Parabolic equations with slow diffusion

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Overview

1. Classical results
2. Main theorem
3. The non-uniqueness example

The model equation

Cauchy problem for the p-Laplacian:

$$\begin{cases} u_t - \Delta_p u = 0 & \text{in }]0, T[\times \mathbb{R}^N \\ u(0, \cdot) = u_0 \end{cases}$$

Concerned with *local weak solutions*, with initial datum attained *strongly* but *locally* in $L^2(\mathbb{R}^N)$.

Discussion

1. Existence of local solutions;
2. Non-uniqueness $\leftrightarrow$ Conditional uniqueness;
3. Propagation of disturbances.

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The heat equation

E1 *Finite energy*: existence and uniqueness in $L^\infty(0, T; L^2(\mathbb{R}^N))$.

E2 Local existence of solutions for initial data obeying

$$|u_0(x)| \leq Ae^{B|x|^2}.$$

Maximal time of existence: $4/B$.

U1 *Controlled growth*: uniqueness holds in

$$\mathcal{G}_B := \left\{ u \in C^\infty(]0, 4/B[\times\mathbb{R}^N) : \exists A : \ |u(x,t)| \leq Ae^{B|x|^2} \right\}.$$

U2 *Non-negative* solutions of a given Cauchy problem are unique.

NU There exists a global, non-trivial solution with $u_0 \equiv 0$.

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Speed of propagation

Suppose $u_0 \in C_c^\infty(\mathbb{R}^N)$, $u_0 \gneqq 0$ and consider the energy solution u of

$$u_t - \Delta_p u = 0, \text{ in } \mathbb{R}^N \times]0, T[, \qquad u(\cdot, 0) = u_0(\cdot). \tag{1}$$

- Fast Diffusion: If $p < 2$ then $u(\cdot, t) > 0$ in $\mathbb{R}^N$ for any small $t > 0$, but $u(\cdot, t) \equiv 0$ for any large t (*exitinction in finite time*, EFT).
- Slow Diffusion: If $p > 2$, the support of $u(\cdot, t)$ stays bounded for any $t > 0$ (*finite speed of propagation*, FSP).

Theorem (Diaz&Veron '85, Bernis '88)

If u is an energy solution of (1) for $p > 2$, then

$$\operatorname{diam}(\operatorname{supp}(u(\cdot, t))) \le \operatorname{diam}(\operatorname{supp}(u_0)) + C\, t^{\frac{1}{N(p-2)+p}} \|u_0\|_{L^1}^{\frac{p-2}{N(p-2)+p}}.$$

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Other models

1. Porous media:

$$u_t = \Delta(|u|^{m-1}u).$$

FSP for $m > 1$; EFT when $0 < m < 1$.

2. Diffusion/Absorption:

$$u_t = \Delta u - |u|^{\alpha-1}u, \qquad \alpha < 1.$$

FSP+EFT: the support shrinks over time, eventually vanishing.

3. Anisotropic diffusion:

$$u_t = \left(|u_x|^{p-2}u_x\right)_x + \left(|u_y|^{q-2}u_y\right)_y, \qquad p, q > 1.$$

FSP *in direction* y if $p > q$; EFT if $\frac{1}{p} + \frac{1}{q} > 1$.

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Aims

1) For *local* solutions, *FSP fails* even in the case $u_t = \Delta_p u$, $p > 2$. However, one can recover it selecting a *branch* of the solution.

Branch of a solution

Let u solve

$$u_t = \mathrm{div}(A(x, u, \nabla u)) \quad \text{in } \mathbb{R}^N \times]0, T[.$$

A branch $\tilde{u}$ of u is a solution of the same equation on $\mathbb{R}^N \times]0, T[$ such that $\tilde{u} = u$ on $\mathrm{supp}(\tilde{u})$.

2) For solutions of anisotropic equations, prove FSP of a suitably selected branch in (possibly optimal) *quantitative form*, separately *in each direction.*

Model anisotropic equation: $A(x, t, z) = \nabla_z(|z_1|^{p_1}, \dots, |z_N|^{p_N})$ for a suitable choice of $p_1, \dots, p_N$ ($p_i \equiv p \rightarrow$ orthotropic p-Laplacian).

