RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS ON MANIFOLDS AND APPLICATIONS TO MOTION PROBLEMS FOR FORCED CONSTRAINED SYSTEMS

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Abstract. We study retarded functional differential equations on manifolds of the type $x'(t) = f(t, x_t)$, where f is a T-periodic vector field. Using the fixed point index theory on ANRs, we prove the existence of T-periodic solutions.

As an application, we show the existence of forced oscillations of motion problems on topologically nontrivial compact constraints. The result is obtained under the assumption that the frictional coefficient is nonzero, and we conjecture that it is still true in the frictionless case.

1. Introduction

Let $M \subseteq \mathbb{R}^k$ be a smooth manifold, possibly with boundary, and let $f : \mathbb{R} \times C((-\infty, 0], M) \to \mathbb{R}^k$ be a continuous map which is T-periodic in the first variable and such that

$$f(t + T, \varphi) = f(t, \varphi) \in T_{\varphi(0)}M\,, \quad \forall\,(t, \varphi) \in \mathbb{R} \times C((-\infty, 0], M),$$

where, given $p \in M$, by $T_pM \subseteq \mathbb{R}^k$ we denote the tangent space of M at p. Such a map will be called a T-periodic *vector field on* M. The vector field f will be said *inward to* M if $f(t, \varphi)$ belongs to the *tangent cone* of M at $\varphi(0)$ for all $(t, \varphi) \in \mathbb{R} \times C((-\infty, 0], M)$.

We are interested in the existence of T-periodic solutions of the following retarded functional differential equation (*RFDE* for short) on M:

$$x'(t) = f(t, x_t). \tag{1.1}$$

Here, given $t \in \mathbb{R}$, we adopt the standard notation $x_t : (-\infty, 0] \to M$ for the function defined by $x_t(\theta) = x(t + \theta)$.

Among the wide bibliography on RFDEs, we cite the works of Gaines and Mawhin ([13]), Nussbaum ([23, 24]) and Mallet-Paret, Nussbaum and Paraskevopoulos ([19]) about equations in Euclidean spaces as well as of Oliva ([25]) about equations on manifolds. For general reference we suggest the monograph by Hale and Verduyn Lunel ([16]).

Our main result (see Theorem 3.3 below) states that, if M is compact with nonzero Euler–Poincaré characteristic, and f is a T-periodic inward vector field on M which is bounded and verifies a suitable Lipschitz-type assumption, then the equation (1.1) admits a T-periodic solution. To prove this result we apply the classical fixed point index theory for locally compact maps on ANRs to a suitable Poincaré-type T-translation operator acting in the Banach space $C([-T, 0], \mathbb{R}^k)$.

The idea of considering the Banach space $C([-T, 0], \mathbb{R}^k)$ instead of the metrizable space $C((-\infty, 0], M)$ spread out from a fruitful discussion with Matteo Franca who pointed out to us that to know a T-periodic solution on the whole past one just need to know it starting from a suitable not too far away past.

This paper is strictly related to our recent ones ([1, 3, 4]) in which we study delay differential equations of the type

$$x'(t) = \widetilde{f}(t, x(t), x(t-1)), \tag{1.2}$$

where $\widetilde{f} : \mathbb{R} \times M \times M \to \mathbb{R}^k$ is a continuous map which is T-periodic in the first variable and tangent to M in the second one; i.e.

$$\widetilde{f}(t+T, p, q) = \widetilde{f}(t, p, q) \in T_pM\,, \quad \forall\,(t, p, q) \in \mathbb{R} \times M \times M.$$

Equation (1.2) is a so-called *constant time lag equation*. In this type of equations the derivative $x'(t)$ depends on the states $x(t)$ and $x(t-1)$, while in the RFDE (1.1) the right-hand side depends on the whole function x_t. Roughly speaking, in equation (1.2) the delay could be uniformly distributed in the whole past $(-\infty, 0]$. In addition, we observe that the equation with constant time lag (1.2) is a special case of the RFDE (1.1); to see this, given $\widetilde{f} : \mathbb{R} \times M \times M \to \mathbb{R}^k$ as above, define the vector field $f : \mathbb{R} \times C((-\infty, 0], M) \to \mathbb{R}^k$ by

$$f(t, \varphi) = \widetilde{f}(t, \varphi(0), \varphi(-1)).$$

Actually in [1, 3, 4] we do not limit ourselves to the study of the existence of T-periodic solutions, but we focus on the structure of the set of pairs (λ, x), where λ is a real parameter and x a T-periodic solution of the equation

$$x'(t) = \lambda \widetilde{f}(t, x(t), x(t-1)),$$

and we obtain global bifurcation results. Here we are merely concerned with existence results, leaving the study of bifurcation to future investigation.

We conclude the paper with an application to motion problems for forced constrained systems. Precisely, we consider the following retarded functional motion equation on a boundaryless manifold $X \subseteq \mathbb{R}^s$:

$$x''_\pi(t) = F(t, x_t) - \varepsilon x'(t), \tag{1.3}$$

where

(1) $x''_\pi(t)$ stands for the tangential part of the acceleration $x''(t) \in \mathbb{R}^s$ at the point $x(t) \in X$,
(2) the frictional coefficient ε is a positive constant,
(3) the applied force $F : \mathbb{R} \times C((-\infty, 0], X) \to \mathbb{R}^s$ is a continuous, T-periodic vector field on X.

