About indefinite Neumann problems with oscillating nonlinear potentials: multiplicity of positive solutions

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Outline

1. Indefinite Neumann boundary value problem (BVP) with oscillating potential
2. Historical overview
3. Main result: multiplicity of positive solutions
4. Applications and further lines of work

Problem formulation

Consider the indefinite Neumann BVP:

$$(\mathcal{N}) \quad \begin{cases} u'' + \boldsymbol{a(t)g(u)} = 0 \\ u(t) > 0, \ \forall t \in [0, T] \\ u'(0) = u'(T) = 0 \end{cases}$$

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Nonlinearity

$g\colon \mathbb{R}^+ := [0,+\infty) \to [0,+\infty)$ is continuous and satisfies

(g_0) $\quad g(0) = 0, \quad g(s) > 0 \ \forall s > 0$

(G_∞) $\quad \liminf_{s\to+\infty} \dfrac{2G(s)}{s^2} = 0 < \limsup_{s\to+\infty} \dfrac{2G(s)}{s^2}$

with $G(s) := \int_0^s g(\xi)d\xi$.

oscillatory potential

Preliminary remarks

Necessary conditions for the existence of solutions to problem $(\mathcal{N})$:

- If g satisfies condition (g_0), then $a(t)$ must change its sign.

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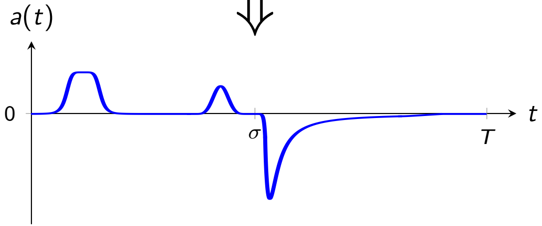

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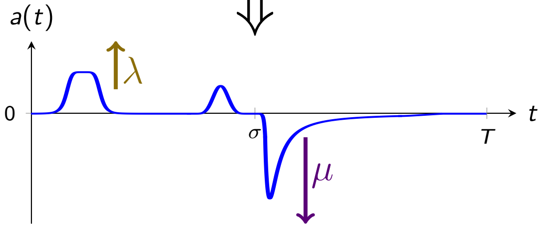

1. Take weights with a “positive hump” followed by a “negative one”.
2. Introduce positive real parameters λ, μ to control $a^+(t)$ and $a^-(t)$.

Framework

Assume that $\exists\, \sigma \in]0, T[$ such that

$$(\star) \quad \begin{array}{l} a(t) \geq 0,\ a(t) \not\equiv 0,\ \forall t \in [0, \sigma], \\ a(t) \leq 0,\ a(t) \not\equiv 0,\ \forall t \in [\sigma, T]. \end{array}$$

Given $\lambda, \mu > 0$, consider $a(t) := \lambda a^{+}(t) - \mu a^{-}(t)$.

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Main purpose

What effects on the dynamics do an indefinite weight term $a(t)$ satisfying $(\star)$ coupled with a positive nonlinearity $g(u)$ oscillating at infinity as in (G_∞) has?

The answer is in the **multiplicity of positive solutions** for the parameter-dependent Neumann problem:

$$(\mathcal{N}_{\lambda,\mu}) \begin{cases} u'' + \big(\lambda a^+(t) - \mu a^-(t)\big) g(u) = 0 \\ u(t) > 0, \quad \forall t \in [0, T] \\ u'(0) = u'(T) = 0 \end{cases}$$

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$$\limsup_{u\to\pm\infty} \frac{2F(x,u)}{u^2} < \lambda_1, \text{ uniformly for } x \in B,$$

where B is a bounded domain, $F(x,u) := \int_0^u f(x,s)\, ds$ and λ_1 is the first eigenvalue of the associated linear problem.

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... **AFTER THAT?** ...

A great deal of works on the solvability of nonlinear BVPs "below the first eigenvalue" with conditions on the primitive $G(u)$ either at 0 or at ∞.

