}_M(Q(\lambda,\cdot),V')$ is defined and

$$\operatorname{ind}_{\widehat{M}}(H(\lambda,\cdot,1),\mathcal{V}') = \operatorname{ind}_M(Q(\lambda,\cdot),V')\,.$$

To see this, let $k\colon \mathcal{V}' \to M$ be as in Step 4 and, given $\lambda \in (0,\sigma']$, define $h_\lambda\colon V' \to \widehat{M}$ by $h_\lambda(p)(\theta) = x^{(\lambda,p^-,1)}(\theta+T)$, $\theta \in [-T,0]$. Clearly, k is a locally compact map since it takes values in the locally compact space M. Moreover, h_λ is actually compact since $h_\lambda(V')$ is contained in $H([0,\sigma]\times\mathcal{V}'\times[0,1])$ which is relatively compact by b') of Step 1. Now, observe that the composition $h_\lambda k$ coincides with $H(\lambda,\cdot,1)$ in $k^{-1}(V')$ and that the set of fixed points of $H(\lambda,\cdot,1)$ in $\overline{\mathcal{V}'}$ is compact by b') of

Step 1 and is contained in $k^{-1}(V')$ by Step 4. Thus, the set of fixed points of $h_\lambda k$ in $k^{-1}(V')$ is compact so that, by applying the commutativity property of the fixed point index to the maps k and h_λ, we get

$$\operatorname{ind}_{\widehat{M}}(h_\lambda k, k^{-1}(V')) = \operatorname{ind}_M(kh_\lambda, h_\lambda^{-1}(\mathcal{V}')).$$

Consequently, since it is easy to verify that the composition kh_λ coincides with $Q(\lambda,\cdot)$ in $h_\lambda^{-1}(\mathcal{V}')$, we obtain

$$\operatorname{ind}_{\widehat{M}}(H(\lambda,\cdot,1), k^{-1}(V')) = \operatorname{ind}_M(Q(\lambda,\cdot), h_\lambda^{-1}(\mathcal{V}'),$$

and, because of Step 4, by the excision property of the index,

$$\operatorname{ind}_{\widehat{M}}(H(\lambda,\cdot,1),\mathcal{V}') = \operatorname{ind}_{\widehat{M}}(H(\lambda,\cdot,1), k^{-1}(V')).$$

To complete the proof of Step 5, let us show that, for λ sufficiently small, $Q(\lambda,p) \neq p$ for $p \in \overline{V'}\backslash h_\lambda^{-1}(\mathcal{V}')$. By contradiction, suppose there exists a sequence $\{(\lambda^n,p^n)\}$ in $[0,\sigma']\times\overline{V'}$ such that $\lambda^n > 0$, $\lambda^n \to 0$, $p^n \in \overline{V'}\backslash h_{\lambda_n}^{-1}(\mathcal{V}')$ and $Q(\lambda^n,p^n) = p^n$. Hence, there exists a sequence $\{\psi^n\}$ in $\mathcal{V}'$ such that $\psi^n(0) = p^n$ and $H(\lambda^n,\psi^n,1) = \psi^n$. Because of b') of Step 1, we may assume that $\psi^n \to \psi^0$, so that, in particular, $p^n \to p^0$, where $p^0 = \psi^0(0)$. Now, by an argument similar to that used in the proof of Theorem 3.2, we get that ψ^0 is constant and $w(p^0) = 0$. Thus, $p^0 \in Z$. Moreover, since $\bigcap_{\lambda>0}(\overline{V'}\backslash h_\lambda^{-1}(\mathcal{V}')) = \partial V'$, we also obtain that p^0 belongs to $\partial V'$, contradicting the choice of V'. Finally, again by excision, we get

$$\operatorname{ind}_M(Q(\lambda,\cdot), h_\lambda^{-1}(\mathcal{V}') = \operatorname{ind}_M(Q(\lambda,\cdot), V'),$$

and, thus, Step 5 is proved.

