Variational approach to some boundary value problems with impulsive effects

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The problem

$$
\begin{cases}
-(p(t)u'(t))' + q(t)u(t) = \lambda f(t,u(t)) \;\; t \in [0,T], \;\; t \neq t_j \\
\\
u(0) = u(T) = 0 \\
\\
\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = \mu I_j(u(t_j)), \;\; j = 1,2,...,n
\end{cases}
\qquad (S_{\lambda,\mu})
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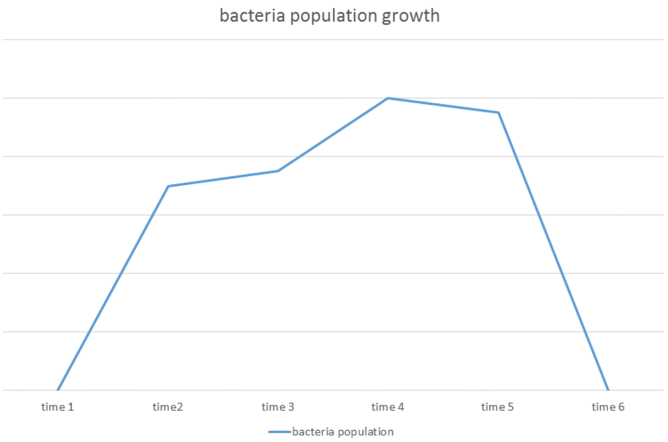
$p \in C^1([0,T],]0, +\infty[)$, $q \in L^\infty([0,T])$ with $\operatorname{ess\,inf}_{t\in[0,T]} q(t) \geq 0$,
$\lambda \in]0, +\infty[$, $\mu \in]0, +\infty[$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$,
$0 = t_0 < t_1 < t_2 < ... < t_n < t_{n+1} = T$,
$\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = \lim_{t\to t_j^+} u'(t) - \lim_{t\to t_j^-} u'(t)$, $I_j : \mathbb{R} \to \mathbb{R}$
are continuous for every $j = 1, 2, ..., n$.

References for Theory

- Benchohra M., Henderson J. and Ntouyas N., *Impulsive Differential Equations and Inclusions*, Hindawi Publ. Corp. (2006).
- Lakshmikantham V., Bainov D.D. and Simeonov P.S., *Theory of Impulsive Differential Equations,* World Scientific, Singapore (1989).

... and for applications

- Chen J., Tisdell C.C. and Yuan R., On the solvability of periodic boundary value problems with impulse, *J. Math. Anal. Appl.* **331** (2007), 902–912.,
- Mawhin J. Topological degree and boundary value problems for nonlinear differential equations, *Topological methods for ordinary differential equations, Lecture Notes in Math.* **1537** Springer, Berlin, 1993, 74–142.
- Qian D. and Li X., Periodic solutions for ordinary differential equations with sublinear impulsive effects, *J. Math. Anal. Appl.* **303** (2005), 288–303.


bacteria population growth
time 1
time2
time 3
time 4
time 5
time 6
bacteria population

In several recent papers impulsive boundary value problems are considered via variational methods

- Bonanno G., Di Bella B. and Henderson J., Existence of solutions to second-order boundary-value problems with small pertubations of impulses, *Electron. J. Differential Equations*, **126**, (2013), 1–14
- Bonanno G., Di Bella B. and Henderson J., Infinitely many solutions for a boundary value problem with impulsive effects, *Bound. Value Probl*, **278**, (2013).
- Bonanno G., Rodriguez-Lopez R. and Tersian S., Existence of solutions to boundary value problem for impulsive fractional differential equations, *Fract. Calc. Appl. Anal.* **17/3** (2014), 717–744.

- Cabada A. and Tersian S., Existence and multiplicity of solutions to boundary value problems for fourth-order impulsive differential equations, *Bound. Value Probl*, **2014:105**, (2014).
- Chen H. and He Z., Variational approach to some damped Dirichlet problems with impulses, *Math. Methods Appl. Sci* **36/18** (2013), 2564–2575.
- Drábek P. and Langerová M., Quasilinear boundary value problem with impulses: variational approach to resonance problem, *Bound. Value Probl*, **2014:64**, (2014)
- Nieto J.J. and O'Regan D., Variational approach to impulsive differential equations, *Nonlinear Anal., RWA* **70** (2009), 680–690.

