UNIVERSITAS STUDIORUM UTINENSIS

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Multiple periodic solutions for one-sided sublinear systems: A refinement of the Poincaré-Birkhoff approach

G. Villari, F.Z.: Dynam. Systems Appl. 25 (2016)

G. Villari, F.Z.: Appl. Math. Lett. 76 (2018)

M. Hayashi, G. Villari, F.Z.: Electron. J. Qual. Theory Differ. Equ. (2018)

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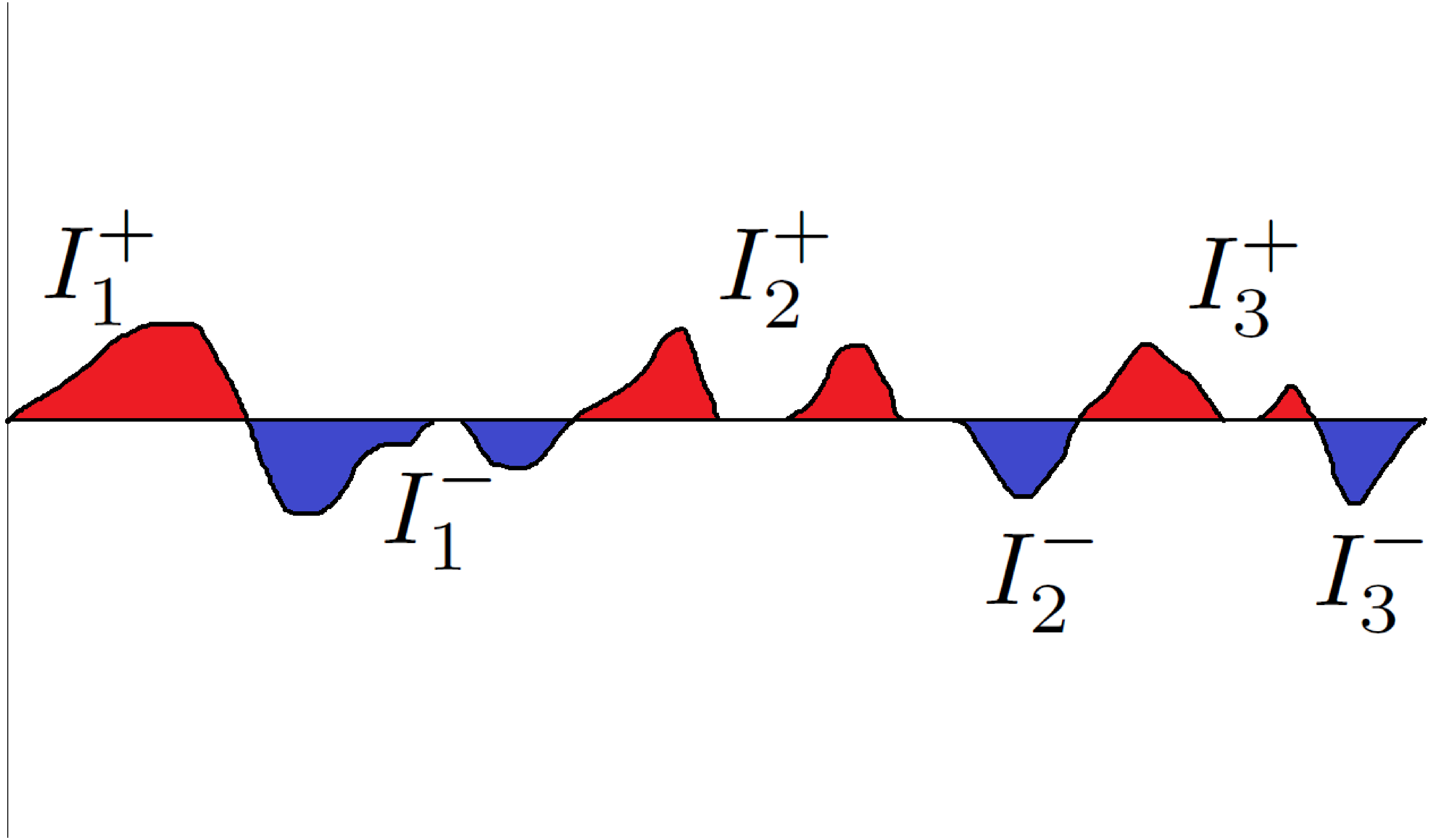
Typical profile of the T-periodic weight function $a(t)$:


I_1^+
I_2^+
I_3^+
I_1^-
I_2^-
I_3^-

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$$a_{\lambda,\mu}(t) := \lambda a^{+}(t) - \mu a^{-}(t), \qquad \lambda, \mu > 0.$$

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We can also deal with more general systems of the form

$$(*) \qquad \begin{cases} x' = h(y) \\ y' = -a_{\lambda,\mu}(t)g(x). \end{cases}$$

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We suppose that $h, g : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz continuous functions satisfying the following assumptions:

