# Nonlinear boundary value problems with variable exponent

Giornate di Equazioni Differenziali Ordinarie: metodi e prospettive

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G. Bonanno, G. D'Aguì, A. Sciammetta, *One-dimensional nonlinear boundary value problems with variable exponent*, Discrete and Continuous Dynamical Systems - Series S **11** (2018), 179-191.

$$\begin{cases} -\left(|u'(x)|^{p(x)-2}u'(x)\right)' + a(x)\,|u(x)|^{p(x)-2}u(x) = \lambda f(x,u(x)) \quad \text{in } ]0,1[, \\ u(0)=u(1)=0, \end{cases} \qquad (D_\lambda^{p(x)})$$

- $a \in L^\infty([0,1])$, with $\operatorname{essinf}_{[0,1]} a \geq 0$,
- $p \in C([0,1])$ Put

$$p^- := \min_{x\in[0,1]} p(x), \qquad p^+ := \max_{x\in[0,1]} p(x)$$

  and assume

$$p^- > 1.$$

- $\lambda$ is a positive real parameter
- $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a nonnegative $L^1-$Carathéodory function, that is:
  1. $x \mapsto f(x,\xi)$ is measurable for every $\xi \in \mathbb{R}$;
  2. $\xi \mapsto f(x,\xi)$ is continuous for almost every $x \in [0,1]$;
  3. for every $s > 0$ there is a function $l_s \in L^1([0,1])$ such that

$$\sup_{|\xi|\leq s} |f(x,\xi)| \leq l_s(x),$$

     for a.e. $x \in [0,1]$.

## Some references

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[3] G. Bonanno, A. Chinnì, *Discontinuous elliptic problems involving the* $p(x)-$*Laplacian*, Math. Nachr. **284** (2011), 639–652.

[4] G. Bonanno, A. Chinnì, *Existence and multiplicity of weak solutions for elliptic Dirichlet problems with variable exponent*, J. Math. Anal. Appl. **418** (2014), 812–827.

[5] X.-L., Q.-H. Zhang, *Existence of solutions for* $p(x)-$*Laplacian Dirichlet problem*, Nonlinear Anal. **52** (2003), 1843–1852.

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## The variable exponent Lebesgue ans Sobolev spaces

$$L^{p(x)}([0,1]) = \left\{ u : [0,1] \to \mathbb{R}, u \text{ is measur. and } \rho_{p(x)}(u) := \int_0^1 |u(x)|^{p(x)}\, dx < +\infty \right\}.$$

$$\|u\|_{L^{p(x)}([0,1])} := \inf \left\{ \eta > 0 : \int_0^1 \left| \frac{u(x)}{\eta} \right|^{p(x)} dx \leq 1 \right\}.$$

$$W^{1,p(x)}([0,1]) := \left\{ u \in L^{p(x)}([0,1]) : u' \in L^{p(x)}([0,1]) \right\}$$

$$\|u\|_{W^{1,p(x)}([0,1])} := \|u\|_{L^{p(x)}([0,1])} + \|u'\|_{L^{p(x)}([0,1])}.$$

Since $p^- > 1$

- $L^{p(x)}([0,1])$ is a separable, reflexive and uniformly convex Banach space;
- $W^{1,p(x)}([0,1])$ is separable, reflexive and uniformly convex a Banach space.

By $W_0^{1,p(x)}([0,1])$ we denote the closure of $C_0^\infty([0,1])$ in $W^{1,p(x)}([0,1])$.

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[1] D.V. Cruz-Uribe, A. Fiorenza,*Variable Lebesgue Spaces*, Applied and Numerical Harmonic Analysis, Springer Basel, Heidelberg 2013.

[2] L. Diening, P. Harjulehto, P. Hästö, M. Ružička, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, Springer-Verlag, Heidelberg 2017.

