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FAST FORCED OSCILLATIONS FOR CONSTRAINED MOTION PROBLEMS WITH DELAY

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Dedicated to Prof. Espedito de Pascale on the occasion of his retirement.

ABSTRACT. We show that a global continuation result for T-periodic solutions of delay differential equations on manifolds proved by the authors in a previous paper still holds when the period T is smaller than the delay. As an application we get an existence result for fast forced oscillations of motion problems with delay on compact, topologically nontrivial, manifolds.

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1. INTRODUCTION

Let $M \subseteq \mathbb{R}^k$ be a smooth differentiable manifold with (possibly empty) boundary (a *∂-manifold* for short). Given $r > 0$, consider the following delay differential equation depending on a real parameter λ:

$$x'(t) = \lambda f\big(t, x(t), x(t-r)\big), \quad \lambda \geq 0, \tag{1.1}$$

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where $f : \mathbb{R} \times M \times M \to \mathbb{R}^k$ is a continuous map which is T-periodic in the first variable and tangent to M in the second one. Namely, f is such that

$$f(t+T,p,\hat{p}) = f(t,p,\hat{p}) \in T_pM\,, \quad \forall\,(t,p,\hat{p}) \in \mathbb{R} \times M \times M,$$

where $T_pM \subseteq \mathbb{R}^k$ denotes the tangent space of M at p.

By a *T-periodic pair* of the above equation we mean a pair (λ, x), where $\lambda \geq 0$ and $x : \mathbb{R} \to M$ is a T-periodic solution of (1.1) corresponding to λ. The set of the T-periodic pairs of (1.1) is regarded as a subset of $[0,+\infty) \times C_T(M)$, where $C_T(M)$ is the set of the continuous T-periodic maps from $\mathbb{R}$ to M with the metric induced by the Banach space $C_T(\mathbb{R}^k)$ of the continuous T-periodic $\mathbb{R}^k$-valued maps (with the standard supremum norm). A T-periodic pair (λ, x) will be called *trivial* if $\lambda = 0$. In this case x is a constant M-valued map and will be identified with a point of M.

In [1] we proved a *continuation* result for the T-periodic solutions of (1.1). Namely, under the assumptions that M is compact with nonzero Euler–Poincaré characteristic, that $T \geq r$, and that f satisfies a natural inward condition along ∂M (when nonempty), we proved the existence of an unbounded connected branch of nontrivial T-periodic pairs whose closure intersects M (regarded as the set of the trivial T-periodic pairs) in the so-called *set of bifurcation points*. This unusual notion of bifurcation goes back to Ambrosetti and Prodi: in [12] they used the expression *atypical bifurcation* (also called *co-bifurcation* in [6]) since, in their case, the set of trivial solutions is the kernel of a linear operator instead of the typical λ-axis.

The continuation result in [1] extends an analogous one of the last two authors for the undelayed case (see [7] and [8]), and is proved by applying the fixed point index theory for locally compact maps on ANRs to a Poincaré-type T-translation operator P_λ acting on the space $C([-r,0],M)$ of the initial conditions for the equation (1.1). Here $C([-r,0],M)$ denotes the space of the continuous M-valued functions defined on the interval $[-r,0]$ with the topology induced by the Banach space $C([-r,0],\mathbb{R}^k)$. Since M is a neighborhood retract of $\mathbb{R}^k$, it is not difficult to show (see e.g. [4]) that $C([-r,0],M)$ is a neighborhood retract of $C([-r,0],\mathbb{R}^k)$. Therefore the classical fixed point index theory applies to the operator P_λ provided that it is locally compact; and this happens if and only if the period T is not smaller than the delay r (see [1]).

Our first purpose, here, is to show how to extend, in a very simple way, the continuation result obtained in [1] just by removing the assumption $T \geq r$ (see Theorem 2.4 below). The idea of how to tackle the case $T < r$ became apparent thinking about a fruitful conversation regarding periodic delay equations with Matteo Franca in which he observed that one knows the entire past of a T-periodic function if one knows its past up to $-T$. We are grateful to Matteo for his precious hint.

