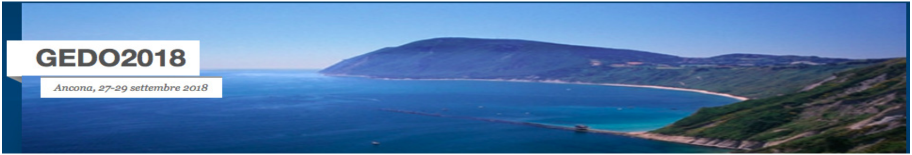

An existence result for a new class of impulsive functional differential equations with delay

Luigi Muglia
Universitá della Calabria

September 28, 2018

A joint work with V. Colao and H.K. Xu

Evolution Equations with non-istantaneous impulses

E. Hernandez and D. O'Regan (*Proc. AMS*, 2013) study the following system:

$$\begin{cases} x'(t) - Ax(t) = f(t, x(t)), \ \ a.e.\ t \in \bigcup_{i=0}^{N} (s_i, t_{i+1}] \\ \\ x(t) = g_i(t, x(t)), \ \ t \in (t_i, s_i], i = 1, \ldots, N \\ \\ x(0) = x_0. \end{cases}$$

since it finds its motivation in moldels of real phenemena in which an impulsive action starts abruptly and stays active on a finite time interval.

Retarded Functional Differential Equation with non-istantaneous impulses.

$$(RFDE)\begin{cases} x'(t) - A(t)x(t) = f(t, x(t), x(\sigma(t))), \;\; a.e.\; t \in \bigcup_{i=0}^{\infty}(s_i, t_{i+1}] \\ \\ x(t) = (Kx)(t), \;\; t \in [-r, 0] \cup \bigcup_{i=1}^{\infty}(t_i, s_i], \end{cases}$$

where:

- $A : [0, +\infty) \to \mathcal{L}(X, X)$.
- $\sigma : [-r, +\infty) \to [0, +\infty)$ with $\sigma(t) \le t$ (*delay*).
- $K : BC\left(\bigcup_{i=1}^{\infty}[t_i, s_i]\right) \to BC\left(\bigcup_{i=1}^{\infty}[t_i, s_i]\right)$ (*non-istantaneous impulse*).

Retarded Functional Differential Equation with non-istantaneous impulses.

$$
(RFDE)\begin{cases} x'(t) - A(t)x(t) = f(t, x(t), x(\sigma(t))), \;\; a.e.\ t \in \bigcup_{i=0}^{\infty}(s_i, t_{i+1}] \\ \\ x(t) = (Kx)(t), \;\; t \in [-r, 0] \cup \bigcup_{i=1}^{\infty}(t_i, s_i], \end{cases}
$$

where:

- $A : [0, +\infty) \to \mathcal{L}(X, X)$.
- $\sigma : [-r, +\infty) \to [0, +\infty)$ with $\sigma(t) \leq t$ (*delay*).
- $K : BC\left(\bigcup_{i=1}^{\infty}[t_i, s_i]\right) \to BC\left(\bigcup_{i=1}^{\infty}[t_i, s_i]\right)$ (*non-istantaneous impulse*).

Retarded Functional Differential Equation with non-istantaneous impulses.

$$(RFDE) \begin{cases} x'(t) - A(t)x(t) = f(t, x(t), x(\sigma(t))), \;\; a.e.\; t \in \bigcup_{i=0}^{\infty} (s_i, t_{i+1}] \\ \\ x(t) = (Kx)(t), \;\; t \in [-r, 0] \cup \bigcup_{i=1}^{\infty} (t_i, s_i], \end{cases}$$

where:

- $A : [0, +\infty) \to \mathcal{L}(X, X)$.
- $\sigma : [-r, +\infty) \to [0, +\infty)$ with $\sigma(t) \leq t$ (*delay*).
- $K : BC\left(\bigcup_{i=1}^{\infty} [t_i, s_i]\right) \to BC\left(\bigcup_{i=1}^{\infty} [t_i, s_i]\right)$ *(non-istantaneous impulse)*.

