Periodic, Dirichlet or homoclinic type solutions for second order boundary value problems

Roberto Livrea
roberto.livrea@unipa.it

Dipartimento di Matematica e Informatica
Università degli Studi di Palermo

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Structure of the talk

First part

An overview of some existence and multiplicity theorems concerning periodic solutions of suitable classes of second order Hamiltonian systems is presented.

Second part

The existence of either at least one Dirichlet type solution or at least one homoclinic positive solution of a nonlinear singular equation is shown.

Second order Hamiltonian system

Find functions $u \in C^1([0,T], \mathbf{R}^N)$ such that $\dot{u}$ is absolutely continuous and

$$(\mathrm{P}) \quad \begin{cases} \ddot{u}(t) = \nabla F(t, u(t)) & \text{a.e. in } [0,T] \\ u(T) - u(0) = \dot{u}(T) - \dot{u}(0) = 0, \end{cases}$$

where $T > 0$ and $F : [0,T] \times \mathbf{R}^N \to \mathbf{R}$, with $N \in \mathbf{N}$, is smooth enough, while $\nabla := \nabla_u$ is the classical gradient with respect to the second variable.

Second order Hamiltonian system

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The solutions of problem (P) can be obtained as critical points of the following functional

$$\varphi(u) := \frac{1}{2} \int_0^T |\dot{u}(t)|^2 dt + \int_0^T F(t, u(t)) dt$$

for every u in the Sobolev space H_T^1 defined by

$$\left\{ u \in L^2([0,T], \mathbf{R}^N) \text{ having weak derivative } \dot{u} \text{ in } L^2([0,T], \mathbf{R}^N) \right\},$$

which is compactly embedded in $C^0([0,T], \mathbf{R}^N)$.

Our nonlinearity

We study problem (P) when

$$F(t,\xi) = \frac{1}{2}A(t)\xi\cdot\xi - \lambda b(t)G(\xi),$$

where $A : [0,T] \to \mathbf{R}^{N\times N}$ is a suitable symmetric matrix-valued function satisfying

$$A(t)\xi\cdot\xi \geq \mu|\xi|^2, \text{ a.e. in } [0,T],\ \forall \xi \in \mathbf{R}^N, \tag{1}$$

with $a_{ij} \in L^\infty([0,T])$, $\mu > 0$, $G \in C^1(\mathbf{R}^N)$, $b \in L^1([0,T])$ is a. e. nonnegative and $\lambda > 0$.

Hence the differential problem becomes

$$(\mathrm{P}_\lambda) \quad \begin{cases} \ddot{u}(t) = A(t)u(t) - \lambda b(t)\nabla G(t,u(t)) & \text{a.e. in } [0,T] \\ u(T)-u(0) = \dot{u}(T) - \dot{u}(0) = 0. \end{cases}$$

A first multiplicity result

Theorem 1 (JMAA '10 Bonanno-L.)

Assume that $G(\xi) \geq G(0) = 0$ *for every* $\xi \in \mathbf{R}^N$ *and let* $\gamma_1, \gamma_2 > 0$, $\bar{\xi} \in \mathbf{R}^N$ *be such that*

(k_1) $\gamma_1 < |\bar{\xi}| < \sqrt{\frac{L}{2}}\gamma_2$,

(k_2) $$\frac{\max_{|\xi| \leq \gamma_1} G(\xi)}{\gamma_1^2} < R\frac{G(\bar{\xi})}{|\bar{\xi}|^2}, \qquad \frac{\max_{|\xi| \leq \gamma_2} G(\xi)}{\gamma_2^2} < \frac{R}{2}\frac{G(\bar{\xi})}{|\bar{\xi}|^2},$$

where

$$R := \frac{L}{1+L}, \qquad L := \frac{1}{c^2 T \sum_{i,j=1}^N \|a_{ij}\|_\infty},$$

while c *is a constant related to the embedding* $H_T^1 \hookrightarrow C^0$.
Then, for every $b \in L^1([0,T]) \setminus \{0\}$ *a.e. nonnegative and for every*
$\lambda \in \Lambda_{\gamma_1,\gamma_2} := \left] \frac{1}{2R\|b\|_1 c^2} \frac{|\bar{\xi}|^2}{G(\bar{\xi})}, \frac{1}{2\|b\|_1 c^2} \min\left\{ \frac{\gamma_1^2}{\max_{|\xi| \leq \gamma_1} G(\xi)}, \frac{\gamma_2^2}{2\max_{|\xi| \leq \gamma_2} G(\xi)} \right\} \right[$
problem (P_λ) *admits at least two **non trivial** solutions* u_1, u_2 *such that*
$\|u_i\|_{C^0} \leq \gamma_2, \; i = 1, 2.$

