Multiplicity of ground states for the scalar curvature equation

F.D., Matteo Franca & Andrea Sfecci

Ancona - September 28th, 2018

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The problem

A perturbative result

Our main results

References Non-existence results If K has a maximum If K has a minimum

Existence & multiplicity of positive solutions with fast decay

$$\Delta u + \mathcal{K}(|x|) u^{\frac{n+2}{n-2}} = 0 \qquad x \in \mathbb{R}^n, \quad n > 2$$

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$$\Delta u + \mathcal{K}(|x|) u^{\frac{n+2}{n-2}} = 0 \qquad x \in \mathbb{R}^n, \quad n > 2$$

u > 0

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u > 0

$$u(x) = O(|x|^{2-n})$$
 as $|x| \to +\infty$

Existence & multiplicity of positive solutions with fast decay

$$\Delta u + \mathcal{K}(|x|) u^{\frac{n+2}{n-2}} = 0 \qquad x \in \mathbb{R}^n, \quad n > 2$$

$$u > 0$$

 $u(x) = O(|x|^{2-n})$ as $|x| \to +\infty$

where

K is a **reciprocally symmetric**, C^1 function of |x| = r:

• 0 <
$$\underline{\mathbf{K}} \leq \mathbf{K}(\mathbf{r}) \leq \overline{\mathbf{K}}, \quad \forall r > 0$$

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$$u > 0$$

 $u(x) = O(|x|^{2-n})$ as $|x| \to +\infty$

where

K is a **reciprocally symmetric**, C^1 function of |x| = r:

•
$$0 < \underline{K} \leq K(r) \leq \overline{K}, \quad \forall r > 0$$

• K decreases in (0,1) and increases in $(1,\infty)$

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Existence & multiplicity of positive solutions with fast decay

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•
$$0 < \underline{\mathbf{K}} \leq \mathbf{K}(\mathbf{r}) \leq \overline{\mathbf{K}}, \quad \forall r > 0$$

• K decreases in (0,1) and increases in $(1,\infty)$

Multiplicity results are ensured when $\overline{K}/\underline{K} < \overline{1 + \varepsilon}$, ε small

The problem

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References Non-existence results If K has a maximum If K has a minimum

$$\Delta u + K(|x|) u^{\frac{n+2}{n-2}} = 0$$
 (E)

The existence of positive solutions with fast decay

is equivalent to

the existence of a metric in \mathbb{R}^n

with scalar curvature K

conformally equivalent to the Euclidean metric

The problem

A perturbative result

Our main results

References Non-existence results If K has a maximum If K has a minimum

Radial symmetry

G. BIANCHI, Comm. Partial Diff. Eqns. (1996-1997)

• K decreases in (0, 1) and increases in $(1, \infty)$

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each solution of (E) is radially symmetric

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Radial symmetry

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• K decreases in (0, 1) and increases in $(1, \infty)$

11

each solution of (E) is radially symmetric

Hence, we equivalently study the radial singular equation:

$$(u'(r) r^{n-1})' + K(r) r^{n-1} u^{\frac{n+2}{n-2}} = 0, \quad r \in (0,\infty),$$
 (P)

obtained by setti

ing
$$r = |x|$$

Non-existence results If K has a maximum

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Radial problem

We concentrate on problem (P):

$$(u'(r) r^{n-1})' + K(r) r^{n-1} u^{\frac{n+2}{n-2}} = 0, \quad r \in (0,\infty),$$

We are interested in REGULAR SOLUTIONS

$$u(0) = d > 0, \qquad u'(0) = 0$$

GROUND STATE SOLUTIONS

u is a regular solution defined in $[0, +\infty)$: $\lim_{r\to\infty} u(r) = 0$

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We concentrate on problem (P):

$$(u'(r) r^{n-1})' + K(r) r^{n-1} u^{\frac{n+2}{n-2}} = 0, \quad r \in (0,\infty),$$

We are interested in REGULAR SOLUTIONS

$$u(0) = d > 0, \qquad u'(0) = 0$$

GROUND STATE SOLUTIONS

u is a regular solution defined in $[0, +\infty)$: $\lim_{r\to\infty} u(r) = 0$

FAST DECAY SOLUTIONS

$$\lim_{r\to+\infty}u(r)r^{n-2}=L>0$$

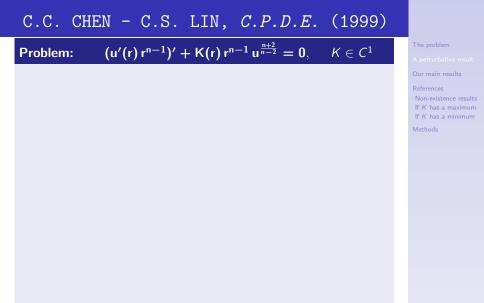
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The problem

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References Non-existence results If K has a maximum If K has a minimum



C.C. CHEN - C.S. LIN, C.P.D.E. (1999)
Problem:
$$(u'(r) r^{n-1})' + K(r) r^{n-1} u^{\frac{n+2}{n-2}} = 0$$
, $K \in C^1$
(K₀)
 $K(r) = [1 + \varepsilon k(r)]$, $0 < k(r) < 1$
If K has a mainum If K has a mainum If K has a mainum