Retarded Functional Differential Equation with non-istantaneous impulses.

$$(RFDE)\begin{cases} x'(t) - A(t)x(t) = f(t, x(t), x(\sigma(t))), \;\; a.e.\ t \in \bigcup_{i=0}^{\infty}(s_i, t_{i+1}] \\ \\ x(t) = (Kx)(t), \;\; t \in [-r, 0] \cup \bigcup_{i=1}^{\infty}(t_i, s_i], \end{cases}$$

where:

- $A : [0, +\infty) \to \mathcal{L}(X, X)$.
- $\sigma : [-r, +\infty) \to [0, +\infty)$ with $\sigma(t) \leq t$ (*delay*).
- $K : BC\left(\bigcup_{i=1}^{\infty}[t_i, s_i]\right) \to BC\left(\bigcup_{i=1}^{\infty}[t_i, s_i]\right)$ (*non-istantaneous impulse*).

Novelty elements with respect to the literature on the topic

- Time dependence of the operator $A(\cdot)$.
- Unboundedness of the time interval.
- Presence of an infinite number of *non-istantaneous* impulses.
- Delay.
- We will mainly rely on topological conditions.

Our aim..

is to prove the existence of *mild* and *strong* solutions for our system by using

- Fixed Point Theory - Evolution Operator Theory

- a Compactness criteria for piecewise continuous and bounded functions.

Novelty elements with respect to the literature on the topic

- Time dependence of the operator $A(\cdot)$.
- Unboundedness of the time interval.
- Presence of an infinite number of *non-istantaneous* impulses.
- Delay.
- We will mainly rely on topological conditions.

Our aim..

is to prove the existence of *mild* and *strong* solutions for our system by using

- Fixed Point Theory - Evolution Operator Theory

- a Compactness criteria for piecewise continuous and bounded functions.

Novelty elements with respect to the literature on the topic

- Time dependence of the operator $A(\cdot)$.
- Unboundedness of the time interval.
- Presence of an infinite number of *non-istantaneous* impulses.
- Delay.
- We will mainly rely on topological conditions.

Our aim..

is to prove the existence of *mild* and *strong* solutions for our system by using

- Fixed Point Theory - Evolution Operator Theory

- a Compactness criteria for piecewise continuous and bounded functions.

Novelty elements with respect to the literature on the topic

- Time dependence of the operator $A(\cdot)$.
- Unboundedness of the time interval.
- Presence of an infinite number of *non-istantaneous* impulses.
- Delay.
- We will mainly rely on topological conditions.

Our aim..

is to prove the existence of *mild* and *strong* solutions for our system by using

- Fixed Point Theory - Evolution Operator Theory

- a Compactness criteria for piecewise continuous and bounded functions.

Novelty elements with respect to the literature on the topic

- Time dependence of the operator $A(\cdot)$.
- Unboundedness of the time interval.
- Presence of an infinite number of *non-istantaneous* impulses.
- Delay.
- We will mainly rely on topological conditions.

Our aim..

is to prove the existence of *mild* and *strong* solutions for our system by using

- Fixed Point Theory - Evolution Operator Theory

- a Compactness criteria for piecewise continuous and bounded functions.

Novelty elements with respect to the literature on the topic

- Time dependence of the operator $A(\cdot)$.
- Unboundedness of the time interval.
- Presence of an infinite number of *non-istantaneous* impulses.
- Delay.
- We will mainly rely on topological conditions.

Our aim..

is to prove the existence of *mild* and *strong* solutions for our system by using

- Fixed Point Theory - Evolution Operator Theory

- a Compactness criteria for piecewise continuous and bounded functions.