Dirichlet BVPs: remark I

$$(\mathcal{D}_1)\ \begin{cases} \Delta u + g(u) = h(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $\Omega \subseteq \mathbb{R}^N$ bounded domain with smooth $\partial\Omega$, $h \in L^\infty(\Omega)$ and $g \in C(\mathbb{R})$

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- The existence of at least one solution is guaranteed for $(\mathcal{D}_1)$ if g satisfies suitable polynomial growth (Sobolev embeddings) and the Hammerstein-type condition (H): $\limsup_{s\to\pm\infty} \frac{2G(s)}{s^2} < \lambda_1^{\mathcal{D}}(\Omega)$.

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- In the one-dimensional case, $\Omega =]0, T[$:
  - replacing (H) with $\liminf_{s\to\pm\infty} \frac{2G(s)}{s^2} < \lambda_1^{\mathcal{D}}(\Omega) = \left(\frac{\pi}{T}\right)^2$, the above result still holds;
  - adding the **"oscillatory assumption at ∞"**

$$\liminf_{s\to+\infty} \frac{2G(s)}{s^2} < \left(\frac{\pi}{T}\right)^2 < \limsup_{s\to+\infty} \frac{2G(s)}{s^2},$$

along with the technical condition $\lim_{s\to+\infty} g(s) = +\infty$, the existence of infinitely many solutions $u(t) > 0 \ \forall t \in]0, T[$ holds.

cf.: Fernandes, Omari, Zanolin, *Differential Integral Equations* (1989).

Dirichlet BVPs: remark II

$$(\mathcal{D}_2) \quad \begin{cases} \Delta u + a(x)g(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

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If $\mathrm{essinf}_\Omega a(x) > 0$, g satisfies (g_0) and **"oscillatory assumption at ∞"**

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then $\exists (u_n)_n$ sequence of solutions of $(\mathcal{D}_2)$ such that

- $u_n(x) \geq 0 \; \forall \, x \in \Omega$
- $\max_{\overline{\Omega}} u_n \to +\infty$.

cf.: Omari, Zanolin, *Comm. Partial Differential Equations* (1996).

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- $u_n(x) > 0$ and $v_n(x) > 0 \; \forall\, x \in \Omega$
- $\lim_n u_n(x)/\operatorname{dist}(x, \partial\Omega) = \lim_n v_n(x)/\operatorname{dist}(x, \partial\Omega) = +\infty$.

cf.: Obersnel, Omari, *J. Math. Anal. Appl.* (2006).

Back to Neumann BVPs, what can we say?

$$(\mathcal{N}_1) \begin{cases} \Delta u + g(u) = h(x) & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$
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If g satisfies the Hammerstein-type condition (H):

$$\limsup_{s \to \pm\infty} \frac{2G(s)}{s^2} < \lambda_1^{\mathcal{N}}(\Omega) = 0,$$

then $\exists\,(w_n)_n, (v_n)_n$ sequences of reals numbers s.t. $w_n \to -\infty$, $g(w_n) \to +\infty$ and $v_n \to +\infty$, $g(v_n) \to -\infty$. This way: **Neumann BVP easy affordable** (with the theory of lower/upper-solutions) and **no compatible with (g_0)**.

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What about Hammerstain-type condition w.r.t. $\lambda_2^{\mathcal{N}}(\Omega)$?

Extensive literature for $(\mathcal{N}_1)$ starting from

Mawhin, Ward, Willem, *Arch. Rational Mech. Anal.* (1986).
Gossez, Omari, *Proc. Amer. Math. Soc.* (1992). *Trans. Amer. Math. Soc.* (1995).

Lot of multiplicity results for $(\mathcal{N}_2)$ with g super-linear or sub-linear

BUT

it looks still not completely explored the case of g satisfying (g_0) and (G_∞) (even in one-dimension).