We prove (see Theorem 4.1 below) that the equation (1.3) admits at least one *forced oscillation*, i.e. a T-periodic solution, provided that the constraint X is compact with nonzero Euler–Poincaré characteristic and the vector field F is bounded and verifies a suitable Lipschitz-type assumption. To get Theorem 4.1 we apply Theorem 3.3 to a RFDE on the noncompact tangent bundle $TX \subseteq \mathbb{R}^{2s}$ which is equivalent to (1.3).

Theorem 4.1 generalizes analogous results given in [2] and [4] for equations with constant time lag (see also [10] for the undelayed case). As far as we know, when the frictional coefficient ε is zero, the problem of the existence of forced oscillations of (1.3) is still open, even in the undelayed case. An affirmative answer, in the undelayed situation, regarding the special constraint $X = S^2$ (the spherical pendulum) can be found in [11] (see also [12] for the extension to the case $X = S^{2n}$).

2. Preliminaries

2.1. **RFDE.** Let M be an arbitrary subset of $\mathbb{R}^k$. We recall the notions of tangent cone and tangent space of M at a given point p in the closure $\overline{M}$ of M. The definition of tangent cone is equivalent to the classical one introduced by Bouligand in [6].

Definition 2.1. A vector $v \in \mathbb{R}^k$ is said to be *inward* to M at $p \in \overline{M}$ if there exist two sequences $\{\alpha_n\}$ in $[0, +\infty)$ and $\{p_n\}$ in M such that

$$p_n \to p \quad \text{and} \quad \alpha_n(p_n - p) \to v.$$

The set C_pM of the inward vectors to M at p is called the *tangent cone* of M at p. The *tangent space* T_pM of M at p is the vector subspace of $\mathbb{R}^k$ spanned by C_pM. A vector v of $\mathbb{R}^k$ is said to be *tangent* to M at p if $v \in T_pM$.

To simplify some statements and definitions we put $C_pM = T_pM = \emptyset$ whenever $p \in \mathbb{R}^k$ does not belong to $\overline{M}$ (this can be regarded as a consequence of Definition 2.1 if one replaces the assumption $p \in \overline{M}$ with $p \in \mathbb{R}^k$). Observe that T_pM is the trivial subspace $\{0\}$ of $\mathbb{R}^k$ if and only if p is an isolated point of M. In fact, if p is a limit point, then, given any $\{p_n\}$ in $M \backslash \{p\}$ such that $p_n \to p$, the sequence $\{\alpha_n(p_n - p)\}$, with $\alpha_n = 1/\|p_n - p\|$, admits a convergent subsequence whose limit is a unit vector.

One can show that in the special and important case when M is a smooth manifold with (possibly empty) boundary ∂M (a *∂-manifold* for short), this definition of tangent space is equivalent to the classical one (see for instance [20], [15]). Moreover, if $p \in \partial M$, C_pM is a closed half-space in T_pM (delimited by $T_p\partial M$), while $C_pM = T_pM$ if $p \in M \backslash \partial M$.

Let, as above, M be a subset of $\mathbb{R}^k$. We denote by D a nontrivial closed real interval with $\max D = 0$; that is, D is either $(-\infty, 0]$ or $[-r, 0]$ with $r > 0$. By $C(D, M)$ we mean the metrizable space of the M-valued continuous functions defined on D with the topology of the uniform convergence on compact subintervals of D.

Given a continuous function $x : J \to M$, defined on a real interval J, and given $t \in \mathbb{R}$ such that $t + D \subseteq J$, we adopt the standard notation $x_t : D \to M$ for the function defined by $x_t(\theta) = x(t + \theta)$.

Let $h : \mathbb{R} \times C(D, M) \to \mathbb{R}^k$ be a continuous map. We say that h is a *vector field on M* if $h(t, \varphi) \in T_{\varphi(0)}M$ for all $(t, \varphi) \in \mathbb{R} \times C(D, M)$. In particular, h will be said *inward* (to M) if $h(t, \varphi) \in C_{\varphi(0)}M$ for all (t, φ). If M is a closed subset of a boundaryless smooth manifold $N \subseteq \mathbb{R}^k$, we will say that h is *away from $N \backslash M$* if $h(t, \varphi) \notin C_{\varphi(0)}(N \backslash M)$ for all $(t, \varphi) \in \mathbb{R} \times C(D, M)$. Notice that this condition is satisfied whenever $\varphi(0)$, which is a point of M, is not in the topological boundary of M relative to N since, in that case, $C_{\varphi(0)}(N \backslash M) = \emptyset$.

In this paper we are interested in retarded functional differential equations (*RFDE* for short) of the type

$$x'(t) = h(t, x_t), \tag{2.1}$$

where $h : \mathbb{R} \times C(D, M) \to \mathbb{R}^k$ is a vector field on M.

By a *solution* of (2.1) we mean a continuous function $x : J \to M$, defined on a real interval J with $\inf J = -\infty$, which verifies eventually the equality $x'(t) = h(t, x_t)$. That is, x is a solution of (2.1) if there exists $\bar{t}$, with $-\infty \le \bar{t} < \sup J$, such that x is C^1 on the subinterval $(\bar{t}, \sup J)$ of J and verifies $x'(t) = h(t, x_t)$ for all $t \in J$ with $t > \bar{t}$.

Observe that, when $D = [-r, 0]$, there is a one-to-one correspondence between our notion of solution and the classical one which can be found e.g. in [16] (see also [25]). The correspondence is the one that assigns to any solution of (2.1) its restriction to the interval $[\bar{t} - r, \sup J)$.

