ON THE EXISTENCE OF FORCED OSCILLATIONS OF RETARDED FUNCTIONAL MOTION EQUATIONS ON TOPOLOGICALLY NONTRIVIAL MANIFOLDS

PIERLUIGI BENEVIERI, ALESSANDRO CALAMAI, MASSIMO FURI, AND MARIA PATRIZIA PERA

Abstract. Using a topological approach, based on the fixed point index theory for locally compact maps on metric ANRs, we prove the existence of forced oscillations for retarded functional motion equations defined on topologically nontrivial compact constraints, provided that the frictional coefficient is nonzero. We do not know if an analogous result holds true in the frictionless case.

Dedicated to Fabio Zanolin on the occasion of his 60th birthday

1. Introduction

Consider a compact boundaryless smooth manifold $M \subseteq \mathbb{R}^s$ and denote by $BU((-\infty,0],M)$ the space of bounded and uniformly continuous maps from $(-\infty,0]$ into M with the topology of the uniform convergence. In this paper we study a *retarded functional motion equation* on M of the type

$$x''_\pi(t) = f(t, x_t) - \varepsilon x'(t), \tag{1.1}$$

where

(1) $x''_\pi(t)$ stands for the tangential part of the acceleration $x''(t) \in \mathbb{R}^s$ at the point $x(t) \in M$,
(2) the frictional coefficient ε is a positive constant,
(3) the applied force $f : \mathbb{R} \times BU((-\infty,0],M) \to \mathbb{R}^s$ is continuous, T-periodic in the first variable and such that $f(t,\varphi) \in T_{\varphi(0)}M$ for all (t,φ), where $T_pM \subseteq \mathbb{R}^s$ stands for the tangent space of M at a point p of M.

We will call *functional field* a continuous map $f : \mathbb{R} \times BU((-\infty,0],M) \to \mathbb{R}^s$ verifying the above tangency condition. In addition, let us recall that, given any map x, defined on a real interval J with $\inf J = -\infty$, and given $t \in J$, x_t denotes the map $\theta \mapsto x(\theta + t)$, defined on $(-\infty,0]$.

The main result of this work, Theorem 4.1 below, shows that the equation (1.1) admits at least one T-periodic solution (a *forced oscillation*), provided that M has nontrivial Euler-Poincaré characteristic and f is bounded and verifies a sort of Lipschitz condition.

This result provides a positive answer to a conjecture recently formulated in [4]. A key tool that allowed us to solve our conjecture is Lemma 3.1 below, proved in [10].

An existence result for a similar problem has been obtained in [1] (see also [2, 3]), with the difference that, in [1], the function f is defined and continuous on $\mathbb{R} \times C((-\infty,0],M)$ endowed with the compact-open topology. The continuity

2010 *Mathematics Subject Classification.* 34C40, 34K13, 37C25, 47H10.
Key words and phrases. Retarded functional differential equations, fixed point index, forced oscillations.

assumption of f on $\mathbb{R}\times C((-\infty,0],M)$ is more restrictive than the hypothesis of continuity on $\mathbb{R}\times BU((-\infty,0],M)$, since the compact-open topology on $C((-\infty,0],M)$ induces on $BU((-\infty,0],M)$ a topology which is weaker than that of uniform convergence. This means that the existence of forced oscillations for (1.1), proved in this paper, is not a byproduct of the analogous result given in [1], whose proof, in addition, does not fit in the present context.

To get our main result we consider a first order *retarded functional differential equation* (*RFDE* for short) on the tangent bundle $TM \subseteq \mathbb{R}^{2s}$, which turns out to be equivalent to the above second order equation (1.1). More precisely, in the first part of the paper we study a first order RFDE of the type

$$x'(t) = g(t, x_t), \tag{1.2}$$

where $g\colon \mathbb{R}\times BU((-\infty,0],N) \to \mathbb{R}^k$ is a functional field over a boundaryless smooth manifold $N \subseteq \mathbb{R}^k$.

Assuming that g is T-periodic in the first variable, we tackle the problem of the existence of T-periodic solutions of equation (1.2). More generally, given a closed subset X of N, we study the existence of *confined* T-periodic solutions, that is, T-periodic solutions having image in X.

The main result of the first part of the paper, Theorem 3.2 below, states that the equation (1.2) admits a confined T-periodic solution provided that X is a compact absolute neighborhood retract (ANR), with nonzero Euler-Poincaré characteristic, and the functional field g satisfies some additional conditions. The proof is given by applying the fixed point index theory for locally compact maps on ANRs to a sort of Poincaré T-translation operator acting in a suitable subset of the Banach space $C([-T,0],\mathbb{R}^k)$.