Main result

By recalling the indefinite Neumann BVP

$$(\mathcal{N}_{\lambda,\mu}) \quad \begin{cases} u'' + (\lambda a^{+}(t) - \mu a^{-}(t))g(u) = 0 \\ u(t) > 0, \quad \forall t \in [0, T] \\ u'(0) = u'(T) = 0 \end{cases}$$

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Theorem

Let $a\colon [0, T] \to \mathbb{R}$ *be bounded piecewise continuous satisfying* $(\star)$.
Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ *be continuous s.t.* (g_0), (G_∞) *and*
$\limsup_{s\to 0^+} g(s)/s < +\infty$.
Then, $\exists\, \lambda^* \geq 0$ *s.t.* $\forall\, \lambda > \lambda^*$, $\forall\, r > 0$, $\forall\, k \in \mathbb{Z}$ *with* $k \geq 1$,
$\exists\, \mu^* = \mu^*(\lambda, r, k) > 0$ *s.t.* $\forall\, \mu > \mu^*$ *problem* $(\mathcal{N}_{\lambda,\mu})$ *has at least* 2^k *solutions which are nonincreasing on* $[0, T]$ *and* $0 < u(t) \leq r\ \forall\, t \in [\sigma, T]$.
If $\limsup_{s\to+\infty} 2G(s)/s^2 = +\infty$, *the result holds with* $\lambda^* = 0$.

cf.: S., Zanolin, *J. Math. Anal. Appl.* (2017)

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Nonlinearity example

Let $\rho, \theta, A, B \in \mathbb{R}$ s.t $k, A > 0, \theta \in [0, 2\pi[$, $|B| < 2A/(\rho^2 + 4)^{1/2}$. We define $\forall s \geq 0$
$\boldsymbol{G(s) := As^2 + Bs^2 \cos(\rho \log(1 + s) + \theta)}$.

Then, $\boldsymbol{g(s) := G'(s)}$ is $C^\infty(\mathbb{R}^+)$ satisfying

- $g(0) = 0, \quad g(s) > 0 \; \forall s > 0$
- $\liminf_{s \to +\infty} 2G(s)/s^2 = 2(A - B) < 2(A + B) = \limsup_{s \to +\infty} 2G(s)/s^2$
- $\lim_{s \to 0^+} g(s)/s = 2(A + B\cos\theta) > 0$

cf.: S., Zanolin, *RIMUT* (2015)

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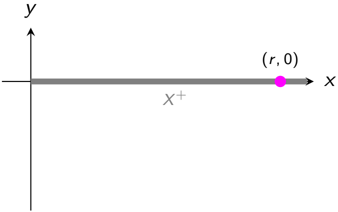
Sketch of the proof via Shooting Method

Solutions $\big(x(\cdot;0,(r,0)),v(\cdot;0,(r,0))\big)$ of the Cauchy problem

$$(\mathcal{S})\ \begin{cases} x' = y \\ y' = -\left(\lambda a^{+}(t) - \mu a^{-}(t)\right) g(x) \end{cases}$$

with initial data

$$\begin{cases} x(0) = r, \quad r > 0 \\ y(0) = 0 \end{cases}$$



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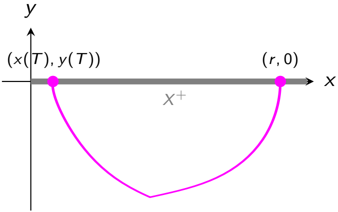

satisfying

$$\Big(x(T;0,(r,0)),\, y(T;0,(r,0))\Big) \in X^+$$

with $X^+ :=]0,+\infty[\times \{0\}$

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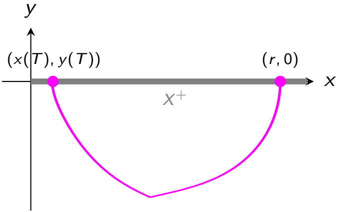

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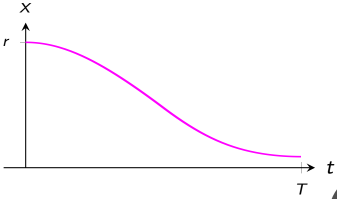

identify solutions $u(\cdot) := x(\cdot;0,(r,0))$ of problem $(\mathcal{N}_{\lambda,\mu})$

Phase plane analysis

$\Phi_0^T : \mathbb{R}^2 \to \mathbb{R}^2$ Poincaré Map (PM) associated with

$$(\mathcal{S}) \begin{cases} x' = y \\ y' = -\left(\lambda a^+(t) - \mu a^-(t)\right) g(x) \end{cases}$$

defined as $\Phi_0^T(x_0, y_0) := \Big(x(T; 0, (x_0, y_0)), y(T; 0, (x_0, y_0))\Big)$