Setting of our (RFDE)

- Let $(X, \|\cdot\|_X)$ be a Banach space and let $\Theta := \{t_0 < \ldots < t_i < \ldots\} \subset \mathbb{R}$ be a sequence such that $t_i \to +\infty$.
- $PC_\Theta([-r, +\infty), X)$)will denote the space of functions $x : [-r, +\infty) \to X$:
  (i) continuous on $[0, +\infty) \setminus \Theta$
  (ii) there exist $\lim_{t \to t_i^-} x(t) = x(t_i)$ and $\lim_{t \to t_i^+} x(t)$ for any $i \in \mathbb{N}$.
- $BPC_\Theta([-r, +\infty), X)$ will denote the subspace of bounded functions in $PC_\Theta([-r, +\infty), X)$ that is a Banach space endowed with the "sup" norm $\|\cdot\|_\infty$.

A *mild solution* to (RFDE) is defined as follows.

Definition

A function $u \in BPC_{\Theta}([-r, +\infty), X)$ is called a *mild solution* if $u(t) = Ku(t)$ on $\cup_{i=1}^{\infty}(t_i, s_i]$, and

$$u(t) = E(t, s_i)(Ku)(s_i) + \int_{s_i}^{t} E(t, s) f(s, u(s), u(\sigma(s)))ds$$

for any $\cup_{i=1}^{\infty}(s_i, t_i]$.

Our approach

Our approach belongs to the fixed point theory and consists in translating (RFDE) into a fixed point problem, which is to find $x \in BPC_{\Theta}([-r,+\infty), X)$ with the property that $x(\cdot) = Tx(\cdot)$ where

$$Tx(t) := \begin{cases} E(t,s_i)(Kx)(s_i) + \displaystyle\int_{s_i}^{t} E(t,s)f(s,x(s),x(\sigma(s)))ds, & t \in \displaystyle\bigcup_{i=0}^{\infty}(s_i,t_{i+1}] \\ (Kx)(t), & \text{otherwise.} \end{cases}$$

where $E(t,s) \in \mathcal{L}(X,X)$ is an element of an evolution system generated by $A(\cdot)$.

We will prove a compactness criterion for $BPC_{\Theta}([-r,+\infty),X)$ in order to apply the following well-known fixed point theorem:

Theorem (Schaefer's Theorem)

Let C be a convex subset of a Banach space E and $0 \in C$. Let $F : C \to C$ be a completely continuous operator, and let

$$\zeta(F) := \{x \in E : x = \lambda F x,\ 0 < \lambda < 1\}.$$

Then either $\zeta(F)$ is unbounded or F has a fixed point.

A compactness criteria in $BPC_\Theta([-r,+\infty),X)$

Lemma

Let $\Omega \subset BPC_\Theta([-r,+\infty),X)$ be bounded. Then Ω is relatively compact if and only if

(C1) *for all $t \in [-r,+\infty) \setminus \Theta$, $\Omega(t)$ is compact in X. Moreover, $\Omega(t_i^+)$ is compact in X for any $i \in \mathbb{N}$,*

(C2) *Ω is quasi-equicontinuous, i.e. for every $u \in \Omega$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $|u(\tau_1) - u(\tau_2)| < \varepsilon$ whenever $|\tau_1 - \tau_2| < \delta$ for $\tau_1, \tau_2 \in (t_k, t_{k+1}]$ for some $k \in \mathbb{N}$ or $\tau_1, \tau_2 \in [-r, 0)$.*

(C3) *$\forall \varepsilon > 0$ there exists $N = N(\varepsilon) > 0$ such that $\chi(\Omega_{|[N,+\infty)}) < \varepsilon$.*

Our Hypotheses

Our assumption will be the following:

(H_1) $A : [0, +\infty) \to \mathcal{L}(X, X)$ is the generator of a compact and uniformly bounded evolution system $\{E(t, s)\}_{0 \leq s \leq t}$,
i.e. for any $0 \leq s \leq t < \infty$, $E(t, s) \in \mathcal{L}(X, X)$ is compact and there exits M_E (not depending on t and s) such that $\|E(t, s)\|_{\mathcal{L}(X,X)} \leq M_E$.