More on the assumptions of Theorem 1

$$\gamma_1 \leq |\bar{\xi}| \leq \sqrt{\frac{L}{2}}\gamma_2$$

$$\frac{\max_{|\xi|\leq\gamma_1} G(\xi)}{\gamma_1^2} < R\frac{G(\bar{\xi})}{|\bar{\xi}|^2}, \qquad \frac{\max_{|\xi|\leq\gamma_2} G(\xi)}{\gamma_2^2} < \frac{R}{2}\frac{G(\bar{\xi})}{|\bar{\xi}|^2}$$

$$R = \frac{L}{1+L}$$

$$L = \frac{1}{c^2 T \sum_{i,j=1}^{N} \|a_{ij}\|_\infty}$$

More on the assumptions of Theorem 1

$$\gamma_1 \le |\bar{\xi}| \le \sqrt{\frac{L}{2}}\gamma_2$$

$$\frac{\max_{|\xi|\le\gamma_1} G(\xi)}{\gamma_1^2} < R\frac{G(\bar{\xi})}{|\bar{\xi}|^2}, \qquad \frac{\max_{|\xi|\le\gamma_2} G(\xi)}{\gamma_2^2} < \frac{R}{2}\frac{G(\bar{\xi})}{|\bar{\xi}|^2}$$

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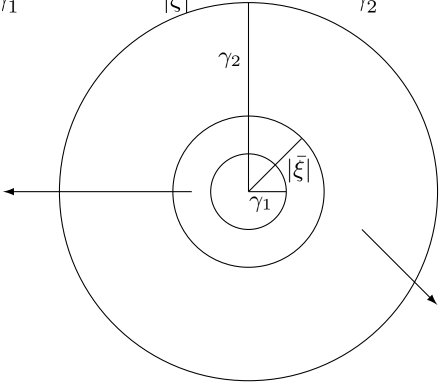
$$L = \frac{1}{c^2 T \sum_{i,j=1}^{N} \|a_{ij}\|_\infty}$$

G has a strong superquadratic growth



G has a subquadratic growth

Example

Put

$$G(\xi)=\begin{cases} 0 & \text{if } \xi\le 0 \\ e^{\xi^4}-1 & \text{if } 0<\xi<\sqrt{2} \\ 8\sqrt{2}e^4\xi-15e^4-1 & \text{if } \sqrt{2}\le\xi<140 \\ \frac{8\sqrt{2}e^4}{3\ 140^2}\xi^3+\rho & \text{if } \xi\ge 140, \end{cases}$$

where $\rho=\frac{2240}{3}\sqrt{2}e^4-15e^4-1$.
Then, for every $b\in L^1([0,1])\setminus\{0\}$ a.e. nonnegative and for every $\lambda\in\frac{3}{4\|b\|_1}\left]\frac{2}{e^4-1},\frac{140^2}{2240\sqrt{2}e^4-30e^4-2}\right[$, problem

$$\begin{cases} \ddot{u}=u-\lambda b(t)G'(u) & \text{a.e. in } [0,1] \\ u(1)-u(0)=\dot{u}(1)-\dot{u}(0)=0 \end{cases}$$

admits at least two non trivial solutions u_1,u_2 such that $\|u_i\|_{C^0}\le 140$ with $i=1,2$.
Indeed, we can apply Theorem 1 when $N=1$, $T=1$, $A(t)=1$, $c=\sqrt{2}$, $L=1/2$, $\gamma_1=1$, $\gamma_2=140$ e $\bar{\xi}=\sqrt{2}$.