For general reference on RFDEs we suggest the monograph by Hale and Verduyn Lunel [16]. For RFDEs with finite delay in Euclidean spaces, we refer also to the works of Gaines and Mawhin [11], Nussbaum [22, 23] and Mallet-Paret, Nussbaum and Paraskevopoulos [19]. For RFDEs with infinite delay in Euclidean spaces, we recommend the article of Hale and Kato [15] and, book by Hino, Murakami and Naito [17] and the recent paper by Oliva and Rocha [26]. Finally, for RFDEs with finite delay on manifolds we cite the papers of Oliva [24, 25].

2. Preliminaries

Given a subset A of $\mathbb{R}^k$, we will denote by $BU((-\infty,0],A)$ the set of bounded and uniformly continuous maps from $(-\infty,0]$ into A with the topology of the uniform convergence. Clearly, $BU((-\infty,0],A)$ is a metric subspace of the Banach space $BU((-\infty,0],\mathbb{R}^k)$ and is complete if and only if A is closed. For brevity, throughout the paper we will use the notation

$$\widetilde{A} := BU((-\infty,0],A).$$

Moreover, the norm in $\mathbb{R}^k$ will be denoted by $|\cdot|$ and the norm in $\widetilde{\mathbb{R}^k}$ by $\|\cdot\|$.

A vector $v \in \mathbb{R}^k$ is said to be *inward* to A at a given point p in the closure $\overline{A}$ of A if there exist two sequences $\{\alpha_n\}$ in $[0,+\infty)$ and $\{p_n\}$ in A such that

$$p_n \to p \quad \text{and} \quad \alpha_n(p_n - p) \to v.$$

The set C_pA of the inward vectors to A at p is called the *tangent cone* of A at p (see [6]). One can easily check that the tangent cone is always closed in $\mathbb{R}^k$. The vector subspace of $\mathbb{R}^k$ spanned by C_pA is the *tangent space* T_pA of A at p, whose elements are the *tangent vectors* to A at p.

To simplify some statements and definitions we put $C_pA = T_pA = \emptyset$ whenever p does not belong to $\overline{A}$ (this can be regarded as a consequence of the definition of inward vector if one replaces the assumption $p \in \overline{A}$ with $p \in \mathbb{R}^k$).

Observe that T_pA is the trivial subspace $\{0\}$ of $\mathbb{R}^k$ if and only if p is an isolated point of A. In fact, if p is a limit point, then, given any $\{p_n\}$ in $A \backslash \{p\}$ such that $p_n \to p$, the sequence $\big\{\alpha_n(p_n - p)\big\}$, with $\alpha_n = 1/|p_n - p|$, admits a convergent subsequence whose limit is a unit vector. On the other hand, if p is an isolated point of A, the unique inward vector is the null one since the unique sequence $\{p_n\}$ in A convergent to p is the constant sequence coinciding with p.

One can show that, in the special and important case when A is a smooth differentiable manifold with (possibly empty) boundary ∂A (a *∂-manifold* for short), this definition of tangent space is equivalent to the classical one (see for instance [14, 20]). Moreover, if $p \in \partial A$, C_pA is a closed half-space in T_pA (delimited by $T_p\partial A$), while $C_pA = T_pA$ if $p \in A \backslash \partial A$.

2.1. **Initial value problem.** Let N be a boundaryless smooth manifold in $\mathbb{R}^k$. We say that a continuous map $g\colon \mathbb{R} \times \widetilde{N} \to \mathbb{R}^k$ is a *retarded functional tangent vector field over* N if $g(t,\varphi) \in T_{\varphi(0)}N$ for all $(t,\varphi) \in \mathbb{R} \times \widetilde{N}$. To simplify the notation, in the sequel we frequently call g a *functional field (over N).*

Let us consider a *retarded functional differential equation* (*RFDE* for short) of the type

$$x'(t) = g(t, x_t), \tag{2.1}$$

where $g\colon \mathbb{R} \times \widetilde{N} \to \mathbb{R}^k$ is a functional field over N. Here, as usual and whenever it makes sense, given $t \in \mathbb{R}$, by $x_t \in \widetilde{N}$ we mean the function $\theta \mapsto x(t + \theta)$.

A *solution* of (2.1) is a function $x\colon J \to N$, defined on an open real interval J with $\inf J = -\infty$, bounded and uniformly continuous on any closed half-line $(-\infty, b] \subset J$, and which verifies eventually the equality $x'(t) = g(t, x_t)$. That is, x is a solution of (2.1) if there exists τ, with $-\infty \le \tau < \sup J$, such that x is C^1 on the subinterval $(\tau, \sup J)$ of J, and verifies $x'(t) = g(t, x_t)$ for all $t \in (\tau, \sup J)$. Observe that the derivative of a solution x may not exist at $t = \tau$. However, the right derivative $D_+x(\tau)$ of x at τ always exists and is equal to $g(\tau, x_\tau)$. Also, notice that, since x is uniformly continuous on any closed half-line $(-\infty, b]$ of J, then $t \mapsto x_t$ is a continuous curve in $\widetilde{N}$.