Evolution Systems

Recall that for first order differential system

$$x'(t) - Ax(t) = 0, \qquad x(\theta) = x_0. \tag{1}$$

where $A := (a_{i,j})$ be a real $n \times n$-matrix, a solution of (1) can be written in the form $x(t) = \exp[(t-\theta)A]x_0$, where

$$\exp(sA) = \sum_{k=1}^{\infty} \frac{1}{k!}(sA)^k.$$

In the non-constant case, the natural extension of the above solution, indicated by $x(t) = \exp\left[\int_\theta^t A(s)ds\right] x_0$, does not work, in general, since the equality

$$\frac{d}{dt}\exp\left[\int_\theta^t A(s)ds\right] = A(t)\exp\left[\int_\theta^t A(s)ds\right]$$

A representation of the solution of (1) in terms of $A(t)$ is given by

$$x(t) = E(t,\theta)x_0 \tag{2}$$

where

$$E(t,\theta) = \lim_{k\to\infty}(I+\int_\theta^t A(t_1)dt_1+\cdots+\int_\theta^t \dots \int_\theta^{t_{k-1}} A(t_1)A(t_2)\cdots A(t_k)dt_k \dots$$

In the non-constant case, the natural extension of the above solution, indicated by $x(t) = \exp\left[\int_\theta^t A(s)ds\right] x_0$, does not work, in general, since the equality

$$\frac{d}{dt}\exp\left[\int_\theta^t A(s)ds\right] = A(t)\exp\left[\int_\theta^t A(s)ds\right]$$

A representation of the solution of (1) in terms of $A(t)$ is given by

$$x(t) = E(t,\theta)x_0 \tag{2}$$

where

$$E(t,\theta) = \lim_{k\to\infty}(I + \int_\theta^t A(t_1)dt_1 + \cdots + \int_\theta^t \ldots \int_\theta^{t_{k-1}} A(t_1)A(t_2)\cdots A(t_k)dt_k \ldots$$

This family $\{E(t,\theta)\}_{0\le\theta\le t}$ is known as *evolution system* generated by $A(t)$.

Some relevant properties of $E(t,\theta)$ immediately follow:

(i) $E(t,t) = I$;

(ii) $E(t,s)E(s,\theta) = E(t,\theta)$ by the uniqueness of the solution of (1);

(iii) the mapping $(t,\theta) \to E(t,\theta)$ is continuous in the uniform norm topology;

(iv) $\dfrac{\partial E(t,\theta)}{\partial t} = A(t)E(t,\theta)$ (by deriving formula (2));

(v) $\dfrac{\partial E(t,\theta)}{\partial \theta} = -E(t,\theta)A(\theta)$ (by deriving formula (2)).

This family $\{E(t,\theta)\}_{0\leq\theta\leq t}$ is known as *evolution system* generated by $A(t)$.

Some relevant properties of $E(t,\theta)$ immediately follow:

(i) $E(t,t) = I$;

(ii) $E(t,s)E(s,\theta) = E(t,\theta)$ by the uniqueness of the solution of (1);

(iii) the mapping $(t,\theta) \to E(t,\theta)$ is continuous in the uniform norm topology;

(iv) $\dfrac{\partial E(t,\theta)}{\partial t} = A(t)E(t,\theta)$ (by deriving formula (2));

(v) $\dfrac{\partial E(t,\theta)}{\partial \theta} = -E(t,\theta)A(\theta)$ (by deriving formula (2)).