Remark 2.2. Any equation of the form (2.1) with $D = [-r, 0]$ can be regarded as an equation of the same type with $D = (-\infty, 0]$, in the sense that to any equation (2.1) with $D = [-r, 0]$ can be associated an equivalent equation of the same type with $D = (-\infty, 0]$. In other words, given a vector field $h : \mathbb{R} \times C([-r, 0], M) \to \mathbb{R}^k$, there exists a vector field $g : \mathbb{R} \times C((-\infty, 0], M) \to \mathbb{R}^k$ such that the equation

$$x'(t) = g(t, x_t) \tag{2.2}$$

has the same set of solutions as (2.1). To see this, it is enough to define $g : \mathbb{R} \times C((-\infty, 0], M) \to \mathbb{R}^k$ by

$$g(t, \varphi) = h(t, \varphi|_{[-r,0]}),$$

for any $(t, \varphi) \in \mathbb{R} \times C((-\infty, 0], M)$.

As a consequence of Remark 2.2, it is not restrictive to study the broader class of RFDE's of the type

$$x'(t) = g(t, x_t), \tag{2.3}$$

where $g : \mathbb{R} \times C((-\infty, 0], M) \to \mathbb{R}^k$ is a vector field on M. Therefore, from now on we will focus on this kind of equations.

2.2. **Initial value problem.** We are now interested in the following initial value problem:

$$\begin{cases} x'(t) = g(t, x_t), & t > 0, \\ x(t) = \eta(t), & t \le 0, \end{cases} \tag{2.4}$$

where M is a subset of $\mathbb{R}^k$, $g : \mathbb{R} \times C((-\infty, 0], M) \to \mathbb{R}^k$ is a vector field on M, and $\eta : (-\infty, 0] \to M$ is a continuous map.

A *solution* of problem (2.4) is a solution $x : J \to M$ of (2.3) such that $\sup J > 0$, $x'(t) = g(t, x_t)$ for $t > \bar{t} = 0$, and $x(t) = \eta(t)$ for $t \le 0$.

The following technical lemma regards the existence of a persistent solution of problem (2.4).

Lemma 2.3. *Let M be a compact subset of a boundaryless smooth manifold $N \subseteq \mathbb{R}^k$, and g a vector field on M which is away from $N \backslash M$. Suppose that g is bounded. Then problem* (2.4) *admits at least one solution defined on the whole real line.*

Proof. We define a suitable extension $\widetilde{g} : \mathbb{R} \times C((-\infty, 0], \mathbb{R}^k) \to \mathbb{R}^k$ of g. Let $U \subseteq \mathbb{R}^k$ be a tubular neighborhood of N and let $\rho : U \to N$ be the associated retraction (if N is an open subset of $\mathbb{R}^k$, then $U = N$ and ρ is the identity). Fix $\delta > 0$ such that $M_\delta = \{p \in U : \operatorname{dist}(p, M) \le \delta\}$ is a compact neighborhood of M in U.

We extend g to a vector field $\widetilde{g} : \mathbb{R} \times C((-\infty, 0], \mathbb{R}^k) \to \mathbb{R}^k$ with the following properties:

i) $\widetilde{g}$ is bounded;
ii) $\widetilde{g}(t, \varphi) = 0$ if $\operatorname{dist}(\varphi(0), M) \ge \delta$;
iii) $\widetilde{g}(t, \varphi) \in T_{\rho(\varphi(0))}N$ for all $(t, \varphi) \in \mathbb{R} \times C((-\infty, 0], \mathbb{R}^k)$ such that $\varphi(0) \in M_\delta$.

Observe that the existence of a map $\widetilde{g}$ satisfying the first two properties is ensured by the Tietze Extension Theorem. In fact, $C((-\infty, 0], M)$ and $\{\varphi \in C((-\infty, 0], \mathbb{R}^k) : \operatorname{dist}(\varphi(0), M) \ge \delta\}$ are two disjoint closed subsets of the metrizable space $C((-\infty, 0], \mathbb{R}^k)$. Moreover, we may assume that $\widetilde{g}$ has the additional property *iii)*. In fact, if this is not the case, it is sufficient to consider the orthogonal projection of $\widetilde{g}(t, \varphi)$ onto the space $T_{\rho(\varphi(0))}N$.

Now, consider the following auxiliary problem depending on $n \in \mathbb{N}$:

$$\begin{cases} x'(t) = \widetilde{g}(t, x_{t-\frac{1}{n}}), & t > 0, \\ x(t) = \eta(t), & t \le 0. \end{cases} \tag{2.5}$$

Clearly problem (2.5) has a solution defined on $(-\infty, 1/n]$ and, given a solution on $(-\infty, \beta]$, one can extend it to the interval $(-\infty, \beta + 1/n]$. Thus, problem (2.5) has a global solution $x^n : \mathbb{R} \to \mathbb{R}^k$. By Ascoli's Theorem we may assume that, as $n \to \infty$, $\{x^n(t)\}$ converges to a continuous function $x(t)$, uniformly on compact subintervals of $\mathbb{R}$.

Observe that problem (2.5) is equivalent to the following integral equation:

$$x(t) = \eta(0) + \int_0^t \widetilde{g}(s, x_{s-\frac{1}{n}})\, ds, \quad t \ge 0.$$

Moreover, for any given $t > 0$, the sequence $\{\widetilde{g}(t, x^n_{t-\frac{1}{n}})\}$ converges to $\widetilde{g}(t, x_t)$. Thus, $\widetilde{g}$ being bounded, from Lebesgue's Dominated Convergence Theorem we get

$$x(t) = \eta(0) + \int_0^t \widetilde{g}(s, x_s)\, ds, \quad t \ge 0.$$

Therefore, $x'(t) = \widetilde{g}(t, x_t)$ for all $t > 0$, and the assertion follows if we prove that $x(t)$ lies entirely in M.