## Lemma (Hölder's inequality)

Let $p$ and $p' \in C([0,1])$ s. t. $p^- > 1$ and $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, for all $x \in [0,1]$. For all $f \in L^{p(x)}([0,1])$ and for all $g \in L^{p'(x)}([0,1])$ one has $fg \in L^1([0,1])$

$$\|fg\|_1 \leq \left(1 + \frac{1}{p^-} - \frac{1}{p^+}\right) \|f\|_{L^{p(x)}([0,1])} \|g\|_{L^{p'(x)}([0,1])}.$$

## Proposition (Poincaré inequality)

Let $p \in C([0,1])$ such that $p^- > 1$. Then for all $u \in W_0^{1,p(x)}([0,1])$ one has

$$\|u\|_\infty \leq \|u'\|_{L^{p(x)}([0,1])} \quad \text{and} \quad \|u\|_{L^{p(x)}([0,1])} \leq \|u'\|_{L^{p(x)}([0,1])}.$$

Moreover, the embedding of $W_0^{1,p(x)}([0,1])$ into $C([0,1])$ is compact.

## Remark

$$\|u\|_\infty \leq \frac{1}{2}\left(1 + \frac{1}{p^-} - \frac{1}{p^+}\right) \|u'\|_{L^{p(x)}([0,1])}, \quad \text{for all} \quad u \in W_0^{1,p(x)}([0,1]).$$

$$\|u\|_{L^{p(x)}([0,1])} \leq \frac{1}{2}\left(1 + \frac{1}{p^-} - \frac{1}{p^+}\right) \|u'\|_{L^{p(x)}([0,1])} \quad \text{for all} \quad u \in W_0^{1,p(x)}([0,1]).$$

## Definition of Globally log-Hölder continuity

Let $\Omega \subseteq \mathbb{R}$. A function $p : \Omega \to \mathbb{R}$ is *locally log-Hölder continuous* on $\Omega$ if there exist $c_1 > 0$ s. t.

$$|p(x) - p(y)| \leq \frac{c_1}{\log\left(e + \frac{1}{|x-y|}\right)}, \quad \text{for all} \quad x, y \in \Omega.$$

We say that $p$ satisfies the *log-Hölder decay condition* if there exist $p_\infty \in \mathbb{R}$ and $c_2 > 0$ s. t.

$$|p(x) - p_\infty| \leq \frac{c_2}{\log\left(e + |x|\right)}, \quad \text{for all} \quad x \in \Omega.$$

Locally log-Hölder continuous + log-Hölder decay condition $\Longrightarrow$ *globally log-Hölder continuous*.

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[1] L. Diening, P. Harjulehto, P. Hästö, M. Ružička, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, Springer-Verlag, Heidelberg 2017.

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[3] X.-L. Fan, J. Shen, D. Zhao, *Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$*, J. Math. Anal. Appl., **262** (2001), 749–760.

[4] O. Kováčik, J. Rákosník, *On the spaces $L^{p(x)}$ and $W^{1,p(x)}$*, Czechoslovak Math. **41** (1991), 592–618.

[5] D.V. Cruz-Uribe, A. Fiorenza, *Variable Lebesgue Spaces*, Applied and Numerical Harmonic Analysis, Springer Basel, Heidelberg 2013.

As in the case with constant exponent, on $W_0^{1,p(x)}([0,1])$ we can consider the norm

$$\|u\|_{W_0^{1,p(x)}([0,1])} := \|u'\|_{L^{p(x)}([0,1])},$$

Now, taking into account that $a \in L^\infty(\Omega)$, with $\operatorname{essinf}_{x\in[0,1]} a(x) \geq 0$, we define on $W_0^{1,p(x)}([0,1])$ the following norm

$$\|u\|_a := \inf \left\{ \sigma > 0 : \int_0^1 \left( \left|\frac{u'(x)}{\sigma}\right|^{p(x)} + a(x) \left|\frac{u(x)}{\sigma}\right|^{p(x)} \right) dx \leq 1 \right\}.$$

## Proposition

Let $p \in C([0,1]$ such that $p^- > 1$. Then, one has

$$\|u\|_{W_0^{1,p(x)}([0,1])} \leq \|u\|_a \leq (1 + \|a\|_\infty)^{\frac{1}{p^-}} \|u\|_{W_0^{1,p(x)}([0,1])}.$$

From Poincaré inequality, previous Proposition and [1, Theorem 1.3] we obtain

$$\|u\|_\infty \leq \|u\|_a$$

for all $u \in W_0^{1,p(x)}([0,1])$.