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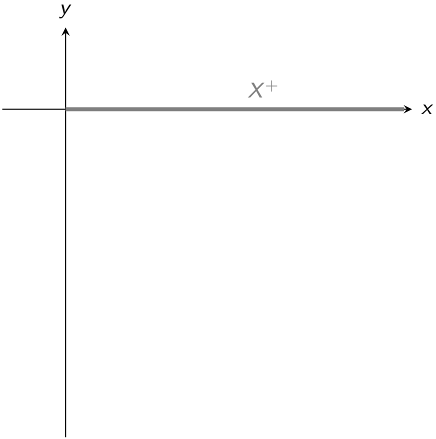
LOOK FOR $\boldsymbol{P \in X^+}$ s.t. $\boldsymbol{\left(\Phi_\sigma^T \circ \Phi_0^\sigma\right)(P) \in X^+}$

Dynamics on $[0, \sigma]$



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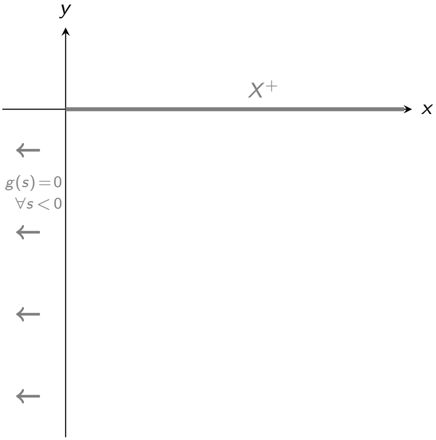
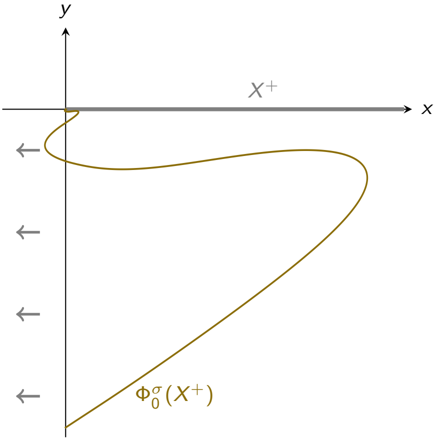
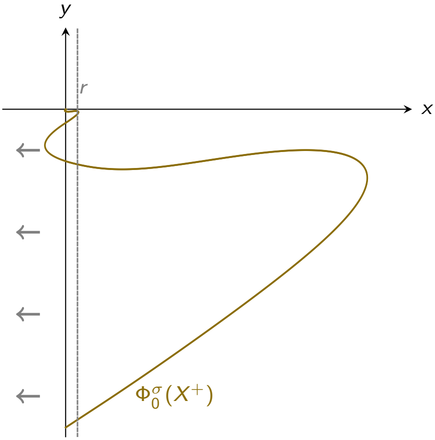


Dynamics on $[0, \sigma]$



$\lambda > \lambda^*$ fixed

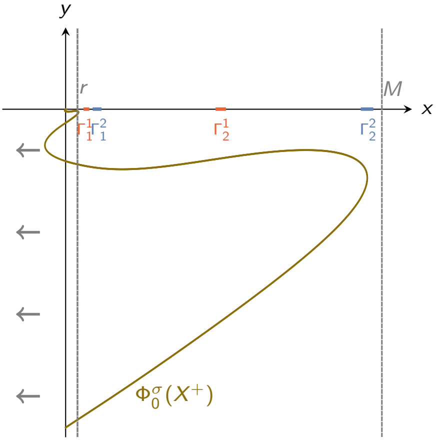
Dynamics on $[0, \sigma]$



$\lambda > \lambda^*$ fixed

Dynamics on $[0, \sigma]$

$\forall k \geq 1 \; \exists \, (\Gamma_k^1)_k, \; (\Gamma_k^2)_k$ subint. of $[r, M]$