Let us show first that $x(t) \in N$ for all $t \ge 0$ (this could be false if $\widetilde{g}$ were an arbitrary continuous extension of g). Clearly $x(t) \in M_\delta$ for all $t \ge 0$ (recall that $\widetilde{g}(t, \varphi) = 0$ if $\varphi(0) \notin M_\delta$). Thus, the C^1 function

$$\sigma(t) = \|x(t) - \rho(x(t))\|^2$$

is well defined for $t \ge 0$ and verifies $\sigma(0) = 0$. Assume, by contradiction, that $x(t) \notin N$ for some $t > 0$. This means that $\sigma(t) > 0$ for some $t > 0$ and, consequently, its derivative must be positive at some $\tau > 0$. That is,

$$\sigma'(\tau) = 2\big\langle x(\tau) - \rho(x(\tau)),\ \widetilde{g}(\tau, x_\tau) - w(\tau)\big\rangle > 0,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^k$, and $w(\tau)$ is the derivative at $t = \tau$ of the curve $t \mapsto \rho(x(t))$. This is a contradiction since both the vectors $\widetilde{g}(\tau, x_\tau)$ and $w(\tau)$ are tangent to N at $\rho(x(\tau))$ and, consequently, orthogonal to $x(\tau) - \rho(x(\tau))$.

It remains to show that $x(t) \in M$ for all $t > 0$. Let $s = \inf\{t > 0 : x(t) \in N \setminus M\}$, and assume by contradiction $s < +\infty$ (here we adopt the convention $\inf \emptyset = +\infty$). Note that $x(s) \in M$ since M is compact. Let $\{t_n\}$ be a sequence converging to s and such that $x(t_n) \in N \backslash M$. We have $t_n > s$ for all n and

$$\lim_{n\to\infty} \frac{x(t_n) - x(s)}{t_n - s} = x'(s) = \widetilde{g}(s, x_s) \in C_{x(s)}(N \backslash M).$$

Now, the function x_s takes values in M and, consequently, we have $g(s, x_s) = \widetilde{g}(s, x_s) \in C_{x(s)}(N \backslash M)$, contradicting the fact that the vector field g is away from $N \backslash M$. □

From now on M will be a compact ∂-manifold in $\mathbb{R}^k$. In this case one may regard M as a subset of a smooth boundaryless manifold N of the same dimension as M (see e.g. [17], [21]). It is not hard to show that a vector field g on M is away from the complement $N \backslash M$ if and only if it is *strictly inward*; meaning that g is inward and $g(t, \varphi) \notin T_{\varphi(0)} \partial M$ for all $(t, \varphi) \in \mathbb{R} \times C((-\infty, 0], M)$ such that $\varphi(0) \in \partial M$.

We say that a subset Q of $C((-\infty, 0], M)$ is a *brush* if there is $\sigma \le 0$ such that

$$\varphi(\theta) = \psi(\theta), \quad \theta \le \sigma$$

for all $\varphi, \psi \in Q$. We will make the following assumption:

(H) Given $\delta > 0$ and any compact brush Q of $C((-\infty,0],M)$, there exists $L \geq 0$ such that
$$\|g(t,\varphi) - g(t,\psi)\| \leq L \sup_{s\leq 0} \|\varphi(s) - \psi(s)\| \tag{2.6}$$
for all $t \in [0,\delta]$ and $\varphi, \psi \in Q$.

Remark 2.4. Assumption (H) extends the one given in [16]. Indeed, in that monograph the authors study equations of the type
$$x'(t) = h(t, x_t),$$
where $h : \mathbb{R} \times C([-r,0],\mathbb{R}^k) \to \mathbb{R}^k$ is Lipschitz in the second variable in each compact subset of $\mathbb{R} \times C([-r,0],\mathbb{R}^k)$. Now, define $g : \mathbb{R} \times C((-\infty,0],\mathbb{R}^k) \to \mathbb{R}^k$ by
$$g(t,\varphi) = h(t, \varphi|_{[-r,0]})$$
and observe that the vector field g clearly verifies (H).

We will use the following folk result, whose proof is given for the sake of completeness.

Lemma 2.5. *Let $\alpha : [0,b] \to \mathbb{R}^k$ be a C^1 function such that $\alpha(0) = 0$ and*
$$\|\alpha'(t)\| \leq c \sup_{0\leq s\leq t} \|\alpha(s)\|, \quad t \in [0,b]$$
for some constant $c \geq 0$. Then, $\alpha(t) = 0$ for all $t \in [0,b]$.

Proof. Let $0 < \delta \leq b$ be such that $\delta c < 1$. Let $\tau \in [0,\delta]$ be such that $\|\alpha(\tau)\| = \max_{0\leq s\leq \delta} \|\alpha(s)\|$. We have
$$\|\alpha(\tau)\| = \|\alpha(\tau) - \alpha(0)\| \leq \tau \sup_{0\leq s\leq \tau} \|\alpha'(s)\| \leq \delta c \|\alpha(\tau)\|.$$
Being $\delta c < 1$, this inequality is verified if and only if $\alpha(\tau) = 0$. Thus $\alpha(t) = 0$ for any $t \in [0,\delta]$, and the assertion follows in a finite number of steps. □

The following proposition regards existence and uniqueness of solutions of problem (2.4) in the case when g is inward, bounded, and verifies (H).

