EXISTENCE RESULTS FOR STURM-LIOUVILLE EQUATIONS WITH MIXED BOUNDARY CONDITIONS

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Sturm-Liouville problems with mixed conditions and involving the ordinary $p-$Laplacian

$$\begin{cases} -(q|u'|^{p-2}u')' + s|u|^{p-2}u = \lambda f(x,u) \quad \text{on} \quad]a,b[\,, \\ \\ u(a) = u'(b) = 0, \end{cases} \qquad (P)$$

- $q, s \in L^{\infty}([a,b])$, $s \not\equiv 0$, $q_0 = \operatorname{ess\,inf}_{[a,b]} q > 0$ and $s_0 = \operatorname{ess\,inf}_{[a,b]} s \geq 0$,
- λ positive real parameter,
- $p > 1$,
-

$$f : [a,b] \times \mathbb{R} \to \mathbb{R}$$

  is a L^1-Carathéodory function

- $f(\cdot, s)$ is measurable for all $s \in \mathbb{R}$,
- $f(x, \cdot)$ is continuous for a.e. $x \in [a,b]$,
- for all $\rho > 0$ one has $e\sup_{|t|\leq\rho} |f(x,t)| \in L^1([a,b])$.

Put

$$F(x,\xi) := \int^{\xi} f(x,t)dt \quad \text{for all} \quad (x,\xi) \in [a,b] \times \mathbb{R}.$$

D. Averna, R. Salvati, *Three solutions for a mixed boundary value problem involving the one-dimensional p−Laplacian*, J. Math. Anal. Appl. **298** (2004), 245–260.

$$\begin{cases} -(|u'|^{p-2}u')' + |u|^{p-2}u = \lambda f(x,u) \quad \text{on} \quad]a,b[\,, \\ u(a) = u'(b) = 0, \end{cases}$$

D. Averna, N. Giovannelli , E. Tornatore, *Existence of three solutions for a mixed boundary value problem with the Sturm-Liouville equation*, Bull. Korean Math. Soc., **49** 6 (2012), 1213–1222.

G. Bonanno - E. Tornatore, *Infinitely many solutions for a mixed boundary value* Annales Polonici Mathematici (2010).

$$\begin{cases} -(qu')' + su = \lambda f(x,u) \quad \text{on} \quad]a,b[\,, \\ u(a) = u'(b) = 0, \end{cases}$$

G. D'Aguì, *Existence results for a mixed boundary value problem with Sturm–Liouville equation*, Adv. Pure Appl. Math. **2** (2011) 237–248.

$$\begin{cases} -(q|u'|^{p-2}u')' + s|u|^{p-2}u = \lambda f(x,u) \quad \text{on} \quad]a,b[\,, \\ u(a) = u'(b) = 0, \end{cases}$$

We want to prove the existence of two non-zero solutions for problem for each λ in an appropriate interval. We use variational methods and multiple critical points theorems.

Denote by

$$X = \{u \in W^{1,p}([a,b]) \,:\, u(a) = 0\}$$

the Sobolev space endowed with the norm

$$\|u\| = \Big(\int_a^b q(x)|u'(x)|^p dx + \int_a^b s(x)|u(x)|^p dx \Big)^{\frac{1}{p}}.$$

for every $u \in X$. $(X, \|\cdot\|)$ is compactly embedded in $(C^0([a,b]), \|\cdot\|_\infty)$ and one has

$$\|u\|_\infty < \left(\frac{b-a}{q_0}\right)^{\frac{p-1}{p}} \|u\| \quad \text{for all} \quad u \in X.$$

Consider the following operators
$\Phi, \Psi : X \to \mathbb{R}$ defined as follows

$$\Phi(u) = \frac{1}{p}\|u\|^p \qquad \Psi(u) = \int_a^b F(x, u(x))dx, \quad \forall u \in X.$$

- Φ is coercive and continuously Gâteaux differentiable and its Gâteaux derivative at point $u \in X$ is defined by

$$\Phi'(u)(v) = \int_a^b q(x)|u'(x)|^{p-2}u'(x)v'(x)dx + \int_a^b s(x)|u(x)|^{p-2}u(x)v(x)dt,$$

  for every $v \in X$.
- Ψ is continuously Gâteaux differentiable and its Gâteaux derivative at point $u \in X$ is defined by

$$\Psi'(u)(v) = \int_a^b f(x, u(x))v(x)dx \quad \forall v \in X,$$

moreover

$$\Phi(0) = \Psi(0) = 0.$$

Definition

$u : [a, b] \to \mathbb{R}$ is a weak solution of problem if $u \in X$ satisfies the following condition

$$\int_a^b q(x)|u'(x)|^{p-2}u'(x)v'(x)dx + \int_a^b s(x)|u(x)|^{p-2}u(x)v(x)dx =$$

$$\lambda \int_a^b f(x, u(x))v(x)dx$$

for all $v \in X$.