A solution of (2.1) is said to be *maximal* if it is not a proper restriction of another solution to the same equation. As in the case of ODEs, Zorn's lemma implies that any solution is the restriction of a maximal solution.

In what follows, given $\eta \in \widetilde{N}$, we will also consider the initial value problem

$$\begin{cases} x'(t) = g(t, x_t), \\ x_0 = \eta\,. \end{cases} \tag{2.2}$$

A solution of (2.2) is a solution $x\colon J \to N$ of (2.1) such that $\sup J > 0$, $x'(t) = g(t, x_t)$ for $t > 0$, and $x_0 = \eta$.

Moreover, given a relatively closed subset X of N, if one takes $\eta \in \widetilde{X}$, then problem (2.2) will be called the *confined problem* and any X-valued solution of (2.2) a *confined solution.* For instance, X could be a ∂-manifold of the type $\{p \in N : F(p) \le 0\}$, where the "cutting function" $F\colon N \to \mathbb{R}$ is smooth, having $0 \in \mathbb{R}$ as a regular value (this is the situation considered in Section 4). Furthermore, N could be an open subset of $\mathbb{R}^k$ and X one of its connected components.

Following [4], we say that the functional field $g\colon \mathbb{R} \times \widetilde{N} \to \mathbb{R}^k$ is *away from N at $p \in X$* if either $g(t,\varphi) \notin C_p(N \backslash X)$ for all (t,φ) with $\varphi(0) = p$ or $g(t,\varphi) = 0$ for all (t,φ) with $\varphi(0) = p$. We point out that this condition is obviously satisfied whenever p, which is a point of X, is not in the topological boundary of X relative to N since, in that case, $C_p(N\backslash X) = \emptyset$. Notice that this condition is also satisfied when $X = N$, since $C_p(\emptyset) = \emptyset$. If g is away from N at any $p \in X$, we say that g is *away from N in X*.

Theorem 2.1 below is a particular case of a global existence result for the confined case (see [4, Theorem 3.9]; see also [1, Lemma 2.1]).

Theorem 2.1 (confined global existence)**.** *Let X be a compact subset of a boundaryless smooth manifold $N \subseteq \mathbb{R}^k$ and $g\colon \mathbb{R} \times \widetilde{N} \to \mathbb{R}^k$ a functional field away from N in X. Assume that $g(\mathbb{R} \times \widetilde{X})$ is bounded. Then, any maximal solution of the confined problem* (2.2) *is defined on the whole real line.*

The continuous dependence of the solutions on initial data is stated in Theorem 2.2 below and is a staightforward consequence of Theorem 4.4 of [4].

Theorem 2.2 (continuous dependence)**.** *Let N be a boundaryless smooth manifold and $g\colon \mathbb{R} \times \widetilde{N} \to \mathbb{R}^k$ a functional field. Assume the uniqueness of the maximal solution of problem* (2.2)*. Then, given $T > 0$, the set*

$$\mathcal{D} = \{\eta \in \widetilde{N} : \text{ the maximal solution of (2.2) is defined up to } T\}$$

is open and the map that associates to any $\eta \in \mathcal{D}$ the restriction to $[0, T]$ of the unique maximal solution of problem (2.2) *is continuous.*

2.2. **Fixed point index.** We recall that a metrizable space X is an *absolute neighborhood retract* (ANR) if, whenever it is homeomorphically embedded as a closed subset C of a metric space Y, there exists an open neighborhood V of C in Y and a retraction $r\colon V \to C$ (see e.g. [5, 13]). Polyhedra and differentiable manifolds are examples of ANRs. Let us also recall that a continuous map between topological spaces is called *locally compact* if it has the property that each point in its domain has a neighborhood whose image is contained in a compact set.

Let X be a metric ANR and consider a locally compact (continuous) X-valued map k defined on a subset $\mathcal{D}(k)$ of X. Given an open subset U of X contained in $\mathcal{D}(k)$, if the set of fixed points of k in U is compact, the pair (k, U) is called *admissible*. It is known that to any admissible pair (k, U) we can associate an integer $\mathrm{ind}_X(k, U)$ −the *fixed point index* of k in U− which satisfies properties analogous to those of the classical Leray–Schauder degree [18]. The reader can see for instance [7, 12, 21, 23] for a comprehensive presentation of the index theory for ANRs. As regards the connection with the homology theory we refer to standard algebraic topology textbooks (e.g. [8, 27]).

