An infinite-dimensional version of the Poincaré – Birkhoff theorem

Alessandro Fonda

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to appear in the Annali della Scuola Normale di Pisa

Jules Henri Poincaré (1854 – 1912)

SUR UN THÉORÈME DE GÉOMÉTRIE.

Par M. **H. Poincaré** (Paris).

Adunanza del 10 marzo 1912.

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Note: Poincaré died on July 17th, 1912

RENDICONTI

DEL

CIRCOLO MATEMATICO

DI PALERMO

DIRETTORE: G. B. GUCCIA.

TOMO XXXIII

(1° SEMESTRE 1912).

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“Théorème de géométrie”

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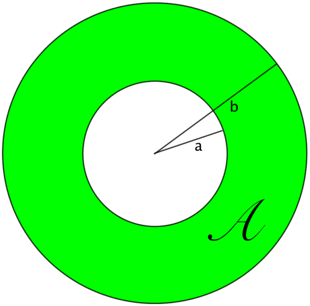

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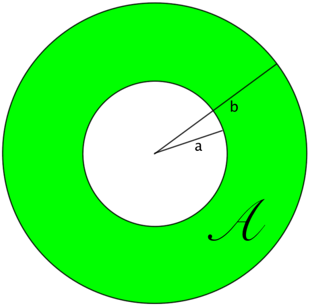

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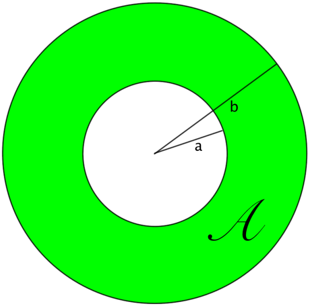

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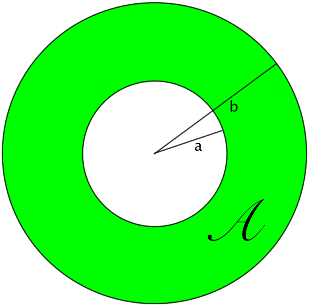

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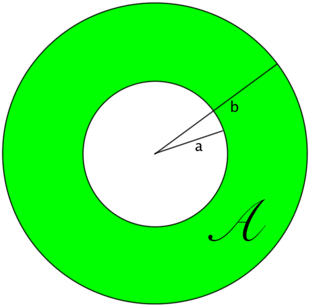

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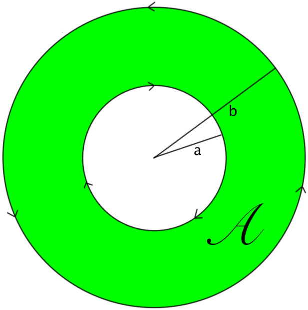

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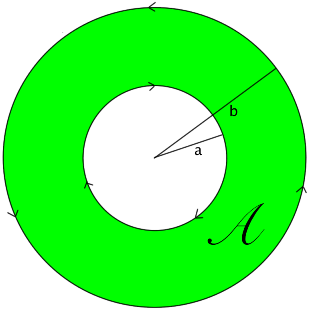

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An equivalent formulation

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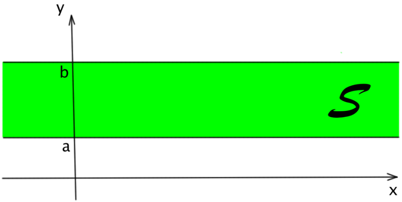

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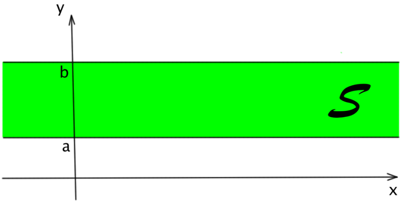

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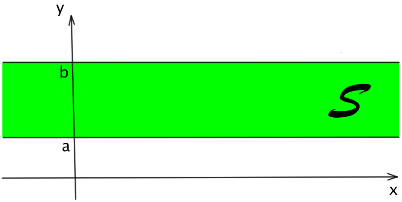

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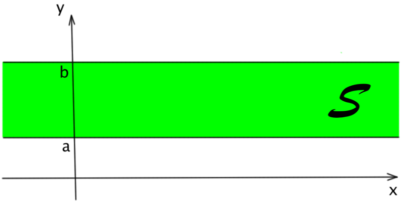

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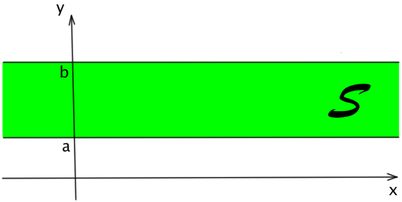

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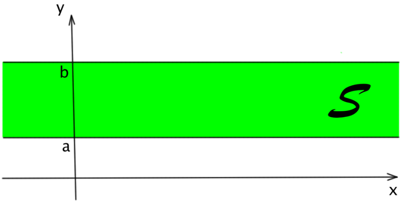