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[1] X.-L. Fan, D. Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. **263** (2001), 424–446.

## Main tool

### Theorem (G. Bonanno 2012)

Let $X$ be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that

$$\frac{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})},$$

and, for each

$$\lambda \in \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)} \right[,$$

the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)^{[r]}$−condition.
Then, for each

$$\lambda \in \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)} \right[,$$

there is $u_\lambda \in \Phi^{-1}(]0, r[)$ such that $I_\lambda(u_\lambda) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}(]0, r[)$ and $I'_\lambda(u_\lambda) = 0$.

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[1] G. Bonanno, *Relations between the mountain pass theorem and local minima*, Adv. Nonlinear Anal. **1** (2012), no. 3, 205–220.

$$\underbrace{I_\lambda(u) = \underbrace{\int_0^1 \frac{1}{p(x)} \left[ \left| u'(x) \right|^{p(x)} + a(x) \left| u(x) \right|^{p(x)} \right] dx}_{\Phi(u)} - \lambda \underbrace{\int_0^1 F(x, u(x)) dx}_{\Psi(u)} .}_{\text{Energy functional}}$$

$$F(x,t) = \int_0^t f(x,\xi) d\xi, \quad \text{for all} \quad (x,t) \in [0,1] \times \mathbb{R}.$$

## Definition

A function $u : [0,1] \to \mathbb{R}$ is a weak solution of problem $(D_\lambda^{p(x)})$ if $u \in X$ satisfies the following condition for all $v \in X$

$$\underbrace{\int_0^1 \left| u'(x) \right|^{p(x)-2} u'(x) v'(x)\, dx + \int_0^1 a(x) \left| u(x) \right|^{p(x)-2} u(x) v(x)\, dx}_{\Phi'(u)(v)} = \lambda \underbrace{\int_0^1 f(x, u(x)) v(x)\, dx}_{\Psi'(u)(v)} .$$

## Lemma

If we assume $f(x,0) \geq 0$ for a.e. $x \in [0,1]$, then the weak solutions of problem $(D_\lambda^{p(x)})$ are nonnegative.

## Main result

### Theorem

Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a nonnegative $L^1$-Carathéodory function. Assume that there exist two positive constants $c$ and $d$, with $d < c$, s. t.

$$\frac{\int_0^1 F(x,c)\ dx}{\min\{c^{p^-}; c^{p^+}\}} < \frac{2p^-}{p^+(4^{p^+} + 2\|a\|_1)} \frac{\int_{\frac{1}{4}}^{\frac{3}{4}} F(x,d)\ dx}{\max\{d^{p^-}; d^{p^+}\}}. \tag{1}$$

Then, for each $\lambda \in \Lambda = \left] \frac{4^{p^+} + 2\|a\|_1}{2p^-} \frac{\max\{d^{p^-}; d^{p^+}\}}{\int_{\frac{1}{4}}^{\frac{3}{4}} F(x,d)\ dx}, \frac{1}{p^+} \frac{\min\{c^{p^-}; c^{p^+}\}}{\int_0^1 F(x,c)\ dx} \right[$, problem $(D_\lambda^{p(x)})$ admits at least one nonnegative and non-zero weak solution $\bar{u}$ s. t. $|\bar{u}(x)| < c$ for all $x \in [0,1]$.

## Sketch of Proof

1. $X = W_0^{1,p(x)}([0,1])$
2. $\Phi(u) := \int_0^1 \frac{1}{p(x)} \left[ |u'(x)|^{p(x)} + a(x)\,|u(x)|^{p(x)} \right] dx$, $\Psi(u) := \int_0^1 F(x,u(x))dx \; \forall\, u \in X$ satisfy all regularity assumptions requested in our main tool and the critical points in $X$ of $I_\lambda = \Phi - \lambda\Psi$ are the weak solutions of $(D_\lambda^{p(x)})$.
3. Put $r = \frac{1}{p^+}\min\{c^{p^-}; c^{p^+}\}$ and

$$\tilde{u}(x) = \begin{cases} 4dx & \text{if } x \in \left[0, \frac{1}{4}\right[, \\ d & \text{if } x \in \left[\frac{1}{4}, \frac{3}{4}\right], \\ 4d(1-x) & \text{if } x \in \left]\frac{3}{4}, 1\right]. \end{cases}$$