The second, and main, purpose of the paper is an application of Theorem 2.4 to second order delay differential equations on manifolds: we show that, in presence of friction, any constrained periodic motion problem with delay admits forced oscillations, provided that the constraint is a smooth compact boundaryless manifold with nonzero Euler–Poincaré characteristic (see Theorem 4.1 below). As we shall see, this application to second order equations, which can be regarded as first order equations on noncompact manifolds, is made possible by two facts: 1) the result for first order equations is given on manifolds with boundary; 2) the presence of friction implies that the speed of any forced oscillation cannot be too large.

Theorem 4.1 generalizes a result of the last two authors regarding the undelayed case (see [9]) as well as a result for slow forced oscillations obtained by the authors in [2]. We ask whether or not the existence of forced oscillations holds true even in the frictionless case, provided that the constraint is compact with nonzero Euler–Poincaré characteristic. We believe the answer to this question is affirmative; but, as far as we know, this problem is still unsolved even in the undelayed case. An affirmative answer, in the undelayed situation, regarding the special constraint S^2 (the spherical pendulum) can be found in [10]. See also [11] for the extension to the case in which the constraint is S^{2n}.

2. GLOBAL CONTINUATION AND BIFURCATION

Let X be an arbitrary subset of $\mathbb{R}^k$. We recall the notions of tangent cone and tangent space of X at a given point p in the closure $\overline{X}$ of X. The definition of tangent cone we give below is equivalent to the classical one introduced by Bouligand in [3].

Definition 2.1. A vector $v \in \mathbb{R}^k$ is said to be *inward* to X at $p \in \overline{X}$ if there exist two sequences $\{\alpha_n\}$ in $[0, +\infty)$ and $\{p_n\}$ in X such that

$$p_n \to p \quad \text{and} \quad \alpha_n(p_n - p) \to v.$$

The set C_pX of the vectors which are inward to X at p is called the *tangent cone* of X at p. The *tangent space* T_pX of X at p is the vector subspace of $\mathbb{R}^k$ spanned by C_pX. A vector v of $\mathbb{R}^k$ is said to be *tangent* to X at p if $v \in T_pX$.

As in the previous section, let $M \subseteq \mathbb{R}^k$ denote a smooth ∂-manifold. It is known that in this case the tangent space T_pM has the same dimension as M for all $p \in M$. Moreover, if p is in the boundary ∂M of M, C_pM is a closed half-space in T_pM delimited by $T_p(\partial M)$. If $p \in M \backslash \partial M$, then C_pM is a vector subspace of $\mathbb{R}^k$ and, consequently, coincides with T_pM. A *tangent vector field* on M is a continuous map $w : M \to \mathbb{R}^k$ such that $w(p) \in T_pM$ for all $p \in M$. If, in particular, w is such that $w(p) \in C_pM$ for all $p \in M$, then w is a tangent vector field on M and is said to be *inward*.

As before, let $f : \mathbb{R} \times M \times M \to \mathbb{R}^k$ be a continuous map which is T-periodic in the first variable and such that $f(t, p, \hat{p}) \in T_pM$ for all $(t, p, \hat{p}) \in \mathbb{R} \times M \times M$. Given a delay $r > 0$, consider the following equation depending on a parameter $\lambda \geq 0$:

$$x'(t) = \lambda f\big(t, x(t), x(t-r)\big). \tag{2.1}$$

An element $p_0 \in M$ will be called a *bifurcation point* of (2.1) if every neighborhood of $(0, p_0)$ in $[0, +\infty) \times C_T(M)$ contains a nontrivial T-periodic pair. The following result provides a necessary condition for a point $p_0 \in M$ to be a bifurcation point.

Proposition 2.2. *Assume that $p_0 \in M$ is a bifurcation point of the equation* (2.1)*. Then the tangent vector field $w : M \to \mathbb{R}^k$ defined by*

$$w(p) = \frac{1}{T} \int_0^T f(t, p, p)\, dt$$

vanishes at p_0.

