Positive solutions to discrete two point nonlinear boundary value problems

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Outline

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Difference equations arise in different research fields (for example, computer science, discrete optimization, economics, population genetics, etc.). Many authors have discussed the existence and multiplicity of solutions for difference problems by exploiting various methods from nonlinear analysis, including the method of upper and lower solutions, fixed point theory, Rabinowitz's global bifurcation theorem, Leray-Schauder degree and critical groups.

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C. Bereanu, J. Mawhin,
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A. Cabada, S. Tersian,
Existence of heteroclinic solutions for discrete $p-$Laplacian problems with a parameter,
Nonlinear Anal. RWA 12 (2011) 2429–2434.

L. Jiang, Z. Zhou,
Three solutions to Dirichlet boundary value problems for p-Laplacian difference equations,
Adv. Diff. Equ. 2008 (2008), 1–10.

D. Q. Jiang, D. ORegan, R. P. Agarwal,
A generalized upper and lower solution method for singular discrete boundary value problems for the onedimensional $p-$Laplacian,
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Overview books

- R. P. Agarwal
  Difference equations and inequalities. Theory, methods, and applications.
  Marcel Dekker, Inc, New York Basel, 2000.

- W. G. Kelly, A. C. Peterson
  Difference Equations: An Introduction with Applications.
  Academic press, San Diego, New York, Basel, 1991

Outline

The discrete problem

G. D'Aguì, J. Mawhin, A. Sciammetta,
Positive solutions for a discrete two point nonlinear boundary value problem with $p-$laplacian,
J. Math.Anal.Appl. 447 (2017), 383–397.

. . . .

Let N be a positive integer, denote with $[1, N]$ the discrete interval $\{1, \dots, N\}$ and consider the following problem

$$\begin{cases} -\Delta\left(\varphi_p\left(\Delta u\left(k-1\right)\right)\right) + q(k)\varphi_p\left(u\left(k\right)\right) = \lambda f\left(k, u\left(k\right)\right) & k \in [1, N], \\ u(0) = u(N+1) = 0, \end{cases} \tag{$D_\lambda^{f,q}$}$$

where $f : [1, N] \times \mathbb{R} \to \mathbb{R}$ is a continuous function,
$\Delta u\left(k-1\right) = u(k) - u(k-1)$ is the forward difference operator, $q(k) \geq 0$ for all $k \in [1, N]$, $\varphi_p(s) = |s|^{p-2}s$, $1 < p < +\infty$ and λ is a positive real parameter.

Abstract

In this paper, under suitable assumptions on the nonlinearity f, we obtain the existence of two positive solutions to problem $(D_\lambda^{f,q})$. Our main tool is a two critical point theorem that is an appropriate combination of a local minimum theorem with the classical and seminal Ambrosetti-Rabinowitz theorem. A crucial assumption of the mountain pass theorem is the Palais-Smale condition. It is satisfied in the applications in an infinite dimensional space by requiring a condition on the nonlinear term stronger than $p-$superlinearity at infinity.

Abstract

In this paper it is proved that the $p-$superlinearity at infinity of the primitive on the nonlinear datum is sufficient to prove the Palais-Smale condition. It is worth noticing that, here, to obtain the existence of two positive solutions, it is enough to assume only an algebraic condition on the nonlinearity , which is more general than the $p-$sublinearity at zero. Essentially, the existence of at least two solutions to $(D_{\lambda}^{f,q})$ is obtained by requiring the $p-$superlinearity at infinity and the $p-$sublinearity at zero on the primitive of f. Moreover, by requiring on the nonlinearity that $f(k,0) \geq 0$, we obtain the existence result of positive solutions by applying a strong maximum principle proved here.

Particular case

Theorem

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$\lim_{t \to 0^+} \frac{f(t)}{t^{p-1}} = +\infty, \quad \textit{and} \quad \lim_{t \to +\infty} \frac{f(t)}{t^{p-1}} = +\infty.$$

Then, the problem

$$\begin{cases} -\Delta\left(\varphi_p\left(\Delta u\left(k-1\right)\right)\right) + q(k)\varphi_p\left(u\left(k\right)\right) = \lambda f\left(u\left(k\right)\right) & k \in [1, N], \\ u(0) = u(N+1) = 0, & \end{cases}$$

for each $\lambda \in \left]0, \frac{2^p}{pN(N+1)^{p-1}} \sup_{c>0} \frac{c^p}{\max_{|\xi| \le c} \int_0^\xi f(t)\,dt}\right[$ *admits at least two positive solutions.*

Outline

Previous papers on discrete framework

P. Candito, G. D'Aguì,
Three solutions for a discrete nonlinear Neumann problem involving the p-Laplacian,
Advances in Difference Equations Article ID 862016, (2010) 1–11

P. Candito, G. D'Aguì,
Three Solutions to A Perturbed Nonlinear Discrete Dirichlet Problem,
J. Math. Anal. Appl., 375,(2011) 594–601.