A first comparison

Remark

Put $F(t,x) = \frac{|x|^2}{2} - \lambda b(t)G(x)$, where G is the function considered in the previous Example and $\lambda > 0$.
Clearly $\lim_{x\to 0} \frac{F(t,x)}{|x|^2} = \frac{1}{2}$, and it can be observed that F does not satisfy the following behavior at zero
there exist $R > 0$ *and* $k \in \mathbf{N}$ *such that, for every* $|x| \leq R$, *a.e. in* $[0,T]$

$$-\frac{1}{2}(k+1)^2\omega^2|x|^2 \leq F(t,x) \leq -\frac{1}{2}k^2\omega^2|x|^2.$$

Hence, the existence of at least two non trivial solutions for problem

$$(\mathrm{P}) \quad \begin{cases} \ddot{u}(t) = \nabla F(t,u(t)) & \text{a.e. in } [0,T] \\ u(T) - u(0) = \dot{u}(T) - \dot{u}(0) = 0 \end{cases}$$

cannot be derived by

H. Brezis and Nirenberg, *Remarks on Finding Critical Points*, Comm. Pure Appl. Math., **44** (1991), 939–963.

Deepening the comparison

Remark

Put $F(t,x) = \frac{|x|^2}{2} - \lambda b(t) G(x)$, where G is as in the previous Example and $\lambda > 0$. If, in addition, $b \in C^0([0,1], \mathbf{R}^+)$, one has that

$$\lim_{x \to +\infty} F(t,x) = -\infty, \quad \lim_{x \to -\infty} F(t,x) = +\infty$$

uniformly with respect to t and the coercivity assumptions required by Brezis-Nirenberg (namely $\lim_{|x| \to +\infty} F(t,x) = +\infty$) and weakened in

C.L. Tang and X.P. Wu, *Periodic solutions for second order systems with not uniformly coercive potential*, J. Math. Anal. Appl. **259** (2001), 386–397.

Y.-W. Ye, C.L. Tang, *Periodic solutions for some nonautonomous second order Hamiltonian systems*, J. Math. Anal. Appl. **344** (2008) 462–471.

do not hold.

Three critical point theorem by Bonanno and Candito – J. Differential Equations (2008)

The proof of Theorem 1 is based on an abstract critical point result obtained in

G. Bonanno and P. Candito, *Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities*, J. Differential Equations, **244** (2008), 3031–3059,

where the existence of at least three distinct critical points for a class of functionals satisfying the following structure

$$\Phi - \lambda\Psi$$

is proved, whenever λ is a positive parameter belonging to a precise interval.

Different conditions

Theorem 2 - An existence result (MaMa '17 Bonanno – L. – Schechter)

Assume that $G(0) = 0$ *and that there exist* $\gamma > 0$, $\bar{\xi} \in \mathbf{R}^N$ *with* $|\bar{\xi}| < \gamma$ *such that*

$$\frac{\max_{|\xi| \leq \gamma} G(\xi)}{\gamma^2} < L \frac{G(\bar{\xi})}{|\bar{\xi}|^2},$$

where $L = \frac{1}{c^2 T \sum_{i,j=1}^N \|a_{ij}\|_\infty}$.
Then, for every $b \in L^1([0,T]) \setminus \{0\}$ *a.e. nonnegative and for every*

$$\lambda \in \frac{1}{2c^2 \|b\|_1} \left] \frac{1}{L} \frac{|\tilde{\xi}|^2}{G(\tilde{\xi})}, \frac{\gamma^2}{\max_{|\xi| \leq \gamma} G(\xi)} \right[$$

problem

$$(P_\lambda) \quad \begin{cases} \ddot{u}(t) = A(t)u(t) - \lambda b(t) \nabla G(t, u(t)) & \text{a.e. in } [0,T] \\ u(T) - u(0) = \dot{u}(T) - \dot{u}(0) = 0 \end{cases}$$

admits at least one nontrivial solution such that $|u(t)| < \gamma$ *for every* $t \in [0,T]$.

Example of G quadratic at zero

If $T = 1$, $N = 2$ and $A(t) = I_{2\times 2}$, the function

$$G(\xi) = |\xi|^2(|\xi|^2 - 1)(|\xi|^2 - 100)e^{|\xi|^2/100} \; \forall \xi \in \mathbf{R}^2$$

satisfies condition

$$\frac{\max_{|\xi|\leq\gamma} G(\xi)}{\gamma^2} < L\frac{G(\bar{\xi})}{|\bar{\xi}|^2},$$

with the choice $\gamma = 10$ and $|\bar{\xi}| < 1$ such that such that

$$\max_{|\xi|\leq 10} G(\xi) = \max_{|\xi|\leq 1} G(\xi) = G(\bar{\xi}).$$

Subquadratic behaviour at zero as a special case

Theorem 3 (MaMa '17 Bonanno – L. – Schechter)