(C_0)

$$h(0)=0,\ h(y)y>0 \textbf{ for all } y\neq 0$$

$$g(0)=0,\ g(x)x>0 \textbf{ for all } x\neq 0$$

$$h_0:=\liminf_{|y|\to 0}\frac{h(y)}{y}>0,\qquad g_0:=\liminf_{|x|\to 0}\frac{g(x)}{x}>0.$$

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$(h_\pm)$ h **is bounded on** $\mathbb{R}^\pm$,

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We also set

$$G(x) := \int_0^x g(\xi)d\xi, \qquad H(y) := \int_0^y h(\xi)d\xi.$$

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$$x'' + a_{\lambda,\mu}(t)g(x) = 0, \qquad \textbf{with} \;\; g(x) = -1 + \exp x.$$

Since system $(*)$ has a Hamiltonian structure, of the form

$$\begin{cases} x' = \frac{\partial \mathcal{H}}{\partial y}(t,x,y) \\ y' = -\frac{\partial \mathcal{H}}{\partial x}(t,x,y) \end{cases}$$

for

$$\mathcal{H}(t,x,y) = a_{\lambda,\mu}(t)G(x) + H(y),$$

the associated Poincaré map is an area-preserving homeomorphism, defined on a open set

$$\Omega := \mathbf{dom}\Phi \subseteq \mathbb{R}^2,$$

with $(0,0) \in \Omega$.

Thus a possible method to prove the existence (and multiplicity) of T-periodic solutions can be based on the Poincaré-Birkhoff “twist” fixed point theorem.

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A typical way to apply this result is to find a suitable annulus around the origin with radii $0 < r_0 < R_0$ such that for some $a < b$ the twist condition

$$(TC) \qquad \begin{cases} \mathbf{rot}_z(T) > b, & \forall\, z \text{ with } ||z|| = r_0 \\ \mathbf{rot}_z(T) < a, & \forall\, z \text{ with } ||z|| = R_0 \end{cases}$$

holds, where $\mathbf{rot}_z(T)$ is the rotation number on the interval $[0, T]$ associated with the initial point $z \in \mathbb{R}^2 \setminus \{(0,0)\}$.

Notice that, due to the assumptions $h(s)s > 0$ and $g(s)s > 0$ for $s \neq 0$, it is convenient to use a formula in which the angular displacement is positive when the rotations around the origin are performed in the clockwise sense.

Under these assumptions, the Poincaré-Birkhoff theorem guarantees that for each integer $j \in [a, b]$, there exist *at least two* T-periodic solutions of system $(*)$, having j as associated rotation number. In virtue of the first condition in (C_0), it turns out that these solutions have precisely $2j$ simple transversal crossings with the y-axis in the interval $[0, T[$.

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Equivalently, for such a periodic solution $(x(t), y(t))$, we have that x has precisely $2j$ simple zeros in the interval $[0, T[$

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Unfortunately, in general, the (forward) global existence of solutions for the initial value problems is not guaranteed.

A classical counterexample can be found in [Coffman and Ullrich (1967)] for the superlinear equation

$$x'' + q(t)x^{2n+1} = 0$$

(with $n \geq 1$), where, even for a positive weight $q(t)$, the global existence of the solutions may fail.

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(with $n \geq 1$), where, even for a positive weight $q(t)$, the global existence of the solutions may fail.

A typical feature of this class of counterexamples is that solutions presenting a blow-up at some time β^-, will make infinitely many winds around the origin as $t \to \beta^-$. It is possible to overcome these difficulties by prescribing the rotation number for large solutions and using some truncation argument on the nonlinearity, as shown in [Hartman (1977)].

The situation is even more complicated in the time intervals where the weight function is negative [Burton and Grimmer (1971), Butler (1976)]. Unless we impose that the vector field in $(*)$ has at most a linear growth at infinity, we cannot prevent (in general) blow-up phenomena.

