Multiplicity of ground states for the scalar curvature equation

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Existence & multiplicity of positive solutions with fast decay

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K is a **reciprocally symmetric**, C^1 function of $|x| = r$:

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- **K decreases in $(0,1)$** and **increases in $(1,\infty)$**

Multiplicity results are ensured when $\overline{\mathbf{K}}/\underline{\mathbf{K}} < 1 + \varepsilon$, ε small

Scalar curvature equation

$$\Delta u + K(|x|)\, u^{\frac{n+2}{n-2}} = 0 \qquad \text{(E)}$$

The existence of positive solutions with fast decay

is equivalent to

the existence of a metric in $\mathbb{R}^n$
with scalar curvature K
conformally equivalent to the Euclidean metric

G. BIANCHI, Comm. Partial Diff. Eqns. (1996-1997)

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each solution of **(E)** is **radially symmetric**

Radial symmetry

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Hence, we equivalently study the radial singular equation:

$$(u'(r)\, r^{n-1})' + K(r)\, r^{n-1}\, u^{\frac{n+2}{n-2}} = 0\,, \qquad r \in (0, \infty), \qquad \mathbf{(P)}$$

obtained by setting $r = |x|$

Radial problem

We concentrate on problem (P):

$$(u'(r)\, r^{n-1})' + K(r)\, r^{n-1}\, u^{\frac{n+2}{n-2}} = 0\,, \qquad r \in (0, \infty),$$

We are interested in REGULAR SOLUTIONS

$$u(0) = d > 0, \qquad u'(0) = 0$$

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FAST DECAY SOLUTIONS

$$\lim_{r\to+\infty} u(r) r^{n-2} = L > 0$$

C.C. CHEN - C.S. LIN, *C.P.D.E.* (1999)

Problem: $(u'(r)\, r^{n-1})' + K(r)\, r^{n-1}\, u^{\frac{n+2}{n-2}} = 0, \quad K \in C^1$

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equation (**P**) admits at least ℓ **G.S. with fast decay**

$u_1, \ldots, u_\ell$:

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$\mathbf{u_j(r)\, r^{\frac{n-2}{2}}}$ has *j local maxima* and $(j-1)$ *local minima*.

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Perturbative result

Our multiplicity result: **Theorem 1**

Assume (K_0) - (K_1) - (K_2) - (K_3)

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In particular, we find the following approximations from below:

n	ε_1	ε_2	ε_3	ε_4	ε_5	ε_6	ε_7	ε_8
3	2	0.910	0.584	0.429	0.339	0.280	0.238	0.207
4	1	0.5	0.333	0.25	0.2	0.166	0.142	0.125
5	0.666	0.347	0.235	0.178	0.143	0.119	0.103	0.090
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The explicit expression of ε_1 and ε_2 is

$$\varepsilon_1 = \frac{2}{n-2}, \qquad \varepsilon_2 = \frac{2}{n}\left[\left(\frac{n}{n-2}\right)^{\frac{n-2}{2}} - 1\right]^{-1}.$$

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ε_3 solves the equation

$$[\mathcal{X}^q(\varepsilon_3) + \mathcal{W}(\varepsilon_3)]^{\frac{2}{q}} = \mathcal{X}^2(\varepsilon_3) + \frac{2}{q}\mathcal{W}(\varepsilon_3),$$

where

$$\mathcal{X}(\varepsilon_3) = \left(\frac{q}{2(\varepsilon_3+1)}\right)^{\frac{1}{q-2}}, \ \mathcal{W}(\varepsilon_3) = 1 + \frac{1}{\varepsilon_3}\left(1 - \frac{q}{2}\right), \ q = \frac{2n}{n-2}.$$

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Problem: $(u'(r)\, r^{n-1})' + K(r)\, r^{n-1}\, u^{\frac{n+2}{n-2}} = 0, \quad K \in C^1$

Replace

(K_0) $$K(r) = [1 + \varepsilon k(r)]\,, \quad 0 < k(r) < 1$$

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Assume $(K_0)'$ - (K_1) - (K_2) - (K_3)

$\forall\, \ell \in \mathbb{N} \quad \exists$ a **computable** $\varepsilon_\ell > 0$:

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Our existence result: **Theorem 2**

Assume

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Moreover, suppose that

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THEN

Equation (**P**) admits at least **one G.S. with fast decay**

References

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EXISTENCE IN PRESENCE OF A MAXIMUM OF K

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- **The scalar curvature equation admits nonradial solutions**

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EXISTENCE IN PRESENCE OF A MAXIMUM OF **K**

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$$\mathbf{K(r) = A + Br^s} \text{ at } \mathbf{r = 0}, \quad \mathbf{K(r) = C + Dr^l} \text{ at } \mathbf{r = \infty}$$

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- **Equation (P) admits a UNIQUE G.S. with f.d.** when

$$\frac{\mathbf{rK'(r)}}{\mathbf{K(r)}} \ \textbf{is decreasing} \text{ in } r \in (0, \infty), \text{ no constant}$$

