UNIVERSITÀ DEGLI STUDI DI TRIESTE

dipartimento di matematica e geoscienze

Global components of positive bounded variation solutions of a one-dimensional indefinite quasilinear Neumann problem

Giornate di Equazioni Differenziali Ordinarie: metodi e prospettive
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A quasilinear elliptic Neumann problem

Consider the problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = f(x,u) & \text{in } \Omega \\ -\dfrac{\nabla u \cdot \nu}{\sqrt{1+|\nabla u|^2}} = \kappa(x) & \text{on } \partial\Omega \end{cases}$$

- Ω is a bounded domain in $\mathbb{R}^N$, with a Lipschitz boundary $\partial\Omega$ and unit outer normal ν
- $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function
- $\kappa \in L^\infty(\partial\Omega)$

Motivations

Classical issues

- Cartesian surfaces with prescribed mean curvature
- Capillarity phenomena for incompressible fluids
- Equilibrium configurations for sessile, or pendant, drops

More recent ones include

- Reaction-diffusion processes with saturation of the flux at high regimes
- Capillarity phenomena for compressible fluids
- Mathematical models in human physiology
- MEMS models with capillarity effects

► LBNL: this problem is challenging also from the purely mathematical point of view.

Features of the operator and consequences

Features

- $|\nabla u| \ll 1$: $$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \approx \mathrm{div}(\nabla u) = \Delta u$$
- $|\nabla u| \gg 1$: $$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \approx \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) = \Delta_1 u$$

Thus:

- the mean curvature operator is not homogeneous
- the mean curvature operator is not uniformly elliptic

Consequences

Even in simple situations, there may occur

- non-existence phenomena
- loss of regularity

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A very elementary, but paradigmatic, example

Consider the 1-D autonomous equation

$$-\left(\frac{u'}{\sqrt{1+u'^2}}\right)' = f(u)$$

- $f : \mathbb{R} \to \mathbb{R}$ continuous, odd, $f(s) > 0$ for $s > 0$
- solutions (u, u') parametrize the level sets of the energy

$$E(u, u') = 1 - \frac{1}{\sqrt{1+u'^2}} + F(u), \qquad F(u) = \int_0^u f$$

- let $(u(0), u'(0)) = (u_0, 0)$

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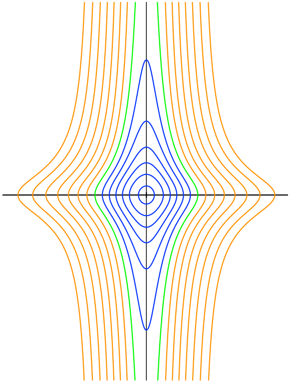
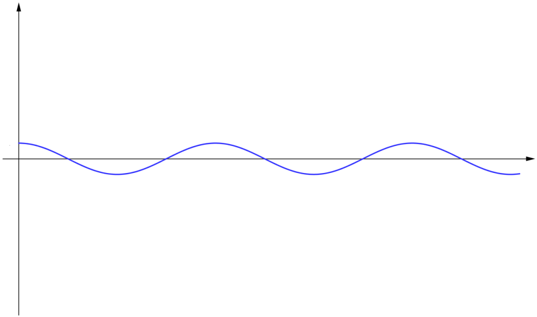
Case: $F(u_0) < 1$

Phase-plane portrait



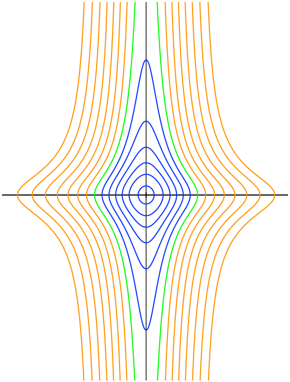
Profile of solutions



Small energy orbits (blue) are compact connected
$\longrightarrow$ regular solutions