Let $G \in C^1(\mathbf{R}^N)$ *be such that* $G(0) = 0$ *and*

(j_1) $\displaystyle\limsup_{\xi \to 0} \frac{G(\xi)}{|\xi|^2} = +\infty.$

Then, for every $b \in L^1([0,T], \mathbf{R}) \setminus \{0\}$ *a.e. nonnegative and for every* $\lambda \in \frac{1}{2c^2\|b\|_1}\left]0, \sup_{\rho>0} \frac{\rho^2}{\max_{|\xi|\le\rho} G(\xi)}\right[$ *problem*

$$
(P_\lambda) \quad \begin{cases} \ddot{u}(t) = A(t)u(t) - \lambda b(t)\nabla G(t, u(t)) & \text{a.e. in } [0,T] \\ u(T) - u(0) = \dot{u}(T) - \dot{u}(0) = 0 \end{cases}
$$

admits at least one non trivial solution u_λ.

Theorem 4 - A further existence result (MaMa '17 Bonanno – L. – Schechter)

Assume that $G \in C^1(\mathbf{R}^N)$ *with* $G(0) = 0$. *Moreover, suppose that there exist* $R > 0$ *and* $\mu > 2$ *such that for every* $|\xi| \geq R$

$$0 < \mu G(\xi) \leq \nabla G(\xi) \cdot \xi. \tag{AR}$$

Then, for every $b \in L^1([0,T], \mathbf{R}) \setminus \{0\}$ *a.e. nonnegative and for every* $\lambda \in \frac{1}{2c^2\|b\|_1} \left]0, \sup_{\rho>0} \frac{\rho^2}{\max_{|\xi|\leq\rho} G(\xi)}\right[$ *problem*

$$(P_\lambda) \quad \begin{cases} \ddot{u}(t) = A(t)u(t) - \lambda b(t)\nabla G(t, u(t)) & \text{a.e. in } [0,T] \\ u(T) - u(0) = \dot{u}(T) - \dot{u}(0) = 0 \end{cases}$$

admits at least one non trivial solution u_λ.

Again two solutions

Theorem 5 (MaMa '17 Bonanno – L. – Schechter)

Let $G \in C^1(\mathbf{R}^N)$ *be such that* $G(0) = 0$ *and*

(j_1) $\limsup_{\xi \to 0} \frac{G(\xi)}{|\xi|^2} = +\infty.$

Moreover, assume that there exist two positive constants $\nu > 2$ *and* r*, such that, for every* $|\xi| \geq r$ *one has*

$$0 < \mu G(\xi) \leq \nabla G(\xi) \cdot \xi. \qquad \text{(AR)}$$

Then, for every $b \in L^1([0,T]) \setminus \{0\}$ *a.e. nonnegative and for every* $\lambda \in \frac{1}{2c^2\|b\|_1} \left]0, \sup_{\rho>0} \frac{\rho^2}{\max_{|\xi|\leq\rho} G(\xi)}\right[$ *problem*

$$(P_\lambda) \quad \begin{cases} \ddot{u}(t) = A(t)u(t) - \lambda b(t)\nabla G(t, u(t)) & \text{a.e. in } [0,T] \\ u(T) - u(0) = \dot{u}(T) - \dot{u}(0) = 0 \end{cases}$$

admits at least two non trivial solutions.

An example

Consider the problem

$$\begin{cases} -\ddot{u} + u = \lambda |u|^{q-2}u + |u|^{r-2}u & \text{a.e. in } [0,1] \\ u(1) - u(0) = \dot{u}(1) - \dot{u}(0) = 0, \end{cases}$$

with $0 < q < 2 < r$.
Put

$$\lambda^* = \left(\frac{r}{r-2} \frac{2-q}{q} \right)^{\frac{2-q}{r-2}} \left[\frac{q}{2} \cdot \frac{r-2}{r-q} \right]^{\frac{r-q}{r-2}}.$$

Then, for every $\lambda \in]0, \lambda^*[$ the previous problem admits at least two nontrivial solutions.

Some comparisons

I. Ekeland and N. Ghoussoub, *Certain new aspects of the calculus of variations in the large*, Bull. Amer. Math. Soc. **39** (2002) 207–265.