We summarize below the main properties of the fixed point index.

i) (*Existence*) If $\mathrm{ind}_X(k, U) \neq 0$, then k admits at least one fixed point in U.
ii) (*Normalization*) If X is compact, then $\mathrm{ind}_X(k, X) = \Lambda(k)$, where $\Lambda(k)$ denotes the Lefschetz number of k.
iii) (*Additivity*) Given two disjoint open subsets U_1, U_2 of U such that any fixed point of k in U is contained in $U_1 \cup U_2$, then $\mathrm{ind}_X(k, U) = \mathrm{ind}_X(k, U_1) + \mathrm{ind}_X(k, U_2)$.
iv) (*Excision*) Given an open subset U_1 of U such that k has no fixed points in $U\backslash U_1$, then $\mathrm{ind}_X(k, U) = \mathrm{ind}_X(k, U_1)$.
v) (*Commutativity*) Let X and Y be metric ANRs. Suppose that U and V are open subsets of X and Y respectively and that $k\colon U \to Y$ and $h\colon V \to X$ are locally compact maps. Assume that one of the sets of fixed points of hk in $k^{-1}(V)$ or kh in $h^{-1}(U)$ is compact. Then the other set is compact as well and $\mathrm{ind}_X(hk, k^{-1}(V)) = \mathrm{ind}_Y(kh, h^{-1}(U))$.
vi) (*Homotopy invariance*) Let $H\colon U \times [0, 1] \to X$ be a locally compact map such that the set $\{(x, \lambda) \in U \times [0, 1] : H(x, \lambda) = x\}$ is compact. Then $\mathrm{ind}_X(H(\cdot, \lambda), U)$ is independent of λ.

3. Existence of periodic solutions

Let $N \subseteq \mathbb{R}^k$ be a boundaryless differentiable manifold and $X \subseteq N$ a compact ANR. Given $T > 0$, denote by $\widehat{X} := C([-T,0],X)$ the metric subspace of $C([-T,0],\mathbb{R}^k)$ of the X-valued continuous function on $[-T,0]$ and by $\widehat{X}_0$ the set $\{\psi \in \widehat{X} : \psi(-T) = \psi(0)\}$. Observe that $\widehat{X}$ is complete since X is closed. Moreover, it is not difficult to show that $\widehat{X}$ is itself an ANR.

Let $g\colon \mathbb{R} \times \widetilde{N} \to \mathbb{R}^k$ be a functional field. Given $T > 0$, assume that g is T-periodic in the first variable. We are interested in proving the existence of X-valued T-periodic solutions of equation (2.1). To this end, let us consider the family of RFDE

$$x'(t) = \lambda\, g(t, x_t) \tag{3.1}$$

depending on the parameter $\lambda \in [0,1]$. Our aim is to define a parametrized Poincaré-type T-translation operator whose fixed points are the restrictions to the interval $[-T,0]$ of the T-periodic solutions of (3.1). For this purpose, we need to introduce a suitable backward extension of the elements of $\widehat{X}$. The properties of such an extension are contained in Lemma 3.1 below, obtained in [10]. In what follows, by a T-periodic map defined on $(-\infty,0]$ (or on $(-\infty,-T]$) we mean the restriction of a T-periodic map on $\mathbb{R}$.

Lemma 3.1. *There exist an open neighborhood U of $\widehat{X}_0$ in $\widehat{X}$ and a continuous map from U to $\widetilde{X}$, $\psi \mapsto \widetilde{\psi}$, with the following properties:*

1) *$\widetilde{\psi}$ is an extension of ψ;*
2) *$\widetilde{\psi}$ is T-periodic on $(-\infty,-T]$;*
3) *$\widetilde{\psi}$ is T-periodic on $(-\infty,0]$, whenever $\psi \in \widehat{X}_0$.*

Let us now state our existence result.

Theorem 3.2. *Let $N \subseteq \mathbb{R}^k$ be a boundaryless smooth manifold and $g\colon \mathbb{R}\times\widetilde{N} \to \mathbb{R}^k$ a T-periodic functional field. Let $X \subseteq N$ be a compact ANR with Euler-Poincaré characteristic $\chi(X) \neq 0$. Assume that g is away from N in X and that $g(\mathbb{R}\times\widetilde{X})$ is bounded. Also assume that, for any $\eta \in \widetilde{X}$, the maximal solution of problem (2.2) is unique. Then, the equation $x'(t) = g(t,x_t)$ has a T-periodic solution in X.*

Proof. Given $\eta \in \widetilde{X}$ and $\lambda \in [0,1]$, let $x(\eta,\lambda,\cdot)$ be the X-valued maximal solution of the parametrized confined problem

$$\begin{cases} x'(t) = \lambda\, g(t,x_t), \\ x_0 = \eta, \end{cases} \tag{3.2}$$

whose global existence is ensured by Theorem 2.1 (observe that λg is still away from N in X even for $\lambda = 0$). Let now U be an open neighborhood of $\widehat{X}_0$ in $\widehat{X}$ as in Lemma 3.1 and consider the homotopy $P\colon U \times [0,1] \to \widehat{X}$ defined by $P(\psi,\lambda)(\theta) = x(\widetilde{\psi},\lambda,T+\theta)$, where $\widetilde{\psi} \in \widetilde{X}$ is the continuous extension of ψ as in Lemma 3.1.