Proof. By assumption there exists a sequence $\{(\lambda_n, x_n)\}$ of T-periodic pairs such that $\lambda_n > 0$, $\lambda_n \to 0$, and $x_n(t) \to p_0$ uniformly on $\mathbb{R}$. Given $n \in \mathbb{N}$, since $x_n(T) = x_n(0)$ and $\lambda_n \neq 0$, we get

$$\int_0^T f\big(t, x_n(t), x_n(t-r)\big)\, dt = 0,$$

and the assertion follows passing to the limit. $\square$

Theorem 2.4 below provides a sufficient condition for the existence of a bifurcation point in M. More precisely, under suitable assumptions on M and f, it asserts that (2.1) admits an unbounded and connected set Σ of nontrivial T-periodic pairs whose closure intersects M (regarded as the set of the trivial T-periodic pairs). Thus, $C_T(M)$ being bounded, Σ is necessarily unbounded with respect to λ, and this ensures the existence of a T-periodic solution of the equation (2.1) for each $\lambda \geq 0$.

We need the following simple result. The proof is straightforward and will be omitted.

Lemma 2.3. *Let $n \in \mathbb{Z}$ be such that $s := r - nT > 0$. Then, given any $\lambda \geq 0$, equation* (2.1) *and*

$$x'(t) = \lambda f\big(t, x(t), x(t-s)\big) \tag{2.2}$$

have the same T-periodic solutions.

Theorem 2.4. *Let $M \subseteq \mathbb{R}^k$ be a compact ∂-manifold with nonzero Euler–Poincaré characteristic, and let $f : \mathbb{R} \times M \times M \to \mathbb{R}^k$ be continuous and such that*

$$f(t+T, p, \hat{p}) = f(t, p, \hat{p}) \in C_p M\,, \quad \forall\, (t, p, \hat{p}) \in \mathbb{R} \times M \times M.$$

Then, the equation (2.1) *admits an unbounded connected set of nontrivial T-periodic pairs whose closure meets the set of the trivial T-periodic pairs. In particular, the equation*

$$x'(t) = f\big(t, x(t), x(t-r)\big)$$

has a T-periodic solution.

Proof. If $T \geq r$, the assertion follows directly from Theorem 4.6 in [1]. If $T < r$, there exists n such that $0 < r - nT \leq T$. Taking $s := r - nT$, Lemma 2.2 ensures that

$$x'(t) = \lambda f\big(t, x(t), x(t-s)\big)$$

has the same set of T-periodic pairs as the equation (2.1); and the assertion follows again from Theorem 4.6 in [1]. $\square$

Observe that from Proposition 2.2 and Theorem 2.4 we can deduce the following well known consequence of the Poincaré–Hopf Theorem: *If w is an inward tangent vector field on a compact ∂-manifold with nonzero Euler–Poincaré characteristic, then w must vanish at some point.*

3. SOME EXAMPLES

In this section we give three examples illustrating how Theorem 2.4 can be applied. In the first one $M \subseteq \mathbb{R}^k$ is the closure of an open ball; in the second one M is an annulus in $\mathbb{R}^{2n+1}$; and in the third one M is a (two dimensional) sphere in $\mathbb{R}^3$. As before, any point $p \in M$ will be identified with the constant function which assigns p to any $t \in \mathbb{R}$. All the maps are tacitly assumed to be continuous.