Note that if $A(t) = A$, then $E(t,\theta) = \exp[(t-\theta)A]$ and therefore properties (i) to (iii) become the well-known properties which characterize the *uniformly continuous semigroup* generated by A (see Pazy 2012). Moreover, by integrating (iv), we find that

$$E(t,\theta) = I + \int_{\theta}^{t} A(\tau)E(\tau,\theta)d\tau. \tag{3}$$

If $\tau_1 \leq \tau_2$ then (3) implies

$$\|E(\tau_2,\theta) - E(\tau_1,\theta)\| \leq \int_{\tau_1}^{\tau_2} \|A(\tau)\|\|E(\tau,\theta)\|d\tau$$

and, if the operators are *uniformly bounded*, i.e. if there exist two positive numbers $M_A,\ M_E$ such that $\|A(t)\| \leq M_A$ and $\|E(t,\theta)\| \leq M_E$, for all $0 \leq \theta < t < +\infty$, then the evolution operator $E(t,\theta)$ is equicontinuous.

Note that if $A(t) = A$, then $E(t,\theta) = \exp[(t-\theta)A]$ and therefore properties (i) to (iii) become the well-known properties which characterize the *uniformly continuous semigroup* generated by A (see Pazy 2012). Moreover, by integrating (iv), we find that

$$E(t,\theta) = I + \int_{\theta}^{t} A(\tau)E(\tau,\theta)d\tau. \tag{3}$$

If $\tau_1 \leq \tau_2$ then (3) implies

$$\|E(\tau_2,\theta) - E(\tau_1,\theta)\| \leq \int_{\tau_1}^{\tau_2} \|A(\tau)\|\|E(\tau,\theta)\|d\tau$$

and, if the operators are *uniformly bounded*, i.e. if there exist two positive numbers $M_A,\ M_E$ such that $\|A(t)\| \leq M_A$ and $\|E(t,\theta)\| \leq M_E$, for all $0 \leq \theta < t < +\infty$, then the evolution operator $E(t,\theta)$ is equicontinuous.

In the setting of an infinite-dimensional Banach space X, in order to represent a solution of (1) by $x(t) = E(t, \theta)x_0$, the boundedness of the linear operator $A(t)$ (for fixed t) plays a relevant role;
even in the autonomous case $A(t) = A$, the boundedness of the operator is a necessary and sufficient condition in order to obtain continuity of the generated semigroup in the uniform norm topology, see Theorem 1.2-Pazy 2012.

In a similar way, if $A(\cdot)$ is a linear and bounded operator on its domain $D(A(\cdot)) \subset X$ and the mapping $t \to A(t)$ is continuous in the uniform operator topology, then the evolution operator $E(t, \theta)$ satisfies (i)-(iv) and (3) (see Theorem 5.2-Pazy 2012).

In the setting of an infinite-dimensional Banach space X, in order to represent a solution of (1) by $x(t) = E(t, \theta)x_0$, the boundedness of the linear operator $A(t)$ (for fixed t) plays a relevant role;
even in the autonomous case $A(t) = A$, the boundedness of the operator is a necessary and sufficient condition in order to obtain continuity of the generated semigroup in the uniform norm topology, see Theorem 1.2-Pazy 2012.

In a similar way, if $A(\cdot)$ is a linear and bounded operator on its domain $D(A(\cdot)) \subset X$ and the mapping $t \to A(t)$ is continuous in the uniform operator topology, then the evolution operator $E(t, \theta)$ satisfies (i)-(iv) and (3) (see Theorem 5.2-Pazy 2012).

Nevertheless, in the setting of infinite-dimensional Banach spaces, unbounded operators $A(t)$ have been widely applied to model several problems related to partial differential equations.
Since by dropping the boundedness, we lose property (iii), we will work with a well-known weaker form of continuity, namely *strong continuity*, i.e. by assuming that the mapping $(t, \theta) \to E(t, \theta)x$ is continuous for all $x \in X$.

We stress that, whenever $A(t)$ is unbounded for some $t \in [0, +\infty)$, a set of sufficient conditions for which $A(\cdot)$ generates a uniformly bounded evolution system satisfying (i), (ii), (iv) and (v), is given by Theorem 4.8-Pazy 2012 and Theorem 6.1-Pazy 2012 for parabolic and hyperbolic equations, respectively.