M. Schechter, *Periodic non-autonomous second order dynamical systems*, J. Differential Equations **223** (2006) 290–302.

Z. Wang, J. Zhang and Z. Zhang, *Periodic solutions of second order non-autonomous Hamiltonian systems with local superquadratic potential*, Nonlinear Anal. **70** (2009) 3672–3681.

The abstract tools

These last results have been obtained by applying suitable versions of some critical points theorems proved in the following papers

G. Bonanno, *Relations between the mountain pass theorem and local minima*, Adv. Nonlinear Anal. **1** (2012), 205–220.

G. Bonanno and G. D'Aguì, *Two non-zero solutions for elliptic Dirichlet problems*, Z. Anal. Anwend. **35** (2016), 449–464.

Further existence and multiplicity results

G. Bonanno, R. Livrea and M. Schecther, *Multiple solutions of second order Hamiltonian systems*, Electron. J. Qual. Theory Differ. Equ. 2017, Paper No. 33, 15 pp.

Linking methods are exploited in order to proof some new multiplicity results.

P. Candito, R. Livrea, L. Sanchez and M. Zamora, *Positive solutions of Dirichlet and homoclinic type for a class of singular equations*, Journal Math. Anal. Appl. vol. 461, (2018) 1561-1584

In the recent literature one finds several results on positive solutions to singular equations that can be put in the form

$$(|u'|^{p-2}u')' = \frac{|u'|^k}{u^\mu} - f(t, u, u'), \tag{2}$$

with $p > 1$ and $k,\ \mu > 0$ (as better described in the following).

Here are some references:
Guo, Wenrui, Weigao - Appl. Math. Lett. 2005
Stanek - Nonlin. Anal. 2009
Zhou, Qin, Xu, Wei - Nonlin. Anal. 2012
Nachmann, Callegari - SIAM J. Appl. Math. 1980
Barenblatt, Bertsch, Chertock and Prostokishin - Proc. Natl. Acad. Sci. USA 2000

Motivation for the study of such problems may be traced back at least to the paper of Nachmann and Callegari and the paper by Barenblatt et al and is given also in some of the more recent articles where this or similar equations are studied. Such problems appear in Physics in connection with (possibly degenerate) parabolic equations from fluid flow theory, in particular involving non-Newtonian models.

In the literature we have mentioned the research is focused on the two-point boundary value problem for (2) in a finite interval $I = [0, T]$, namely the problem of finding a positive function $u(t)$, solving (2) in $(0, T)$ and satisfying the boundary conditions

$$u(0) = 0, \;\; u(T) = 0.$$

For brevity, we shall refer to such solutions as *Dirichlet type solutions*.

According to Barenblatt et al (2000), the height of groundwater level $h(t,x)$ in a portion of porous medium with infiltration (t is time, x the horizontal spacial coordinate along the impermeable bed) is found to satisfy the following degenerate parabolic equation

$$\frac{\partial h}{\partial t} = h\frac{\partial^2 h}{\partial x^2} - \lambda\left(\frac{\partial h}{\partial x}\right)^2$$

which inspires several interesting ODE problems. Indeed, trying to find solutions of the form

$$h(x,t) = t^{-1}w(x)$$

one can observe that w must be a solution of

$$w'' = \lambda\frac{w'^2}{w} - 1$$

and w must vanish on the boundary of some interval, to express the continuity of the groundwater level.

According to Zhou et al (2012), the groundwater flow in a water-absorbing fissurized porous rock with non-Newtonian filtration leads to study the equation

$$(|u'|^{p-2}u')' = \frac{|u'|^p}{u} - c.$$

This led those authors to consider the more general equation

$$(|u'|^{p-2}u')' = \frac{|u'|^p}{u^\mu} - f(t,u,u')$$

having shown that that the condition $\mu < p$ is necessary and sufficient for the existence of positive solutions to the Dirichlet problem in a finite interval, while the condition $1 \leq \mu < p$ is necessary and sufficient for the existence of positive "periodic" solutions.

$$(|u'|^{p-2}u')' = \frac{|u'|^k}{u^\mu} - f(t,u,u')$$

Our purpose is to extend in some way the range of powers k, μ that have been considered in the literature. Moreover, we include new information about the appearance of the T-periodic solutions.
We found that the order relation between μ and k determines the type of solutions that one can expect: roughly speaking