G. D'Aguì, P. Candito,
Constant-sign solutions for a nonlinear Neumann problem involving the discrete p-Laplacian,
Opuscula Math. 34, 4 (2014), 683–690.

G. Bonanno, G. D'Aguì, P. Candito,
Positive solutions for a nonlinear parameter-depending algebraic system,
Electronic Journal of Differential Equations, (2015), 17, 1–14.

Previous papers on discrete framework

G. Bonanno, G. D'Aguì, P. Candito,
Variational methods on finite dimensional Banach spaces and discrete problems,
Advanced Nonlinear Studies 14, (2014), 915–939.

In this paper, existence and multiplicity results for a class of second-order difference equations are established. In particular, the existence of at least one positive solution without requiring any asymptotic condition at infinity on the nonlinear term is presented and the existence of two positive solutions under a superlinear growth at infinity of the nonlinear term is pointed out. The approach is based on variational methods and, in particular, on a local minimum theorem and its variants. It is worth noticing that, in this paper, some classical results of variational methods are opportunely rewritten by exploiting fully the finite dimensional framework in order to obtain novel results for discrete problems.

Outline

Preliminaries

Consider the N-dimensional Banach space

$$S = \{u : [0, N+1] \to \mathbb{R} \quad \text{such that} \quad u(0) = u(N+1) = 0\}, \tag{1}$$

endowed with the norm

$$\|u\| = \left(\sum_{k=1}^{N+1} \left(|\Delta u\,(k-1)|^p + q(k)|u(k)|^p \right) \right)^{\frac{1}{p}}, \tag{2}$$

for any $q(k) \geq 0$ for all $k \in [1, N]$ and for all $p > 1$. In the sequel, we will use the following inequality

$$\max_{k \in [1,N]} |u(k)| \leq \frac{(N+1)^{\frac{p-1}{p}}}{2} \|u\|, \tag{3}$$

for every $u \in S$.

Preliminaries

The following property provides numerical estimates for the equivalence between the norm $\|\cdot\|$ and the classical Hölder norm

$$\|u\|_p = \left(\sum_{k=1}^{N} |u(k)|^p\right)^{\frac{1}{p}}.$$

Theorem

For every $u \in S$, one has

$$\frac{2}{N+1}\|u\|_p \leq \|u\| \leq (2^p + \overline{q})^{\frac{1}{p}}\|u\|_p, \tag{4}$$

where

$$\overline{q} = \max_{k\in[1,N]} q(k).$$

Preliminaries

Let $f : [1, N] \times \mathbb{R} \to \mathbb{R}$ be a continuous function.
Put

$$F(k,t) = \int_0^t f(k,\xi)d\xi \quad \text{for all} \quad (k,t) \in [1,N] \times \mathbb{R},$$

and define the functional $I_\lambda : S \to \mathbb{R}$ for all $\lambda > 0$ by

$$I_\lambda = \Phi - \lambda\Psi,$$

where

$$\Phi(u) := \frac{\|u\|^p}{p}, \quad \text{and} \quad \Psi(u) := \sum_{k=1}^{N} F(k, u(k)). \tag{5}$$

Preliminaries

Clearly, Φ and Ψ are two functionals of class $C^1(S,\mathbb{R})$ whose Gâteaux derivatives at the point $u \in S$ are given by

$$\Phi'(u)(v) = \sum_{k=1}^{N+1} \varphi_p\left(\Delta u\left(k-1\right)\right) \Delta v\left(k-1\right) + q(k)\left|u\left(k\right)\right|^{p-2} u\left(k\right) v\left(k\right),$$

and

$$\Psi'(u)(v) = \sum_{k=1}^{N+1} f\left(k, u\left(k\right)\right) v(k),$$

for all u, $v \in S$.

Palais-Smale condition

Here and in the sequel we suppose $f(k,0) \geq 0$ for all $k \in [1,N]$. We assume that $f(k,x) = f(k,0)$ for all $x < 0$ and for all $k \in [1,N]$. Put

$$L_\infty(k) := \liminf_{s \to +\infty} \frac{F(k,s)}{s^p}, \quad L_\infty := \min_{k \in [1,N]} L_\infty(k).$$

We have the following result.