C.C. C	HEN - C.S. LIN, <i>C.P.D.E</i> . (1999)	
Problem:	$(u'(r) r^{n-1})' + K(r) r^{n-1} u^{\frac{n+2}{n-2}} = 0, K \in C^1$	The problem A perturbative result
(K ₀)	$K(r) = [1 + \varepsilon k(r)], 0 < k(r) < 1$	Our main results References Non-existence results If K has a maximum If K has a minimum
(K ₁)	$\mathcal{K}(r) = \mathcal{K}\left(rac{1}{r} ight) ext{for } 0 < r \leq 1$	Methods

C.C. C	HEN - C.S. LIN, <i>C.P.D.E.</i> (1999)	
Problem:	$(u'(r) r^{n-1})' + K(r) r^{n-1} u^{\frac{n+2}{n-2}} = 0, K \in C^1$	The problem A perturbative result
(K ₀)	K (r) = [1 + ε k (r)], 0 < k(r) < 1	Our main results References Non-existence results If K has a maximum If K has a minimum
(K ₁)	$\mathcal{K}(r) = \mathcal{K}\left(rac{1}{r} ight) ext{for } 0 < r \leq 1$	Methods
(K ₂)	$K'(r) \leq 0 \ \ for \ 0 < r \leq 1, K'(r) \not\equiv 0$	

C.C. C	HEN - C.S. LIN, <i>C.P.D.E.</i> (1999)	
Problem:	$(u'(r) r^{n-1})' + K(r) r^{n-1} u^{\frac{n+2}{n-2}} = 0, K \in C^1$	The problem A perturbative result
(K ₀)	$K(r) = [1 + \varepsilon k(r)], 0 < k(r) < 1$	Our main results References Non-existence results If K has a maximum If K has a minimum
(K ₁)	$K(r) = K\left(rac{1}{r} ight)$ for $0 < r \le 1$	Methods
(K ₂)	$K'(r) \leq 0 \text{ for } 0 < r \leq 1, K'(r) \neq 0$	
(K ₃)	$K(r) = K(0) - Ar^{I} + h(r), A > 0,$ $0 < I < \frac{n-2}{2}$	
	$\lim_{r\to 0} \frac{ \mathbf{h}(r) }{r^{i}} + \frac{ \mathbf{h}'(r) }{r^{i-1}} = 0.$	(°

C.C. CHEN - C.S. LIN, C.P.D.E. (1999)

Assume $(K_0) - (K_1) - (K_2) - (K_3)$

The problem

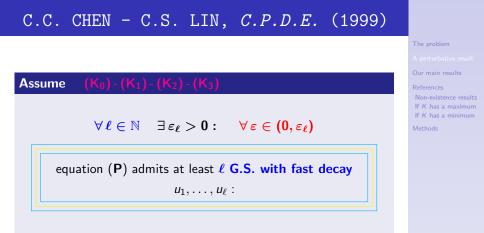
A perturbative result

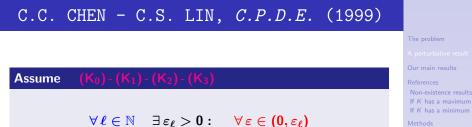
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equation (P) admits at least ℓ G.S. with fast decay

 U_1, \ldots, U_{ℓ} :

 $\mathbf{u}_{\mathbf{i}}(\mathbf{r}) \mathbf{r}^{\frac{n-2}{2}}$ has j local maxima and (j-1) local minima.

C.C. CHEN - C.S. LIN, C.P.D.E. (1999) Assume $(K_0) - (K_1) - (K_2) - (K_3)$ The problem A perturbative result Our main results References Non-existence results

 $\forall \ell \in \mathbb{N} \quad \exists \varepsilon_{\ell} > 0: \quad \forall \varepsilon \in (0, \varepsilon_{\ell})$

equation (P) admits at least ℓ G.S. with fast decay

 u_1,\ldots,u_ℓ :

 $u_j(r) r^{\frac{n-2}{2}}$ has <u>j</u> local maxima and (j-1) local minima.



Our multiplicity result: Theorem 1

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Assume $(K_0) - (K_1) - (K_2) - (K_3)$

All the constants ε_{ℓ} can be **explicitely computed**.

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All the constants ε_{ℓ} can be **explicitely computed**. In particular, we find the following approximations from below:

		ε_2						
		0.910						
		0.5						
		0.347						
6	0.5	0.266	0.182	0.138	0.111	0.093	0.080	0.070

The problem

A perturbative result

Our main results

References Non-existence results If *K* has a maximum

All the constants ε_{ℓ} can be **explicitely computed**. In particular, we find the following approximations from below:

		ε_2						
3	2	0.910	0.584	0.429	0.339	0.280	0.238	0.207
		0.5						
		0.347						
6	0.5	0.266	0.182	0.138	0.111	0.093	0.080	0.070

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The problem

A perturbative result

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All the constants ε_{ℓ} can be **explicitely computed**. In particular, we find the following approximations from below:

		ε_2						
3	2	0.910	0.584	0.429	0.339	0.280	0.238	0.207
		0.5						
5	0.666	0.347	0.235	0.178	0.143	0.119	0.103	0.090
6	0.5	0.266	0.182	0.138	0.111	0.093	0.080	0.070