Case: $F(u_0) = 1$

Phase-plane portrait



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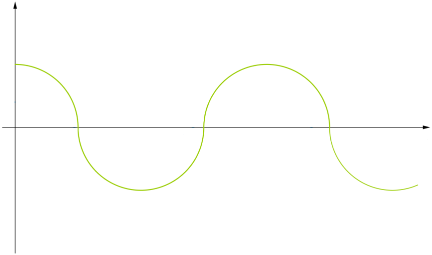
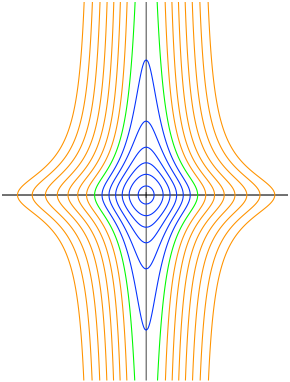


There is a first orbit (green) disconnected and asymptotic to the vertical axis
$\longrightarrow$ continuous non-regular solution

Case: $F(u_0) > 1$

Phase-plane portrait



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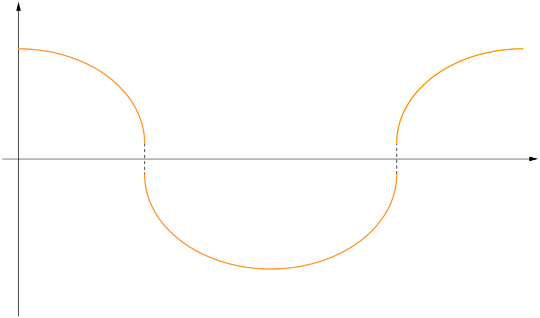

Large energy orbits (orange) are all disconnected and asymptotic to distinct vertical lines $\longrightarrow$ singular (discontinuous) solutions

Notion of solution

- Even in 1-D, non-regular solutions, having vertical tangents, or jump discontinuities, appear and coexist with regular solutions
- An appropriate notion of solution is needed to describe these patters:
  if a function exhibits jumps, its distributional derivative is a measure having a singular component w.r. to the Lebesgue measure $\longrightarrow$ bounded variation function
- A brief overview of the literature:
  - generalized solution: Giaquinta, Giusti (early seventies)
  - pseudo-solution: Temam, Lischnewski (early seventies) (see also Ekeland, Ladyzhenskaya, Ural'tseva, Marcellini, Miller, Kawohl, Kutev, ...)
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Bounded variation functions

- A function $v \in BV(\Omega)$ if $v \in L^1(\Omega)$ and its distributional gradient is a vector valued Radon measure with finite total variation $\int_\Omega |Dv|$.
- If $v \in BV(\Omega)$,

$$Dv = (Dv)^{\mathrm{a}} dx + (Dv)^{\mathrm{s}}$$

  is the Lebesgue-Nikodym decomposition of the Radon measure Dv in its absolutely continuous part $(Dv)^{\mathrm{a}} dx$, with density function $(Dv)^{\mathrm{a}}$, and its singular part $(Dv)^{\mathrm{s}}$, with respect to the Lebesgue measure in $\mathbb{R}^N$.

  Moreover, $\frac{Dv}{|Dv|}$ stands for the density function of Dv with respect to its absolute variation $|Dv|$.

Notion of BV solution

BV solution: Anzellotti (1985)

$u \in BV(\Omega)$ is a BV solution of

$$\begin{cases} -\mathrm{div}\left(\nabla u/\sqrt{1+|\nabla u|^2}\right) = f(x,u) & \text{in } \Omega \\ -\nabla u \cdot \nu/\sqrt{1+|\nabla u|^2} = \kappa(x) & \text{on } \partial\Omega \end{cases}$$

if, for every $\phi \in BV(\Omega)$ such that $|D\phi^s| \ll |Du^s|$,

$$\int_\Omega \frac{(Du)^a\,(D\phi)^a}{\sqrt{1+|(Du)^a|^2}}\,dx + \int_\Omega \frac{Du^s}{|Du^s|}\,D\phi^s$$
$$= \int_\Omega f(x,u)\phi\,dx - \int_{\partial\Omega} \kappa\phi\,d\mathcal{H}_{N-1}.$$

Comparison with $W^{1,1}$ solutions

$W^{1,1}$ solution

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if, for every $\phi \in W^{1,1}(\Omega)$,

$$\int_\Omega \frac{Du\,D\phi}{\sqrt{1+|Du|^2}}\,dx = \int_\Omega f(x,u)\phi\,dx - \int_{\partial\Omega} \kappa\phi\,d\mathcal{H}_{N-1}.$$

Euler equation in $BV(\Omega$

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This is the Euler equation in $BV(\Omega)$ of the functional

$$\int_\Omega \sqrt{1+|(Du)^a|^2}\,dx + \int_\Omega |Du^s| - \int_\Omega F(x,u)\,dx,$$

with $F(x,s) = \int_0^s f(x,t)\,dt$.