Lemma

If $L_\infty > 0$ then I_λ satisfies $(PS)-$condition and it is unbounded from below for all $\lambda \in \left] \frac{2^p + \overline{q}}{pL_\infty}, +\infty \right[$.

Strong Maximum Principle

We point out the following result that we will use to obtain positive solutions for our problem. The first is the following strong maximum principle.

Theorem

Fix $u \in S$ such that either

$$u(k) > 0 \quad \text{or} \quad -\Delta\left(\varphi_p\left(\Delta u\left(k-1\right)\right)\right) + q(k)\varphi_p\left(u\left(k\right)\right) \geq 0, \tag{6}$$

for all $k \in [1, N]$.
Then, either $u > 0$ in $[1, N]$ or $u \equiv 0$.

Lemma

If $f(k, 0) \geq 0$, any critical point of the functional I_λ^+ is a positive solution of problem $(D_\lambda^{f,q})$.

Outline

Two positive solutions.

In this section, we present our main results. To be precise, we establish an existence result of at least two positive solutions.

Two positive solutions.

Theorem

Let $f : [1, N] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(k, 0) \geq 0$ for all $k \in [0, N]$. Assume also that there exist two positive constants c and d with $d < c$ such that

$$\frac{\sum_{k=1}^{N} \max_{|\xi| \leq c} F(k, \xi)}{c^p} < \frac{2^p}{(N+1)^{p-1}} \min \left\{ \frac{1}{2+Q} \frac{\sum_{k=1}^{N} F(k, d)}{d^p}, \frac{L_\infty}{2^p + \overline{q}} \right\}. \tag{7}$$

Then, for each $\lambda \in \bar{\Lambda}$ with

$$\bar{\Lambda} = \left] \max \left\{ \frac{2+Q}{p} \frac{d^p}{\sum_{k=1}^{N} F(k, d)}, \frac{2^p + \overline{q}}{pL_\infty} \right\}, \frac{2^p}{p(N+1)^{p-1}} \frac{c^p}{\sum_{k=1}^{N} \max_{|\xi| \leq c} F(k, \xi)} \right[,$$

the problem admits at least two positive solutions.

Outline

Proof: main tool

G. Bonanno, G. D'Aguì
Two non-zero solutions for elliptic Dirichlet problems,
Z. Anal. Anwendungen, 35 (2016), 449–464

Proof: main tool

Theorem

Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two functionals of class C^1 such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that

$$\frac{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}, \tag{8}$$

and, for each

$$\lambda \in \Lambda = \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)} \right[,$$

the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)-$condition and it is unbounded from below.
Then, for each $\lambda \in \Lambda$, the functional I_λ admits at least two nonzero critical points $u_{\lambda,1}$, $u_{\lambda,2}$ such that $I(u_{\lambda,1}) < 0 < I(u_{\lambda,2})$.

Proof.

Fix $\lambda \in \bar{\Lambda}$ and observe that from (7) one has that $L_\infty > 0$ and $\bar{\Lambda}$ is non-degenerate. Then, by the previous Lemma , the functional I_λ satisfies the $(PS)-$condition for each $\lambda > \dfrac{2^p + \overline{q}}{pL_\infty}$, and it is unbounded from below.

Hence,

$$\frac{\sup\limits_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{r} \leq \frac{p(N+1)^{p-1}}{2^p} \frac{\sum\limits_{k=1}^{N} \max\limits_{|\xi| \leq c} F(k, \xi)}{c^p}. \tag{9}$$

Now, define $\tilde{u} \in \mathbb{R}^{N+2}$ be such that $\tilde{u}(k) = d$ for all $k \in [1, N]$ and $\tilde{u}(0) = \tilde{u}(N+1) = 0$. Clearly, $\tilde{u} \in S$. It is easy to see that

$$\Phi(\tilde{u}) = \frac{(2+Q)d^p}{p}, \tag{10}$$

and hence, one has

$$\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} = \frac{p}{2+Q}\,\frac{\sum_{k=1}^{N} F(k,d)}{d^p}. \tag{11}$$

Therefore,one has

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}. \tag{12}$$

Therefore, Theorem "on two critical points" ensures that the functional admits at least two non-zero critical points and then, for all $\lambda \in \bar{\Lambda} \subset \Lambda$, these are positive solutions the problem.