The explicit expression of
$$\varepsilon_1$$
 and ε_2 is
 $\varepsilon_1 = \frac{2}{n-2}, \qquad \varepsilon_2 = \frac{2}{n} \left[\left(\frac{n}{n-2} \right)^{\frac{n-2}{2}} - 1 \right]^{-1}.$

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Our main results

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All the constants ε_{ℓ} can be **explicitely computed**. In particular, we find the following approximations from below:

						ε_{6}		
3	2	0.910	0.584	0.429	0.339	0.280	0.238	0.207
4	1	0.5	0.333	0.25	0.2	0.166	0.142	0.125
5	0.666	0.347	0.235	0.178	0.143	0.119	0.103	0.090
6	0.5	0.266	0.182	0.138	0.111	0.093	0.080	0.070

The explicit expression of
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Our main results

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All the constants ε_{ℓ} can be **explicitely computed**. In particular, we find the following approximations from below:

				ε_4				
				0.429				
				0.25				
				0.178				
6	0.5	0.266	0.182	0.138	0.111	0.093	0.080	0.070

$$\begin{split} \varepsilon_{3} \text{ solves the equation} \\ \left[\mathcal{X}^{q}(\varepsilon_{3}) + \mathcal{W}(\varepsilon_{3})\right]^{\frac{2}{q}} &= \mathcal{X}^{2}(\varepsilon_{3}) + \frac{2}{q}\mathcal{W}(\varepsilon_{3}) \,, \\ \text{where} \\ \mathcal{X}(\varepsilon_{3}) &= \left(\frac{q}{2(\varepsilon_{3}+1)}\right)^{\frac{1}{q-2}}, \ \mathcal{W}(\varepsilon_{3}) = 1 + \frac{1}{\varepsilon_{3}}\left(1 - \frac{q}{2}\right), \ q = \frac{2n}{n-2} \,. \end{split}$$

The problem

A perturbative result

Our main results

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All the constants ε_{ℓ} can be **explicitely computed**. In particular, we find the following approximations from below:

				ε_4				
3	2	0.910	0.584	0.429	0.339	0.280	0.238	0.207
4	1	0.5	0.333	0.25	0.2	0.166	0.142	0.125
				0.178				
6	0.5	0.266	0.182	0.138	0.111	0.093	0.080	0.070

$$\begin{split} \varepsilon_{3} \text{ solves the equation} \\ \left[\mathcal{X}^{q}(\varepsilon_{3}) + \mathcal{W}(\varepsilon_{3})\right]^{\frac{2}{q}} &= \mathcal{X}^{2}(\varepsilon_{3}) + \frac{2}{q}\mathcal{W}(\varepsilon_{3}) \,, \\ \text{where} \\ \mathcal{X}(\varepsilon_{3}) &= \left(\frac{q}{2(\varepsilon_{3}+1)}\right)^{\frac{1}{q-2}}, \ \mathcal{W}(\varepsilon_{3}) = 1 + \frac{1}{\varepsilon_{3}}\left(1 - \frac{q}{2}\right), \ q = \frac{2n}{n-2} \,. \end{split}$$

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$$n=4 \implies \varepsilon_\ell = rac{1}{\ell}$$

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All the constants ε_{ℓ} can be **explicitely computed**. In particular, we find the following approximations from below:

			ε_3					
			0.584					
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			0.235					
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$$n=4 \implies \varepsilon_\ell = rac{1}{\ell}$$

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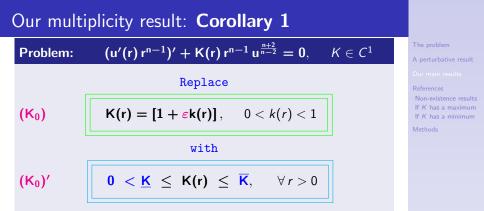
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A perturbative result

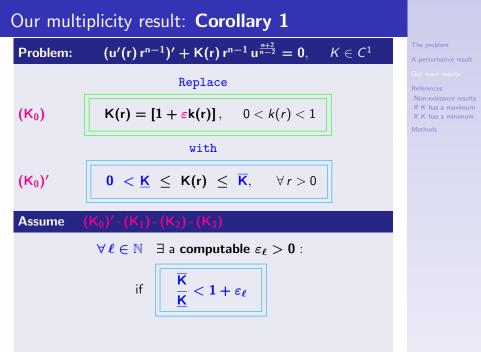
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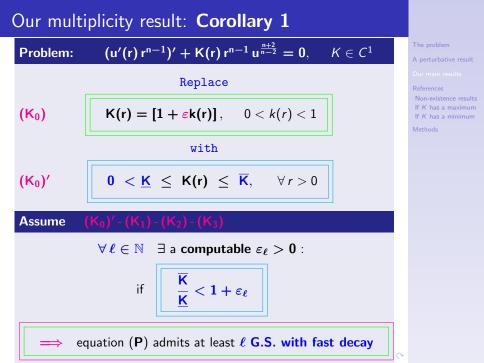
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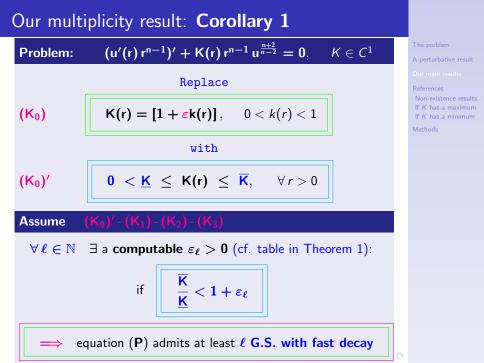