Let us now go back to the proof of our lemma. Step 1 and Step 2 above imply that there exist $\varepsilon' > 0$ and an open subset $\mathcal{V}'$ of $\widehat{M}$, containing $\mathcal{V}\cap Z^-$, with $\overline{\mathcal{V}'} \subseteq \mathcal{V}$ and such that, if $0 < \lambda \le \varepsilon'$, then $\operatorname{ind}_{\widehat{M}}(H(\lambda,\cdot,\mu),\mathcal{V}')$ is defined and is independent of $\mu \in [0,1]$. Moreover, in case reducing ε', by Step 3 and by assumption b), it follows that, for $\lambda \in (0,\varepsilon']$, the fixed points of $H(\lambda,\cdot,0) = P(\lambda,\cdot)$ in $\mathcal{V}$ are a compact subset of $\mathcal{V}'$. Therefore, by the excision property and the homotopy invariance of the index, we get

$$\operatorname{ind}_{\widehat{M}}(P(\lambda,\cdot),\mathcal{V}) = \operatorname{ind}_{\widehat{M}}(P(\lambda,\cdot),\mathcal{V}') = \operatorname{ind}_{\widehat{M}}(H(\lambda,\cdot,0),\mathcal{V}') = \operatorname{ind}_{\widehat{M}}(H(\lambda,\cdot,1),\mathcal{V}').$$

On the other hand, by Step 5, if $\lambda > 0$ is sufficiently small, we have

$$\operatorname{ind}_{\widehat{M}}(H(\lambda,\cdot,1),\mathcal{V}') = \operatorname{ind}_M(Q(\lambda,\cdot),V').$$

Moreover, as shown in [9],

$$\operatorname{ind}_M(Q(\lambda,\cdot),V') = \deg(-w,V').$$

Finally, notice that $\deg(-w,V)$ is well-defined since $V\cap Z$ is compact being homeomorphic to $\mathcal{V}\cap Z^-$. Also observe that there are no zeros of w in $V\backslash\overline{V'}$. Thus, by the excision property of the degree, we obtain

$$\deg(-w,V') = \deg(-w,V).$$

This shows that, for small values of $\lambda > 0$, $\operatorname{ind}_{\widehat{M}}(P(\lambda,\cdot),\mathcal{V}) = \deg(-w,V)$. The assertion of the lemma now follows by applying the homotopy invariance of the fixed point index to $P(\lambda,\cdot)$ on $\mathcal{V}$. □

Lemma 3.10 below, whose proof makes use the following Wyburn's type topological lemma, is another important step in the construction of the proof of Theorem 3.3.

Lemma 3.9 ([10]). *Let K be a compact subset of a locally compact metric space Y. Assume that any compact subset of Y containing K has nonempty boundary. Then $Y\backslash K$ contains a connected set whose closure is noncompact and intersects K.*

Before presenting Lemma 3.10, we introduce the sets

$$S = \{(\lambda, \psi) \in D : P(\lambda, \psi) = \psi\} \text{ and } S_+ = \{(\lambda, \psi) \in S : \lambda > 0\},$$

and we recall that $Z \subseteq M$ denotes the set of zeros of the tangent vector field w.

Lemma 3.10. *Let Y be a locally compact open subset of $(\{0\} \times Z^-) \cup S_+$. Assume that $K := Y \cap (\{0\} \times Z^-)$ is compact, and that $\deg(w, V) \neq 0$, where $V \subseteq M$ is an isolating neighborhood of $\{p \in M : (0, p^-) \in K\}$. Then the pair (Y, K) verifies the assumptions of Lemma 3.9.*