Consider the functional

$$I_\lambda = \Phi - \lambda\Psi$$

a critical point for functional I_λ is exactly a weak solution for problem .

The following theorem of Bonanno and D'Aguì, ensures the existence of at least two critical points.

We consider the Banach space X, and we observe that all regularity assumptions on the functionals Φ and Ψ are satisfied.

Theorem (Bonanno -D'Aguì (2016))

$\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. *Assume that there are* $r \in \mathbb{R}$ *and* $\tilde{u} \in X$, *with* $0 < \Phi(\tilde{u}) < r$, *such that*

$$\frac{\sup\limits_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})},$$

and, for each $\lambda \in \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)} \right[$, *the functional* $I_\lambda = \Phi - \lambda \Psi$ *satisfies* (PS)*-condition and it is unbounded from below.*

Then, for each $\lambda \in \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)} \right[$, *the functional* I_λ *admits at least two non-zero critical points* $u_{\lambda,1}$, $u_{\lambda,2}$ *such that* $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$.

G. Bonanno, G. D'Aguì, *Two non-zero solutions for elliptic Dirichlet problems*, Z. Anal. Anwend. **35** (2016), 449–464.

Main result

Theorem

Assume that there exist four positive constants c, d, μ and R with $\mu > p$ and $d < c$, such that

(i) $\int_a^{\frac{a+b}{2}} F(x,t)dx > 0 \quad \forall t \in [0,d]$,

(ii) $$\frac{\int_a^b \max_{|\xi|\le c} F(x,\xi)dx}{c^p} < \left(\frac{q_0}{2}\right)^{p-1} k \frac{\int_{\frac{a+b}{2}}^b F(x,d)dx}{d^p},$$
where $k = \frac{1}{||q||_\infty + \frac{p+2}{p+1}\left(\frac{b-a}{2}\right)^p ||s||_\infty}$,

(iii) $0 < \mu F(x,t) \le tf(x,t)$ *for all* $x \in [a,b]$ *and for all* $|t| \ge R$.

Then, for each

$$\lambda \in \left] \frac{2^{p-1}d^p}{p(b-a)^{p-1}k\int_{\frac{a+b}{2}}^b F(x,d)dx}, \frac{q_0^{p-1}c^p}{p(b-a)^{p-1}\int_a^b \max_{|\xi|\le c} F(x,\xi)dx} \right[$$

problem (P) has at least two non-zero weak solutions.

Proof

The proof relies on the following steps

1. we choose the function $\bar{u} \in X$ defined by putting

$$\bar{u}(t) := \begin{cases} \dfrac{2d}{b-a}(x-a) & \textit{if } x \in \left[a, \dfrac{a+b}{2}\right[, \\ \\ d & \textit{if } x \in \left[\dfrac{a+b}{2}, b\right]. \end{cases}$$

2.

$$0 \leq \Phi(\bar{u}) \leq \frac{1}{p}\left(\frac{2}{b-a}\right)^{p-1}\left[||q||_\infty + \frac{p+2}{p+1}\left(\frac{b-a}{2}\right)^p ||s||_\infty\right] d^p$$

3. since $0 < d < c$ and using (ii), we obtain

$$d^p < \left(\frac{q_0}{2}\right)^{p-1} kc^p.$$

4. Put $r = \dfrac{q_0^{p-1} c^p}{p(b-a)^{p-1}}$ we have

$$\Phi(\bar{u}) < r$$

$$\Psi(\bar{u}) \geq \int_{\frac{a+b}{2}}^{b} F(x,d)dx$$

5.

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{c^p} \leq \frac{\int_0^1 \max_{|\xi|\leq c} F(x,\xi)dx}{c^p} < \frac{\Psi(\bar{u})}{\Phi(\bar{u})}$$

6. taking into account (iii) by standard computations, for each $\lambda > 0$ the functional $\Phi - \lambda\Psi$ is unbounded from below and satisfies the $(PS)-$condition.

All assumptions of theorem (Bonanno-D'Aguì(2016)) are satisfied, so for each

$$\lambda \in \left] \frac{2^{p-1}d^p}{p(b-a)^{p-1}k\int_{\frac{a+b}{2}}^{b} F(x,d)dx}, \frac{q_0^{p-1}c^p}{p(b-a)^{p-1}\int_a^b \max_{|\xi|\leq c} F(x,\xi)dx} \right[$$

problem (P) has at least two non-zero weak solutions.