By an argument similar to that used in [2, Proposition 3.2], we get that $\psi \in U$ is a fixed point of $P(\cdot,\lambda)$, $\lambda \in [0,1]$, if and only if it is the restriction to $[-T,0]$ of a T-periodic solution of (3.1).

Let us show that P is admissible for the fixed point index.

P is continuous. Consider the problem

$$\begin{cases} x'(t) = \mu\, g(t,x_t), \\ \mu'(t) = 0, \\ x_0 = \eta, \\ \mu(0) = \lambda. \end{cases} \tag{3.3}$$

The continuity of P follows immediately by Lemma 3.1 and by applying Theorem 2.2 to the auxiliary problem (3.3).

The image of P is contained in a compact subset of $\widehat{X}$. By assumption, there exists $c > 0$ such that $|g(t,\varphi)| \le c$ for any $(t,\varphi) \in \mathbb{R} \times \widetilde{X}$. Hence, $P(U \times [0,1])$ is contained in the set $K = \{y \in \widehat{X} : |y'(t)| \le c\}$ which is compact by Ascoli's theorem, since X is bounded and $\widehat{X}$ complete.

The set $\{(\psi,\lambda) \in U \times [0,1] : P(\psi,\lambda) = \psi\}$ is compact. Observe that, for any $\lambda \in [0,1]$, the set $\{\psi \in U : P(\psi,\lambda) = \psi\}$ is contained in $K \cap \widehat{X}_0$ that is clearly a compact subset of U.

The three steps proved above imply that P is an admissible homotopy in U. Consequently, by the homotopy invariance of the fixed point index, we get

$$\mathrm{ind}_{\widehat{X}}(P(\cdot,1),U) = \mathrm{ind}_{\widehat{X}}(P(\cdot,0),U).$$

Now, observe that $P(\cdot,0)$ sends U onto the subset of $\widehat{X}_0 \subseteq U$ of the constant X-valued functions, which will be identified with X itself. According to this identification, the restriction $P(\cdot,0)|_X$ coincides with the identity I_X of X. Therefore, by the commutativity and normalization properties of the fixed point index, we get

$$\mathrm{ind}_{\widehat{X}}(P(\cdot,0),U) = \mathrm{ind}_X(P(\cdot,0)|_X, X) = \Lambda(I_X).$$

As well-known, the Lefschetz number $\Lambda(I_X)$ coincides with the Euler-Poincaré characteristic $\chi(X)$ of X that, by assumption, is nonzero. Hence,

$$\mathrm{ind}_{\widehat{X}}(P(\cdot,1),U) = \chi(X) \neq 0,$$

which implies that $P(\cdot,1)$ has a fixed point in U. Thus, as previously observed, this is equivalent to the existence of a T-periodic solution of equation (2.1), as claimed. □

Remark 3.3. We believe that the above existence result is still valid without the uniqueness assumption on the solutions of the initial value problem.

Remark 3.4. A functional field $g\colon \mathbb{R} \times \widetilde{N} \to \mathbb{R}^k$ is said to be *compactly Lipschitz* (for short, *c-Lipschitz*) if, given any compact subset Q of $\mathbb{R} \times \widetilde{N}$, there exists $L \ge 0$ such that

$$|g(t,\varphi) - g(t,\psi)| \le L\|\varphi - \psi\|$$

for all $(t,\varphi),(t,\psi) \in Q$. Moreover, we will say that g is *locally c-Lipschitz* if for any $(\tau,\eta) \in \mathbb{R} \times \widetilde{N}$ there exists an open neighborhood of (τ,η) in which g is c-Lipschitz. In spite of the fact that a locally Lipschitz map is not necessarily (globally) Lipschitz, one could actually show that if g is locally c-Lipschitz, then it is also (globally) c-Lipschitz. As a consequence, if g is C^1 or, more generally, locally Lipschitz in the second variable, then it is additionally c-Lipschitz. In [4] we proved that if g is a c-Lipschitz functional field, then problem (2.2) has a unique maximal solution for any $\eta \in \widetilde{N}$. For a characterisation of compact subsets of $\widetilde{N}$ see e.g. [9, Part 1, IV.6.5].