Chen-He obtained, by using the Mountain Pass theorem, for the following damped Dirichlet problem with impulses

$$\begin{cases} -u''(t)+q(t)u(t)=f(t,u(t)) \;\; t\in[0,T], \;\; t\neq t_j \\ u(0)=u(T)=0 \\ \Delta u'(t_j)=u'(t_j^+)-u'(t_j^-)=I_j(u(t_j)), \;\; j=1,2,...,n \end{cases} \tag{1}$$

the existence of at least one solution.

For a Dirichlet boundary value problem involving the one dimensional $p-$Laplace operator of type

$$\begin{cases} -(|u'(x)|^{p-2}u'(x))' - \lambda|u(x)|^{p-2}u(x) = f(x) \;\; x \in [0,1], \;\; t \neq t_j \\ \\ u(0) = u(1) = 0 \\ \\ \Delta_p u'(t_j) = |u'(t_j^+)|^{p-2}u'(t_j^+) - |u'(t_j^-)|^{p-2}u'(t_j^-) = I_j(u(t_j)), \;\; j = 1,2,...,n \end{cases} \tag{2}$$

Drábek-Langerová proved the existence of at least one solution .The proof is variational and relies on the linking theorem. .

Nieto and O'Regan obtained, by using the Mountain Pass theorem, for the following nonlinear Dirichlet problem with impulses

$$\begin{cases} -u''(t) + \lambda u(t) = f(t, u(t)) \ \ a.e.\, t \in [0, T] \\ u(0) = u(T) = 0 \\ \Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = I_j(u(t_j)), \ \ j = 1, 2, ..., n \end{cases} \tag{3}$$

the existence of at least one solution.

The solution

A function u is called a solution of $(S_{\lambda,\mu})$ if

- $u \in \left\{ w \in C([0,T] : \ w_{|[t_j,t_{j+1}]} \in H^2([t_j,t_{j+1}]), \forall j = 0,...,n \right\}$

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- satisfy the impulsive conditions $\Delta u'(t_j) = \mu I_j(u(t_j))$
- the boundary conditions $u(0) = u(T) = 0$ are verified.

If f is continuous then the classical solution $u \in C^2([t_j,t_{j+1}]), j = 0,1,...n$ and satisfies the equation in $(S_{\lambda,\mu})$ for all $t \in [0,T] \setminus \{t_1,...t_n\}$.

In the Sobolev space $H_0^1(0,T)$, consider the inner product

$$(u,v) = \int_0^T p(t)u'(t)v'(t)\,dt + \int_0^T q(t)u(t)v(t)\,dt\ ,$$

which induces the norm

$$\|u\| = \left(\int_0^T p(t)(u'(t))^2\,dt + \int_0^T q(t)(u(t))^2\,dt\right)^{1/2}$$

where $p \in C^1([0,T],]0,+\infty[)$, $q \in L^\infty([0,T])$ with $\operatorname{ess\,inf}_{t\in[0,T]} q(t) \geq 0$.

A function $u \in H^1_0(0,T)$ is said to be a weak solution of $(S_{\lambda,\mu})$ if u satisfies

$$\begin{aligned}\int_0^T p(t)u'(t)v'(t)\,dt \quad &+ \quad \int_0^T q(t)u(t)v(t)dt \\ &-\lambda\int_0^T f(t,u(t))v(t)\,dt + \mu\sum_{j=1}^n p(t_j)I_j(u(t_j))v(t_j) = 0,\end{aligned} \tag{4}$$

for any $v \in H^1_0(0,T)$.

Lemma

If u is a is weak solution of $(S_{\lambda,\mu})$, then $u \in H_0^1(0,T)$ is a classical solution of $(S_{\lambda,\mu})$.

The nonlinearity

$$f : [0,1] \times \mathbb{R} \to \mathbb{R}$$

(a) $t \to f(t,x)$ is measurable $\forall\, x \in \mathbb{R}$;

(b) $x \to f(t,x)$ is continuous a.e. $t \in [0,T]$;

(c) $\forall\, \rho > 0 \quad \exists \quad l_\rho \in L^1([0,T])$ t.c.

$$\sup_{|x| \leq \rho} |f(t,x)| \leq l_\rho(t) \quad \text{a.e.} \;\; t \in [0,T] \,.$$

Lemma

Assume that $f(t,x) \geq 0$ *for all* $(t,x) \in [0,T] \times \mathbb{R}$ *and* $I_j(x) \leq 0$ *for all* $x \in \mathbb{R}$, $j = 1, ...n$. *If* u *is a classical solution of* $(S_{\lambda,\mu})$, *then* $u(t) \geq 0$ *for all* $t \in [0,T]$.