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Main result

Theorem (Düzgün, M., Vespri '18)

Let $\bar{p}$ be the harmonic mean of the p_i's and suppose

$$2 < \min\{p_1, \ldots, p_N\} \le \max\{p_1, \ldots, p_N\} < \bar{p}\left(1 + \frac{1}{N}\right).$$

Let u be a local solution of $u_t = \operatorname{div}(A(x, u, \nabla u))$ with L^2 nontrivial initial datum supported in a cube of edge R_0.
Then there is a branch $\tilde{u} \neq 0$ of u s. t.

$$\operatorname{supp}(\tilde{u}(\cdot, t)) \subseteq \prod_{j=1}^{N} [-R_j(t), R_j(t)],$$

$$R_j(t) = 2\, R_0 + C t^{\frac{N\,(\bar{p}-p_j)+\bar{p}}{\lambda\, p_j}} \|u_0\|_1^{\frac{\bar{p}}{p_j}\frac{p_j-2}{\lambda}}, \qquad \lambda = N(\bar{p}-2)+\bar{p}.$$

Comments

1. The assumptions on the leading term $A(x, u, \nabla u)$ are of "measurable coefficients" type. Contrary to other approaches, *no monotonicity* assumptions (hence no comparison principle).
2. It is proved that the estimated speed is *optimal* for large t, separately in each direction.
3. The statement can be generalized when only *some* of the diffusion rates lie in the slow diffusion range $p_i > 2$, but in this case *optimality of the speed is unknown.*
4. The condition $p_i \leq \bar{p}(1 + 1/N)$ for all i is an actual threshold: formally, the corresponding velocity of propagation vanishes if it is violated. Antontsev-Shmarev examples suggest that this may be the case, but the proof is missing.

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Self-similar solutions

Aim: when $p > 2$, find nontrivial solutions to $u_t = \Delta_p u$ on $\mathbb{R}\times\,]0, +\infty[$ with vanishing initial datum.

$$Ansatz: \qquad u(x,t) = t^{-\alpha} U(x\, t^{\beta}), \quad \alpha, \beta > 0$$

$$(p-2)\alpha = 1 + p\beta \quad \Rightarrow \quad \left(|U'|^{p-2} U'\right)' - \beta\, s\, U' + \alpha U = 0$$

where derivation is with respect to $s := x\, t^{\beta}$.

$$V = |U'|^{p-2} U' \quad \to \quad \begin{cases} U' = |V|^{\frac{2-p}{p-1}} V \\ V' = -\alpha U + \beta\, s\, |V|^{\frac{2-p}{p-1}} V \end{cases}$$

Non trivial solution found if $U \neq 0$ for $s \geq 1$ and

$$U(1) = V(1) = 0 \quad \Leftrightarrow \quad U(1) = U'(1) = 0$$

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Heuristics

Define $|t|^{\frac{p-2}{p-1}}t =: t^\gamma$, $\gamma \in]0,1[$, *increasing*. Set for simplicity $\alpha = \beta = 1$.

$$\begin{cases} U' = V^\gamma \\ V' = -U + sV^\gamma \end{cases} \qquad s \geq 1, \quad U(1) = V(1) = 0.$$

1. The system has *anti-dissipative* features thanks to the monotonicity of $V \mapsto V^\gamma$: Peano phenomenon has chances.
2. Sub-linear growth, hence local solutions extend to global solutions.
3. "Linearizes" near $s = 1$, $(U, V) = (0, 0)$ as

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Proof

Peano-Picard existence theorem in a suitable convex set of

$$X = \left\{ \mathbf{x} \in C^0([1, 1+\delta], \mathbb{R}^2) : \mathbf{x}(1) = \mathbf{0} \right\}.$$

Led by previous heuristics, define

$$C_{a,b} := \left\{ (x_1, x_2) \in X : \begin{cases} 0 \le x_1(s) \le b(s-1)^{\frac{1}{1-\gamma}} \\ a(s-1)^{\frac{1}{1-\gamma}} \le x_2(s) \le b(s-1)^{\frac{1}{1-\gamma}} \end{cases} \right\}.$$

For suitable $b > a > 0$ and small δ, the set $C_{a,b}$ is invariant for

$$T((x_1, x_2))(s) = \int_1^s \left(x_2^\gamma(\tau), -x_1(\tau) + x_2^\gamma(\tau) \right) \, d\tau,$$

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Thank you for your attention!