Positive solutions

Several (even recent) results concerning the existence and multiplicity of (classical or BV) solutions of

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are available in the literature under different configurations of the prescribed curvature f,

YET very little is known about positive, classical or BV, solutions,

in spite of the fact that this topic has been widely investigated in the literature, starting with the late eighties, in the semilinear case or in the quasilinear case, but mainly devoted to the p-laplacian, or variations thereof: Bandle, Pozio, Tesei; Berestycki, Capuzzo Dolcetta, Nirenberg; Alama, Tarantello; ... .

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With J. López-Gómez and S. Rivetti we began this study, starting from the simplest 1-D non-autonomous model

$$\begin{cases} -\left(\dfrac{u'}{\sqrt{1+u'^2}}\right)' = \lambda a(x) f(u) & \text{in }]0,1[\\ u'(0) = u'(1) = 0 \end{cases}$$

- $a \in L^1(0,1)$
- $f : [0,+\infty[\to [0,+\infty[$ continuous
- $\lambda > 0$ (inverse diffusivity coefficient)

Notation

- $u > 0$ if $u \geq 0$ and $u \not\equiv 0$
- $u \gg 0$ if $\text{ess inf } u > 0$

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Program

Aim to study

- existence, or non-existence
- multiplicity
- regularity (partial, or complete)
- structure of the set of positive solutions

Approach

Combination of

- elliptic regularization
- critical point theory, or topological degree
- bifurcation methods
- ODE techniques

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Necessary conditions for existence

I necessary condition

- $\exists$ BV solution $u > 0$
- $a \not\equiv 0$
- $f(s) > 0$ for $s > 0$

$\Longrightarrow$ $a(\cdot)$ changes sign $\longrightarrow$ the problem is indefinite in sign

II necessary condition

- $\exists$ BV solution $u \gg 0$
- $a \not\equiv 0$
- $f(0) = 0$ and $f'(s) > 0$ for $s > 0$

$\Longrightarrow$ $a^+ \not\equiv 0$ and $\int_0^1 a < 0$

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Sufficient conditions for existence

Assumptions on the weight $a(x)$

(a) $a(x) > 0$ a.e. in an interval and $\int_0^1 a < 0$

Assumptions on the potential $F(s) = \int_0^s f$

Recall: $(u'/\sqrt{1+u'^2})' \approx u''$ at 0, $(u'/\sqrt{1+u'^2})' \approx (u'/|u'|)'$ at ∞.

at 0			at $+\infty$		
(Q_0)	$F(s) \approx s^2$		(L_∞)	$F(s) \approx s$	
(superQ_0)	$F(s) \approx s^p$	$p > 2$	(superL_∞)	$F(s) \approx s^q$	$q > 1$
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Sufficient conditions for existence, multiplicity, regularity

Existence, non-existence, multiplicity of positive BV solutions

Assume (a). Then:

- (superQ_0) and (superL_∞) $\Longrightarrow$ $\forall\,\lambda > 0 \;\; \exists \geq 1$ pos. BV sol.
- (subQ_0) and (subL_∞) $\Longrightarrow$ $\forall\,\lambda > 0 \;\; \exists \geq 1$ pos. BV sol.
- (subQ_0) and (superL_∞) $\Longrightarrow$ $\forall\lambda \ll 1 \; \exists \geq 2$ pos. BV sols and $\forall\lambda \gg 1 \;\; \nexists$ pos. BV sol.
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Partial regularity