The abstract tools

Our results have been obtained by applying two abstract critical point theorems obtained in

Bonanno G.,
Relations between the mountain pass theorem and local minima,
Adv. Nonlinear Anal., Volume 1, (2012) 205–220

Bonanno G. and D'Aguì G.,
Two non-zero solutions for elliptic Dirichlet problems,
Zeitschrift für Analysis und ihre Anwendungen, Volume 35, (2016) 449–464

$$E_\lambda = \Phi - \lambda\Psi$$

- *Assume that there are* $r \in \mathbb{R}$ *and* $\tilde{u} \in X$, *with* $0 < \Phi(\tilde{u}) < r$, *such that*

$$\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \tag{2.1}$$

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$$\frac{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \qquad (2.1)$$

- $\lambda \in \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)} \right[$, *the functional* $E_\lambda = \Phi - \lambda\Psi$ *satisfies* (PS)*-condition and it is unbounded from below.*

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- Then, for each $\lambda \in \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup\limits_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)} \right[$, the functional E_λ admits at least two non-zero critical points $u_{\lambda,1}, u_{\lambda,2}$ such that $E_\lambda(u_{\lambda,1}) < 0 < E_\lambda(u_{\lambda,2})$.

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- Then, for each $\lambda \in \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}\right[$, there is $u_{0,\lambda} \in \Phi^{-1}(]0,r[)$ such that $E_\lambda(u_{0,\lambda}) \leq E_\lambda(u),\ \forall u \in \Phi^{-1}(]0,r[)$ and $E'_\lambda u_{0,\lambda} = 0$.

Definition (1)

Let $(X, \|\cdot\|)$, we say that the functional E_λ satisfies the *Palais-Smale condition*, in short (PS)-condition, if any sequence $\{u_k\} \subseteq X$ such that

- $\{E(u_k)\}$ is bounded;
- $\lim_{k\to+\infty} \|E'_\lambda(u_k)\|_{X^*} = 0;$

has a convergent subsequence.

Definition (2)

Let $(X, \|\cdot\|)$ be a real Banach space and $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals; put $E = \Phi - \Psi$ and fix $r \in \mathbb{R}$. We say that the functional E satisfies the *Palais-Smale condition cut off upper r*, in short $(PS)^{[r]}$-condition, if any sequence $\{u_k\} \subseteq X$ such that

- $\{E(u_k)\}$ is bounded;
- $\lim_{k \to +\infty} \|E'(u_k)\|_{X^*} = 0$;
- $\Phi(u_k) < r$ for each $k \in \mathbb{N}$;

has a convergent subsequence.

If $r = +\infty$ it coincides with the classical (PS)-condition.

Definition (3)

We say that $u \in X$ is a critical point of E_λ when $E'_\lambda(u) = 0_{X^*}$, that is, $E'_\lambda(u)(v) = 0 \ \forall v \in X$.

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As consequence, the critical points of E_λ are the weak solutions of the problem $(S_{\lambda,\mu})$.

The first result guarantees the existence of one non trivial solution to problem $(S_{\lambda,\mu})$.

Assumptions

1. (h_1) there exist constants α, $\beta > 0$ and $\sigma \in [0,1[$ such that
$$|I_j(x)| \leq \alpha + \beta |x|^{\sigma} \quad \text{for any } x \in \mathbb{R},\ j = 1, 2, ..., n\,.$$
2. $\tilde{p} := \sum_{j=1}^{n} p(t_j)$,
3. $k := \frac{6p^*}{12\|p\|_\infty + T^2\|q\|_\infty}$,
4. $\Gamma_c := \frac{\alpha}{c} + \left(\frac{\beta}{\sigma+1}\right) c^{\sigma-1}$, where α, β, σ are given by (h_1) and c is a positive constant.