4. Retarded functional motion equations

Let $M \subseteq \mathbb{R}^s$ be a boundaryless smooth manifold and let

$$TM = \big\{(q,v) \in \mathbb{R}^s \times \mathbb{R}^s :\ q \in M,\ v \in T_qM\big\}$$

be the tangent bundle of M. Given $q \in M$, let $(T_qM)^\perp \subseteq \mathbb{R}^s$ denote the normal space of M at q. Since $\mathbb{R}^s = T_qM \oplus (T_qM)^\perp$, any vector $u \in \mathbb{R}^s$ can be uniquely decomposed into the sum of the parallel (or tangential) component $u_\pi \in T_qM$ of u at q and the normal component $u_\nu \in (T_qM)^\perp$ of u at q.

Consider the retarded functional motion equation on the constraint M

$$x''_\pi(t) = f(t, x_t) - \varepsilon x'(t), \tag{4.1}$$

where $x''_\pi(t)$ stands for the parallel component of the acceleration $x''(t) \in \mathbb{R}^s$ at the point $x(t)$, the parameter $\varepsilon > 0$ is the frictional coefficient, and the map $f\colon \mathbb{R}\times\widetilde{M} \to \mathbb{R}^s$ is a functional field, T-periodic in the first variable. Any T-periodic solution of (4.1) is called a *forced oscillation*.

Theorem 4.1 below gives a positive answer to the conjecture presented by the authors in [4].

Theorem 4.1. *Let M be a compact boundaryless smooth manifold with nonzero Euler-Poincaré characteristic, and let $f\colon \mathbb{R} \times \widetilde{M} \to \mathbb{R}^k$ be a T-periodic functional field on M. Assume that f is locally Lipschitz in the second variable and has bounded image. Then, the equation (4.1) has a forced oscillation.*

Proof. Let us observe first that the equation (4.1) can be equivalently written as

$$x''(t) = r(x(t), x'(t)) + f(t, x_t) - \varepsilon x'(t), \tag{4.2}$$

where $r\colon TM \to \mathbb{R}^s$ is a smooth map (the so-called reactive force or inertial reaction) satisfying the following properties:

(a) $r(q, v) \in (T_qM)^\perp$ for any $(q, v) \in TM$;
(b) r is quadratic in the second variable;
(c) given $(q, v) \in TM$, $r(q, v)$ is the unique vector such that $(v, r(q, v))$ belongs to $T_{(q,v)}(TM)$;
(d) any C^2 curve $\gamma\colon (a, b) \to M$ verifies the condition $\gamma''_\nu(t) = r(\gamma(t), \gamma'(t))$ for any $t \in (a, b)$, i.e. for each $t \in (a, b)$, the normal component $\gamma''_\nu(t)$ of $\gamma''(t)$ at $\gamma(t)$ equals $r(\gamma(t), \gamma'(t))$.

Now, let us transform the second order equation (4.2) into the first order system

$$\begin{cases} x'(t) = y(t), \\ y'(t) = r(x(t), y(t)) + f(t, x_t) - \varepsilon y(t). \end{cases} \tag{4.3}$$

System (4.3) is actually a first order RFDE on the noncompact manifold TM, since it can be written as

$$(x'(t), y'(t)) = G(t, (x_t, y_t)),$$

where the map $G\colon \mathbb{R} \times \widetilde{TM} \to \mathbb{R}^s \times \mathbb{R}^s$ is the T-periodic functional field over TM given by

$$G(t, (\varphi, \psi)) = (\psi(0), r(\varphi(0), \psi(0)) + f(t, \varphi) - \varepsilon\psi(0)).$$

It is easy to see that equation (4.2) and system (4.3) are equivalent in the sense that a function $x\colon J \to M$ is a solution of (4.2) if and only if the pair $(x, x')\colon J \to TM$ is a solution of (4.3).

Given $c > 0$, consider the closed subset

$$X_c = \big\{(q, v) \in TM : |v| \le c\big\}$$

of TM. It is not difficult to show that X_c is a ∂-manifold in $\mathbb{R}^s \times \mathbb{R}^s$ with boundary

$$\partial X_c = \big\{(q, v) \in X_c : |v| = c\big\}.$$

Moreover, since M is a deformation retract of X_c, then the two spaces are homotopically equivalent. Thus, $\chi(X_c) = \chi(M)$, so that $\chi(X_c) \neq 0$.

Observe now that $G(\mathbb{R} \times \widetilde{X}_c)$ is a bounded subset of $\mathbb{R}^s \times \mathbb{R}^s$, since f is bounded by assumption and X_c is compact.