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Our existence result: Theorem 2

Assume

(K₀)
$$K(r) = [1 + \varepsilon k(r)], \quad 0 < k(r) < 1$$

(K₁) K(r) = K
$$\left(\frac{1}{r}\right)$$
 for $0 < r \le 1$

Remove

- (K₂) $K'(r) \le 0$ for $0 < r \le 1$, $K'(r) \ne 0$
- (K₃) $K(r) = K(0) Ar^{l} + h(r), \quad A > 0,$

$$0 < l < \frac{n-2}{2}, \qquad \lim_{r \to 0} \frac{|h(r)|}{r^{l}} + \frac{|h'(r)|}{r^{l-1}} = 0$$

The problem

A perturbative result

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Our existence result: Theorem 2

Assume

(K₀) K(r) =
$$[1 + \varepsilon k(r)], \quad 0 < k(r) < 1$$

(K₁) K(r) = K
$$\left(\frac{1}{r}\right)$$
 for $0 < r \le 1$

Moreover, suppose that

$$0 < \varepsilon \leq \varepsilon_1 := rac{2}{n-2}$$

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Our existence result: Theorem 2

Assume

(K₀) K(r) =
$$[1 + \varepsilon k(r)], \quad 0 < k(r) < 1$$

(K₁) K(r) = K
$$(\frac{1}{r})$$
 for $0 < r \le 1$

Moreover, suppose that

$$0 \ < \ arepsilon \ \le \ arepsilon_1 := rac{2}{n-2}$$

THEN

Equation (P) admits at least one G.S. with fast decay

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NON-EXISTENCE OF G.S. WITH F.D.

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NON-EXISTENCE OF G.S. WITH F.D.

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• K(r) < 0

G. BIANCHI, H. EGNELL, A.R.M.A (1993)

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NON-EXISTENCE OF G.S. WITH F.D.

• K(r) < 0

G. BIANCHI, H. EGNELL, A.R.M.A (1993)

• K(r) is monotone

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NON-EXISTENCE OF G.S. WITH F.D.

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K increasing \implies changing sign solutions

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NON-EXISTENCE OF G.S. WITH F.D.

• K(r) < 0

G. BIANCHI, H. EGNELL, A.R.M.A (1993)

• K(r) is monotone

- K increasing \implies changing sign solutions
- K decreasing \implies slow decay solutions

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G. BIANCHI, H. EGNELL, A.R.M.A (1993)

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K increasing \implies changing sign solutions

K decreasing \implies slow decay solutions

W.M. NI, Indiana Univ. Math. J. (1982) W.Y. DING, W.M. NI, Duke Math. J. (1985) Y. LI, W.M. NI, Duke Math. J. (1988) N. KAWANO, W.M. NI, S. YOTSUTANI, J.M. Soc. Japan (1990) A perturbative result Our main results References

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EXISTENCE IN PRESENCE OF A MAXIMUM OF K

• The scalar curvature equation admits nonradial solutions

E.S. NOUSSAIR, S. YAN, Nonlinear Anal. (2001)

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EXISTENCE IN PRESENCE OF A MAXIMUM OF K

- The scalar curvature equation admits nonradial solutions
- Equation (P) admits at least one G.S. with f.d. when $K(r) = A + Br^{s} \text{ at } r = 0, \quad K(r) = C + Dr^{l} \text{ at } r = \infty$

 $A, C \ge 0 \qquad B, D > 0 \qquad I < \mathbf{0} < \mathbf{s}$

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EXISTENCE IN PRESENCE OF A MAXIMUM OF K

- The scalar curvature equation admits nonradial solutions
- Equation (P) admits at least one G.S. with f.d. when

 $K(\mathbf{r}) = \mathbf{A} + \mathbf{B}\mathbf{r}^{\mathbf{s}} \text{ at } \mathbf{r} = \mathbf{0}, \quad K(\mathbf{r}) = \mathbf{C} + \mathbf{D}\mathbf{r}^{\mathbf{I}} \text{ at } \mathbf{r} = \infty$ $A, C > 0 \qquad B, D > 0 \qquad \mathbf{I} < \mathbf{0} < \mathbf{s}$

Y. LI, W.M. NI, Duke Math. J. (1988) K.S. CHENG, J.L. CHERN, J.D.E. (1992) E. YANAGIDA, S. YOTSUTANI, Nonlinear Anal. (1994) E. YANAGIDA, S. YOTSUTANI, J.D.E. (1995) A = C = 0 The problem A perturbative result Our main results References Non-existence results If K has a minimum Mathede

EXISTENCE IN PRESENCE OF A MAXIMUM OF K

- The scalar curvature equation admits nonradial solutions
- Equation (P) admits at least one G.S. with f.d. when