- $a(\cdot)$ one sign a.e. in $]\alpha, \beta[$ $\Longrightarrow$ $u \in W^{2,1}_{\text{loc}}(\alpha, \beta) \cap W^{1,1}(\alpha, \beta)$
- $a(x) \geq 0$ a.e. in $]\alpha, \beta[$ and $a(x) \leq 0$ a.e. in $]\beta, \gamma[$ $\Longrightarrow$ $u \in W^{2,1}_{\text{loc}}(\alpha, \gamma)$ or, else, $u(\beta^-) \geq u(\beta^+)$ and $u'(\beta^-) = u'(\beta^+) = -\infty$

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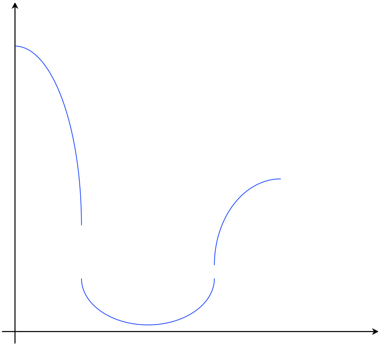
Strict positivity and regularity

Strict positivity

- $a(\cdot)$ changes sign finitely many times, f loc. Lipschitz in $[0, +\infty[$

  $\Longrightarrow \quad u \gg 0$ and u is a special function of bounded variation

Profile of the BV solutions



Existence of strong solutions

Assume (a). Then:

- $F(s) \approx s^p, 1 < p < 2,$ at 0 $\implies$ $\forall\, \lambda \ll 1$
  $\exists \geq 1$ strong solution $u > 0$
- $F(s) \approx s^p,\ p > 2,$ at 0 $\implies$ $\forall\, \lambda \gg 1$
  $\exists \geq 1$ strong solution $u \gg 0$
- $F(s) \approx s^2,$ at 0 $\implies$ $\forall\, \lambda$ close to the principal positive EV λ_0
  $\exists \geq 1$ strong solution $u \gg 0$

Remark. The assumption (a) implies that

$$\begin{cases} -u'' = \lambda a(x) u & \text{in }]0,1[, \\ u'(0) = u'(1) = 0 \end{cases}$$

has two principal eigenvalues $\lambda = 0$ and $\lambda = \lambda_0 > 0$.

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- $F(s) \approx s^p, 1 < p < 2,$ at 0 $\implies$ $\forall\, \lambda \ll 1$ $\exists \geq 1$ strong solution $u > 0$
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- $F(s) \approx s^2,$ at 0 $\implies$ $\forall\, \lambda$ close to the principal positive EV λ_0 $\exists \geq 1$ strong solution $u \gg 0$

Remark. The assumption (a) implies that

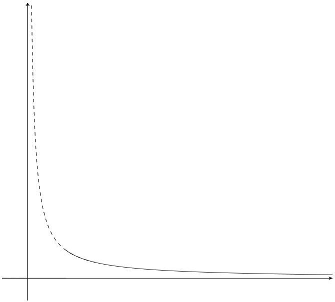
$$\begin{cases} -u'' = \lambda a(x) u & \text{in }]0,1[, \\ u'(0) = u'(1) = 0 \end{cases}$$

has two principal eigenvalues $\lambda = 0$ and $\lambda = \lambda_0 > 0$.

Topological structure: qualitative bifurcation diagrams

The above results can be grafically summarized as follows:

$$F(s) \approx s^p,\ p > 2,\ \text{ at } 0$$
$$F(s) \approx s^q,\ q > 1,\ \text{ at } +\infty$$



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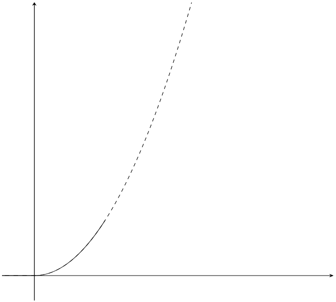

- $\|u\|_\infty$ is plotted, in ordinates, versus λ, in abscissas
- dotted line denotes BV solutions, solid line denotes strong solutions

Topological structure: qualitative bifurcation diagrams

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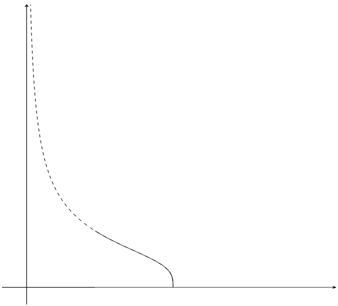