Nevertheless, in the setting of infinite-dimensional Banach spaces, unbounded operators $A(t)$ have been widely applied to model several problems related to partial differential equations.
Since by dropping the boundedness, we lose property (iii), we will work with a well-known weaker form of continuity, namely *strong continuity*, i.e. by assuming that the mapping $(t, \theta) \to E(t, \theta)x$ is continuous for all $x \in X$.

We stress that, whenever $A(t)$ is unbounded for some $t \in [0, +\infty)$, a set of sufficient conditions for which $A(\cdot)$ generates a uniformly bounded evolution system satisfying (i), (ii), (iv) and (v), is given by Theorem 4.8-Pazy 2012 and Theorem 6.1-Pazy 2012 for parabolic and hyperbolic equations, respectively.

In the above cited results, a common hypothesis is that $D(A(\cdot)) = D$, i.e. the domain does not depend on t, and that D is dense in X. This assumption is widely used in many papers in the literature (see e.g. Aizicovici-Staicu NoDea 2007, Benedetti-Rubbioni TMNA 2008, Cardinali-Rubbioni NA 2012, Li et.al. MCM 2009, Xiao et. al. NA 2005).

We will also assume the compactness of the family $\{E(t,\theta)\}_{0\leq\theta\leq t}$, which is a fair common hypothesis when working with the lack of boundedness of the operator A (see above references). Sufficient conditions to ensure the compactness of an evolution system are given for example in Kartsatos's papers Proc. AMS 1995 and Math. Ann. 1995.

In the above cited results, a common hypothesis is that $D(A(\cdot)) = D$, i.e. the domain does not depend on t, and that D is dense in X. This assumption is widely used in many papers in the literature (see e.g. Aizicovici-Staicu NoDea 2007, Benedetti-Rubbioni TMNA 2008, Cardinali-Rubbioni NA 2012, Li et.al. MCM 2009, Xiao et. al. NA 2005).

We will also assume the compactness of the family $\{E(t,\theta)\}_{0\leq\theta\leq t}$, which is a fair common hypothesis when working with the lack of boundedness of the operator A (see above references). Sufficient conditions to ensure the compactness of an evolution system are given for example in Kartsatos's papers Proc. AMS 1995 and Math. Ann. 1995.

Our Hypotheses

Our assumption will be the following:

(H_1) $A : [0, +\infty) \to \mathcal{L}(X, X)$ is the generator of a compact and uniformly bounded evolution system $\{E(t, s)\}_{0 \le s \le t}$, pause

(H_2) $f : [0, +\infty) \times X \times X \to X$ is L^1-Caratheodory that is:

1. $f(t, \cdot, \cdot) : X \times X \to X$ is continuous for a.e. $t \in [0, +\infty)$;
2. for any fixed $x, y \in X$, $f(\cdot, x, y)$ is strongly measurable on $[0, +\infty)$;
3. for any $R > 0$ there exists $p_R \in L^1[0, +\infty)$ such that

$$|f(t, x, y)| \le p_R(t), \qquad \|x\|_X, \|y\|_X \le R.$$

(H_3) $\sigma : [0, +\infty) \to [-r, +\infty)$ is a continuous and increasing function such that $\sigma(t) \le t$ for any t in the domain.

(H_4) $K : BC\left(\cup_{i=0}^{\infty}[t_i, s_i], X\right) \to BC\left(\cup_{i=0}^{\infty}[t_i, s_i], X\right)$ is a completely continuous operator; moreover there exists $M_K > 0$ such that $\|K(u)\|_\infty \le M_K$, for all $u \in BC\left(\cup_{i=0}^{\infty}[t_i, s_i], X\right)$.