 $K(\mathbf{r}) = \mathbf{A} + \mathbf{B}\mathbf{r}^{\mathbf{s}} \text{ at } \mathbf{r} = \mathbf{0}, \quad K(\mathbf{r}) = \mathbf{C} + \mathbf{D}\mathbf{r}^{\mathbf{I}} \text{ at } \mathbf{r} = \infty$ $A, C > 0 \qquad B, D > 0 \qquad \mathbf{I} < \mathbf{0} < \mathbf{s}$

Y. LI, W.M. NI, Duke Math. J. (1988) K.S. CHENG, J.L. CHERN, J.D.E. (1992) E. YANAGIDA, S. YOTSUTANI, Nonlinear Anal. (1994) E. YANAGIDA, S. YOTSUTANI, J.D.E. (1995) C.S. LIN, S.S. LIN, Appl. Anal (1990) G. BIANCHI, H. EGNELL, Math. Z. (1992) G. BIANCHI, H. EGNELL, A.R.M.A (1993) E. YANAGIDA, S. YOTSUTANI, J.D.E. (1995) A, C > 0

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EXISTENCE IN PRESENCE OF A MAXIMUM OF K

- The scalar curvature equation admits nonradial solutions
- Equation (P) admits at least one G.S. with f.d. when

 $K(r) = A + Br^{s}$ at r = 0, $K(r) = C + Dr^{l}$ at $r = \infty$

 $A, C \ge 0 \qquad B, D > 0 \qquad I < \mathbf{0} < \mathbf{s}$

• Equation (P) admits a UNIQUE G.S. with f.d. when

 $\frac{\mathbf{r}\mathbf{K'(r)}}{\mathbf{K(r)}} \text{ is decreasing in } r \in (0,\infty), \text{no constant}$

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Uniqueness when K has a unique maximum

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- E. YANAGIDA, S. YOTSUTANI, A.R.M.A. (1993)
- N. KAWANO, E. YANAGIDA, S. YOTSUTANI, Funkl. Ekvac. (1993)
- E. YANAGIDA, S. YOTSUTANI, A.R.M.A. (1996)



EXISTENCE IN PRESENCE OF A MAXIMUM OF K

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• A <u>classification</u> of the regular solutions is available

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F.D., M. FRANCA, Comm. Math. Phys. (2016)

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CLASSIFICATION AND UNIQUENESS IN PRESENCE OF A UNIQUE MAXIMUM OF ${\sf K}$

Assume that
$$\frac{rK'(r)}{K(r)}$$
 is decreasing in $r \in (0, \infty)$.

Denote by $\mathbf{u}(\mathbf{r}; \mathbf{d})$ the regular solution of (P): $\mathbf{u}(\mathbf{0}) = \mathbf{d} > 0, \quad \mathbf{u}'(\mathbf{0}) = \mathbf{0}.$

$$\Rightarrow \exists$$
 a unique d^{*}: $u(r, d^*)$ is a G.S. with f.d.

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Assume that
$$\frac{rK'(r)}{K(r)}$$
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Denote by $\mathbf{u}(\mathbf{r}; \mathbf{d})$ the regular solution of (P): $\mathbf{u}(\mathbf{0}) = \mathbf{d} > 0, \quad \mathbf{u}'(\mathbf{0}) = \mathbf{0}.$

 $\implies \exists a unique d^*: u(r, d^*) is a G.S. with f.d.$ $u(r, d) is a G.S. with slow decay \quad \forall d < d^*$

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CLASSIFICATION AND UNIQUENESS IN PRESENCE OF A UNIQUE MAXIMUM OF K

Assume that
$$\frac{rK'(r)}{K(r)}$$
 is decreasing in $r \in (0, \infty)$.

Denote by $\mathbf{u}(\mathbf{r}; \mathbf{d})$ the regular solution of (P): $\mathbf{u}(\mathbf{0}) = \mathbf{d} > 0, \quad \mathbf{u}'(\mathbf{0}) = \mathbf{0}.$

 $\implies \exists a unique d^*: u(r, d^*) \text{ is a G.S. with f.d.}$ $u(r, d) \text{ is a G.S. with slow decay} \quad \forall d < d^*$ $u(r, d) \text{ is a changing sign solution} \quad \forall d > d^*$

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MULTIPLICITY IN PRESENCE OF A MAXIMUM OF ${\sf K}$

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MULTIPLICITY IN PRESENCE OF A MAXIMUM OF K

• If K has several critical points,

the number of solutions depends on the number of critical points

G. BIANCHI, H. EGNELL, Prog. Nonlinear Diff. Eqns Appl. (1992) R. JOHNSON, X.B. PAN, Y.F. YI, Indiana Math. J. (1994) S. YAN, J.D.E. (2000)

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D. CAO, E. NOUSSAIR, S. YAN, Calc. Var. PDEs (2002)

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MULTIPLICITY IN PRESENCE OF A MAXIMUM OF K

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G. BIANCHI, H. EGNELL, Prog. Nonlinear Diff. Eqns Appl. (1992)
R. JOHNSON, X.B. PAN, Y.F. YI, Indiana Math. J. (1994)
S. YAN, J.D.E. (2000)
D. CAO, E. NOUSSAIR, S. YAN, Calc. Var. PDEs (2002)

If K has a local maximum,

the s.c.e. admits infinitely many non-radial positive solutions

J. WEI, S. YAN, J. Funct. Anal. (2010)

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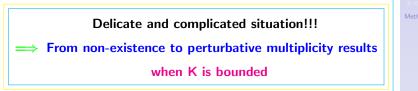
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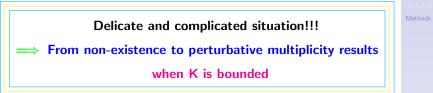
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Non-existence results

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Non-existence of G.S. with f.d.