Proposition 2.6. *Let $M \subseteq \mathbb{R}^k$ be a compact ∂-manifold and g an inward vector field on M. Suppose that g is bounded. Then, problem (2.4) admits a solution defined on the whole real line. Moreover, if g verifies (H), then the solution is unique.*

Proof. As already pointed out, we may regard M as a subset of a smooth boundaryless manifold N of the same dimension as M. Define $\nu : M \to \mathbb{R}^k$ as follows. Given $p \in \partial M$, let $\mu(p)$ be the unique unit vector belonging to $C_pM \cap (T_p\partial M)^\perp$. Then, extend $\mu : \partial M \to \mathbb{R}^k$ by Tietze's Theorem to a map from M to $\mathbb{R}^k$ and consider its orthogonal projection $\nu(p)$ onto the space T_pM for any $p \in M$. For any $n \in \mathbb{N}$ define the strictly inward vector field $g_n : \mathbb{R} \times C((-\infty,0],M) \to \mathbb{R}^k$ by $g_n(t,\varphi) = g(t,\varphi) + \frac{1}{n}\nu(\varphi(0))$, and let $x^n : \mathbb{R} \to M$ be a solution of the initial value problem
$$\begin{cases} x'(t) = g_n(t, x_t), & t > 0, \\ x(t) = \eta(t), & t \leq 0, \end{cases}$$
whose existence is ensured by Lemma 2.3. As in the proof of that lemma, one can show that $\{x^n(t)\}$ has a subsequence which converges uniformly on compact subintervals of $\mathbb{R}$ to a solution of problem (2.4) defined on the whole real line.

Assume now that g verifies (H). Let $x^1, x^2 : \mathbb{R} \to M$ be two solutions of problem (2.4), and let $b > 0$. Then, the set $\{x_t^i \in C((-\infty, 0], M) : \ t \in [0, b], i = 1, 2\}$ is a compact brush, being the image of two curves in $C((-\infty, 0], M)$. Thus, for any $t \in [0, b]$ we have

$$\|g(t, x_t^2) - g(t, x_t^1)\| \le L \sup_{s\le 0} \|x_t^2(s) - x_t^1(s)\| = L \sup_{s\le t} \|x^2(s) - x^1(s)\| = L \sup_{0\le s\le t} \|x^2(s) - x^1(s)\|.$$

Let now $y = x^2 - x^1$. We have $\|y(t)\| = 0$ for $t \le 0$ and

$$\|y'(t)\| = \|g(t, x_t^2) - g(t, x_t^1)\| \le L \sup_{0\le s\le t} \|y(s)\|, \quad t \in [0, b].$$

Hence, the assertion follows from Lemma 2.5. □

2.3. **Fixed point index.** Here we summarize the main properties of the fixed point index in the context of absolute neighborhood retracts (ANRs). Let X be a metric ANR and consider a locally compact (continuous) X-valued map k defined on a subset $\mathcal{D}(k)$ of X. Given an open subset U of X contained in $\mathcal{D}(k)$, if the set of fixed points of k in U is compact, the pair (k, U) is called *admissible*. It is known that to any admissible pair (k, U) we can associate an integer $\mathrm{ind}_X(k, U)$ - the *fixed point index* of k in U - which satisfies properties analogous to those of the classical Leray–Schauder degree [18]. The reader can see for instance [5], [14], [22] or [24] for a comprehensive presentation of the index theory for ANRs. As regards the connection with the homology theory we refer to standard algebraic topology textbooks (e.g. [7], [26]).

We summarize for the reader's convenience the main properties of the index.

i) (*Existence*) If $\mathrm{ind}_X(k, U) \neq 0$, then k admits at least one fixed point in U.
ii) (*Normalization*) If X is compact, then $\mathrm{ind}_X(k, X) = \Lambda(k)$, where $\Lambda(k)$ denotes the Lefschetz number of k.
iii) (*Additivity*) Given two disjoint open subsets U_1, U_2 of U such that any fixed point of k in U is contained in $U_1 \cup U_2$, then $\mathrm{ind}_X(k, U) = \mathrm{ind}_X(k, U_1) + \mathrm{ind}_X(k, U_2)$.
iv) (*Excision*) Given an open subset U_1 of U such that k has no fixed points in $U\backslash U_1$, then $\mathrm{ind}_X(k, U) = \mathrm{ind}_X(k, U_1)$.
v) (*Commutativity*) Let X and Y be metric ANRs. Suppose that U and V are open subsets of X and Y respectively and that $k : U \to Y$ and $h : V \to X$ are locally compact maps. Assume that one of the sets of fixed points of hk in $k^{-1}(V)$ or kh in $h^{-1}(U)$ is compact. Then the other set is compact as well and $\mathrm{ind}_X(hk, k^{-1}(V)) = \mathrm{ind}_Y(kh, h^{-1}(U))$.
vi) (*Homotopy invariance*) Let $H : [0, 1] \times U \to X$ be a locally compact map such that the set $\{(\lambda, x) \in [0, 1] \times U : H(\lambda, x) = x\}$ is compact. Then $\mathrm{ind}_X(H(\lambda, \cdot), U)$ is independent of λ.

3. Existence of periodic solutions

From now on we will adopt the following notation. By M we mean a compact ∂-manifold in $\mathbb{R}^k$. Given $T > 0$, by $C_0([-T, 0], M)$ we mean the (complete) metric space of the continuous functions $\varphi : [-T, 0] \to M$ such that $\varphi(-T) = \varphi(0)$, endowed with the metric induced by the Banach space $C([-T, 0], \mathbb{R}^k)$.