Theorem (5)

Suppose that (h_1) *is satisfied. Furthermore, assume that there exist two positive constants* c, d*, with* $d < c$*, such that*

(a_1) $F(t,\xi) \geq 0$ *for all* $(t,\xi) \in \left([0, \frac{T}{4}] \cup [\frac{3T}{4}, T]\right) \times [0,d]$;

(a_2) $$\frac{\int_0^T \max_{|\xi| \leq c} F(t,\xi)\, dt}{c^2} < k \frac{\int_{T/4}^{3T/4} F(t,d)\, dt}{d^2}.$$

Then, for every

$$\lambda \in \Lambda := \left] \frac{2p^*}{kT} \frac{d^2}{\int_{T/4}^{3T/4} F(t,d)\, dt}, \frac{2p^*}{T} \frac{c^2}{\int_0^T \max_{|\xi|\leq c} F(t,\xi)\, dt} \right[,$$

there exists $\delta :=$

$$\frac{1}{T\tilde{p}} \min \left\{ \frac{2p^* c^2 - \lambda T \int_0^T \max_{|\xi|\leq c} F(t,\xi)\, dt}{c^2 \Gamma_c}, \frac{k\lambda T \int_{T/4}^{3T/4} F(t,d)\, dt - 2p^* d^2}{d^2 \Gamma_{(d/\sqrt{k})}} \right\}$$

such that, for each $\mu \in]0,\delta[$ *the problem* $(S_{\lambda,\mu})$ *has at least* **one** *non-trivial classical solution.*

Chen-He, in order to obtain one solution, assume a conditon on F at infinity:

there exists a constant a such that

$$\lim_{|u|\to+\infty} \frac{\max_{[0,T]} F(t,u)}{|u|^2} \leq a$$

Let $g : \mathbb{R} \to]0, +\infty[$ be a continuous function, put

$$G(\xi) = \int_0^{\xi} g(x)\, dx$$

for all $\xi \in \mathbb{R}$, and $k^* := \dfrac{6(e^{-T/4} - e^{-3T/4})}{(12 + T^2)(e^T - 1)}$.

Corollary (1)

Suppose that (h_1) *is satisfied. Furthermore, assume that there exist two positive constants* c, d, *with* $d < c$, *such that*

(b) $\dfrac{G(c)}{c^2} < k^* \dfrac{G(d)}{d^2}$.

Then, for every $\lambda \in \left] \dfrac{2}{k^* T(e^T - 1)} \dfrac{d^2}{G(d)}, \dfrac{2}{T(e^T - 1)} \dfrac{c^2}{G(c)} \right[$ *, there exists*

$$\delta = \frac{1}{T} \min \left\{ \frac{2e^{-T} c^2 - \lambda T(1 - e^{-T}) G(c)}{c^2 \Gamma_c}, \frac{k^* \lambda T(e^{-T/4} - e^{-3T/4}) G(d) - 2e^{-T} d^2}{d^2 \Gamma_{(d/\sqrt{k})}} \right\}$$

such that,

for each $\mu \in]0, \delta[$ *the problem*

$$\begin{cases} -u''(t) + u'(t) + u(t) = \lambda g(u(t)) \quad t \in [0, T], \quad t \neq t_j \\ \\ u(0) = u(T) = 0 \\ \\ \Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = \mu I_j(u(t_j)), \quad j = 1, 2, ..., n \end{cases} \tag{5}$$

has at least one non-trivial classical solution.

To establish the existence of two solutions to problem $(S_{\lambda,\mu})$ we assume that the nonlinear term f satisfies the Ambrosetti-Rabinowitz condition:

assume that there exist $\nu > 2$ and $R > 0$ such that

$$0 < \nu F(t,\xi) \leq \xi f(t,\xi) \quad \forall\, t \in [0,T],\ |\xi| \geq R\,. \tag{AR}$$

Theorem (6)

Suppose that (h_1), (a_1), (a_2) *and* (AR) *are satisfied.*

Then, for every $\lambda \in \Lambda$, *there exists* δ, *where* Λ *and* δ *are introduced in the statement of Theorem 5, such that, for each* $\mu \in]0, \delta[$, *the problem* $(S_{\lambda,\mu})$ *has at least* **two** *distinct non-trivial solutions.*

Subquadratic behaviour at zero

Theorem (7)

Let $f : \mathbb{R} \to \mathbb{R}$ *be a continuous and non-negative function. Assume* (AR) *and* $\lim_{\xi \to 0^+} \frac{f(\xi)}{\xi} = +\infty$. *Put* $\lambda^* = \frac{2}{T^2} \sup_{c>0} \frac{c^2}{F(c)}$.
Then, for all $I_j : \mathbb{R} \to \mathbb{R}$ *continuous functions,* $j = 1, 2, ..., n$, *satisfying* (h_1), *the following problem*

$$\begin{cases} -u''(t) + u(t) = \lambda f(u(t)) \quad t \in [0, T], \quad t \neq t_j \\ u(0) = u(T) = 0 \\ \Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = \mu I_j(u(t_j)), \quad j = 1, 2, ..., n \end{cases} \qquad (S_{\lambda,\mu})$$

admits at least two non-trivial solutions for all $\lambda \in]0, \lambda^*[$,
$\mu \in \left]0, \frac{1}{T\Gamma_c} \left(2 - \lambda T^2 \frac{F(c)}{c^2}\right)\right[$.