Clearly, $\tilde{u} \in W_0^{1,p(x)}([0,1])$. From $d < c+ \,(1) \Longrightarrow 0 < \Phi(\tilde{u}) < r$ and

$$\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \geq \frac{2p^-}{4^{p^+} + 2\|a\|_1} \frac{\int_{\frac{1}{4}}^{\frac{3}{4}} F(x,d)\,dx}{\max\{d^{p^-}; d^{p^+}\}} > \frac{p^+ \int_0^1 F(x,c)\,dx}{\min\{c^{p^-}; c^{p^+}\}} \geq \frac{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}{r}.$$

4. Hence

$$\lambda \in \Lambda \subseteq \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)} \right[,$$

and our conclusion is achieved.

## Some consequences

### Theorem

Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a nonnegative $L^1$-Carathéodory function. Assume that there exist two distinct positive constants $c$ and $d$, with $d \leq 1 \leq c$ such that

$$\frac{\int_0^1 F(x,c)\ dx}{c^{p^-}} < \frac{2p^-}{p^+(4^{p^+} + 2\|a\|_1)} \frac{\int_{\frac{1}{4}}^{\frac{3}{4}} F(x,d)\ dx}{d^{p^-}}.$$

Then, for each $\lambda \in \left] \frac{4^{p^+} + 2\|a\|_1}{2p^-} \frac{d^{p^-}}{\int_{\frac{1}{4}}^{\frac{3}{4}} F(x,d)\ dx}, \frac{1}{p^+} \frac{c^{p^-}}{\int_0^1 F(x,c)\ dx} \right[$, problem $(D_\lambda^{p(x)})$ admits at least one nonnegative and non-zero weak solution $\bar{u}$ such that $|\bar{u}(x)| < c$ for all $x \in [0,1]$.

## Some consequences

Let $\alpha \in L^1([0,1])$ be such that $\alpha(x) \geq 0$ a.e. $x \in [0,1]$, $\alpha \not\equiv 0$, and let $g : \mathbb{R} \to \mathbb{R}$ be a continuous nonnegative function. Consider the following Dirichlet boundary value problem

$$(AD_\lambda^{p(x)}) \begin{cases} -\left(|u'(x)|^{p(x)-2} u'(x)\right)' + a(x)\,|u(x)|^{p(x)-2} u(x) = \lambda\alpha(x)g(u(x)) \quad \text{in } ]0,1[, \\ u(0) = u(1) = 0. \end{cases}$$

$$G(t) = \int_0^t g(\xi)\,d\xi, \quad \text{for all} \quad t \in \mathbb{R}, \quad K = \frac{2p^-}{p^+(4^{p^+} + 2\|a\|_1)} \frac{\int_{\frac{1}{4}}^{\frac{3}{4}} \alpha(x)\,dx}{\|\alpha\|_1}.$$

### Theorem

Assume that there exist two positive constants $c, d$, with $d < c \leq 1$, such that

$$\frac{G(c)}{c^{p^+}} < K \frac{G(d)}{d^{p^-}}.$$

Then, for each $\lambda \in$

$$\left] \frac{1}{K} \frac{1}{p^+\|\alpha\|_1} \frac{d^{p^-}}{G(d)}, \frac{1}{p^+\|\alpha\|_1} \frac{c^{p^+}}{G(c)} \right[,$$

problem $(AD_\lambda^{p(x)})$ admits at least one nonnegative and non-zero weak solution $\bar{u}$ such that $|\bar{u}(x)| < c$ for all $x \in [0,1]$.