Proof. First of all, observe that, by Lemma 3.7, S is closed in D and locally compact. In addition, K is clearly nonempty being $\deg(w, V) \neq 0$. Now, let G be an open subset of D such that

$$G \cap \big((\{0\} \times Z^-) \cup S_+\big) = Y.$$

To prove the assertion, suppose by contradiction that there exists a compact open neighborhood C of K in Y. Consequently, we can find an open subset W of G such that $\overline{W} \subseteq G$ and $C = W \cap Y = \overline{W} \cap Y$. Therefore, denoted by G_0 the slice

$$G_0 = \{\psi \in \widehat{M} : (0, \psi) \in G\},$$

we have that $G_0 \cap Z^-$ is a compact subset of $\widehat{M}$ and is contained in the open slice $W_0 \subseteq \overline{W}_0 \subseteq G_0$ of W at $\lambda = 0$. Let $\mathcal{V}$ be an open subset of W_0 such that $\mathcal{V} \subseteq \overline{\mathcal{V}} \subseteq W_0$ and $\mathcal{V} \cap Z^- = W_0 \cap Z^-$. Since C is compact and because of the local compactness of P, we may suppose that $P(W)$ is relatively compact. Consequently, there exists $\varepsilon > 0$ such that

1. $[0, \varepsilon] \times \overline{\mathcal{V}} \subseteq W$;
2. $P(\lambda, \psi) \neq \psi$ for $\psi \in \overline{W}_\lambda \backslash \mathcal{V}$ and $0 < \lambda \leq \varepsilon$ (here, as usual, W_λ denotes the slice $\{\psi \in \widehat{M} : (\lambda, \psi) \in W\}$).

Notice that $P([0, \varepsilon] \times \mathcal{V})$ is relatively compact. This follows easily from the above condition 1 and the relative compactness of $P(W)$.

We can now apply Lemma 3.8 and the excision properties of the fixed point index and of the degree, obtaining, for any $0 < \lambda \leq \varepsilon$,

$$\operatorname{ind}_{\widehat{M}}(P(\lambda, \cdot), W_\lambda) = \operatorname{ind}_{\widehat{M}}(P(\lambda, \cdot), \mathcal{V}) = \deg(-w, V), \tag{3.5}$$

where $V = \{p \in M : p^- \in \mathcal{V}\}$. Observe that V is an isolating neighborhood of $\{p \in M : (0, p^-) \in K\}$. Thus, by formula (2.1), by the above equalities (3.5) and the assumption $\deg(w, V) \neq 0$, we get

$$\operatorname{ind}_{\widehat{M}}(P(\lambda, \cdot), W_\lambda) \neq 0, \quad 0 < \lambda \leq \varepsilon.$$

Since C is compact, by the generalized homotopy invariance property of the fixed point index, we get that $\operatorname{ind}_{\widehat{M}}(P(\lambda, \cdot), W_\lambda)$ does not depend on $\lambda > 0$. Hence,

$$\operatorname{ind}_{\widehat{M}}(P(\lambda, \cdot), W_\lambda) \neq 0, \quad \forall \lambda > 0.$$

On the other hand, because of the compactness of C, for some positive $\overline{\lambda}$ the slice $C_{\overline{\lambda}} = \{\psi \in W_{\overline{\lambda}} : P(\overline{\lambda}, \psi) = \psi\}$ is empty. Thus,

$$\operatorname{ind}_{\widehat{M}}(P(\overline{\lambda}, \cdot), W_{\overline{\lambda}}) = 0,$$

and we have a contradiction. Therefore, (Y, K) verifies the assumptions of Lemma 3.9 and the proof is complete. □

Proof of Theorem 3.3. Let $\rho\colon [0, +\infty) \times C_T(M) \to [0, +\infty) \times \widehat{M}_*$ be the isometry given by $\rho(\lambda, x) = (\lambda, \psi)$, where ψ is the restriction of x to the interval $[-T, 0]$. As previously, let $X \subseteq [0, +\infty) \times C_T(M)$ denote the set of the T-periodic pairs of (3.1) and, as in Lemma 3.10, let S be the set of the pairs (λ, ψ) such that $P(\lambda, \psi) = \psi$.