Example 4.

Consider the following problem:

$$\begin{cases} -u''(t) + u(t) = \lambda f(u(t)) \;\; a.e. \;\; [0,1] \\ u(0) = u(1) = 0 \\ \Delta u'(t_1) = u'(t_1^+) - u'(t_1^-) = \mu I(u(t_1)) \end{cases} \qquad (6)$$

where $|I(x)| \leq \frac{3}{2}\sqrt{|x|}$ and

$$f(x) = \begin{cases} 12x - x^2 \;\; if \;\; 0 < x < 12 \\ 0 \qquad\qquad otherwise. \end{cases}$$

Then, $F(x) = \begin{cases} 0 & if \;\; x < 0 \\ 6x^2 - \frac{x^3}{3} & if \;\; 0 \leq x \leq 12 \\ 288 & if \;\; x > 12. \end{cases}$

Let $c = 20$ and $d = 6$. It is easily verified that all conditions in Theorem 5 are satisfied. Then, problem (6) has at least one non-trivial solution for each $\lambda \in \left]\frac{13}{6}, \frac{25}{9}\right[=]2.1667, 2.7778[$ and for each $\mu \in]0, \delta[$ where

$$\delta = \min\left\{\sqrt{5}\left(4 - \frac{36}{25}\lambda\right), 2\sqrt[4]{78}\left(\frac{6\lambda}{13} - 1\right)\right\} = h(\lambda).$$

For instance, for $\lambda = 2.4$ we have $\delta = 0.5724$. If $\lambda = 2.6$ then $\delta =$0.6400. The graph of the function h is presented on Figure 1. Note that the equation $\sqrt{5}\left(4 - \frac{36}{25}\lambda\right) = 2\sqrt[4]{78}\left(\frac{6\lambda}{13} - 1\right)$ has unique solution $\lambda_0 = 2.4966$ and the maximum value of $h(\lambda)$ in the interval]2.1667, 2.7778[is $\delta_0 = h(\lambda_0) = 0.9053$.

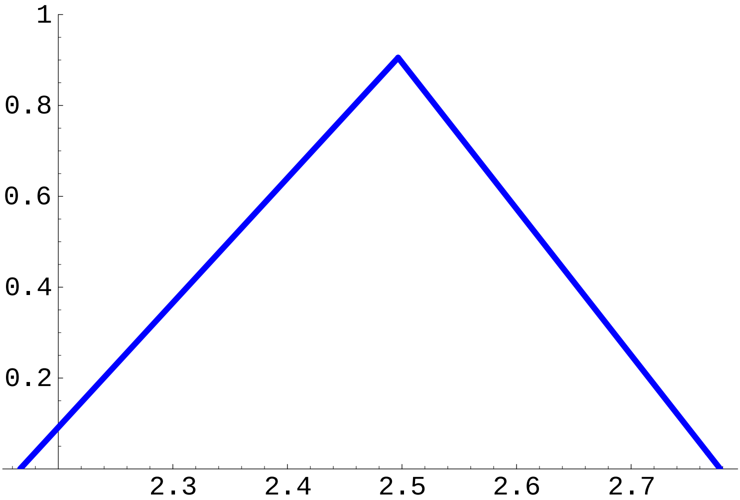

Figura: 1

Example 5

Owing to Theorem 7, the following superlinear problem:

$$\begin{cases} -u''(t)+u(t) = \sqrt[3]{|u(t)|}+|u(t)|^3 \;\; a.e. \;\; [0,1] \\ \\ u(0)=u(1)=0 \\ \\ \Delta u'(t_1) = u'(t_1^+)-u'(t_1^-) = \log(|u(t_1)|+1) \end{cases} \tag{7}$$

admits two non-zero solutions. Indeed, in this case we can choose $\lambda = \mu = 1$.

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