Example 3.1. Let $f : \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ be T-periodic in the first variable, and assume that the inner product $\langle f(t,p,\hat{p}), p\rangle$ is negative for $\|p\|$ large and all $(t,\hat{p}) \in \mathbb{R} \times \mathbb{R}^k$. Let us prove that the equation

$$x'(t) = \lambda f\big(t, x(t), x(t-1)\big) \tag{3.1}$$

admits a connected branch of T-periodic pairs $(\lambda, x) \in (0,+\infty) \times C_T(\mathbb{R}^k)$ which is unbounded with respect to λ and whose closure in $[0,+\infty) \times C_T(\mathbb{R}^k)$ contains a pair of the type $(0, p_0)$ with $p_0 \in \mathbb{R}^k$. Thus p_0 is a bifurcation point of (3.1) and, by Proposition 2.2, one has $w(p_0) = 0$, where $w : \mathbb{R}^k \to \mathbb{R}^k$ is the average wind velocity defined by

$$w(p) = \frac{1}{T}\int_0^T f(t,p,p)\,dt.$$

By assumption, there exists $R > 0$ such that $\langle f(t,p,\hat{p}), p\rangle$ is negative for $\|p\| = R$ and all $(t,\hat{p}) \in \mathbb{R} \times \mathbb{R}^k$. Let $M = \overline{B(0,R)}$, where $B(0,R)$ denotes the open ball in $\mathbb{R}^k$ centered at 0 with radius R. Clearly, the vector $f(t,p,\hat{p})$ points inward M for each $(t,p,\hat{p}) \in \mathbb{R} \times M \times M$. Moreover, $\chi(M) = 1$ since M is contractible. Hence, Theorem 2.4 applies to the equation (3.1).

Example 3.2. Let $k \in \mathbb{N}$ be odd and let $f : \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ be T-periodic in the first variable. Assume that $f(t,p,\hat{p})$ is centrifugal for $\|p\| > 0$ small and centripetal for $\|p\|$ large. Let us show that the equation

$$x'(t) = f\big(t, x(t), x(t-1)\big)$$

has a T-periodic solution $x(t)$ satisfying the condition $x(t) \neq 0$ for all $t \in \mathbb{R}$. Incidentally, observe that the above equation admits the trivial solution since, f being continuous, as a consequence of the centrifugal hypothesis on f we must have $f(t, 0, \hat{p}) = 0$ for all $(t, \hat{p}) \in \mathbb{R} \times \mathbb{R}^k$.

Because of the centrifugal and centripetal assumptions, there exist $R_1, R_2 > 0$, with $R_1 < R_2$, such that for all $(t, \hat{p}) \in \mathbb{R} \times \mathbb{R}^k$ the inner product $\langle f(t, p, \hat{p}), p\rangle$ is positive when $\|p\| = R_1$ and negative when $\|p\| = R_2$. Let M be the annulus $\overline{B(0, R_2)} \backslash B(0, R_1)$. Clearly, the vector $f(t, p, \hat{p})$ points inward M for any $(t, p, \hat{p}) \in \mathbb{R} \times M \times M$. Moreover, $\chi(M) = 2$ since M is homotopically equivalent to the even dimensional sphere S^{k-1}. Hence, Theorem 2.4 implies that the equation

$$x'(t) = f\big(t, x(t), x(t-1)\big)$$

has a T-periodic solution lying on the annulus M.

In the above example, the assumption that the dimension k is odd cannot be removed. In fact, if k is any even natural number, we may define a centrifugal-centripetal vector field $f : \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ by

$$f(t, p, \hat{p}) = Ap + (1 - \|p\|)p,$$

where A is the $k \times k$ matrix associated with the linear operator

$$(p_1, p_2, \ldots, p_k) \mapsto (-p_2, p_1, \ldots, -p_k, p_{k-1}).$$

Observe that f is an autonomous tangent vector field on $\mathbb{R}^k$ which does not depend on the third variable. Therefore, given any $T > 0$, it may be regarded as T-periodic. However, all the periodic solutions of

$$x' = Ax + (1 - \|x\|)x$$

have period 2π since are also solutions of the linear differential equation $x' = Ax$. In fact, because of the centrifugal-centripetal property of f, they must lie on the unit sphere S^{k-1}.