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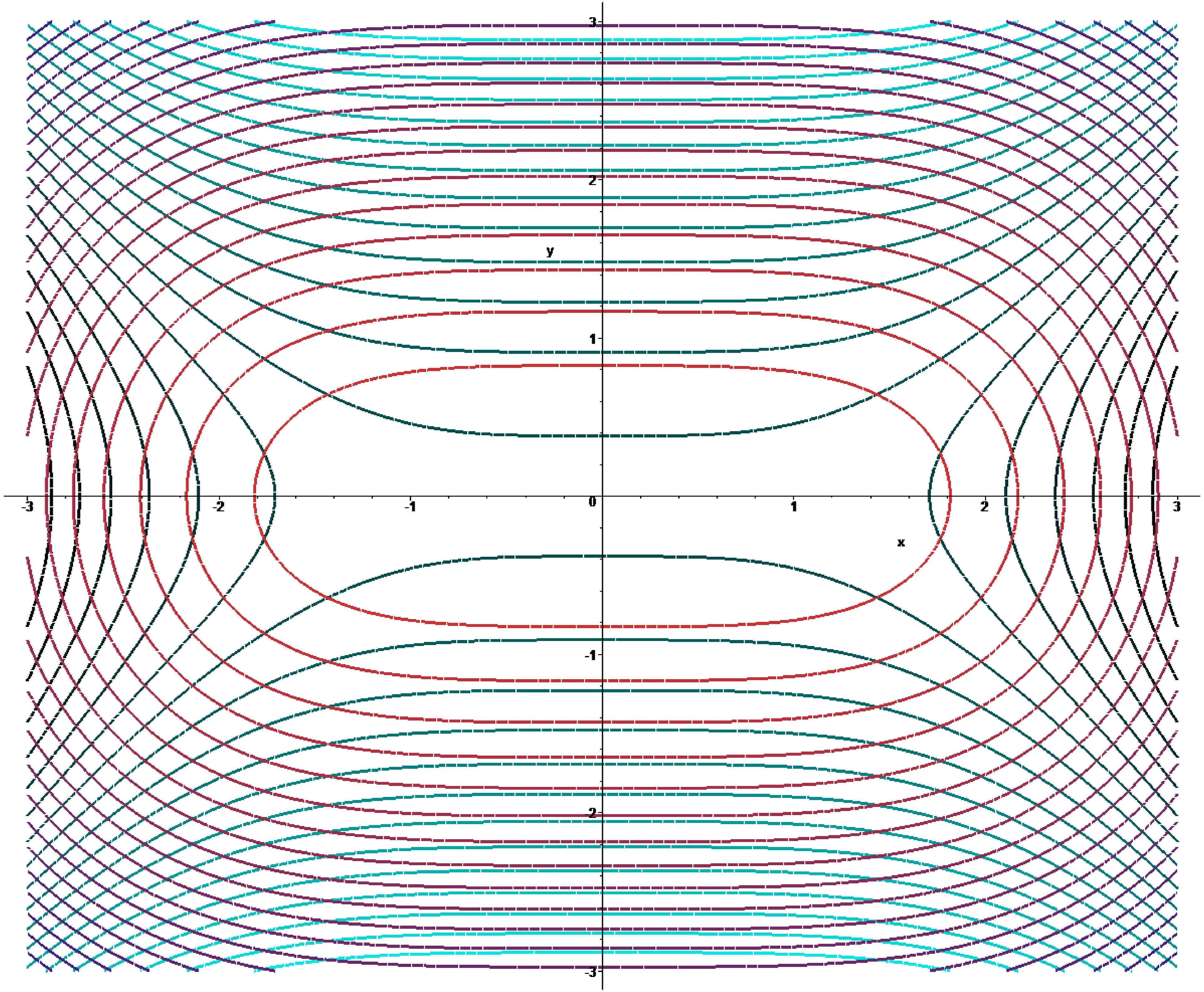
With these premises, the following result can be proved.

Theorem 1 ***Let*** $g, h : \mathbb{R} \to \mathbb{R}$ ***be locally Lipschitz continuous functions satisfying*** (C_0) ***and at least one between*** (h_∞) ***and*** (g_∞). ***Assume, moreover, the global continuability for the solutions of*** $(*)$. ***Then, for each positive integer*** k, ***there exists*** $\Lambda_k > 0$ ***such that for each*** $\lambda > \Lambda_k$ ***and*** $j = 1, \ldots, k$, ***the system*** $(*)$ ***has at least two*** T***-periodic solutions*** (x, y) ***with*** x ***having exactly*** $2j$***-zeros in the interval*** $[0, T[$.

Notice that in the above result we do not require any condition on the parameter $\mu > 0$. On the other hand, we have to assume the global continuability of the solutions, which in general is not guaranteed.
Quite the opposite, for the next result we do not require the Poincaré map to be defined on the whole plane, although now the parameter μ plays a crucial role.

Notice that in the above result we do not require any condition on the parameter $\mu > 0$. On the other hand, we have to assume the global continuability of the solutions, which in general is not guaranteed.
Quite the opposite, for the next result we do not require the Poincaré map to be defined on the whole plane, although now the parameter μ plays a crucial role.
For simplicity, we present the statement in the case in which $a(t)$ has a positive hump followed by a negative one.

Theorem 2 ***Let*** $g, h : \mathbb{R} \to \mathbb{R}$ ***be locally Lipschitz continuous functions satisfying*** (C_0) ***and at least one between*** (h_∞) ***and*** (g_∞). ***Then, for each positive integer*** k, ***there exists*** $\Lambda_k > 0$ ***such that for each*** $\lambda > \Lambda_k$ ***there exists*** $\mu^* = \mu^*(\lambda)$ ***such that for each*** $\mu > \mu^*$ ***and*** $j = 1, \ldots, k$, ***the system*** $(*)$ ***has at least four*** T***-periodic solutions*** (x, y) ***with*** x ***having exactly*** $2j$***-zeros in the interval*** $[0, T[$.