Observe that S is actually contained in $[0,+\infty)\times\widehat{M}_*$. Taking into account Lemma 3.6, X and S correspond under ρ. Analogously to the definition of S_+, let us denote

$$X_+ = \{(\lambda, x) \in X : \lambda > 0\}.$$

In addition, consider

$$Z^T = \{p^- \in C_T(M) : w(p) = 0\}.$$

Theorem 3.2 implies that $(\{0\}\times Z^T)\cup X_+$ is a closed subset of X. Therefore, it is locally compact, since so is X according to Lemma 3.1. Now, consider

$$Y^T = \Omega \cap \left((\{0\}\times Z^T)\cup X_+\right).$$

Observe that Y^T is locally compact, being open in $(\{0\}\times Z^T)\cup X_+$. Then,

$$Y := \rho(Y^T)$$

is locally compact and open in $(\{0\}\times Z^-)\cup S_+$. Denote by K^T and K the subsets of Y^T and Y defined as

$$K^T = \{(\lambda, x) \in Y^T : \lambda = 0\} \quad \text{and} \quad K = \rho(K^T).$$

Now, observe that $j^{-1}(\Omega)$ is an isolating neighborhood of

$$\{p \in M : (0, p^-) \in K\}.$$

Since $\deg(w, j^{-1}(\Omega)) \neq 0$, we can apply Lemma 3.10, concluding that (Y, K) verifies the assumptions of Lemma 3.9. Therefore, also (Y^T, K^T) verifies the same assumptions, since the pairs (Y, K) and (Y^T, K^T) correspond under the isometry ρ. Therefore, Lemma 3.9 implies that $Y^T\backslash K^T$ contains a connected set Γ whose closure (in Y^T) is noncompact and intersects K^T. Now, observe that, according to Theorem 3.2, Y^T is closed in Ω. Thus the closures of Γ in Y^T and in Ω coincide. This concludes the proof. □

We give now some consequences of Theorem 3.3. The first one is in the spirit of a celebrated result due to P. H. Rabinowitz [31].

Corollary 3.11 (Rabinowitz type global bifurcation result). *Let M and f be as in Theorem 3.3. Assume that M is closed in $\mathbb{R}^k$ and that f sends bounded subsets of $\mathbb{R}\times\widetilde{M}$ into bounded subsets of $\mathbb{R}^k$. Let V be an open subset of M such that* $\deg(w, V) \neq 0$*, where w is the mean value tangent vector field defined in formula* (3.2)*. Then, equation* (3.1) *has a connected subset of nontrivial T-periodic pairs whose closure contains some $(0, p^-)$, with $p \in V$, and is either unbounded or goes back to some $(0, q^-)$, where $q \notin V$.*

Proof. Let Ω be the open set obtained by removing from $[0,+\infty)\times C_T(M)$ the closed set $\{(0, q^-) : q \notin V\}$. In other words,

$$\Omega = \big([0,+\infty)\times C_T(M)\big) \setminus \big(\{0\}\times (M\setminus V)^-\big).$$

Observe that $\overline{\Omega}$ is complete, due to the closedness of M. Consider, by Theorem 3.3, a connected set $\Gamma \subseteq \Omega$ of nontrivial T-periodic pairs with noncompact closure (in Ω) and intersecting $\{0\}\times C_T(M)$ in a subset of $\{(0, p^-) \in \Omega : w(p) = 0\}$. Suppose that Γ is bounded. From Remark 3.4 it follows that $\overline{\Gamma}\setminus\overline{\Gamma}_\Omega$, where $\overline{\Gamma}_\Omega$ denotes the closure of Γ in Ω, is nonempty and hence contains a point $(0, q^-)$ which does not belong to Ω, that is, such that $q \notin V$. □

Remark 3.12. The assumption of Corollary 3.11 above on the existence of an open subset V of M such that $\deg(w, V) \neq 0$ is clearly satisfied in the case when w has an isolated zero with nonzero index. For example, if $w(p) = 0$ and w is C^1 with injective derivative $w'(p)\colon T_pM \to \mathbb{R}^k$, then p is an isolated zero of w and its index is either 1 or -1. In fact, in this case, $w'(p)$ sends T_pM into itself and, consequently, its determinant is well defined and nonzero. The index of p is just the sign of this determinant (see e.g. [24]).