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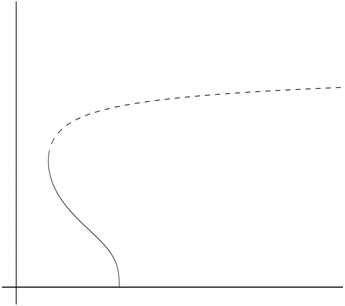

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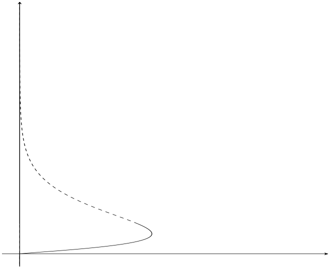

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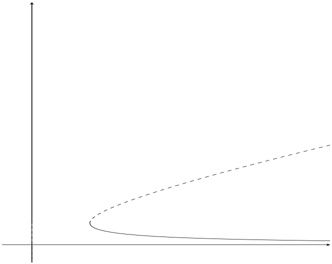

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Questions

Does really the solution set contain connected components as those depicted in the preceding diagrams?

Using the Crandall-Rabinowitz theorem and the invariance under homotopy of the topological degree, we can get some partial information, namely, the existence of connected components of strong solutions, in the three cases:

- F quadratic, $F(s) \approx s^2$, at 0
- F subquadratic, $F(s) \approx s^p$, $1 < p < 2$, at 0
- F superquadratic, $F(s) \approx s^p$, $p > 2$, at 0.

Yet, no information is provided with reference to the following questions:

- ⋆ What is the structure of the set of the singular BV solutions?
- ⋆ Does the solution set contain global connected components including both regular and singular solutions?

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- ⋆ Does the solution set contain global connected components including both regular and singular solutions?

Global bifurcation

To answer (partially) these questions, we focus on the following simple, but significant, case:

Assumptions

- the weight a changes sign once in $[0,1]$, namely, we assume $a \in L^{\infty}(0,1)$ satisfies $\int_0^1 a(x)\,dx < 0$ and there is $z \in]\,0,1\,[$ such that $a(x) > 0$ a.e. in $]0,z[$ and $a(x) < 0$ a.e. in $]z,1[$
- the potential F is quadratic at 0 and has polynomial growth, namely, we assume $f \in C^1(\mathbb{R})$ satisfies $f(s)s > 0$ for all $s \neq 0$, $f'(0) = 1$, and, for some constants $\kappa > 0$ and $p > 2$, $|f'(s)| \leq \kappa\,(|s|^{p-2} + 1)$ for all $s \in \mathbb{R}$.

Global bifurcation

Aims

1. to establish the existence of global connected components of the set of the positive BV solutions, emanating from the line of the trivial solutions at the two principal eigenvalues 0 and λ_0 of the linearized problem around 0
2. to prove that the solutions in these components are regular, as long as they are small, while they may develop jump singularities at the node of the weight function a, as they become larger, thus showing the coexistence along the same component of both regular and singular solutions

Notation

- $\mathcal{S}^+ = \{(\lambda, u) \in [0, +\infty) \times BV(0, 1) : u \text{ is a positive solution}\} \cup \{(0, 0), (\lambda_0, 0)\}$

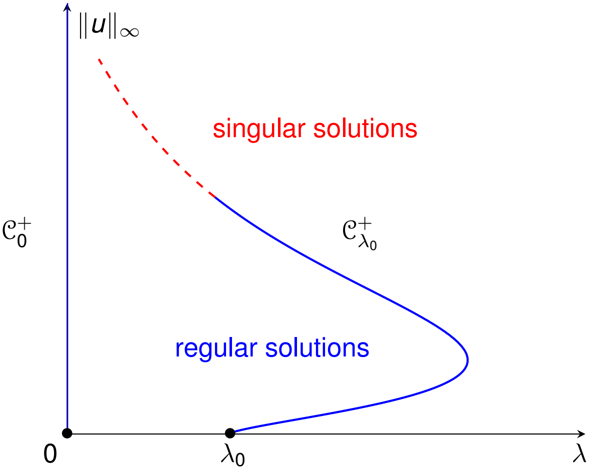
The global bifurcation theorem

There exist two subsets $\mathcal{C}_0^+$ and $\mathcal{C}_{\lambda_0}^+$ of $\mathcal{S}^+$ such that