y
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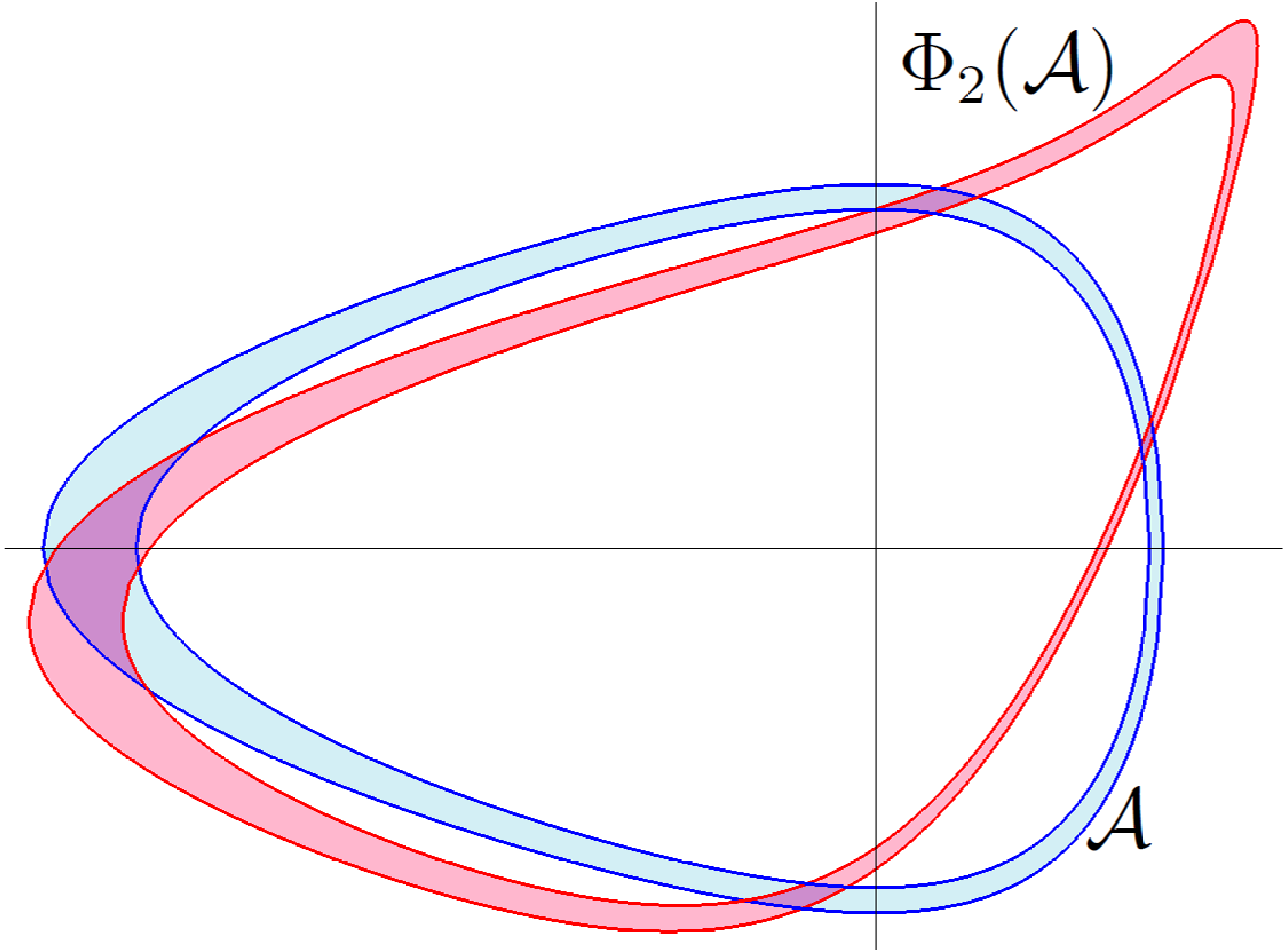
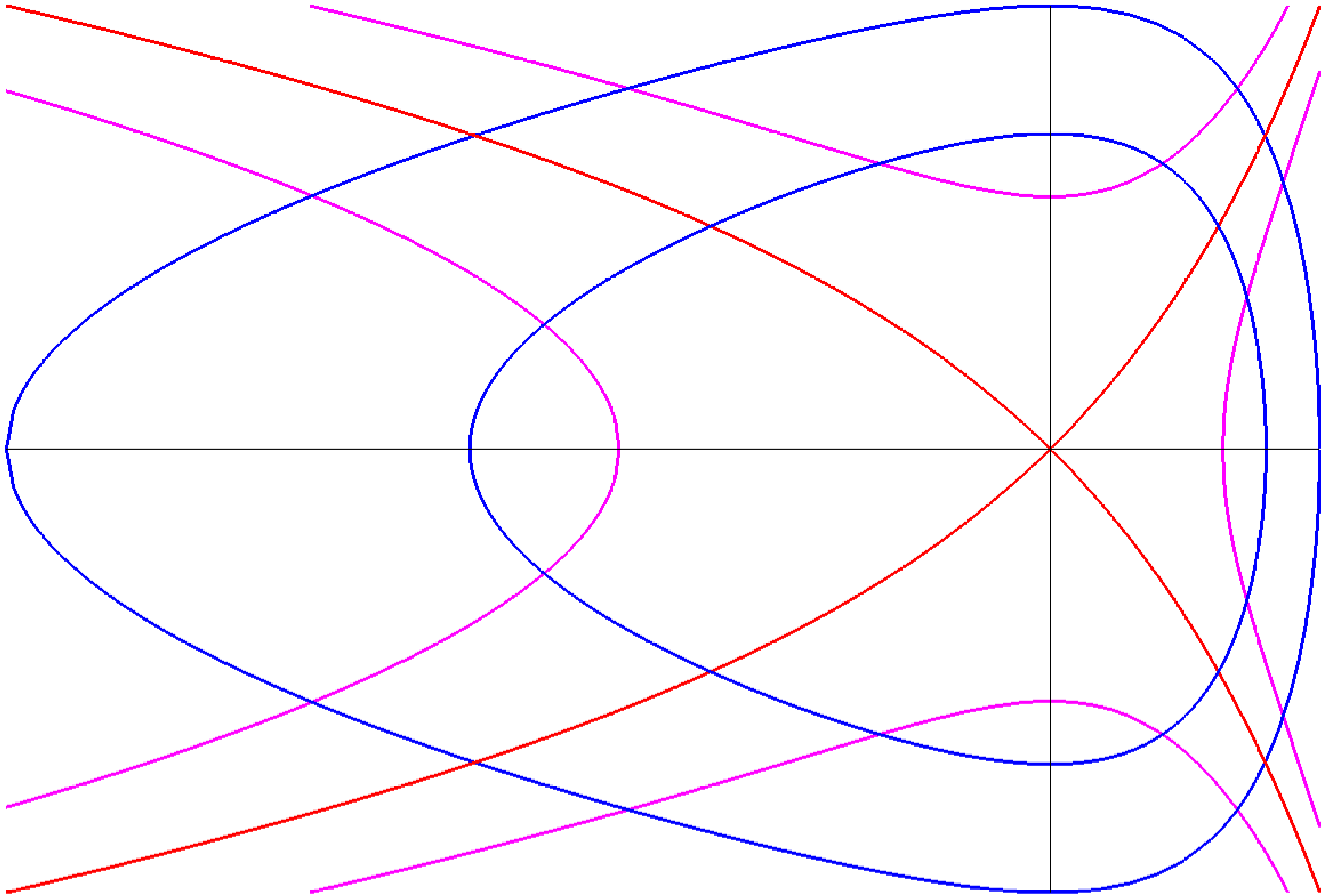