Since M is an ANR, it is not difficult to show (see e.g. [8]) that the metric space $C_0([-T, 0], M)$ is an ANR as well. For the sake of simplicity, from now on, the metric space $C_0([-T, 0], M)$ will be denoted by $\widetilde{M}_0$.

Let $f : \mathbb{R} \times C((-\infty, 0], M) \to \mathbb{R}^k$ be an inward vector field on M which is T-periodic in the first variable. Assume that f is bounded and verifies (H). We are interested in the existence of a T-periodic solution of the RFDE

$$x'(t) = f(t, x_t).$$

Given $\varphi \in \widetilde{M_0}$, we will denote by $\widehat{\varphi}$ the unique element of $C((-\infty, 0], M)$ obtained by considering the T-periodic backward extension of the function φ; i.e. $\widehat{\varphi}$ is defined as follows:

$$\widehat{\varphi}(\theta) = \varphi(\theta + nT) \quad \text{if } \theta \in [-(n+1)T, -nT],\ n \in \mathbb{N}.$$

Observe that $\widetilde{M_0}$ is bounded and closed as a subset of the Banach space $C([-T, 0], \mathbb{R}^k)$. Hence, $\widetilde{M_0}$ being an ANR, there exist a bounded open subset U of $C([-T, 0], \mathbb{R}^k)$ containing $\widetilde{M_0}$ and a retraction ρ of U onto $\widetilde{M_0}$.

Now, given $\lambda \in [0, +\infty)$ consider the operator

$$P_\lambda : U \to C([-T, 0], \mathbb{R}^k)$$

defined as $P_\lambda(\psi)(s) = x(s + T)$, where x is the unique solution, ensured by Proposition 2.6, of the following initial value problem:

$$\begin{cases} x'(t) = \lambda f(t, x_t), & t > 0, \\ x(t) = \widehat{\rho(\varphi)}(t), & t \le 0. \end{cases} \tag{3.1}$$

The following two propositions regard some crucial properties of P_λ.

The proof of Proposition 3.1 is straightforward and, therefore, it is omitted.

Proposition 3.1. *The set of fixed points of P_λ is contained in $\widetilde{M_0}$. Moreover, the fixed points of P_λ correspond to the T-periodic solutions of the equation*

$$x'(t) = \lambda f(t, x_t)$$

in the following sense: ψ is a fixed point of P_λ if and only if it is the restriction to $[-T, 0]$ of a T-periodic solution.

Proposition 3.2. *The map $P : [0, 1] \times U \to C([-T, 0], \mathbb{R}^k)$, defined by $(\lambda, \psi) \mapsto P_\lambda(\psi)$, is continuous with compact image.*

Proof. To show that P is continuous, let $\{\psi_n\}$ be a sequence in U which converges to ψ, and let $\{\lambda_n\}$ be a sequence in $[0, 1]$ converging to λ. Since ρ is continuous, we have $\rho(\psi_n) \to \rho(\psi)$. Thus, $\widehat{\rho(\psi_n)}(\theta) \to \widehat{\rho(\psi)}(\theta)$ uniformly for $\theta \in (-\infty, 0]$.

Now, let $x^n : \mathbb{R} \to M$ be the unique solution (ensured by Proposition 2.6) of the initial value problem

$$\begin{cases} x'(t) = \lambda_n f(t, x_t), & t > 0, \\ x(t) = \widehat{\rho(\psi_n)}(t), & t \le 0. \end{cases}$$

As in the proof of Lemma 2.3, one can show that every subsequence of $\{x^n(t)\}$ has a subsequence which converges uniformly on compact subintervals of $\mathbb{R}$ to the unique solution $x(t)$ of problem (3.1). Therefore, $x^n(t) \to x(t)$ uniformly on compact subintervals of $\mathbb{R}$ and, consequently, $P(\lambda_n, \psi_n) \to P(\lambda, \psi)$. This shows that the map P is continuous.

The compactness of the image of P follows from Ascoli's Theorem. □

We are now ready to establish our existence result.

Theorem 3.3. *Let M be a compact ∂-manifold with nonzero Euler–Poincaré characteristic, and $f : \mathbb{R} \times C((-\infty, 0], M) \to \mathbb{R}^k$ an inward vector field on M which is T-periodic in the first variable. Suppose that f is bounded and verifies (H). Then, the equation*

$$x'(t) = f(t, x_t)$$

admits a T-periodic solution.

Proof. First we observe that, by Propositions 3.1 and 3.2, the set $\{(\lambda, \psi) \in [0, 1] \times U : P(\lambda, \psi) = \psi\}$ is a compact subset of $[0, 1] \times \widetilde{M}_0$. Hence, the fixed point index $\mathrm{ind}_E(P_\lambda, U)$, where $E = C([-T, 0], \mathbb{R}^k)$, is well defined and independent of $\lambda \in [0, 1]$.