Example 3.3. Consider the following system of delay differential equations:

$$\begin{cases} x_1'(t) = -x_2(t)x_3(t-1) \\ x_2'(t) = x_1(t)x_3(t-1) - x_3(t)\sin t \\ x_3'(t) = x_2(t)\sin t\,. \end{cases}$$

Let us show that this system has a 2π-periodic solution lying on the unit sphere S^2 of $\mathbb{R}^3$.

Let $f : \mathbb{R} \times S^2 \times S^2 \to \mathbb{R}^3$ be defined by

$$f(t, p, \hat{p}) = (-p_2\hat{p}_3,\; p_1\hat{p}_3 - p_3 \sin t,\; p_2 \sin t),$$

where $p = (p_1, p_2, p_3)$ and $\hat{p} = (\hat{p}_1, \hat{p}_2, \hat{p}_3)$ belong to S^2. Notice that f is tangent to S^2, since $\langle f(t, p, \hat{p}), p\rangle = 0$ for all $(t, \hat{p}) \in \mathbb{R} \times S^2$. Moreover, it is 2π-periodic with respect to $t \in \mathbb{R}$. We need to prove that the equation

$$x'(t) = f\big(t, x(t), x(t-1)\big)$$

admits a 2π-periodic solution (on S^2). This is a consequence of Theorem 2.4, since $\chi(S^2) = 2$.

4. APPLICATION TO CONSTRAINED MOTION PROBLEMS

Let $N \subseteq \mathbb{R}^s$ be a smooth boundaryless manifold and let $\phi : \mathbb{R} \times N \times N \to \mathbb{R}^s$ be a continuous map which is T-periodic in the first variable and tangent to N in the second one. That is,

$$\phi(t+T, q, \hat{q}) = \phi(t, q, \hat{q}) \in T_qN, \quad \forall\, (t, q, \hat{q}) \in \mathbb{R} \times N \times N.$$

Consider the following second order delay differential equation on N:

$$x''_\pi(t) = \phi\big(t, x(t), x(t-r)\big) - \varepsilon x'(t), \tag{4.1}$$

where, regarding (4.1) as a motion equation,

1. $x''_\pi(t)$ stands for the tangential part of the acceleration $x''(t) \in \mathbb{R}^s$ at the point $x(t)$;
2. the frictional coefficient ε is nonnegative;
3. $r > 0$ is the delay.

By a *solution* of (4.1) we mean a continuous function $x : J \to N$, defined on a (possibly unbounded) real interval, with length greater than r, which is of class C^2 on the subinterval $(\inf J + r, \sup J)$ of J and verifies

$$x''_\pi(t) = \phi\big(t, x(t), x(t-r)\big) - \varepsilon x'(t)$$

for all $t \in J$ with $t > \inf J + r$. A *forced oscillation of* (4.1) is a solution which is T-periodic and globally defined on $J = \mathbb{R}$.

We want to show that equation (4.1) admits at least one forced oscillation, provided that the frictional coefficient ε is nonzero and the constraint N is compact with nonzero Euler–Poincaré characteristic (see Theorem 4.1 below). The existence of a T-periodic solution of (4.1) will be deduced from Theorem 2.4. The possibility of reducing (4.1) to a first order equation is due to the fact that any second order differential equation on N is equivalent to a first order system on the tangent bundle TN of N. The difficulty arising from the noncompactness of TN will be removed by restricting the search for T-periodic solutions to a convenient compact manifold with boundary contained in TN. The choice of such a manifold is suggested by *a priori* estimates on the set of all the possible T-periodic solutions of equation (4.1). These estimates are made possible by the compactness of N and the presence of the positive frictional coefficient ε.

Given $q \in N$, let $(T_qN)^\perp$ denote the normal space of N at q. Since $\mathbb{R}^s = T_qN \oplus (T_qN)^\perp$, any vector $u \in \mathbb{R}^s$ can be uniquely decomposed into the sum of the *parallel* (or *tangential*) *component* $u_\pi \in T_qN$ *of* u *at* q and the *normal component* $u_\nu \in (T_qN)^\perp$ *of* u *at* q. By

$$TN = \{(q, v) \in \mathbb{R}^s \times \mathbb{R}^s : \ q \in N, \ v \in T_qN\}$$

we denote the *tangent bundle of* N, which is a smooth manifold containing a natural copy of N via the embedding $q \mapsto (q, 0)$. The natural projection of TN onto N is just the restriction (to TN as domain and to N as codomain) of the projection of $\mathbb{R}^s \times \mathbb{R}^s$ onto the first factor.