• G. BIANCHI, H. EGNELL, Math. Z. (1992)

Existence of G.S. with f.d.

• G. BIANCHI, H. EGNELL, Math. Z. (1992) & A.R.M.A (1993)

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Multiplicity of G.S. with f.d. - Perturbative results

• C.C. CHEN, C.S. LIN, Comm. Partial Diff. Eqns (1999)

$$\begin{split} \mathsf{K}(|\mathsf{x}|) &= \mathsf{1} + \varepsilon \mathsf{k}(|\mathsf{x}|), \quad 0 < k(|\mathsf{x}|) < 1, \quad k \in C^1, \\ & \textit{under conditions} \ (K_1) - (K_2) - (K_3) \end{split}$$

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IN PRESENCE OF A MINIMUM OF K

Multiplicity of G.S. with f.d. - Perturbative results

 C.C. CHEN, C.S. LIN, Comm. Partial Diff. Eqns (1999)
 K(|x|) = 1 + εk(|x|), 0 < k(|x|) < 1, k ∈ C¹, under conditions (K₁) - (K₂) - (K₃)

• L.S. LIN, Z.L. LIU, J. Funct. Anal. (2009)

 $\mathsf{K}(|\mathsf{x}|) = \mathbf{1} + \boldsymbol{\varepsilon} \mathsf{k}(|\mathsf{x}|), \quad k(r) < 0, \quad k(0) = \mathbf{0} = k(\infty), \quad k \in C,$

 $k(r)\leq -Ar^{\mu}$ at $r=0,\ k(r)\leq -Ar^{-\mu}$ at $r=+\infty,\ A,\mu>0,$

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Multiplicity of G.S. with f.d. - Perturbative results

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 $k(r) \leq -Ar^{\mu}$ at $r=0, \ k(r) \leq -Ar^{-\mu}$ at $r=+\infty, \ A, \mu > 0,$

• I. FLORES, M. FRANCA, J.D.E. (2015) $K(|\mathbf{x}|) = k(|\mathbf{x}|^{\varepsilon}), \quad k(|\mathbf{x}|) \text{ bounded}, \quad k \in C^{1}$ The problem A perturbative resu Our main results

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Our approach is based on

The Fowler transformation

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Our approach is based on

- The Fowler transformation
- Invariant manifold theory

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Our approach is based on

- The Fowler transformation
- Invariant manifold theory
- Phase plane analysis

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Our approach is based on

- The Fowler transformation
- Invariant manifold theory
- Phase plane analysis
- Concept of barrier sets

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Our approach is based on

- The Fowler transformation
- Invariant manifold theory
- Phase plane analysis
- Concept of barrier sets

Fowler transformation:

$$x(t) = u(r)r^{\alpha}$$

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Our approach is based on

- The Fowler transformation
- Invariant manifold theory
- Phase plane analysis
- Concept of barrier sets

Fowler transformation:

$$x(t) = u(r)r^{\alpha}, \quad y(t) = \alpha u(r)r^{\alpha} + u'(r)r^{\alpha+2}$$

$$\alpha = \frac{n-2}{2}, \quad r = e^t, \quad \mathcal{K}(t) = \mathcal{K}(e^t)$$

R.H. FOWLER, Quart. J. Math. (1931).

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Fowler transformation

By Fowler transformation, (P) is equivalent to

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \alpha^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -\mathcal{K}(t) x^{\frac{n+2}{n-2}} \end{pmatrix} \qquad (5)$$

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Assumption (K₁)
$$\iff$$
 $K(r) = K\left(\frac{1}{r}\right)$ for $0 < r \le 1$

As immediate consequence of assumption (K_1)

$$\mathcal{K}(-t) = \mathcal{K}(t) \quad \forall \, t \in \mathbb{R}.$$

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An easy characterization of the solutions

 $|\mathbf{x}(t) > 0$ in $(-\infty, 0)$, $\mathbf{y}(0) = 0 \implies \mathbf{u}(r)$ is a G.S. with f.d.

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As immediate consequence of assumption (K_1)

$$\mathcal{K}(-t) = \mathcal{K}(t) \quad \forall t \in \mathbb{R}.$$

An easy characterization of the solutions

x(t) > 0 in $(-\infty, 0), y(0) = 0$

$$\implies$$
 u(r) is a G.S. with f.d.