- $\mathcal{C}_0^+$ and $\mathcal{C}_{\lambda_0}^+$ are maximal in $\mathcal{S}^+$ with respect to the inclusion, are connected in $\mathbb{R} \times BV(0,1)$ (endowed with the topology of the strict convergence), and are unbounded in $\mathbb{R} \times L^p(0,1)$;
- $(0,0) \in \mathcal{C}_0^+$ and $(\lambda_0, 0) \in \mathcal{C}_{\lambda_0}^+$;
- $\{(0,r) : r \in [0,+\infty)\} \subseteq \mathcal{C}_0^+$;
- if $(\lambda, u) \in \mathcal{C}_0^+ \cup \mathcal{C}_{\lambda_0}^+$ and $u \neq 0$, then $u \gg 0$;
- if $(\lambda, 0) \in \mathcal{C}_0^+ \cup \mathcal{C}_{\lambda_0}^+$ for some $\lambda > 0$, then $\lambda = \lambda_0$;
- either $\mathcal{C}_0^+ \cap \mathcal{C}_{\lambda_0}^+ = \emptyset$,

  or $(\lambda_0, 0) \in \mathcal{C}_0^+$ and $(0,0) \in \mathcal{C}_{\lambda_0}^+$ and, in such case, $\mathcal{C}_0^+ = \mathcal{C}_{\lambda_0}^+$;
- there exists a neighborhood U of $(0,0)$ in $\mathbb{R} \times L^p(0,1)$ such that $\mathcal{C}_0^+ \cap U$ consists of strong solutions;
- there exists a neighborhood V of $(\lambda_0, 0)$ in $\mathbb{R} \times L^p(0,1)$ such that $\mathcal{C}_{\lambda_0}^+ \cap V$ consists of strong solutions.

Global bifurcation diagrams



Global branches emanating from the two principal eigenvalues 0 and λ_0: supercritical bifurcation at $(0, \lambda_0)$, F superlinear at infinity.

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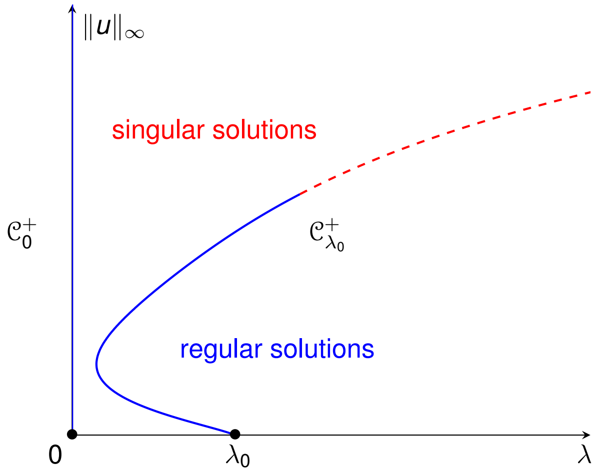

Global branches emanating from the two principal eigenvalues 0 and λ_0: subcritical bifurcation at $(0, \lambda_0)$, F sublinear at infinity.

A quick look at the proof

To get most of the conclusions a number of (at least, for us!) non-trivial technical issues must be previously overcome. Among them count:

- searching for the most appropriate global bifurcation setting: the lack of regularity of the solutions forces us to work in the frame of the Lebesgue spaces L^p, with p finite, where the cone of positive functions has empty interior
- reformulating the problem as a suitable fixed point equation for a compact operator
- proving the differentiability of the associated operator
- solving the problem of the preservation of the positivity of the solutions along the components; this is achieved through a (rather delicate) topological argument relying on some technical convergence results for sequences of BV solutions, with respect to the strict topology of $BV(0, 1)$.

Thank you for your attention!