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$$A, C \geq 0 \qquad B, D > 0 \qquad \mathbf{l < 0 < s}$$

- **Equation (P) admits a UNIQUE G.S. with f.d.** when

$$\frac{\mathbf{rK'(r)}}{\mathbf{K(r)}} \ \textbf{is decreasing} \text{ in } r \in (0, \infty), \text{no constant}$$

Uniqueness when **K** has a **unique maximum**

EXISTENCE IN PRESENCE OF A MAXIMUM OF **K**

- **The scalar curvature equation admits nonradial solutions**
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E. Yanagida, S. Yotsutani, *A.R.M.A.* (1993)
N. Kawano, E. Yanagida, S. Yotsutani, *Funkl. Ekvac.* (1993)
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F.D., M. Franca, *Comm. Math. Phys.* (2016)

CLASSIFICATION AND UNIQUENESS
IN PRESENCE OF A UNIQUE MAXIMUM OF $\mathbf{K}$

Assume that $\frac{rK'(r)}{K(r)}$ is decreasing in $r \in (0, \infty)$.

Denote by $\mathbf{u(r;d)}$ the **regular** solution of $(\mathbf{P})$:

$$\mathbf{u(0) = d} > 0, \qquad \mathbf{u'(0) = 0}.$$

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MULTIPLICITY IN PRESENCE OF A MAXIMUM OF K

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MULTIPLICITY IN PRESENCE OF A MAXIMUM OF **K**

- **If K has several critical points,**

the number of solutions depends on the number of critical points

G. Bianchi, H. Egnell, *Prog. Nonlinear Diff. Eqns Appl.* (1992)
R. Johnson, X.B. Pan, Y.F. Yi, *Indiana Math. J.* (1994)
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- **If K has a local maximum,**

the s.c.e. admits infinitely many non-radial positive solutions

J. Wei, S. Yan, *J. Funct. Anal.* (2010)

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IN PRESENCE OF A MINIMUM OF K

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Delicate and complicated situation!!!

$\Longrightarrow$ **From non-existence to perturbative multiplicity results**

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Non-existence of G.S. with f.d.

- G. Bianchi, H. Egnell, *Math. Z.* (1992)

Existence of G.S. with f.d.

- G. Bianchi, H. Egnell, *Math. Z.* (1992) & *A.R.M.A* (1993)

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Multiplicity of G.S. with f.d. — Perturbative results

- C.C. Chen, C.S. Lin, *Comm. Partial Diff. Eqns* (1999)

$$\mathbf{K}(|\mathbf{x}|) = \mathbf{1} + \varepsilon \mathbf{k}(|\mathbf{x}|), \quad 0 < k(|x|) < 1, \quad k \in C^1,$$

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$$k(r) \le -Ar^{\mu} \text{ at } r = 0, \quad k(r) \le -Ar^{-\mu} \text{ at } r = +\infty, \quad A, \mu > 0,$$

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- I. Flores, M. Franca, *J.D.E.* (2015)

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Methods & Fowler transformation

Our approach is based on

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Fowler transformation:

$$x(t) = u(r)r^{\alpha}, \quad y(t) = \alpha u(r)r^{\alpha} + u'(r)\, r^{\alpha+1}$$

$$\alpha = \frac{n-2}{2}, \quad r = e^t, \quad \mathcal{K}(t) = K(e^t)$$

R.H. Fowler, *Quart. J. Math.* (1931).

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By Fowler transformation, (P) is equivalent to

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \alpha^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -\mathcal{K}(t)\, x^{\frac{n+2}{n-2}} \end{pmatrix} \qquad (S)$$

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An easy characterization of the solutions

$\mathbf{x(t) > 0}$ **in** $\mathbf{(-\infty, 0)}$, $\mathbf{y(0) = 0}$ $\Longrightarrow$ **u(r) is a G.S. with f.d.**

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$\mathbf{x(t) > 0}$ **in** $\mathbf{(-\infty, 0)}$, $\mathbf{y(0) = 0}$ $\Longrightarrow$ **u(r) is a G.S. with f.d.**

$\Longrightarrow$ $x(t)$ is even, $y(t)$ is odd

Idea of the proof

Let $u(r;d)$ be the solution of (P): $u(0) = d$, $u'(0) = 0$

$(x(t;d), y(t;d))$ denotes the corresponding trajectory of (S)

Denote by $T_\ell(d)$ the ℓ^{th} zero of $y(t;d)$

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$\mathbf{u(r)}$ is a G.S. with f.d. if

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$$\Updownarrow$$

$$\exists \ell \in \mathbb{N},\ \exists \mathbf{d^* > 0} : \quad \mathbf{T_\ell(d^*) = 0}$$

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x has ℓ local maxima and $(\ell - 1)$ local minima

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$\mathbf{y}$ has exactly $\mathbf{(2\ell - 1)}$ zeroes

Shooting argument

Our problem:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \alpha^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -\mathcal{K}(t)\, x^{\frac{n+2}{n-2}} \end{pmatrix}$$

$\forall\, \ell \in \mathbb{N}$, we look for $\mathbf{d^*} > \mathbf{0}$: $\mathbf{T}_\ell(\mathbf{d^*}) = \mathbf{0}$

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Assume $(\mathbf{K_0})$. Suppose that $\mathbf{0} < \varepsilon \leq \varepsilon_\mathbf{1}$.