It is known that, associated with $N \subseteq \mathbb{R}^s$, there exists a unique smooth map $\nu : TN \to \mathbb{R}^s$, called the *reactive force* (or *inertial reaction*), with the following properties:

(a) $\nu(q, v) \in (T_qN)^\perp$ for any $(q, v) \in TN$;
(b) ν is quadratic in the second variable;
(c) given $(q, v) \in TN$, $\nu(q, v)$ is the unique vector such that $(v, \nu(q, v))$ belongs to $T_{(q,v)}(TN)$;
(d) any C^2 curve $\gamma : (a, b) \to N$ verifies the condition $\gamma''_\nu(t) = \nu(\gamma(t), \gamma'(t))$ for any $t \in (a, b)$, i.e. for each $t \in (a, b)$, the normal component $\gamma''_\nu(t)$ of $\gamma''(t)$ at $\gamma(t)$ equals $\nu(\gamma(t), \gamma'(t))$.

The map ν is strictly related to the second fundamental form on N and may be interpreted as the reactive force due to the constraint N.

By condition (d) above, equation (4.1) can be equivalently written as

$$x''(t) = \nu(x(t), x'(t)) + \phi(t, x(t), x(t - r)) - \varepsilon x'(t). \tag{4.2}$$

Notice that, if the above equation reduces to the so-called *inertial equation*

$$x''(t) = \nu(x(t), x'(t)),$$

one obtains the geodesics of N as solutions.

Equation (4.2) can be written as a first order differential system on TN as follows:

$$\begin{cases} x'(t) = y(t) \\ y'(t) = \nu(x(t), y(t)) + \phi(t, x(t), x(t - r)) - \varepsilon y(t). \end{cases}$$

This makes sense since, for any $(t, (q, v), (\hat{q}, \hat{v})) \in \mathbb{R} \times TN \times TN$, both the vectors $(v, \nu(q, v))$ and $(0, \phi(t, q, \hat{q}) - \varepsilon v)$ belong to the subspace $T_{(q,v)}(TN)$ of $\mathbb{R}^s \times \mathbb{R}^s$ (see, for example, [5] for more details).

The following result is a consequence of Theorem 2.4.

Theorem 4.1. *Assume that N is compact with nonzero Euler–Poincaré characteristic. Then the equation* (4.1) *has a forced oscillation, provided that the frictional coefficient ε is nonzero.*

Proof. As we already pointed out, the equation (4.1) is equivalent to the following first order system on TN:

$$\begin{cases} x'(t) = y(t) \\ y'(t) = \nu\big(x(t), y(t)\big) + \phi\big(t, x(t), x(t-r)\big) - \varepsilon y(t). \end{cases} \tag{4.3}$$

Define $f : \mathbb{R} \times TN \times TN \to \mathbb{R}^s \times \mathbb{R}^s$ by

$$f\big(t, (q, v), (\hat{q}, \hat{v})\big) = \big(v, \nu(q, v) + \phi(t, q, \hat{q}) - \varepsilon v\big).$$

Notice that f is T-periodic in the first variable and tangent to TN in the second one. Given $c > 0$, set

$$M_c = (TN)_c = \big\{(q, v) \in N \times \mathbb{R}^s : v \in T_q N,\ \|v\| \le c\big\}.$$

It is not difficult to show that M_c is a compact manifold in $\mathbb{R}^s \times \mathbb{R}^s$ with boundary

$$\partial M_c = \big\{(q, v) \in N \times \mathbb{R}^s : v \in T_q N,\ \|v\| = c\big\}.$$

Observe that

$$T_{(q,v)}(M_c) = T_{(q,v)}(TN)$$

for all $(q, v) \in M_c$. Moreover, $\chi(M_c) = \chi(N)$ since M_c and N are homotopically equivalent (N being a deformation retract of TN).