Consequences

Consider the following problem

$$\begin{cases} -(|u'(x)|^{p-2}u'(x))' + |u(x)|^{p-2}u(x) = \lambda f(u(x)) \quad \text{on} \quad]0,1[\,, \\ u(0) = u'(1) = 0, \end{cases} \tag{1}$$

and put $F(t) = \int_0^t f(\xi)d\xi$ for all $t \in \mathbb{R}$.

Theorem

Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function such that

$$\lim_{t \to 0^+} \frac{f(t)}{t^{p-1}} = +\infty,$$

and assume that there are two positive constants $\mu > p$ and $R > 0$ with $|t| \geq R$ such that

$$0 < \mu F(t) < tf(t).$$

Then, for each $\lambda \in]0, \lambda^*[$ where $\lambda^* = \dfrac{2^p(p+1)}{2^p(p+1)+p+2} \sup_{c>0} \dfrac{c^p}{F(c)}$, the problem (1) *has two nonnegative and non zero classical solutions.*

Example

Example

Consider $p = 3$ and the function

$$f(t) = \begin{cases} \frac{3}{2}\sqrt{t} + 5t^4 & t \geq 0 \\ 0 & t < 0. \end{cases}$$

We observe that it is enought to pick for instance $\mu = 4$ and (AR)-condition is verified. Due to Theorem, for each $\lambda \in \left]0, \frac{2}{21}\sqrt[7]{54}\right[$ the problem

$$\begin{cases} -(|u'|u')' + |u|u = \lambda f(u) \quad \text{on} \quad]0,1[\,, \\ \\ u(0) = u'(1) = 0, \end{cases}$$

admits at least two non-zero and nonnegative weak solutions.

mixed boundary value system with $(p_1, \cdots, p_m)$-Laplacian

$$\begin{cases} -(|u_1'|^{p_1-2}u_1')' = \lambda F_{u_1}(t, u_1, \cdots, u_m) & \text{in }]0,1[\\ \vdots & \\ -(|u_m'|^{p_m-2}u_m')' = \lambda F_{u_m}(t, u_1, \cdots, u_m) & \text{in }]0,1[\\ u_i(0) = u_i'(1) = 0 & i = 1, \cdots, m \end{cases}$$

$m \geq 2$, $p_i > 1$ $(1 \leq i \leq m)$, λ is a positive real parameter,

$u : [0,1] \to \mathbb{R}^m$ $(m \geq 2)$

$F : [0,1] \times \mathbb{R}^m \to \mathbb{R}$

- F is a C^1-Carathéodory function
- $F(t, 0, \cdots, 0) = 0$ for every $t \in [0,1]$
- for every $\rho > 0$

$$\sup_{|(x_1,\cdots,x_m)| \le \rho} |F_{u_i}(t, x_1, \cdots, x_m)| \in L^1([0,1]), \qquad i = 1, \cdots, m,$$

where F_{u_i} denotes the partial derivative of F respect on u_i $(i = 1, \cdots, m)$.

Under suitable assumptions on F, we want to prove the existence of multiple weak solutions for this problem for each λ in an appropriate interval.
We use variational methods and multiple critical points theorems

Denote by $X = \prod_{i=1}^{m} X_{p_i}$ where

$$X_{p_i} = \{u \in W^{1,p_i}([0,1]), \quad u(0) = 0\}, \quad p_i \geq 1$$

endowed with the norm

$$\|u\| = \sum_{i=1}^{m} ||u_i||_{p_i}^{p_i} = \sum_{i=1}^{m} \left(\int_0^1 |u_i'(t)|^{p_i} dt \right)^{\frac{1}{p_i}}$$

for every $u \in X$.

Consider the following operators
$\Phi, \Psi : X \to \mathbb{R}$

$$\Phi(u) := \sum_{i=1}^{m} \frac{||u_i||_{p_i}^{p_i}}{p_i}$$

- Φ is continuous and convex
- Φ is coercive, weakly sequentially lower semicontinuous
- continuously Gâteaux differentiable and the Gâteaux derivative at point $u = (u_1, \cdots, u_m) \in X$ is defined by

$$\Phi'(u)(v) = \int_0^1 \sum_{i=1}^{m} |u_i'(t)|^{p_i - 2} u_i'(t) v_i'(t) dt$$

for every $v = (v_1, \cdots, v_m) \in X$.

$$\Psi(u) = \int_0^1 F(t, u(t))dt \qquad \forall u \in X.$$

- Ψ is weakly sequentially upper semicontinuous
- Ψ is continuously Gâteaux differentiable and the Gâteaux derivative at point $u = (u_1, \cdots, u_m) \in X$ is defined by

$$\Psi'(u)(v) = \int_0^1 \sum_{i=1}^{m} F_{u_i}(x, u(t))v_i(t)dt$$

for every $v = (v_1, \cdots, v_m) \in X$.