$\exists$ a small $d_\ell > 0$: $\mathbf{T}_\ell(\mathbf{d}_\ell) > \mathbf{0}$

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$\exists$ a small $d_\ell > 0$: $\mathbf{T_\ell(d_\ell) > 0}$

Assume $(\mathbf{K_0})$-$(\mathbf{K_2})$-$(\mathbf{K_3})$.

$\exists$ a large $D_\ell > 0$: $\mathbf{T_\ell(d_\ell) < 0}$

C.C. CHEN, C.S. LIN, *Comm. PDEs* & *Math. Ann.* (1999)

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The thesis would follow from the continuity of $\mathbf{T}_\ell$

Continuity of $\mathbf{T}_\ell$

Assume $(\mathbf{K}_0)$-$(\mathbf{K}_2)$.

Suppose that $\mathbf{0} < \varepsilon \leq \varepsilon_\ell$, where ε_ℓ is a **computable** constant.

$$T_\ell(d) \leq 0 \quad \Longrightarrow \quad \mathbf{T}_\ell\mathbf{(d)} \text{ is continuous}$$

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$$T_\ell(d) \leq 0 \implies \mathbf{T_\ell(d)\ \text{is continuous}}$$

$$\boxed{\text{Shooting argument}} \implies \exists\, d_\ell^* \in (d_\ell, D_\ell) : \mathbf{T_\ell(d_\ell^*) = 0}$$

□

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Shooting argument $\Longrightarrow \exists\, d^*_\ell \in (d_\ell, D_\ell) :$ $\mathbf{T}_\ell(\mathbf{d}^*_\ell) = \mathbf{0}$

□

The proof of continuity of $\mathbf{T}_\ell$ is based on a **barrier argument** ensuring that $(x(t;d), y(t;d))$ intersects the x-axis transversally

Barrier sets

Consider the autonomous system, by setting $\mathcal{K}(t) \equiv 1 + c$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \alpha^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -(1+c)\, x^{\frac{n+2}{n-2}} \end{pmatrix}.$$

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It admits a unique critical point $\mathbf{P}^*(c) = (P_x^*(c), 0)$ s.t. $P_x^*(c) > 0$

$$P_x^*(c) = \left(\frac{(n-2)^2}{4(c+1)} \right)^{\frac{n-2}{4}}$$

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It admits a unique critical point $\mathbf{P}^*(c) = (P_x^*(c), 0)$ s.t. $P_x^*(c) > 0$

Introduce the corresponding energy function

$$H_c(x,y) := \frac{y^2}{2} - \alpha^2 \frac{x^2}{2} + (1+c)\frac{x^q}{q}, \qquad \text{where } q := \frac{2n}{n-2}.$$

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Recalling that $1 < \mathcal{K}(t) < 1 + \varepsilon \implies 0 < c < \varepsilon$

We define a *spiral* γ rotating around $\mathbf{P}^*(\varepsilon)$ and $\mathbf{P}^*(0)$, for ε small

γ is made up by the level curves of H_ε in $\{y \geq 0\}$ and by the level curves of H_0 in $\{y \leq 0\}$

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Recalling that $1 < \mathcal{K}(t) < 1 + \varepsilon \implies 0 < c < \varepsilon$

We define a *spiral* γ rotating around $\mathbf{P}^*(\varepsilon)$ and $\mathbf{P}^*(0)$, for ε small

The *spiral* γ controls the behaviour of the trajectories (x,y) of (S):

$$\lim_{t \to -\infty} (x(t), y(t)) \longrightarrow (0,0).$$

Barrier sets

Definition

$\mathbf{A}_\ell := (A_\ell, 0)$ is the ℓ^{th} intersection between γ and the **x**-axis.

The *spiral* γ rotating around $\mathbf{P}^*(\varepsilon)$ and $\mathbf{P}^*(0)$ is well defined if

$$A_2 < \ldots < A_{2i} < \ldots < P_x^*(\varepsilon) < P_x^*(0) < \ldots < A_{2i+1} < \ldots < A_1$$

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The critical value ε_ℓ is the only value which satisfies

$\mathbf{A}_\ell = \mathbf{P}_\mathbf{x}^*(\varepsilon_\ell)$ **if ℓ is even,** $\mathbf{A}_\ell = \mathbf{P}_\mathbf{x}^*(0)$ **if ℓ is odd.**

GRAZIE!!!