 \implies x(t) is even, y(t) is odd

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If K has a maximum If K has a minimum

Let u(r; d) be the solution of (P): u(0) = d, u'(0) = 0

(x(t; d), y(t; d)) denotes the corresponding trajectory of (S)

Denote by $T_{\ell}(d)$ the ℓ^{th} zero of y(t; d)

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(x(t; d), y(t; d)) denotes the corresponding trajectory of (S)

Denote by $T_{\ell}(d)$ the ℓ^{th} zero of y(t; d)

u(r) is a G.S. with f.d. if

$$\begin{aligned} \mathbf{x}(\mathbf{t}) > 0 \text{ in } (-\infty, 0), \ \mathbf{y}(0) = 0 \\ \\ & \\ \\ \exists \, \ell \in \mathbb{N}, \ \exists \, \mathbf{d}^* > 0 : \quad \mathsf{T}_{\ell}(\mathbf{d}^*) = 0 \end{aligned}$$

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Idea of the proof

Let u(r; d) be the solution of (P): u(0) = d, u'(0) = 0

(x(t; d), y(t; d)) denotes the corresponding trajectory of (S)

Denote by $T_{\ell}(d)$ the ℓ^{th} zero of y(t; d)

u(r) is a G.S. with f.d. if

$$x(t) > 0$$
 in $(-\infty, 0), y(0) = 0$

介

$$\exists \, \ell \in \mathbb{N}, \; \exists \, d^* > 0: \quad \mathsf{T}_\ell(d^*) = 0$$

x has ℓ local maxima and $(\ell - 1)$ local minima

Our main results References

Non-existence results If K has a maximum If K has a minimum

Idea of the proof

Let u(r; d) be the solution of (P): u(0) = d, u'(0) = 0

(x(t; d), y(t; d)) denotes the corresponding trajectory of (S)

Denote by $T_{\ell}(d)$ the ℓ^{th} zero of y(t; d)

u(r) is a G.S. with f.d. if

$$x(t) > 0$$
 in $(-\infty, 0), y(0) = 0$

 $\exists \, \ell \in \mathbb{N}, \; \exists \, d^* > 0: \quad \mathsf{T}_\ell(d^*) = 0$

y has exactly
$$(2\ell - 1)$$
 zeroes

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Our problem:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \alpha^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -\mathcal{K}(t) x^{\frac{n+2}{n-2}} \end{pmatrix}$$

 $\forall \ell \in \mathbb{N}, \text{ we look for } \mathsf{d}^* > \mathsf{0}: \mathsf{T}_{\ell}(\mathsf{d}^*) = \mathsf{0}$

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Our problem:

$$\left(\begin{array}{c} \dot{x} \\ \dot{y} \end{array}\right) = \left(\begin{array}{c} 0 & 1 \\ \alpha^2 & 0 \end{array}\right) \left(\begin{array}{c} x \\ y \end{array}\right) + \left(\begin{array}{c} 0 \\ -\mathcal{K}(t) x^{\frac{n+2}{n-2}} \end{array}\right)$$

$$\forall \, \ell \in \mathbb{N}, \text{ we look for } d^* > 0: \ \mathsf{T}_\ell(d^*) = 0$$

Assume (K₀). Suppose that $0 < \varepsilon \leq \varepsilon_1$.

 \exists a small $d_{\ell} > 0$:

$$\mathsf{T}_\ell(\mathsf{d}_\ell)\,>\,0$$

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The problem A perturbative result Our main results References Non-existence results If K has a maximum If K has a minimum

Our problem:

$$\left(\begin{array}{c} \dot{x} \\ \dot{y} \end{array}\right) = \left(\begin{array}{c} 0 & 1 \\ \alpha^2 & 0 \end{array}\right) \left(\begin{array}{c} x \\ y \end{array}\right) + \left(\begin{array}{c} 0 \\ -\mathcal{K}(t) x^{\frac{n+2}{n-2}} \end{array}\right)$$

$$\forall\,\ell\in\mathbb{N}, \text{ we look for } \ d^*>0: \ \mathsf{T}_\ell(d^*)=0$$

Assume (K₀). Suppose that $0 < \varepsilon \leq \varepsilon_1$.

$$\exists$$
 a small $d_{\ell} > 0$:

$$\mathsf{T}_{\ell}(\mathsf{d}_{\ell}) > \mathbf{0}$$

Assume $(K_0)-(K_2)-(K_3)$.

$$\exists$$
 a large $D_\ell > 0$:

$$\mathsf{T}_\ell(\mathsf{d}_\ell)\,<\,\mathsf{0}$$

C.C. CHEN, C.S. LIN, Comm. PDEs & Math. Ann. (1999)

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Our problem:

$$\left(\begin{array}{c} \dot{x} \\ \dot{y} \end{array}\right) = \left(\begin{array}{c} 0 & 1 \\ \alpha^2 & 0 \end{array}\right) \left(\begin{array}{c} x \\ y \end{array}\right) + \left(\begin{array}{c} 0 \\ -\mathcal{K}(t) x^{\frac{n+2}{n-2}} \end{array}\right)$$

$$\forall\,\ell\in\mathbb{N}, \text{ we look for } d^*>0: \ \mathsf{T}_\ell(d^*)=0$$

Assume (K₀). Suppose that $0 < \varepsilon \leq \varepsilon_1$.