Now, if $\lambda = 0$, given $\psi \in U$, problem (3.1) becomes

$$\begin{cases} x'(t) = 0, & t > 0, \\ x(t) = \widehat{\rho(\psi)}(t), & t \le 0. \end{cases}$$

Any solution of this problem for $t \ge 0$ is constantly equal to $\rho(\psi)(0)$. It follows that

$$P_0(\psi)(s) = \rho(\psi)(0), \quad s \in [-T, 0].$$

Hence, P_0 sends U into the subset of the constant M-valued functions (which can be identified with M), and its restriction $P_0|_M : M \to M$ coincides with the identity I_M of M. By the commutativity and normalization properties of the fixed point index we get

$$\mathrm{ind}_E(P_0, U) = \mathrm{ind}_M(P_0, M) = \Lambda(I_M) = \chi(M) \neq 0.$$

Therefore, $\mathrm{ind}_E(P_1, U) \neq 0$ and the existence property implies that the operator P_1 has a fixed point. This completes the proof. □

We close this section with the following example.

Example 3.4. Let $g : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^k$ be a continuous map which is T-periodic in the first variable and locally Lipschitz in the second one. Consider the equation

$$x'(t) = g(t, x(t)) + \int_{-\infty}^{t} \mathrm{e}^{s-t} x(s)\, \mathrm{d}s,$$

which is of the form $x'(t) = f(t, x_t)$, where $f : \mathbb{R} \times C((-\infty, 0], \mathbb{R}^k) \to \mathbb{R}^k$ is the vector field defined by

$$f(t, \varphi) = g(t, \varphi(0)) + \int_{-\infty}^{0} \mathrm{e}^{\theta} \varphi(\theta)\, \mathrm{d}\theta.$$

Assume that there exists $c > 0$ such that $\langle g(t, v), v \rangle \le 0$ for $\|v\| = c$ and all $t \in \mathbb{R}$. Let $M = \overline{B(0, c)}$, where $B(0, c)$ denotes the open ball in $\mathbb{R}^k$ centered at 0 with radius c. Then, $\chi(M) = 1$ since M is contractible. Now, the restriction of the map f to $\mathbb{R} \times C((-\infty, 0], M)$ is a T-periodic inward vector field on the ∂-manifold M. Moreover, it is easy to check that this restriction is bounded and verifies (H). Hence, Theorem 3.3 applies to the equation yielding the existence of a T-periodic solution.

Let us observe that an analogous existence result was obtained using different techniques by Gaines and Mawhin (see [13]).

4. Applications to second order delay differential equations on manifolds

In this section we apply the results obtained above to some motion problems for forced constrained systems.

Let $X \subseteq \mathbb{R}^s$ be a boundaryless manifold. Given $q \in X$, let $(T_qX)^\perp \subseteq \mathbb{R}^s$ denote the normal space of X at q. Since $\mathbb{R}^s = T_qX \oplus (T_qX)^\perp$, any vector $u \in \mathbb{R}^s$ can be uniquely decomposed into the sum of the *parallel* (or *tangential*) *component* $u_\pi \in T_qX$ *of* u *at* q and the *normal component* $u_\nu \in (T_qX)^\perp$ *of* u *at* q. By

$$TX = \{(q,v) \in \mathbb{R}^s \times \mathbb{R}^s : \; q \in X, \; v \in T_qX\}$$

we denote the *tangent bundle of* X, which is a smooth manifold containing a natural copy of X via the embedding $q \mapsto (q,0)$. The natural projection of TX onto X is just the restriction (to TX as domain and to X as codomain) of the projection of $\mathbb{R}^s \times \mathbb{R}^s$ onto the first factor.

Given a vector field $F : \mathbb{R} \times C((-\infty,0],X) \to \mathbb{R}^s$ which is T-periodic in the first variable, consider the following retarded functional motion equation on X:

$$x''_\pi(t) = F(t,x_t) - \varepsilon x'(t), \tag{4.1}$$

where

i) $x''_\pi(t)$ stands for the parallel component of the acceleration $x''(t) \in \mathbb{R}^s$ at the point $x(t)$;
ii) the frictional coefficient ε is a positive real constant.

By a *solution* of (4.1) we mean a continuous function $x : J \to X$, defined on a real interval J with $\inf J = -\infty$, which verifies eventually the equality (4.1). That is, x is a solution of (4.1) if there exists $-\infty \le \bar{t} < \sup J$ such that x is C^2 on the subinterval $(\bar{t}, \sup J)$ of J and verifies

$$x''_\pi(t) = F(t,x_t) - \varepsilon x'(t)$$

for all $t \in J$ with $t > \bar{t}$. A *forced oscillation of* (4.1) is a solution which is T-periodic and globally defined on $J = \mathbb{R}$.

It is known that, associated with $X \subseteq \mathbb{R}^s$, there exists a unique smooth map $R : TX \to \mathbb{R}^s$, called the *reactive force* (or *inertial reaction*), with the following properties:

(a) $R(q,v) \in (T_qX)^\perp$ for any $(q,v) \in TX$;
(b) R is quadratic in the second variable;
(c) any C^2 curve $\gamma : (a,b) \to X$ verifies the condition

$$\gamma''_\nu(t) = R(\gamma(t),\gamma'(t)), \quad \forall t \in (a,b),$$

i.e., for each $t \in (a,b)$, the normal component $\gamma''_\nu(t)$ of $\gamma''(t)$ at $\gamma(t)$ equals $R(\gamma(t),\gamma'(t))$.

The map R is strictly related to the second fundamental form on X and may be interpreted as the reactive force due to the constraint X.

By properties (a) and (c) above, equation (4.1) can be equivalently written as

$$x''(t) = R(x(t),x'(t)) + F(t,x_t) - \varepsilon x'(t). \tag{4.2}$$

Notice that, if the above equation reduces to the so-called *inertial equation*

$$x''(t) = R(x(t),x'(t)),$$

one obtains the geodesics of X as solutions.