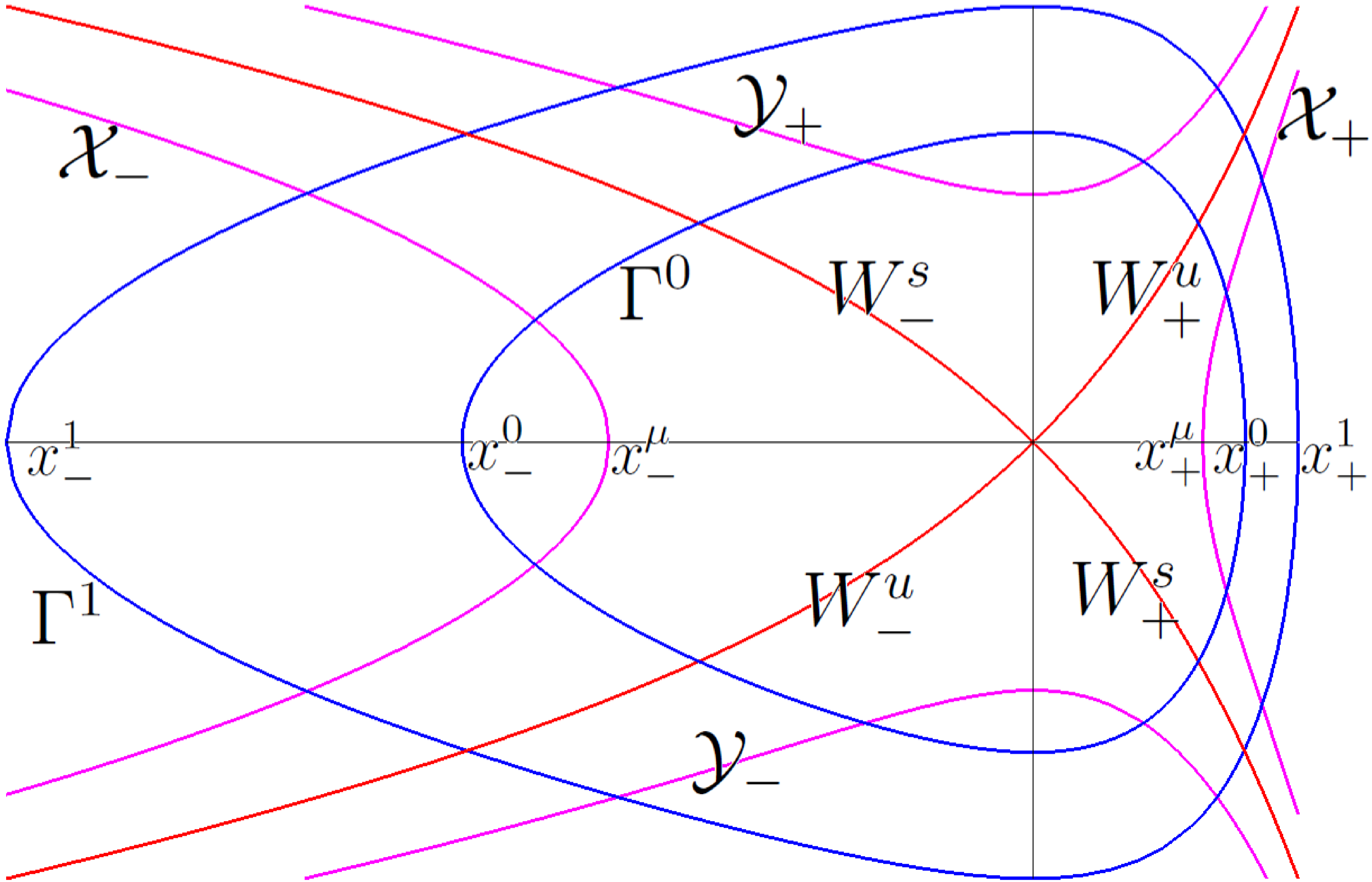
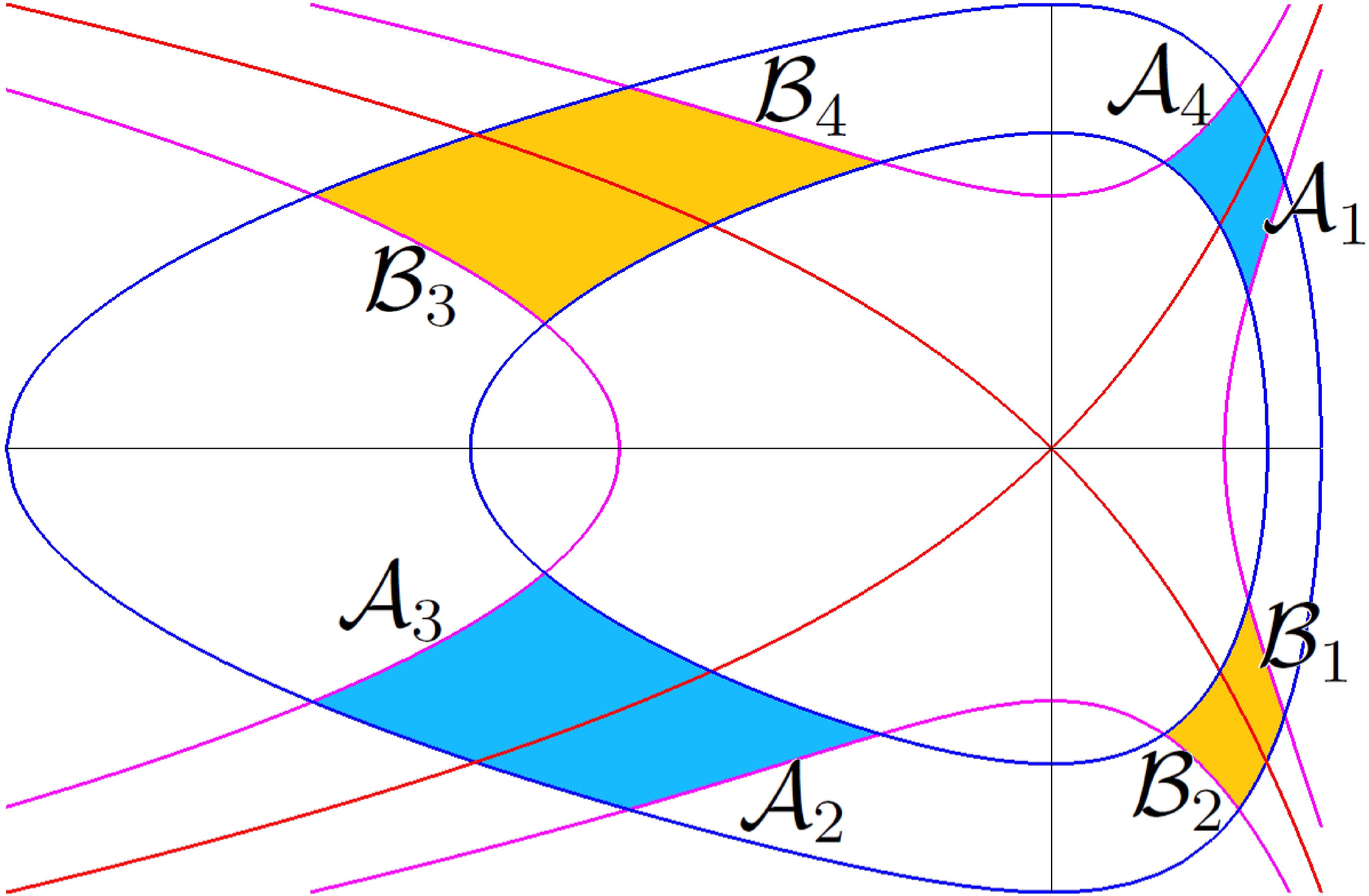