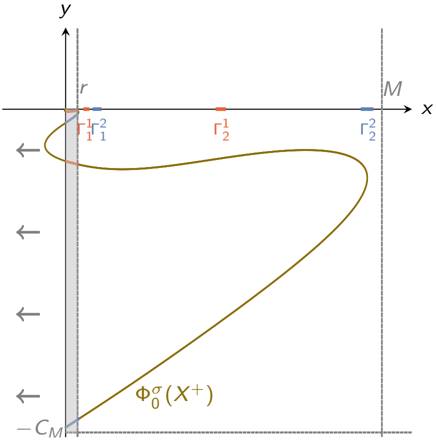


$\lambda > \lambda^*$ fixed

Dynamics on $[0, \sigma]$

$\forall k \geq 1 \; \exists \, (\Gamma_k^1)_k, \, (\Gamma_k^2)_k$ subint. of $[r, M]$

s.t. $\Phi_0^\sigma(\Gamma_k^1), \Phi_0^\sigma(\Gamma_k^2) \subset [0, r] \times [-C_M, 0[$

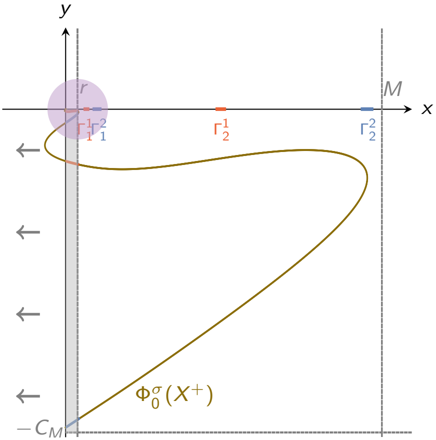


$\lambda > \lambda^*$ fixed

Dynamics on $[0, \sigma]$

$\forall k \geq 1 \; \exists \, (\Gamma_k^1)_k, (\Gamma_k^2)_k$ subint. of $[r, M]$

s.t. $\Phi_0^\sigma(\Gamma_k^1), \Phi_0^\sigma(\Gamma_k^2) \subset [0, r] \times [-C_M, 0[$

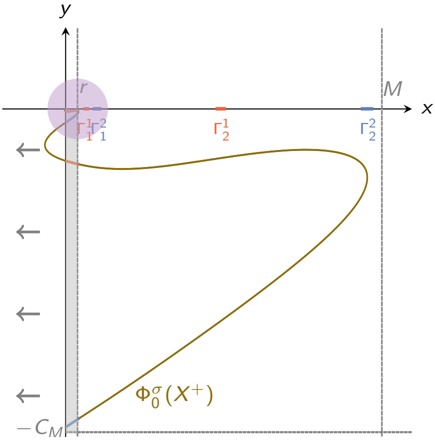


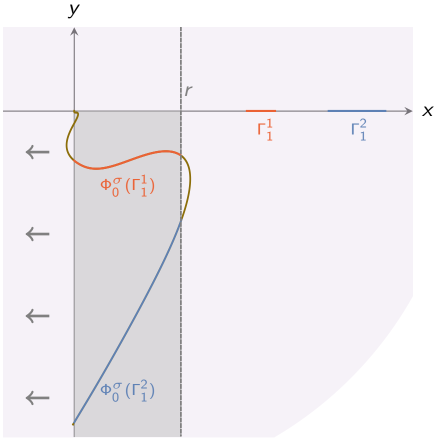
$\lambda > \lambda^*$ fixed

Dynamics on $[0, \sigma]$ Dynamics on $[\sigma, T]$

$\forall k \geq 1 \; \exists \, (\Gamma^1_k)_k, \, (\Gamma^2_k)_k$ subint. of $[r, M]$

s.t. $\Phi_0^\sigma(\Gamma^1_k), \Phi_0^\sigma(\Gamma^2_k) \subset [0, r] \times [-C_M, 0[$