The next consequence of Theorem 3.3 provides an existence result for T-periodic solutions already obtained in [4]. Moreover, it improves an analogous result in [2], in which the map f is continuous on $\mathbb{R} \times C((-\infty, 0], M)$, with the compact-open topology in $C((-\infty, 0], M)$. In fact, such a coarse topology makes the assumption of the continuity of f a more restrictive condition than the one we require here.

Corollary 3.13. *Let M and f be as in Theorem 3.3. Assume that f sends bounded subsets of $\mathbb{R} \times \widetilde{M}$ into bounded subsets of $\mathbb{R}^k$. In addition, suppose that M is compact with Euler-Poincaré characteristic $\chi(M) \neq 0$. Then, equation* (3.1) *has a connected unbounded set of nontrivial T-periodic pairs whose closure meets $\{0\} \times C_T(M)$. Therefore, since $C_T(M)$ is bounded, equation* (3.1) *has a T-periodic solution for any $\lambda \geq 0$.*

Proof. Choose $V = M$. By the Poincaré-Hopf theorem we have

$$\deg(w, M) = \chi(M) \neq 0,$$

where w is the mean value tangent vector field defined in formula (3.2). The assertion follows from Corollary 3.11. □

Corollary 3.14 below is a kind of continuation principle in the spirit of a well known result due to Jean Mawhin for ODEs in $\mathbb{R}^k$ [22, 23], and extends an analogous one for ODEs on differentiable manifolds [10]. In what follows, by a *T-periodic orbit* of $x'(t) = \lambda f(t, x_t)$ we mean the image of a T-periodic solution of this equation.

Corollary 3.14 (Mawhin type continuation principle)**.** *Let M and f be as in Theorem 3.3 and let w be the mean value tangent vector field defined in formula* (3.2)*. Assume that f sends bounded subsets of $\mathbb{R} \times \widetilde{M}$ into bounded subsets of $\mathbb{R}^k$. Let V be a relatively compact open subset of M and assume that*

1. $w(p) \neq 0$ *along the boundary ∂V of V;*
2. $\deg(w, V) \neq 0$*;*
3. *for any $\lambda \in (0, 1]$, the T-periodic orbits of $x'(t) = \lambda f(t, x_t)$ lying in $\overline{V}$ do not meet ∂V.*

Then, the equation

$$x'(t) = f(t, x_t)$$

has a T-periodic orbit in V.

Proof. Define $\Omega = [0, 1) \times C_T(V)$. Observe that $C_T(\overline{V}) = \overline{C_T(V)}$. Therefore,

$$\partial\Omega = \big(\{1\} \times C_T(\overline{V})\big) \cup \big([0, 1) \times C_T(\overline{V}) \setminus C_T(V)\big).$$

According to Theorem 3.3, call Γ a connected subset of Ω of nontrivial T-periodic pairs of equation $x'(t) = \lambda f(t, x_t)$, whose closure in Ω is noncompact and intersects $\{0\} \times C_T(M)$ in a subset of $\{(0, p^-) \in \Omega : w(p) = 0\}$.

As V has compact closure in M, then the closure of Ω in $[0, +\infty) \times C_T(M)$ is complete, being

$$\overline{\Omega} = [0, 1] \times C_T(\overline{V}).$$

Since f sends bounded subsets of $\mathbb{R} \times \widetilde{M}$ into bounded subsets of $\mathbb{R}^k$, recalling Remark 3.4, one has that the closure $\overline{\Gamma}$ of Γ in the whole space (which coincides with the closure in $\overline{\Omega}$) must intersect $\partial\Omega$.

Now, because of the above condition 3, $\overline{\Gamma}$ cannot contain elements of $(0, 1) \times C_T(\overline{V}) \setminus C_T(V)$. In addition, condition 1 and Theorem 3.2 imply that $\overline{\Gamma}$ does not contain elements of $\{0\} \times \big(C_T(\overline{V}) \setminus C_T(V)\big)$. Therefore, the nonempty set

$\overline{\Gamma} \cap \partial\Omega$ is composed by pairs of the form $(1, x)$, where x is a T-periodic solution of $x'(t) = f(t, x_t)$ whose image is contained in V. □

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