- if $\mu < k$, (2) has "Dirichlet solutions"
- if $\mu \geq k$, positive homoclinic solutions appear (that is, solutions defined in the whole real line, vanishing at $\pm\infty$)

Dirichlet type solution

Let us write the equation as

$$u'' = \frac{|u'|^{2\beta}}{u^\mu} - f(t, u, u'). \tag{3}$$

In the statements we assume

(H) $f : [0, T] \times \mathbf{R}^+ \times \mathbf{R} \to \mathbf{R}^+$ is continuous and bounded function such that

$$m_f := \inf f > 0, \qquad M_f := \sup f < +\infty.$$

Theorem 6

Let $2\beta > \mu$. Then equation (3) has a positive solution of Dirichlet type on the interval $(0, T)$.

Looking for a T-periodic solution

$$u'' = \frac{|u'|^{2\beta}}{u^\mu} - f(t,u,u') \tag{3}$$

Theorem 7

If $\mu = \beta$, then equation (3) has a T-periodic, positive solution provided that one of the following conditions holds

(i) $\beta \geq 1$,

(ii) $\beta \in (0,1)$ and
$$M_f \leq (2\beta)^{\frac{\beta}{1-\beta}}(1-\beta). \tag{4}$$

Looking for a T-periodic solution

$$u'' = \frac{|u'|^{2\beta}}{u^{\mu}} - f(t,u,u') \tag{3}$$

Theorem 8

If $2\beta > \mu > \beta$, then the equation has a T-periodic solution (of Dirichlet type), provided one of the following conditions holds

(i) $\mu \geq 1$

(ii) $\mu \in (0,1)$ and either

$$M_f \leq \min\left\{\frac{8}{T^2}, (2\beta)^{\frac{\beta}{1-\beta}}(1-\beta)\right\} \tag{5}$$

or

$$T \leq 2\sqrt{2} M_f^{-\frac{1+\mu-2\beta}{2(\mu-\beta)}} \left(\frac{2\beta(1-\beta)}{1-\mu}\right)^{\frac{\beta}{2(\mu-\beta)}} (1-\beta)^{\frac{1-\beta}{2(\mu-\beta)}}. \tag{6}$$

Looking for a T-periodic solution

$$u'' = \frac{|u'|^{2\beta}}{u^\mu} - f(t,u,u') \tag{3}$$

Theorem 9

If $\beta > \mu \geq 1$, then the equation has a T-periodic positive solution (of Dirichlet type).

This result is a consequence of the following

Proposition

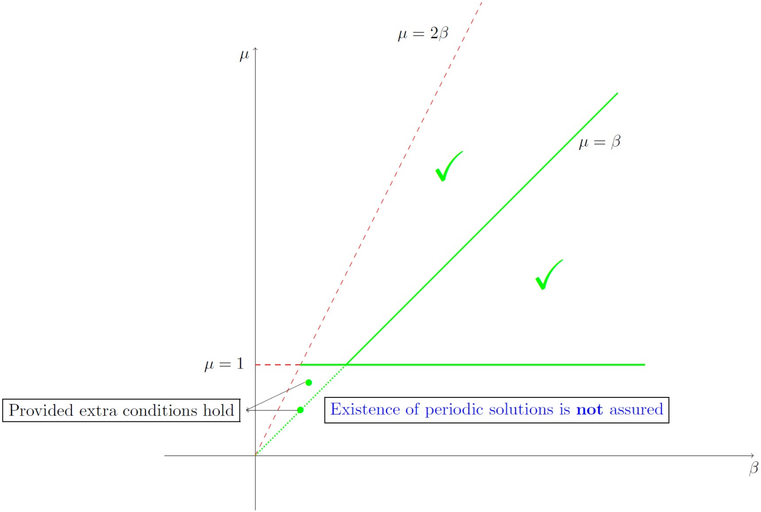
If $\mu \geq 1$ and u is a positive Dirichlet type solution of (3), then $u'(0) = 0 = u'(T)$

Arguing by contradiction. If $u'(0) > 0$ there is $t_u > 0$ such that $u'(t)$ is between $\frac{u'(0)}{2}$ and $2u'(0)$. Hence,

$$\int_0^T \frac{|u'(s)|^{2\beta}}{u^\mu(s)} ds \geq \left(\frac{u'(0)}{2}\right)^{2\beta} \int_0^{t_u} \frac{ds}{u^\mu(s)} \geq \left(\frac{u'(0)}{2}\right)^{2\beta} \frac{1}{2u'(0)} \int_0^{u(t_u)} \frac{dy}{y^\mu},$$

while from the equation itself follows that $\int_0^T \frac{|u'(s)|^{2\beta}}{u^\mu(s)} ds \leq TM_f < +\infty$.