Now assume that the frictional coefficient ε is nonzero. We claim that if c is large enough, then f points inward along the boundary of M_c. To see this, let $(q, v) \in \partial M_c$ be fixed, and observe that the inward half-subspace of $T_{(q,v)}(M_c)$ is

$$C_{(q,v)}(M_c) = \big\{(\dot{q}, \dot{v}) \in T_{(q,v)}(TN) : \langle v, \dot{v}\rangle \le 0\big\},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^s$. We have to show that if c is large enough then $f\big(t, (q, v), (\hat{q}, \hat{v})\big)$ belongs to $C_{(q,v)}(M_c)$ for any $t \in \mathbb{R}$ and $(\hat{q}, \hat{v}) \in TN$; that is,

$$\langle v, \nu(q, v) + \phi(t, q, \hat{q}) - \varepsilon v\rangle = \langle v, \nu(q, v)\rangle + \langle v, \phi(t, q, \hat{q})\rangle - \varepsilon\langle v, v\rangle \le 0$$

for any $t \in \mathbb{R}$ and $(\hat{q}, \hat{v}) \in TN$. Now, $\langle v, \nu(q, v)\rangle = 0$ since $\nu(q, v)$ belongs to $(T_q N)^\perp$. Moreover, $\langle v, v\rangle = c^2$ since $(q, v) \in \partial M_c$, and

$$\langle v, \phi(t, q, \hat{q})\rangle \le \|v\| \|\phi(t, q, \hat{q})\| \le K\|v\|,$$

where

$$K = \max\big\{\|\phi(t, q, \hat{q})\| : (t, q, \hat{q}) \in \mathbb{R} \times N \times N\big\}.$$

Thus,

$$\langle v, \nu(q, v) + \phi(t, q, \hat{q}) - \varepsilon v\rangle \le Kc - \varepsilon c^2.$$

This shows that, if we choose $c > K/\varepsilon$, then f points inward along ∂M_c, as claimed. Therefore, given $c > K/\varepsilon$, Theorem 2.4 implies that system (4.3) admits a T-periodic solution in M_c, and this completes the proof. □

It is evident from this proof that the above result holds true even if we replace

$$\phi(t, q, \hat{q}) - \varepsilon v$$

by a T-periodic force $\psi\big(t,(q,v),(\hat{q},\hat{v})\big) \in T_qN$ satisfying the following assumption: *there exists $c > 0$ such that $\langle \psi\big(t,(q,v),(\hat{q},\hat{v})\big), v\rangle \leq 0$ for any*

$$\big(t,(q,v),(\hat{q},\hat{v})\big) \in \mathbb{R} \times TN \times TN$$

such that $\|v\| = c$.

We point out that, in the above theorem, the condition $\chi(N) \neq 0$ cannot be dropped. Consider for example the equation

$$\theta''(t) = a - \varepsilon\theta'(t), \quad t \in \mathbb{R}, \tag{4.4}$$

where a is a nonzero constant and $\varepsilon > 0$. Equation (4.4) can be regarded as a second order ordinary differential equation on the unit circle $S^1 \subseteq \mathbb{C}$, where θ represents the angular coordinate. In this case, a solution $\theta(\cdot)$ of (4.4) is periodic of period $T > 0$ if and only if for some $k \in \mathbb{Z}$ it satisfies the boundary conditions

$$\begin{cases} \theta(T) - \theta(0) = 2k\pi \\ \theta'(T) - \theta'(0) = 0. \end{cases}$$

Notice that the applied force a represents a nonvanishing autonomous vector field on S^1. Thus, it is periodic of arbitrary period. However, simple calculations show that any periodic solution of (4.4) has period $T = 2\pi\varepsilon/a$.

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