Let us prove that if c is sufficiently large, then G is away from TM in X_c. To this end, write X_c by means of the inner product $\langle\cdot,\cdot\rangle$ in $\mathbb{R}^s$, as $\big\{(q, v) \in TM :$

$\langle v, v\rangle \leq c^2\big\}$ and observe first that the tangent cone of X_c at $(q, v) \in \partial X_c$ is the half subspace of $T_{(q,v)}X_c$ given by

$$C_{(q,v)}X_c = \big\{(\dot q, \dot v) \in T_{(q,v)}(TM) : \langle v, \dot v\rangle \leq 0\big\}.$$

Analogously,

$$C_{(q,v)}(TM\backslash X_c) = \big\{(\dot q, \dot v) \in T_{(q,v)}(TM) : \langle v, \dot v\rangle \geq 0\big\}.$$

Take any $t \in \mathbb{R}$ and any pair $(\varphi, \psi) \in \widetilde{X}_c$ with $|\psi(0)| = c$ and consider the inner product

$$\langle \psi(0), r(\varphi(0), \psi(0)) + f(t, \varphi) - \varepsilon\psi(0)\rangle = \langle \psi(0), r(\varphi(0), \psi(0))\rangle + \langle \psi(0), f(t, \varphi)\rangle - \varepsilon\langle \psi(0), \psi(0)\rangle.$$

Now,

$$\langle \psi(0), r(\varphi(0), \psi(0))\rangle = 0,$$

since $r(\varphi(0), \psi(0))$ belongs to $(T_{\varphi(0)}M)^{\perp}$. Moreover,

$$\langle \psi(0), f(t, \varphi)\rangle \leq |\psi(0)|\, |f(t, \varphi)| \leq K|\psi(0)|,$$

where K is such that $|f(t, \varphi)| \leq K$ for all $(t, \varphi) \in \mathbb{R} \times \widetilde{M}$. Finally,

$$\langle \psi(0), \psi(0)\rangle = c^2,$$

since $(\varphi(0), \psi(0)) \in \partial X_c$. Therefore, by choosing $c > K/\varepsilon$, we get

$$\big\langle \psi(0), r(\varphi(0), \psi(0)) + f(t, \varphi) - \varepsilon\psi(0)\big\rangle \leq Kc - \varepsilon c^2 < 0.$$

This shows that $G(t, (\varphi, \psi)) \notin C_{(q,v)}(TM\backslash X_c)$ for all $(t, (\varphi, \psi))$ with $(\varphi(0), \psi(0)) = (q, v) \in \partial X_c$. Thus, G is away from TM in X_c as claimed.

Consequently, we are reduced to the context of Theorem 3.2 with $\mathbb{R}^k = \mathbb{R}^s \times \mathbb{R}^s$, $N = TM$, $g = G$ and the confining set X given by the compact ∂-manifold X_c.

Moreover, since f is locally Lipschitz in the second variable and r is smooth, then G is locally Lipschitz as well. Therefore, taking into account Remark 3.4, we get that the initial value problem

$$\begin{cases} (x'(t), y'(t)) = G(t, (x_t, y_t)), \\ (x_0, y_0) = (\varphi, \psi) \end{cases} \tag{4.4}$$

has a unique maximal solution for any $(\varphi, \psi) \in \widetilde{TM}$.

Thus, we can apply Theorem 3.2 to the first order equation $(x'(t), y'(t)) = G(t, (x_t, y_t))$, obtaining that system (4.3) has a T-periodic solution and, equivalently, that the motion equation (4.1) has a forced oscillation. □

Remark 4.2. We believe that the assertion of Theorem 4.1 still holds without the Lipschitz assumption.

Remark 4.3. In the frictionless case (i.e. $\varepsilon = 0$) we do not know whether or not the equation

$$x''_\pi(t) = f(t, x_t) \tag{4.5}$$

has a forced oscillation. As far as we know, the problem of the existence of forced oscillations of (4.5) is still open, even in the undelayed situation. In the particular case of the spherical pendulum, i.e. $X = S^2$, or, more generally, in the case of the even dimensional pendulum (i.e. $X = S^{2n}$), the existence of forced oscillations for equation (4.5) has been proved by the authors in [3], assuming the stronger hypothesis of the continuity of the functional field f on $\mathbb{R} \times C((-\infty, 0], X)$.

References

[1] P. Benevieri, A. Calamai, M. Furi and M.P. Pera, *Retarded functional differential equations on manifolds and applications to motion problems for forced constrained systems*, Adv. Nonlinear Stud., **9** (2009), 199–214.

[2] P. Benevieri, A. Calamai, M. Furi and M.P. Pera, *A continuation result for forced oscillations of constrained motion problems with infinite delay*, to appear in Adv. Nonlinear Stud.

[3] P. Benevieri, A. Calamai, M. Furi and M.P. Pera, *On the existence of forced oscillations for the spherical pendulum acted on by a retarded periodic force*, J. Dyn. Diff. Equat., **23** (2011), 541–549.