Definition

A function $u = (u_1, \cdots, u_m) \in X$ is said a weak solution to system if

$$\int_0^1 \sum_{i=1}^m |u_i'(t)|^{p_i-2} u_i'(t) v_i'(t) dt = \lambda \int_0^1 \sum_{i=1}^m F_{u_i}(t, u_1(t), \cdots, u_m(t)) v_i(t) dt$$

for every $v = (v_1, \cdots, v_m) \in X$.

Consider the functional $I = \Phi - \lambda\Psi$

- a critical point for functional $I = \Phi - \lambda\Psi$ is exactly a weak solution for system .
- the functional is weakly sequentially lower semicontinuous

The following theorem of Bonanno, ensures the existence of at least two distinct critical points.

Theorem (Bonanno (2012))

Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$.
Assume that there exist $r \in \mathbb{R}$ and $\bar{u} \in X$, with $0 < \Phi(\bar{u}) < r$, such that

(j) $\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}{r} < \frac{\Psi(\bar{u})}{\Phi(\bar{u})}$

(jj) *for each $\lambda \in \Lambda := \left]\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}\right[$ the functional $I = \Phi - \lambda\Psi$ is unbounded from below and satisfies (PS)-condition.*

Then, for each $\lambda \in \Lambda$, the functional $I = \Phi - \lambda\Psi$ has at least two distinct critical points in X.

G. Bonanno, *Relations between the mountain pass theorem and local minima*, Adv. Nonlnear Anal. **1** 3 (2012) 205–220.

In the following result we use the previous result and also the Ambrosetti-Rabinowitz condition.

Theorem

We suppose that $F_{u_i}(t,0,\cdots,0) \neq 0$ for every $t \in [0,1]$ $(i = 1,\cdots,m)$. We assume that there exist four constants c, d, $\mu \geq \max_{1\leq i\leq m}\{p_i\}$ and R such that

(h_1) $\int_0^{\frac{1}{2}} F(t,x)dt \geq 0$ *for every* $(t,x) \in [0,1] \times R_+$ *where* $R_+ = \{x = (x_1,\cdots,x_m) \in \mathbb{R}^m : 0 \leq x_i \leq d \quad i = 1,\cdots,m\}$;

(i_2)

$$0 < \mu F(t,x_1,\cdots,x_m) \leq \sum_{i=1}^{m} x_i F_{x_i}(t,x_1,\cdots,x_m)$$

for all $t \in [0,1]$ *and* $|x| > R$;

(i_3)

$$\sum_{i=1}^{m} d^{p_i} < \frac{c}{k}$$

$$\frac{\int_0^1 \max_{\xi\in Q(c)} F(t,\xi_i,\cdots,\xi_m)dt}{c} < \frac{\int_{\frac{1}{2}}^1 F(t,d,\cdots,d)dt}{\sum_{i=1}^{m} \frac{2^{p_i-1}}{p_i} d^{p_i}}.$$

Theorem

Then, for each $\lambda \in \left] \frac{\sum_{i=1}^{m} \frac{2^{p_i}-1}{p_i} d^{p_i}}{\int_{\frac{1}{2}}^{1} F(t,d,\cdots,d)dt}, \frac{c}{\int_0^1 \max_{\xi \in Q(c)} F(t,\xi_1,\cdots,\xi_m)dt} \right[$, *the system has at least two non trivial weak solutions.*

$$Q(r) = \{\xi = (\xi_1, \cdots, \xi_m) \in \mathbb{R}^m \,/\, \sum_{i=1}^{m} \frac{|\xi_i|^{p_i}}{p_i} \leq r\}$$

$$\overline{k} = \max_{1 \leq i \leq m}\{\frac{2^{p_i}-1}{p_i}\}, \quad \underline{k} = \min_{1 \leq i \leq m}\{\frac{2^{p_i}-1}{p_i}\}.$$

References

G.D'Aguì - A. Sciammetta - E. Tornatore, *Two non-zero solutions for Sturm-Liouville equations with mixed boundary conditions*, pre-print

D. Averna, E. Tornatore, *Ordinary* $(p_1, \ldots, p_m)-$*Laplacian system with mixed boundary value*, Nonlinear Anal. Real World Appl. **28** (2016), 20-31.