Structure of the proof

Regularity of the bounded variation solutions

1. Characterize the existence of the strong solutions of

$$(H) \qquad -\left(\frac{u'}{\sqrt{1+u'^2}}\right)' = h(x) \quad \text{in }]0,1[, \qquad u'(0)=0,\ u'(1)=0,$$

with $h \in L^1(0,1)$:

problem (H) has a strong solution if and only if there exists a constant $\kappa \in]0,1[$ such that

$$\left|\int_0^1 h\chi_B \, dx\right| \leq \kappa \int_0^1 |D\chi_B|$$

for every Caccioppoli set $B \subseteq]0,1[$.

In particular: when $\|h\|_{L^1} < 1$, any bounded variation solution must be strong.

Structure of the proof

Regularity of the bounded variation solutions – continued

2. Analyze the fine regularity properties of the non-regular bounded variation solutions of problem (H):

when the set of the nodal points of h is discrete, the only singularities that a bounded variation solution of problem (H) can exhibit are jumps, which must be located at the interior points where h changes sign.

In particular: the presence of a Cantor part in the distributional derivative of any bounded variation solution of problem (H) is ruled out, that is, the solutions are special functions of bounded variation.

Structure of the proof

An auxiliary problem

$$(A) \qquad -\left(\frac{u'}{\sqrt{1+u'^2}}\right)' + k(u) = h(x) \quad \text{in }]0,1[, \quad u'(0)=0, \; u'(1)=0,$$

where

- $k:\mathbb{R}\to\mathbb{R}$ is a function of class C^1, strictly increasing and odd, which satisfies

$$k'(0)=1 \quad \text{and} \quad \lim_{|s|\to+\infty}\frac{k'(s)}{|s|^{p-2}}=1, \text{ for some } p\geq 2,$$

- $h \in L^q(0,1)$, with $q=\frac{p}{p-1}$.

Structure of the proof

The associated solution operator

The solution operator

$$\mathcal{P} : L^q(0,1) \to L^p(0,1),$$

which maps h onto the unique BV solution $u = \mathcal{P}h$ of (A), is completely continuous)

The Fréchet derivative at 0

$\mathcal{P}$ is differentiable at $h = 0$ and its derivative is given by the linear operator

$$D\mathcal{P}(0) : L^q(0,1) \to L^p(0,1),$$

which sends any function h onto the unique solution $u \in W^{2,q}(0,1)$ of

$$-u'' + u = h(x) \quad \text{in }]0,1[, \qquad u'(0) = 0, \; u'(1) = 0 \qquad \text{(linearization of } (A))$$

Structure of the proof

Fixed point reformulation

The original problem can be reformulated as an operator equation

$$\mathcal{N}(\lambda, u) = 0 \quad \text{in} \quad L^p(0,1),$$

with $\mathcal{N}$ a compact perturbation of the identity.
Due to the differentiability of $\mathcal{P}$ at $u = 0$, $\mathcal{N}$ can be decomposed as

$$\mathcal{N}(\lambda, u) = \mathcal{L}(\lambda)u + \mathcal{R}(\lambda, u),$$

with $\mathcal{L}(\lambda)$ Fredholm of index 0 and $\mathcal{R}(\lambda, u) = o(\|u\|_p)$.

Unilateral bifurcation

Thus, we are within the functional setting suited for applying a generalized version of the Rabinowitz unilateral bifurcation theorem, at both principal eigenvalues 0 and λ_0 of the linearization at 0.
This yields the existence of two connected components of the solution set emanating from $(0,0)$ and $(\lambda_0, 0)$, respectively, and constituted of positive regular solutions near the bifurcation points.

Structure of the proof

Unbounded subcomponents of positive solutions

Since we are working in $L^p(0,1)$, we cannot guarantee that the whole obtained components consist of positive solutions.

Thus, the remainder of the proof is basically devoted to prove that each of these components contains an unbounded subcomponent $\mathcal{C}_0^+$ and $\mathcal{C}_{\lambda_0}^+$, respectively, consisting of (strictly) positive solutions.

This is achieved through a (rather delicate) topological argument relying on some technical convergence results for sequences of BV solutions, with respect to the strict topology of $BV(0,1)$.

The special nodal structure of the weight function $a(\cdot)$ plays here a relevant role.

Thank you for your attention!