$\Phi_2(\mathcal{A})$
$\mathcal{A}$





$\mathcal{X}_-$
$\mathcal{Y}_+$
$\mathcal{X}_+$
Γ^0
W^s_-
W^u_+
x^1_-
x^0_-
x^μ_-
x^μ_+
x^0_+
x^1_+
Γ^1
W^u_-
W^s_+
$\mathcal{Y}_-$


$\mathcal{B}_4$
$\mathcal{A}_4$
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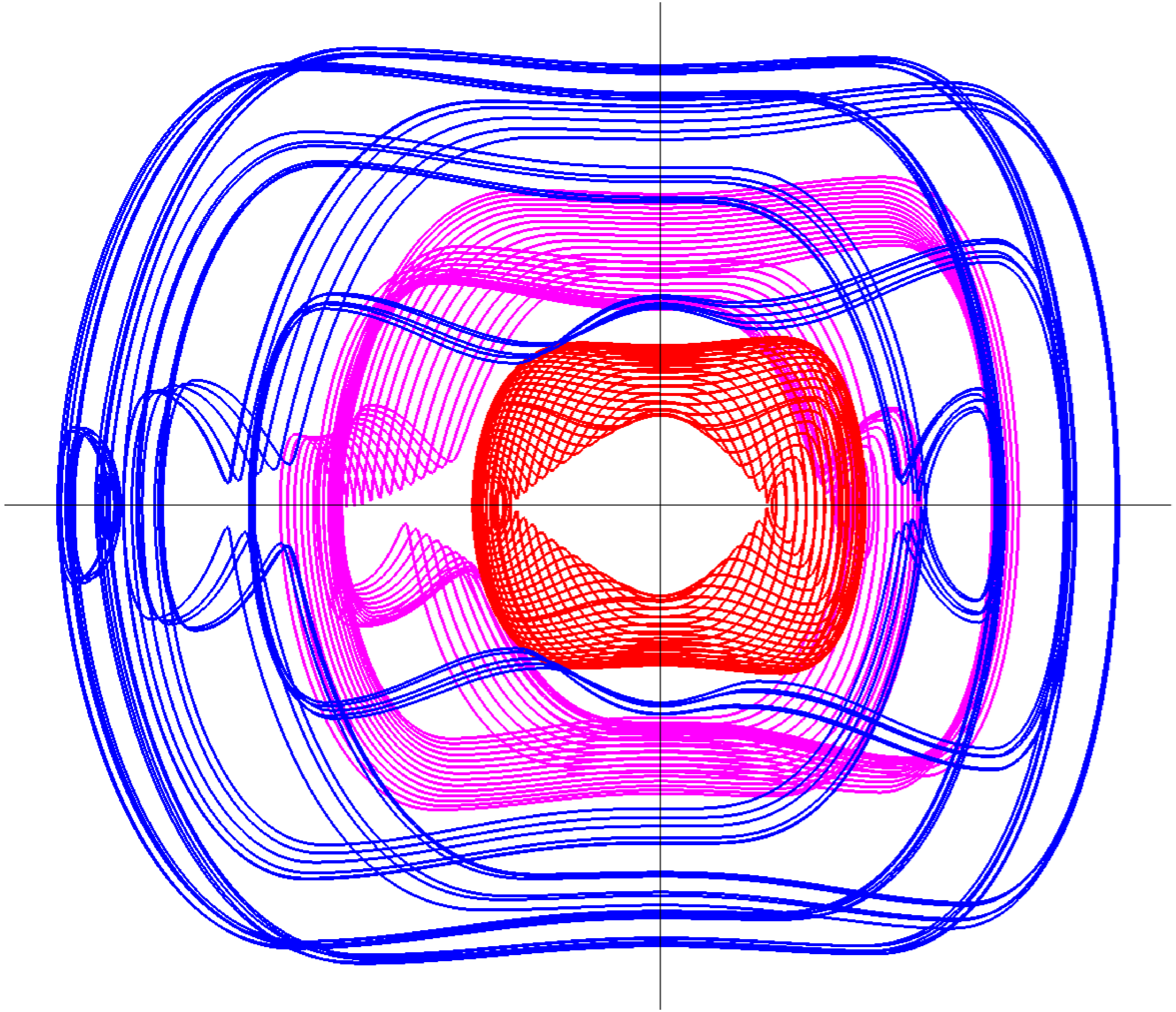
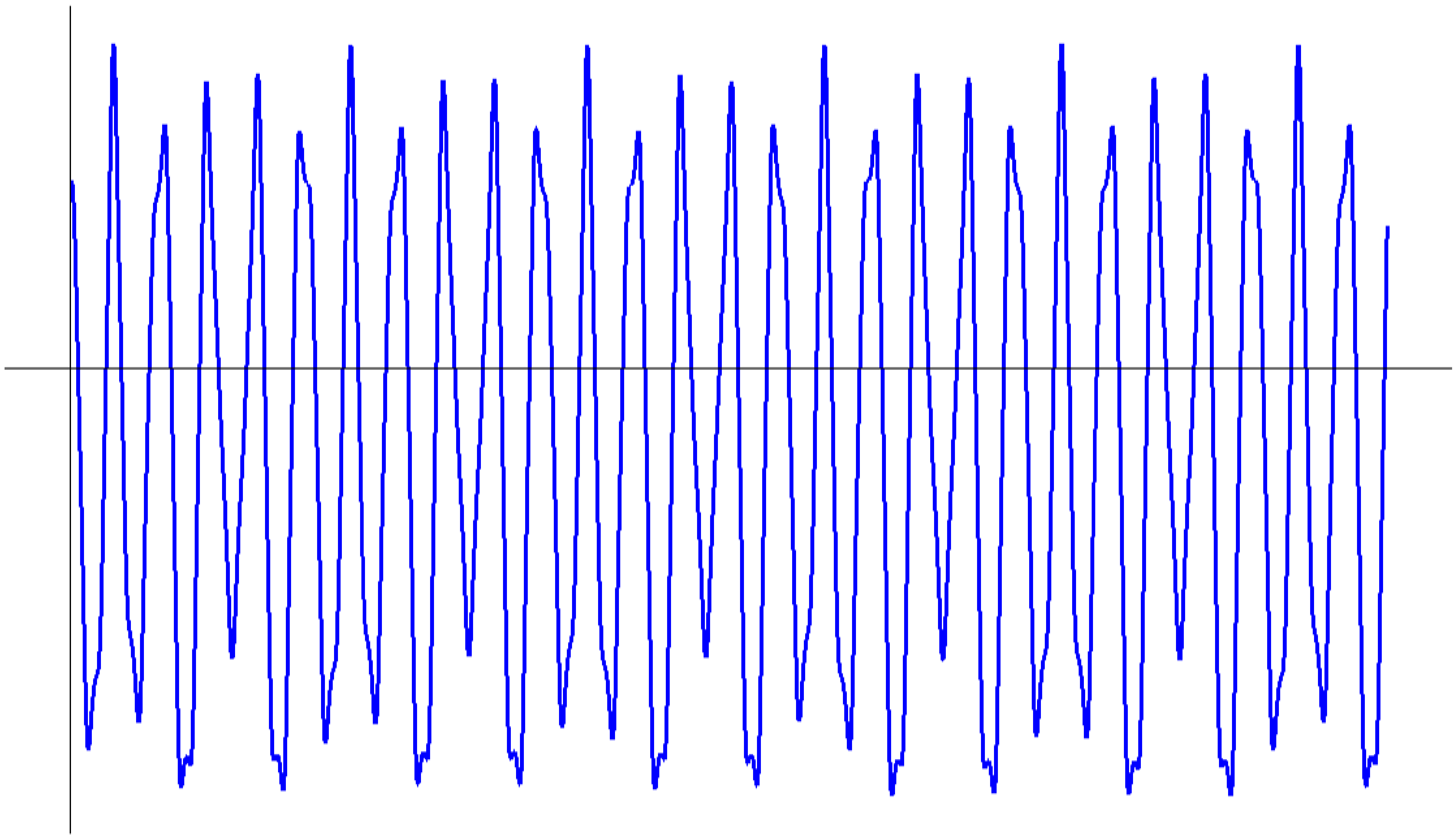
$\mathcal{B}_2$





Some references on the Poincaré-Birkhoff theorem

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Poincaré (1912), Birkhoff (1913), Brown & Neumann (1977), Neumann (1977), Hartman (1977), Jacobowitz (1976-1977), W. Ding (1982-1983), Franks (1988), Rebelo (1997), Dalbono & Rebelo (2002), Margheri, Rebelo & Z. (2002), Qian & Torres (2005), Moser & Zehnder (2005), Martins & Ureña (2007), Le Calvez & Wang (2010), Fonda, Sabatini & Z. (2012), Fonda & Ureña (2017), . . . more coming.

Some references on bend-twist maps

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T. Ding (2007-2012), Pascoletti & Z. (2011-2012-2013), Kirillov & Starkov (2013), Wang, Liu & Qian (2016).

Some references on the stretching along the paths technique

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Papini & Z. (2000-2002-2004-2007), Pascoletti, Pireddu & Z. (2008), Pireddu (2009), Margheri, Rebelo & Z. (2010), Sovrano (2016), Papini, Villari & Z. (2017-2018).

Thank you for your attention !