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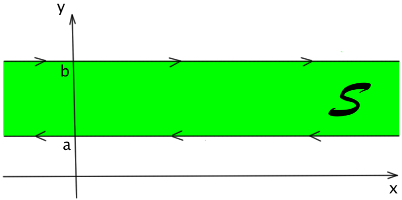

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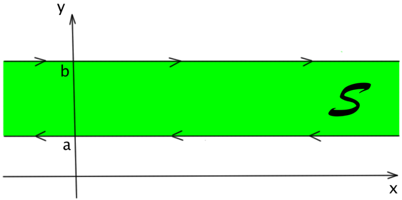

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Then, $\mathcal{P}$ has at least two geometrically distinct fixed points.

George David Birkhoff (1884 – 1944)

The Poincaré – Birkhoff theorem

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Applications to the existence of periodic solutions were provided by:

Bartsch, Bonheure, Boscaggin, Butler, Calamai, Corsato, Dalbono, Del Pino, T. Ding, Dondè, Fabry, Feltrin, Garrione, Gidoni, Hartman, Manásevich, Margheri, Mawhin, Omari, Sabatini, Sfecci, Smets, Toader, Torres, Wang, Zanini, Zanolin, ...

Generalizing the Poincaré – Birkhoff theorem (in the framework of Hamiltonian systems)

Consider the system

$$\dot{x} = \frac{\partial H}{\partial y}(t, x, y)\,, \qquad \dot{y} = -\frac{\partial H}{\partial x}(t, x, y)\,,$$

and assume that the Hamiltonian $H(t, x, y)$ is T-periodic in t.

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However,

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> [Moser and Zehnder, Notes on Dynamical Systems, 2005]

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Then, there are at least $N+1$ distinct T-periodic solutions, with

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The periodicity in $x_1, \ldots, x_N$ permits to define the action functional on the product of a Hilbert space E and the N-torus $\mathbb{T}^N$:

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The result then follows from the fact that

$$\mathrm{cat}(\mathbb{T}^N) = N + 1\,.$$

Some applications:

A. Fonda and A. Sfecci,
Periodic solutions of weakly coupled superlinear systems,
Journal of Differential Equations (2016)

A. Fonda, M. Garrione and P. Gidoni,
Periodic perturbations of Hamiltonian systems,
Advances in Nonlinear Analysis (2016)

A. Fonda and A. Sfecci,
Multiple periodic solutions of Hamiltonian systems confined in a box,
Discrete and Continuous Dynamical Systems (2017)

A. Fonda and P. Gidoni,
An avoiding cones condition for the Poincaré–Birkhoff theorem,
Journal of Differential Equations (2017)

A. Fonda and R. Toader,
Subharmonic solutions of Hamiltonian systems displaying some kind of sublinear growth, Advances in Nonlinear Analysis (2017)

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Moreover, the Hamiltonian $\nabla H : \mathbb{R} \times \ell^2 \times \ell^2 \to \ell^2 \times \ell^2$ satisfies:

- at most linear growth
- Lipschitz continuity on bounded sets.

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Here, $x = (x_1, \dots, x_N, \dots) \in \ell^2$ and $y = (y_1, \dots, y_N, \dots) \in \ell^2$.

We define:

- the projection $P_N : \ell^2 \to \mathbb{R}^N$ as

$$P_N(\xi_1, \xi_2, \dots) = (\xi_1, \xi_2, \dots, \xi_N);$$

The infinite-dimensional case

Consider the system

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- the function $H_N : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ as

$$H_N(t, u, v) = H(t, I_N u, I_N v) .$$

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Twist condition: for every sufficiently large integer N, the solutions of

$$u' = \nabla_v H_N(t,u,v)\,, \qquad v' = -\nabla_u H_N(t,u,v)\,,$$

with starting point $(u(0), v(0)) \in \partial\mathcal{S}_N$ satisfy

$$(\star) \qquad u_k(T) - u_k(0) \begin{cases} < 0\,, & \text{if } v_k(0) = a_k\,, \\ > 0\,, & \text{if } v_k(0) = b_k\,. \end{cases}$$

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Theorem

In the above setting, there exists at least one T-periodic solution, with

$$(x(0), y(0)) \in \mathcal{S}_\infty\,.$$

An application

Consider the system

$$x_k'' + \frac{\partial \mathcal{V}}{\partial x_k}(t, x_1, \ldots, x_k, \ldots) = e_k(t), \qquad k = 1, 2, \ldots,$$

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Theorem

Assume that $(\tau_k)_k$ belongs to ℓ^2 and that there exists $(M_k)_k$ in ℓ^2 such that

$$\left| \frac{\partial \mathcal{V}}{\partial x_k}(t, x) \right| \le M_k\,, \qquad \text{for every } (t, x) \in [0, T] \times \ell^2.$$

Then, there is at least one T-periodic solution.

Grazie per l’attenzione!