[4] P. Benevieri, A. Calamai, M. Furi and M.P. Pera, *On General Properties of Retarded Functional Differential Equations on Manifolds*, to appear in Discrete Cont. Dynam. Systems.

[5] K. Borsuk, Theory of retracts, Polish Sci. Publ., Warsaw, 1967.

[6] G. Bouligand, Introduction à la géométrie infinitésimale directe, Gauthier-Villard, Paris, 1932.

[7] R.F. Brown, The Lefschetz fixed point theorem, Scott, Foresman and Co., Glenview, Ill.-London, 1971.

[8] A. Dold, Lectures on algebraic topology, Springer-Verlag, Berlin, 1972.

[9] N. Dunford and J.T. Schwartz, Linear Operators, Wiley & Sons, Inc., New York, 1957.

[10] M. Furi, M.P. Pera and M. Spadini, *Periodic Solutions of Functional Differential Perturbations of Autonomous Differential Equations*, Commun. Appl. Anal., **15** (2011), 381–394.

[11] R. Gaines and J. Mawhin, Coincidence degree and nonlinear differential equations, Lecture Notes in Math., **568**, Springer Verlag, Berlin, 1977.

[12] A. Granas, *The Leray-Schauder index and the fixed point theory for arbitrary ANRs*, Bull. Soc. Math. France, **100** (1972), 209–228.

[13] A. Granas and J. Dugundji, Fixed point theory, Springer-Verlag, New York, 2003.

[14] V. Guillemin and A. Pollack, Differential Topology, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1974.

[15] J.K. Hale and J. Kato, *Phase Space for Retarded Equations with Infinite Delay*, Funkc. Ekvac., **21** (1978), 11–41.

[16] J.K. Hale and S.M. Verduyn Lunel, Introduction to Functional Differential Equations, Springer Verlag, New York, 1993.

[17] Y. Hino, S. Murakami and T. Naito, Functional-differential equations with infinite delay, Lecture Notes in Math., **1473**, Springer Verlag, Berlin, 1991.

[18] J. Leray and J. Schauder, *Topologie et équations fonctionnelles*, Ann. Sci. École Norm. Sup., **51** (1934), 45–78.

[19] J. Mallet-Paret, R.D. Nussbaum and P. Paraskevopoulos, *Periodic solutions for functional-differential equations with multiple state-dependent time lags*, Topol. Methods Nonlinear Anal., **3** (1994), 101–162.

[20] J.M. Milnor, Topology from the differentiable viewpoint, Univ. Press of Virginia, Charlottesville, 1965.

[21] R.D. Nussbaum, *The fixed point index for local condensing maps*, Ann. Mat. Pura Appl., **89** (1971), 217–258.

[22] R.D. Nussbaum, *Periodic solutions of some nonlinear autonomous functional differential equations*, Ann. Mat. Pura Appl., **101** (1974), 263–306.

[23] R.D. Nussbaum, *The fixed point index and fixed point theorems*, Topological methods for ordinary differential equations (Montecatini Terme, 1991), Lecture Notes in Math., **1537**, Springer, Berlin, 1993, 143–205.

[24] W.M. Oliva, *Functional differential equations on compact manifolds and an approximation theorem*, J. Differential Equations, **5** (1969), 483–496.

[25] W.M. Oliva, *Functional differential equations–generic theory*. Dynamical systems (Proc. Internat. Sympos., Brown Univ., Providence, R.I., 1974), Vol. I, Academic Press, New York, 1976, 195–209.

[26] W.M. Oliva and C. Rocha, *Reducible Volterra and Levin–Nohel Retarded Equation with Infinite Delay*, J. Dyn. Diff. Equat., **22** (2010), 509–532.

[27] E. Spanier, Algebraic Topology, Mc Graw-Hill Series in High Math., New York, 1966.

Pierluigi Benevieri
Dipartimento di Sistemi e Informatica
Università degli Studi di Firenze
Via S. Marta 3
I-50139 Firenze, Italy
and

Instituto de Matemática e Estatística
Universidade de São Paulo
Rua do Matão 1010, São Paulo, 05508-090, Brasil

Alessandro Calamai
Dipartimento di Ingegneria Industriale e Scienze Matematiche
Università Politecnica delle Marche
Via Brecce Bianche
I-60131 Ancona, Italy.

Massimo Furi and Maria Patrizia Pera
Dipartimento di Sistemi e Informatica
Università degli Studi di Firenze
Via S. Marta 3
I-50139 Firenze, Italy

e-mail addresses:
pierluigi.benevieri@unifi.it
calamai@dipmat.univpm.it
massimo.furi@unifi.it
mpatrizia.pera@unifi.it