Our Hypotheses

Our assumption will be the following:

(H_1) $A : [0, +\infty) \to \mathcal{L}(X, X)$ is the generator of a compact and uniformly bounded evolution system $\{E(t, s)\}_{0 \le s \le t}$, pause

(H_2) $f : [0, +\infty) \times X \times X \to X$ is L^1-Caratheodory that is:

1. $f(t, \cdot, \cdot) : X \times X \to X$ is continuous for a.e. $t \in [0, +\infty)$;
2. for any fixed $x, y \in X$, $f(\cdot, x, y)$ is strongly measurable on $[0, +\infty)$;
3. for any $R > 0$ there exists $p_R \in L^1[0, +\infty)$ such that

$$|f(t, x, y)| \le p_R(t), \qquad \|x\|_X, \|y\|_X \le R.$$

(H_3) $\sigma : [0, +\infty) \to [-r, +\infty)$ is a continuous and increasing function such that $\sigma(t) \le t$ for any t in the domain.

(H_4) $K : BC\left(\cup_{i=0}^{\infty}[t_i, s_i], X\right) \to BC\left(\cup_{i=0}^{\infty}[t_i, s_i], X\right)$ is a completely continuous operator; moreover there exists $M_K > 0$ such that $\|K(u)\|_\infty \le M_K$, for all $u \in BC\left(\cup_{i=0}^{\infty}[t_i, s_i], X\right)$.

Our Hypotheses

Our assumption will be the following:

(H_1) $A : [0, +\infty) \to \mathcal{L}(X, X)$ is the generator of a compact and uniformly bounded evolution system $\{E(t, s)\}_{0 \le s \le t}$, pause

(H_2) $f : [0, +\infty) \times X \times X \to X$ is L^1-Caratheodory that is:

1. $f(t, \cdot, \cdot) : X \times X \to X$ is continuous for a.e. $t \in [0, +\infty)$;
2. for any fixed $x, y \in X$, $f(\cdot, x, y)$ is strongly measurable on $[0, +\infty)$;
3. for any $R > 0$ there exists $p_R \in L^1[0, +\infty)$ such that

$$|f(t, x, y)| \le p_R(t), \qquad \|x\|_X, \|y\|_X \le R.$$

(H_3) $\sigma : [0, +\infty) \to [-r, +\infty)$ is a continuous and increasing function such that $\sigma(t) \le t$ for any t in the domain.

(H_4) $K : BC\left(\cup_{i=0}^{\infty}[t_i, s_i], X\right) \to BC\left(\cup_{i=0}^{\infty}[t_i, s_i], X\right)$ is a completely continuous operator; moreover there exists $M_K > 0$ such that $\|K(u)\|_\infty \le M_K$, for all $u \in BC\left(\cup_{i=0}^{\infty}[t_i, s_i], X\right)$.

By using the previous Lemma, we prove the following:

Proposition

Assume that (H_1) *and* (H_2) *hold. Then*

$$Tx(t) := \begin{cases} E(t,s_i)(Kx)(s_i) + \int_{s_i}^{t} E(t,s) f(s,x(s),x(\sigma(s)))ds, & t \in \bigcup_{i=0}^{\infty} (s_i, t_{i+1}] \\ (Kx)(t), & \textit{otherwise.} \end{cases}$$

is completely continuous.

Strong and Mild solutions.

Proposition

If the evolution system $\{E(t,s)\}_{0\leq s\leq t}$ is differentiable (i.e. satisfies (iv) and (v) of Section 2), a point x in $BPC_\Theta([-r,+\infty),X)$ is a fixed point of the operator T if and only if it is a strong solution of Problem (1).

To prove our main result, we will use the following assumption

(H_2^*) f satisfies (H_2) and there exists $\Psi : [0,+\infty) \to [1,+\infty)$ is an increasing function, satisfying

$$\int_0^{+\infty} \frac{ds}{\Psi(s)} = +\infty \tag{4}$$

and

$$\|f(t,x,y)\|_X \leq \Psi(\|x\|_X + \|y\|_X)p(t),$$

for some $p \in L^1[0+\infty)$.

Strong and Mild solutions.