Exponents regions for the existence of a T-periodic solution



Result for homoclinics $\quad (|u'|^{p-2}u')' = \frac{|u'|^k}{u^\mu} - f(t,u,u')$

Theorem

Let $1 < p < \infty$, $k > 1$ *and* $\mu \geq k$, *and assume in addition*

i) *there exist positive constants* α, γ, r *and a positive, continuous function* $\beta : \mathbf{R} \times \mathbf{R}^+ \to \mathbf{R}$, *with* $0 < \alpha \leq \beta$, *such that*

$$0 < r < k \tag{7}$$

and

$$\alpha \leq f(t,x,y) \leq \beta(t,x) + \gamma |y|^r, \tag{8}$$

for every $(t,x,u) \in \mathbf{R} \times [0,+\infty) \times \mathbf{R}$ *and moreover*

$$\sup_{t \in \mathbf{R}} \max_{x \in [0,M]} \beta(t,x) < \infty \tag{9}$$

for every $M > 0$.

Then for every $M > 0$ *equation* (2) *admits at least one positive homoclinic solution* u *such that* $\max u = M$.

Towards Dirichlet solutions

The method of lower and upper solutions works by means of a truncation argument. Hence, the problem moves to the construction of lower and upper solutions to the equation

$$u'' = \frac{|u'|^{2\beta}}{u^\mu} - f(t, u, u'). \qquad (3).$$

We consider the auxiliary equation (where $\alpha > 0$)

$$u'' = \frac{|u'|^{2\beta}}{u^\mu} - \alpha. \qquad (10)$$

We take advantage of the fact that the equation is autonomous and perform a change of variables that leads to a first order differential equation.
Given $M > 0$, the classical theory of the Cauchy problem ensures that the solution of

$$u'' = \frac{|u'|^{2\beta}}{u^\mu} - \alpha, \qquad u(0) = M, \qquad u'(0) = 0 \qquad (11)$$

is even and it is defined in some interval $(-\tau_\alpha(M), \tau_\alpha(M))$ and $u(-\tau_\alpha(M)^+) = 0 = u(\tau_\alpha(M)^-)$.

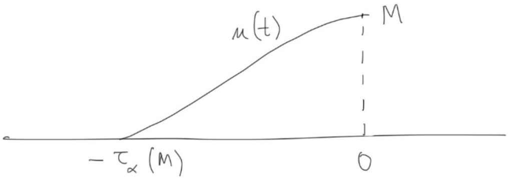
We pass to a first order Cauchy problem, strictly connected to (11), setting

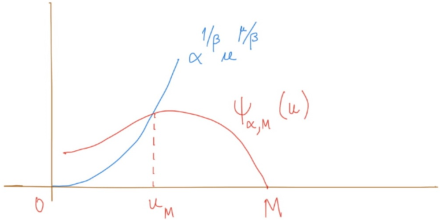
$$u'^2 = \psi(u) \quad \text{for } \ t \in (-\tau_\alpha(M), 0], \tag{12}$$

then ψ solves a first order problem, and conversely:

$$\psi' = 2\left[\frac{\psi^\beta(u)}{u^\mu} - \alpha\right], \qquad \psi(M) = 0 \tag{13}$$

Comparing the solution $\psi_{\alpha,M}$ of this problem with the function $\alpha^{1/\beta}u^{\mu/\beta}$ we see that the two graphs meet in a unique point with abscissa $u_M \in (0, M)$, $\psi_{\alpha,M}$ being concave in (u_M, M), so that





$$\sqrt{\Psi_{\alpha,M}} = u'$$

The time-map properties

Put

$$\tau_\alpha(M) := \int_0^M \frac{du}{\sqrt{\psi_{\alpha,M}(u)}}. \tag{14}$$

We have, always assuming $\mu < 2\beta$,

Lemma 1

$\tau_\alpha(M)$ is finite, and in particular u is defined in the finite interval $(-\tau_\alpha(M), \tau_\alpha(M))$.