## Some consequences

Consider following problem

$$\begin{cases} -\left(|u'(x)|^{p(x)-2}u'(x)\right)' = \lambda g(u(x)) \quad \text{in } ]0,1[, \\ u(0) = u(1) = 0, \end{cases} \qquad (A_\lambda)$$

### Theorem

Let $g : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function such that

$$\int_0^4 g(\xi)d\xi < \frac{p^-}{p^+}\frac{4^{2p^-}}{4^{p^+}}\int_0^{\frac{1}{4}} g(\xi)d\xi.$$

Then, for each $\lambda \in \left]\frac{4^{p^+}}{p^-4^{p^-}}\frac{1}{\int_0^{\frac{1}{4}} g(\xi)d\xi}, \frac{4^{p^-}}{p^+}\frac{1}{\int_0^4 g(\xi)d\xi}\right[$, the problem $(A_\lambda)$ admits at least one non-zero weak solution $\bar{u}$ such that $0 \le \bar{u}(x) < 4$ for all $x \in [0,1]$.

## Some consequences

### Example

$d = \frac{1}{4} < c = 4$, $p(x) = \frac{x^4}{10} + 5$ for all $x \in [0,1]$ and

$$g(\xi) = \begin{cases} (10\xi)^4 & \text{if } \; 0 \leq \xi \leq \frac{1}{4}, \\ \left(\dfrac{5}{8\xi}\right)^4 & \text{if } \; \frac{1}{4} < \xi < 4, \\ h(\xi) & \text{if } \; \xi \geq 4, \end{cases}$$

where $h : [4, +\infty[ \to \mathbb{R}$ is an arbitrary function. Owing to previous Theorem, the problem

$$\begin{cases} -\left(|u'(x)|^{p(x)-2} u'(x)\right)' = g(u(x)) & \text{in } \; ]0,1[, \\ u(0) = u(1) = 0, \end{cases}$$

admits at least one non-zero weak solution $u$ such that $0 \leq u(x) < 4$. Indeed

$$\int_0^4 g(\xi)d\xi < \frac{p^-}{p^+}\frac{4^{2p^-}}{4^{p^+}}\int_0^{\frac{1}{4}} g(\xi)d\xi \text{ and } \frac{4^{p^+}}{p^- 4^{p^-}}\frac{1}{\int_0^{\frac{1}{4}} g(\xi)d\xi} < 1 < \frac{4^{p^-}}{p^+}\frac{1}{\int_0^4 g(\xi)d\xi}.$$

We explicitly observe that the function $f$ is not $(p^- - 1)-$sublinear at zero since one has

$$\lim_{t \to 0^+} \frac{g(t)}{t^{p^- - 1}} = 10^4 < +\infty.$$

## Some consequences

$$(AD_\lambda^{p(x)})\begin{cases} -\left(|u'(x)|^{p(x)-2}u'(x)\right)' + a(x)\,|u(x)|^{p(x)-2}u(x) = \lambda\alpha(x)g(u(x)) \quad \text{in } ]0,1[,\\ u(0)=u(1)=0. \end{cases}$$

### Theorem

Assume that

$$\lim_{t\to 0^+}\frac{g(t)}{t^{p^- -1}} = +\infty,$$

Then, for each $\lambda \in \left]0, \frac{1}{p^+\|\alpha\|_1}\max\left\{\sup_{0<c<1}\frac{c^{p^+}}{\int_0^c g(\xi)\,d\xi}; \sup_{c\geq 1}\frac{c^{p^-}}{\int_0^c g(\xi)\,d\xi}\right\}\right[$, problem $(AD_\lambda^{p(x)})$ admits at least one non-zero and nonnegative weak solution.