Equation (4.2) can be written as a RFDE on TX as follows:

$$\begin{cases} x'(t) = y(t), \\ y'(t) = R(x(t), y(t)) + F(t, x_t) - \varepsilon y(t). \end{cases}$$

This makes sense since the map

$$G : \mathbb{R} \times C((-\infty, 0], TX) \to \mathbb{R}^s \times \mathbb{R}^s, \quad G(t, (\varphi, \psi)) = (\psi(0), R(\varphi(0), \psi(0)) + F(t, \varphi) - \varepsilon\psi(0)) \tag{4.3}$$

is a vector field on TX. Indeed, observe that the condition

$$G(t, (\varphi, \psi)) \in T_{(\varphi(0), \psi(0))}TX$$

is verified for all $(t, (\varphi, \psi)) \in \mathbb{R} \times C((-\infty, 0], TX)$ (see, for example, [9] for more details).

Theorem 4.1 below extends two results obtained in [2] and [4]. The proof is based on Theorem 3.3 above.

Theorem 4.1. *Let $X \subseteq \mathbb{R}^s$ be a compact boundaryless manifold whose Euler–Poincaré characteristic $\chi(X)$ is different from zero, and $F : \mathbb{R} \times C((-\infty, 0], X) \to \mathbb{R}^s$ a vector field which is T-periodic in the first variable. Suppose that F is bounded and verifies (H). Then, the equation* (4.1) *has a forced oscillation.*

Proof. As we already pointed out, the equation (4.1) is equivalent to the following first order system on TX:

$$\begin{cases} x'(t) = y(t), \\ y'(t) = R(x(t), y(t)) + F(t, x_t) - \varepsilon y(t). \end{cases} \tag{4.4}$$

Define $G : \mathbb{R} \times C((-\infty, 0], TX) \to \mathbb{R}^s \times \mathbb{R}^s$ as in (4.3). Then, G is a T-periodic vector field on TX.

Given $c > 0$, define

$$M_c = \big\{(q, v) \in TX : \|v\| \le c\big\}.$$

It is not difficult to show that $M_c \subseteq TX$ is a compact ∂-manifold in $\mathbb{R}^s \times \mathbb{R}^s$ with boundary

$$\partial M_c = \big\{(q, v) \in M_c : \|v\| = c\big\}.$$

Now, let G_c be the restriction of the map G to $\mathbb{R} \times C((-\infty, 0], M_c)$. Clearly, G_c is a T-periodic vector field on M_c which verifies (H). Let us show that G_c is bounded. Indeed, the map F is bounded by assumption, and the compactness of M_c implies that the restriction of the map $(q, v) \mapsto (v, R(q, v) - \varepsilon v)$ to M_c is bounded as well. Therefore G_c is bounded, being the sum of two bounded maps.

We claim that, if $c > 0$ is large enough, then G_c is inward on M_c. To see this, observe that the inward half-subspace of $T_{(q,v)}(M_c) = T_{(q,v)}(TX)$ at $(q, v) \in \partial M_c$ is

$$T^-_{(q,v)}(M_c) = \big\{(\dot q, \dot v) \in T_{(q,v)}(TX) : \langle v, \dot v\rangle \le 0\big\},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^s$. Thus we have to show that, if $c > 0$ is large enough, then $G_c(t, (\varphi, \psi))$ belongs to $T^-_{(\varphi(0),\psi(0))}(M_c)$ for any $t \in \mathbb{R}$ and any pair $(\varphi, \psi) \in C((-\infty, 0], M_c)$ such that $(\varphi(0), \psi(0)) \in \partial M_c$. That is, we need to prove that, for any t and any pair (φ, ψ) with $\|\psi(0)\| = c$, we have

$$\big\langle \psi(0), R(\varphi(0), \psi(0)) + F(t, \varphi) - \varepsilon\psi(0)\big\rangle = \big\langle \psi(0), R(\varphi(0), \psi(0))\big\rangle + \big\langle \psi(0), F(t, \varphi)\big\rangle - \varepsilon\big\langle \psi(0), \psi(0)\big\rangle \le 0.$$

To see this, observe that $\big\langle \psi(0), R(\varphi(0), \psi(0))\big\rangle = 0$ since $R(\varphi(0), \psi(0))$ belongs to $(T_{\varphi(0)}X)^\perp$. Moreover, $\big\langle \psi(0), \psi(0)\big\rangle = c^2$ since $(\varphi(0), \psi(0)) \in \partial M_c$, and

$$\big\langle \psi(0), F(t, \varphi)\big\rangle \le \|\psi(0)\|\|F(t, \varphi)\| \le K\|\psi(0)\|,$$

where K is such that $\|F(t,\varphi)\| \leq K$ for all $(t,\varphi) \in \mathbb{R} \times C((-\infty,0],X)$. Thus,

$$\big\langle \psi(0), R(\varphi(0),\psi(0)) + F(t,\varphi) - \varepsilon\psi(0)\big\rangle \leq Kc - \varepsilon c^2.$$

This shows that, if we choose $c > K/\varepsilon$, then G_c is a strictly inward vector field on M_c, as claimed.

Finally, observe that $\chi(M_c) = \chi(X) \neq 0$ since M_c and X are homotopically equivalent (X being a deformation retract of TX), and $\chi(X) \neq 0$ by assumption. Therefore, given $c > K/\varepsilon$, we can apply Theorem 3.3 with $M = M_c$ and $f = G_c$, and we get that system (4.4) admits a T-periodic solution in M_c. This completes the proof. □

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