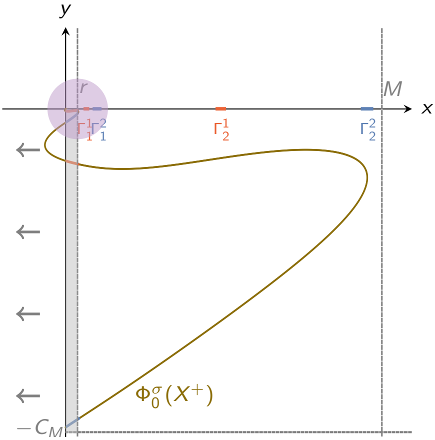


$\lambda > \lambda^*$ fixed

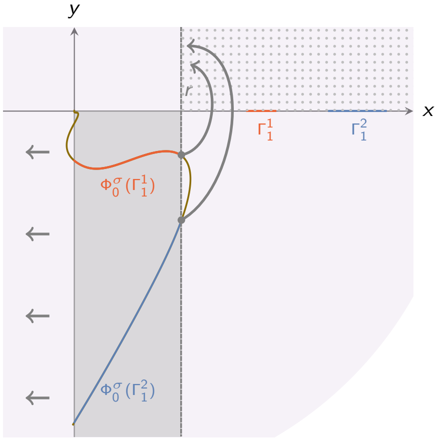
Dynamics on $[0, \sigma]$ — Dynamics on $[\sigma, T]$

$\forall k \geq 1 \; \exists \, (\Gamma^1_k)_k, \, (\Gamma^2_k)_k$ subint. of $[r, M]$

s.t. $\Phi^\sigma_0(\Gamma^1_k), \Phi^\sigma_0(\Gamma^2_k) \subset [0, r] \times [-C_M, 0[$



$\lambda > \lambda^*$ fixed

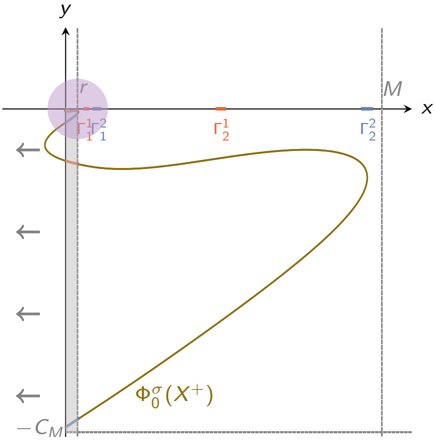


$\mu > \mu^*(\lambda, k, r)$ fixed

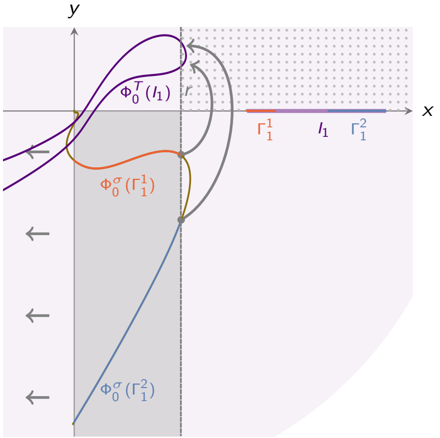
Dynamics on $[0, \sigma]$ Dynamics on $[\sigma, T]$

$\forall k \geq 1 \, \exists \, (\Gamma^1_k)_k, (\Gamma^2_k)_k$ subint. of $[r, M]$

s.t. $\Phi^\sigma_0(\Gamma^1_k), \Phi^\sigma_0(\Gamma^2_k) \subset [0, r] \times [-C_M, 0[$



$\lambda > \lambda^*$ fixed

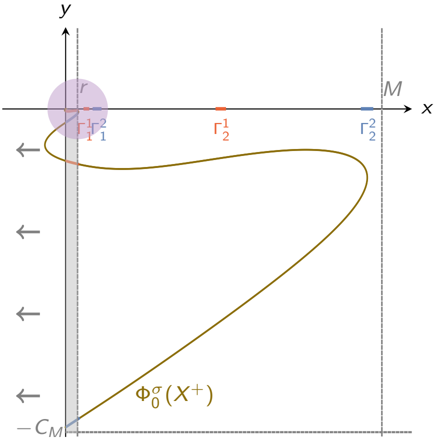


$\mu > \mu^*(\lambda, k, r)$ fixed

Dynamics on $[0, \sigma]$

$\forall k \geq 1 \; \exists \, (\Gamma_k^1)_k, \, (\Gamma_k^2)_k$ subint. of $[r, M]$

s.t. $\Phi_0^\sigma(\Gamma_k^1), \Phi_0^\sigma(\Gamma_k^2) \subset [0, r] \times [-C_M, 0[$