### Example

$$\begin{cases} -\left(|u'(x)|^{x^2+2}u'(x)\right)' + |u(x)|^{x^2+2}u(x) = x^4[u(x)]^2 \quad \text{in } ]0,1[,\\ u(0)=u(1)=0. \end{cases}$$

$p(x)=x^2+4$, $\lim_{t\to 0^+}\frac{g(t)}{t^{p^- -1}} = \lim_{t\to 0^+}\frac{t^2}{t^{4-1}} = +\infty$, $\lambda^* \geq \frac{1}{p^+\|\alpha\|_1}\frac{1}{\int_0^1 g(\xi)\,d\xi} = 3.$

## Infinitely many solutions

G. D'Aguì, A. Sciammetta, *Infinitely many solutions to elliptic problems with variable exponent and nonhomogeneous Neumann conditions*, Nonlinear Analysis: Theory, Methods and Applications, Volume 75, Issue 14, (2012), 5612–5619.

$$\begin{cases} -\Delta_{p(x)}u(x) + \alpha(x)\,|u(x)|^{p(x)-2}\,u(x) = \lambda f\,(x, u(x)) & \text{in } \Omega, \\ |\nabla u(x)|^{p(x)-2}\dfrac{\partial u}{\partial \nu} = \mu g\,(\gamma\,(u(x))) & \text{on } \partial\Omega, \end{cases} \qquad (P_{\lambda,\mu})$$

- $\Delta_{p(x)}u(x) = \operatorname{div}\left(|\nabla u(x)|^{p(x)-2}\,\nabla u(x)\right)$ is the $p(x)-$Laplacian operator;
- $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary;
- $p \in C(\bar{\Omega})$ with $N < p^- := \inf_{x\in\Omega} p(x) \leq p(x) \leq p^+ := \sup_{x\in\Omega} p(x)$;
- $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $g : \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function,
- $\lambda > 0$ and $\mu \geq 0$;
- $\alpha \in L^\infty(\Omega)$, with $essinf_\Omega \alpha > 0$;
- $\nu$ is the outer unit normal to $\partial\Omega$;
- $\gamma : W^{1,p(x)}(\Omega) \to L^{p(x)}(\partial\Omega)$ is the trace operator.

### Main tool

G. Bonanno, *A critical point theorem via Ekeland variational principle*, Nonlinear Anal. 75 (2012), 2992–3007.

$$A := \liminf_{\xi \to +\infty} \frac{\int_\Omega \max_{|t|<\xi} F(x,t)dx}{\xi^{p^-}}, \quad B := \limsup_{\xi \to +\infty} \frac{\int_\Omega F(x,\xi)dx}{\xi^{p^+}}, \quad \lambda_1 = \frac{\|\alpha\|_1}{p^- B}, \quad \lambda_2 = \frac{1}{p^+ m^{p^-} A},$$

$$k_{p^-} \leq 2^{\frac{p^- - 1}{p^-}} \max\left\{ \left( \frac{1}{\|\alpha\|_1} \right)^{\frac{1}{p^-}}, \frac{d}{N^{\frac{1}{p^-}}} \left( \frac{p^- - 1}{p^- - N} m(\Omega) \right)^{\frac{p^- - 1}{p^-}} \frac{\|\alpha\|_\infty}{\|\alpha\|_1} \right\},$$

where $\|\alpha\|_1$ is the usual norm in $L^1(\Omega)$, $m = k_{p^-}(1 + m(\Omega))$.

## Theorem

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ an $L^1$-Carathéodory function. Assume that

$$\liminf_{\xi \to +\infty} \frac{\int_\Omega \max_{|t|<\xi} F(x,t)dx}{\xi^{p^-}} < \frac{p^-}{p^+ m^{p^-} \|\alpha\|_1} \limsup_{\xi \to +\infty} \frac{\int_\Omega F(x,\xi)dx}{\xi^{p^+}}.$$

Then, for each $\lambda \in ]\lambda_1, \lambda_2[$, for each nonnegative continuous function $g : \mathbb{R} \to \mathbb{R}$ such that

$$G_\infty = \limsup_{\xi \to +\infty} \frac{G(\xi)}{\xi^{p^-}} < +\infty,$$

and for each $\mu \in [0, \delta[$, with $\delta = \dfrac{1 - m^{p^-} p^+ \lambda A}{m^{p^-} p^+ G_\infty a(\partial\Omega)}$, where $a(\partial\Omega) = \int_{\partial\Omega} d\sigma$, the problem $(P_{\lambda,\mu})$ admits a sequence of weak solutions which is unbounded in $W^{1,p(x)}(\Omega)$.

**Thank you for your kind attention**