The proof is done by computing

$$\tau_\alpha(M) \leq \frac{1}{\sqrt{\alpha^{1/\beta}}} \int_0^{u_M} \frac{du}{u^{\mu/2\beta}} + \sqrt{\frac{M-u_M}{\psi_{\alpha,M}(u_M)}} \int_{u_M}^M \frac{du}{\sqrt{M-u}}$$

$$= \frac{1}{\sqrt{\alpha^{1/\beta}}} \frac{2\beta}{2\beta-\mu} u_M^{(2\beta-\mu)/2\beta} + 2\frac{M-u_M}{\sqrt{\psi_{\alpha,M}(u_M)}} < +\infty.$$

The time-map properties

Lemma 2

$$\lim_{M\to+\infty} \tau_\alpha(M) = +\infty; \ \lim_{M\to 0^+} \tau_\alpha(M) = 0.$$

The proof is done considering separately the cases $0 < \mu \leq \beta$ and $\beta < \mu < 2\beta$ and is slightly more tricky.

Lemma 3

$\tau_{\alpha_1}(M) > \tau_{\alpha_2}(M)$, provided $\alpha_1 < \alpha_2$.

Lemma 4

For every $M_1 > 0$ there exists $M_2 > M_1$ such that $\tau_{\alpha_1}(M_1) = \tau_{\alpha_2}(M_2)$. Moreover, the inequality

$$\psi_{\alpha_2,M_2}(u) > \psi_{\alpha_1,M_1}(u)$$

for $u \in (0, M_1)$ holds.

The lower and upper solutions

Putting together all the above arguments and arguing with

$$\alpha_1 = m_f, \qquad \alpha_2 = M_f$$

one can construct a couple of positive lower and upper solutions $\sigma_1,\ \sigma_2$ with

$$\sigma_1(t) \leq \sigma_2(t) \quad \forall t \in (0, T),$$

$$\sigma_i(0) = \sigma_i(T) = 0 \quad i = 1,\ 2,$$

that lead to the Dirichlet type solution.

Homoclinics: idea of the proof

Shooting: given $M > 0$ we will consider the following Cauchy problem

$$\begin{cases} (|u'|^{p-2}u')' - \dfrac{|u'|^{\,k}}{u^{\mu}} + f(t,u,u') = 0, \\ u(0) = M, \\ u'(0) = 0. \end{cases} \tag{15}$$

A crucial step is to show that a solution cannot vanish in finite time. Taking a solution $w(t)$ defined and positive in any bounded interval (a,b) with $a < 0 < b$ suppose $w(a^+) = 0$. Using the assumptions about the growth of $f(t,w,w')$ we show that

$$\frac{|w'|^k}{w^{\mu}} \in L^1(a,b)$$

and then letting k' denote the conjugate exponent of k and t is such that $w(\tau) < 1$ for every $\tau \in [a,t]$, we have

$$
\begin{aligned}
\int_a^t |w'(\tau)|\, d\tau &= \int_a^t \frac{|w'(\tau)|}{|w^{\mu/k}(\tau)|} |w^{\mu/k}(\tau)|\, d\tau \\
&\leq \left(\int_a^t \frac{|w'(\tau)|^k}{w^{\mu}(\tau)}\, d\tau\right)^{1/k} \left(\int_a^t w^{\mu k'/k}(\tau)\, d\tau\right)^{1/k'} \\
&\leq \left(\int_a^{\tilde{b}} \frac{|w'(\tau)|^k}{w^{\mu}(\tau)}\, d\tau\right)^{1/k} \left(\int_a^t w^{k'}(\tau)\, d\tau\right)^{1/k'}.
\end{aligned}
$$

This implies that, for every $\tau \in (a, t)$,
$w(t) \leq \int_a^t |w'(\tau)|\, d\tau \leq c_2 \left(\int_a^t w^{k'}(\tau)\, d\tau\right)^{1/k'}$, so that

$$
w^{k'}(t) \leq c_2^{k'} \int_a^t w^{k'}(\tau)\, d\tau,
$$

where c_2 is a constant independent from t. Thus, Gronwall's lemma leads to $w(\tau) = 0$ for every $\tau \in [a, t]$ which is absurd.