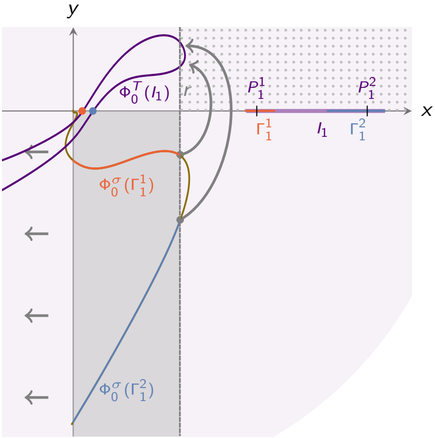


$\lambda > \lambda^*$ fixed

Dynamics on $[\sigma, T]$

$\forall k \geq 1 \; \exists \, (P_k^1)_k, \, (P_k^2)_k, \; P_k^i \in \Gamma_k^i, \, i = 1, 2$

s.t. $\Phi_0^T(P_k^1), \Phi_0^T(P_k^2) \in X^+$



$\mu > \mu^*(\lambda, k, r)$ fixed

Full dynamics $[0, T]$

Summing up:

- $\exists\, 2k$ points $P_j^1, P_j^2 \in X^+$ s.t. $\Phi_0^T(P_j^1), \Phi_0^T(P_j^2) \in X^+ \;\forall\, j = 1 \dots, k$;
- the solutions $(x(t), y(t)) = \left(x(T; 0, P_j^i), y(T; 0, P_j^i)\right)$ of the Cauchy problem

$$\begin{cases} x' = y \\ y' = -\left(\lambda a^+(t) - \mu a^-(t)\right) g(x) \\ (x(0), y(0)) = P_j^i, \quad \forall\, i = 1, 2 \;\forall\, j = 1 \dots, k \end{cases}$$

  satisfy $0 < x(t) < M$ and $y(t) \leq 0 \;\forall t \in [0, T]$;
- $\exists\, 2k$ (positive) solutions of $(\mathcal{N}_{\lambda,\mu})$.

Radially symmetric solutions

In $\mathbb{R}^N$ for $N \geq 2$, we consider the problem

$$(\mathcal{P}) \quad \begin{cases} \Delta u + (\lambda w^+(x) - \mu w^-(x))\, g(u) = 0 & \text{in } \Omega, \\ u(x) > 0 & \text{in } \overline{\Omega}, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an **open ball or** an **open annulus**, $w \in L^1(\Omega)$ is **radially symmetric**, i.e. $\exists\, \mathcal{Q}$ L^1-function s.t. $w(x) = \mathcal{Q}(|x|)$.

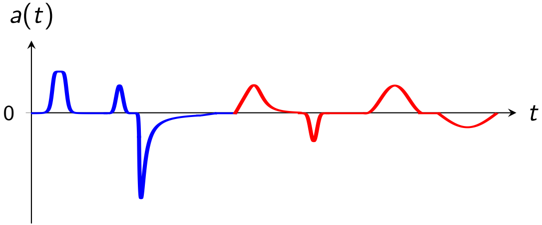
Theorem

Let $\mathcal{Q}(|x|)$ satisfies conditions $(\star)$ adapted to the case of the open ball or the open annulus. Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be continuous satisfying (g_0), (G_∞) and $\limsup_{s\to 0^+} g(s)/s < +\infty$.
Then, $\exists\, \lambda^* \geq 0$ such that, $\forall\, \lambda > \lambda^*\, \forall\, k \in \mathbb{Z}\; k \geq 1$, $\exists\, \mu^* = \mu^*(\lambda, k) > 0$ such that $\forall\, \mu > \mu^*$ problem $(\mathcal{P})$ has at least $2k$ radially symmetric solutions.

cf.: S., Zanolin, *J. Math. Anal. Appl.* (2017)

Further directions

- Increase the number of “positive humps” and “negative humps” in the weight term to detect possible **complex behaviors** for Neumann BVPs with an oscillating nonlinear potential.



- Deal with different boundary conditions (e.g. mixed boundary conditions).

Thank you for your attention!