Proposition

If the evolution system $\{E(t,s)\}_{0\leq s\leq t}$ is differentiable (i.e. satisfies (iv) and (v) of Section 2), a point x in $BPC_{\Theta}([-r,+\infty),X)$ is a fixed point of the operator T if and only if it is a strong solution of Problem (1).

To prove our main result, we will use the following assumption

(H_2^*) f satisfies (H_2) and there exists $\Psi : [0,+\infty) \to [1,+\infty)$ is an increasing function, satisfying

$$\int_0^{+\infty} \frac{ds}{\Psi(s)} = +\infty \tag{4}$$

and

$$\|f(t,x,y)\|_X \leq \Psi(\|x\|_X + \|y\|_X)p(t),$$

for some $p \in L^1[0+\infty)$.

Theorem

Assume that (H_1), (H_2^*), (H_3) *and* (H_4) *hold. Then Problem (1) admits a mild solution in* $BPC_\Theta([-r,+\infty),X)$.

Corollary

Assume that (H_1), (H_2^*), (H_3) *and* (H_4) *hold. If the evolution system is differentiable, Problem (1) admits a strong solution in* $BPC_\Theta([-r,+\infty),X)$.

Remark

A close proof to that Theorem applied to the set of fixed point of the operator T, permits to obtain a uniform bound for the solutions of problem (1); in particular the solutions are uniformly bounded by

$$2(1+M_E)M_K + \|p\|_1\Psi(2(1+M_E)M_K).$$

Theorem

Assume that (H_1), (H_2^*), (H_3) *and* (H_4) *hold. Then Problem (1) admits a mild solution in* $BPC_\Theta([-r,+\infty),X)$.

Corollary

Assume that (H_1), (H_2^*), (H_3) *and* (H_4) *hold. If the evolution system is differentiable, Problem (1) admits a strong solution in* $BPC_\Theta([-r,+\infty),X)$.

Remark

A close proof to that Theorem applied to the set of fixed point of the operator T, permits to obtain a uniform bound for the solutions of problem (1); in particular the solutions are uniformly bounded by

$$2(1+M_E)M_K + \|p\|_1 \Psi(2(1+M_E)M_K).$$

Theorem

Assume that (H_1), (H_2^*), (H_3) *and* (H_4) *hold. Then Problem (1) admits a mild solution in* $BPC_\Theta([-r,+\infty),X)$.

Corollary

Assume that (H_1), (H_2^*), (H_3) *and* (H_4) *hold. If the evolution system is differentiable, Problem (1) admits a strong solution in* $BPC_\Theta([-r,+\infty),X)$.

Remark

A close proof to that Theorem applied to the set of fixed point of the operator T, permits to obtain a uniform bound for the solutions of problem (1); in particular the solutions are uniformly bounded by

$$2(1+M_E)M_K + \|p\|_1 \Psi(2(1+M_E)M_K).$$

A seemingly more general result concerning the existence of not necessarily bounded solutions is obtained by assuming the following instead of (H_2).

(H_2^{loc}) $f : [0, +\infty) \times X \times X \to X$ is such that $f(t, \cdot, \cdot) : X \times X \to X$ is continuous for a.e. $t \in [0, +\infty)$, for any fixed $x, y \in X$, $f(\cdot, x, y)$ is strongly measurable on $[0, +\infty)$ and satisfies $|f(t, x, y)| \leq L(\|x\|_X, \|y\|_X)p(t)$ for some $p \in L^1_{loc}[0, +\infty)$, where $L : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is increasing in each individual variable.

Corollary

Suppose that (H_1) and (H_3) hold. Moreover suppose that (H_2^{loc}) holds with $L(x, y) = \Psi(\|x\| + \|y\|)$, where $\Psi : [0, +\infty) \to [1, +\infty)$ is an increasing function, satisfying (4); then Problem (1) admits a solution in $PC_\Theta([-r, +\infty), X)$.

Thank you for your attention.