 $\exists D_{\ell} > 0$:

 $\exists d_{\ell}$

$$> 0:$$
 $T_{\ell}(d_{\ell}) > 0$

Assume $(K_0)-(K_2)-(K_3)$.

$$\mathsf{T}_{\ell}(\mathsf{d}_{\ell}) < \mathsf{0}$$

The thesis would follow from the continuity of ${\sf T}_\ell$

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Continuity of \textbf{T}_{ℓ}

Assume (K_0) - (K_2) .

Suppose that $0 < \varepsilon \leq \varepsilon_{\ell}$, where ε_{ℓ} is a **computable** constant.

 $T_{\ell}(d) \leq 0 \implies T_{\ell}(d)$ is continuous

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Continuity of $\boldsymbol{\mathsf{T}}_\ell$

Assume (K_0) - (K_2) .

Suppose that $0 < \varepsilon \leq \varepsilon_{\ell}$, where ε_{ℓ} is a **computable** constant.

 $T_{\ell}(d) \leq 0 \implies T_{\ell}(d)$ is continuous

Shooting argument
$$\Rightarrow \exists d_{\ell}^* \in (d_{\ell}, D_{\ell}) : \mathbf{T}_{\ell}(\mathbf{d}_{\ell}^*) = \mathbf{0}$$

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Continuity of $\boldsymbol{\mathsf{T}}_\ell$

Assume (K_0) - (K_2) .

Suppose that $0 < \varepsilon \leq \varepsilon_{\ell}$, where ε_{ℓ} is a **computable** constant.

 $T_{\ell}(d) \leq 0 \implies T_{\ell}(d)$ is continuous

Shooting argument
$$\exists d_{\ell}^* \in (d_{\ell}, D_{\ell}) : \underline{\mathsf{T}_{\ell}(\mathsf{d}_{\ell}^*) = 0}$$

The proof of continuity of T_{ℓ} is based on a **barrier argument** ensuring that (x(t; d), y(t; d)) intersects the x-axis transversally

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$$\left(\begin{array}{c} \dot{x} \\ \dot{y} \end{array}\right) = \left(\begin{array}{c} 0 & 1 \\ \alpha^2 & 0 \end{array}\right) \left(\begin{array}{c} x \\ y \end{array}\right) + \left(\begin{array}{c} 0 \\ -(1+c)x^{\frac{n+2}{n-2}} \end{array}\right).$$

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It admits a unique critical point $\mathbf{P}^*(c) = (P^*_x(c), 0)$ s.t. $P^*_x(c) > 0$

$$P_{x}^{*}(c) = \left(rac{(n-2)^{2}}{4(c+1)}
ight)^{rac{n-1}{4}}$$

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Introduce the corresponding energy function

$$H_c(x,y) := rac{y^2}{2} - lpha^2 rac{x^2}{2} + (1+c)rac{x^q}{q}, \quad ext{where } q := rac{2n}{n-2}.$$

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Recalling that $1 < \mathcal{K}(t) < 1 + \varepsilon \implies 0 < c < \varepsilon$

We define a spiral γ rotating around $P^*(\varepsilon)$ and $P^*(0)$, for ε small

 $\begin{array}{l} \gamma \text{ is made up by the level curves of } H_{\varepsilon} \text{ in } \{y \geq 0\} \\ \qquad \text{ and by the level curves of } H_0 \text{ in } \{y \leq 0\} \end{array}$

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$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \alpha^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -(1+c)x^{\frac{n+2}{n-2}} \end{pmatrix}.$$

It admits a unique critical point $\mathbf{P}^*(c) = (P^*_x(c), 0)$ s.t. $P^*_x(c) > 0$

Introduce the corresponding energy function

$$H_c(x,y) := \frac{y^2}{2} - \alpha^2 \frac{x^2}{2} + (1+c) \frac{x^q}{q}, \quad \text{where } q := \frac{2n}{n-2}.$$

Recalling that $1 < \mathcal{K}(t) < 1 + \varepsilon \implies 0 < \mathbf{c} < \epsilon$

We define a *spiral* γ rotating around $\mathbf{P}^*(\varepsilon)$ and $\mathbf{P}^*(\mathbf{0})$, for ε small The *spiral* γ controls the behaviour of the trajectories (x, y) of (S): $\lim_{t \to \infty} (x(t), y(t)) \longrightarrow (0, 0).$ A perturbative result Our main results References Non-existence results If K has a maximum

Definition

 $A_{\ell} := (A_{\ell}, 0)$ is the ℓ^{th} intersection between γ and the x-axis.

The spiral γ rotating around $P^*(\varepsilon)$ and $P^*(0)$ is well defined if

 $A_2 < \ldots < A_{2i} < \ldots < P_x^*(\varepsilon) < P_x^*(0) < \ldots < A_{2i+1} < \ldots < A_1$

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The critical value ε_{ℓ} is the only value which satisfies

$$\underline{\mathsf{A}_{\ell}=\mathsf{P}_{\mathsf{x}}^{*}(\varepsilon_{\ell})} \text{ if } \ell \text{ is even}, \qquad \underline{\mathsf{A}_{\ell}=\mathsf{P}_{\mathsf{x}}^{*}(0)} \text{ if } \ell \text{ is odd}.$$

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