Outline

Asymptotic case

Corollary

Assume that f is a continuous function such that $f(k,0) \geq 0$ for all $k \in [0,N]$ and

$$\limsup_{t\to 0^+} \frac{F(k,t)}{t^p} = +\infty, \quad \text{and} \quad \lim_{t\to+\infty} \frac{F(k,t)}{t^p} = +\infty, \tag{13}$$

for all $k \in [0,N]$, and put $\lambda^* = \dfrac{2^p}{p(N+1)^{p-1}} \sup\limits_{c>0} \dfrac{c^p}{\sum_{k=1}^{N} \max_{|\xi|\leq c} F(k,\xi)}$.

Then, for each $\lambda \in]0,\lambda^*[$, the problem admits at least two positive solutions.

Outline

Gelfand problem

It is possible to study the classical discrete Gelfand problem by using our results. In particular, we obtain the existence of two positive solutions in a suitable interval of parameters λ for such a problem

$$\begin{cases} -\Delta^2 u(k-1) = \lambda e^{u(k)} & k \in [1, N], \\ u(0) = u(N+1) = 0. \end{cases} \qquad (G_\lambda)$$

We recall that, in "R. P. Agarwal, D. O'Regan, Appl. Math. Lett. (1997)", the authors have proved that problem di Gelfand admits at least one nonnegative solution for all $\lambda \in \Lambda_G = \left[0, \dfrac{8}{e(N+1)(N+2)}\right[$. Therefore, it is worth noticing that, under the same assumption, we obtain two positive solutions for all $\lambda \in \Lambda^* = \left]0, \dfrac{2}{N(N+1)} \max_{c>0} \dfrac{c^2}{e^c - 1}\right[$.

Comparing the two intervals Λ_G and Λ^* we observe that $\sup \Lambda_G < \sup \Lambda^*$ for $N = 1$, instead $\sup \Lambda_G > \sup \Lambda^*$ for $N \geq 2$, but, two positive solutions are provided instead of one.

Example

The following simple example allows us to estimate the sharpness of the estimate on the λ-interval in Theorem in the beginning.

Take $f(u) = |u|^{p-\frac{5}{2}}u + |u|^{p-\frac{3}{2}}u$, which satisfies the conditions (1.1) at 0^+ and $+\infty$ of Theorem 1.1, and $q(k) = 0$. So, for $u > 0$, $f(u) = u^{p-\frac{3}{2}} + u^{p-\frac{1}{2}}$.

For $N = 1$, the problem is

$$-\Delta(\varphi_p(\Delta u(0)) = \lambda f(u(1)), \quad u(0) = u(2) = 0 \tag{14}$$

or, explicitly (simple computations), for positive solutions,

$$2\varphi_p(u(1)) = \lambda f(u(1)),$$

i.e. letting $u(1)^{1/2} = v$,

$$2v = \lambda(1 + v^2). \tag{15}$$

The solutions are

$$v = \frac{1 \pm \sqrt{1-\lambda^2}}{\lambda}.$$

Consequently *problem (14) has two distinct positive solutions if and only if*

$$\lambda \in]0, 1[\,.$$

Let us compute in this simple situation the extremity of the interval for λ given by our Theorem.
We call $g(c) := \frac{c^p}{\max_{|\xi|\leq c} \int_0^\xi f(t)\,dt}$ and we obtain

$$\frac{2^p}{p2^{p-1}} \max_{c>0} g(c) = \frac{2}{p}\frac{1}{2}\left(p^2 - \frac{1}{4}\right)^{1/2} = \left(1 - \frac{1}{4p^2}\right)^{1/2}.$$

This shows that the estimate for the upper bound of the interval for λ giving two positive solutions is sharper and sharper for p large.

Remark

Among discrete nonlinear equations it is important to cite the discrete nonlinear Schrödinger equation, which represents a well-known nonintegrable model which has a number of applications in molecular physics, nonlinear optics and other fields. We refer to

B. Malomed, M. I. Weinstein,
Soliton dynamics in the discrete nonlinear Schrödinger equation,
Phys. Lett. A 220 (1996), 1-3, 91–96.

M.I.Weinstein
Excitation thresholds for nonlinear localized modes on lattices
Nonlinearity 12 (1999) 3, 673-691.

and references therein for more details about this topics. In such kind of results the existence of one solution is guaranteed as minimizer of the variational problem, instead with our main result , we get two nontrivial solutions for the given problem. The first solution is obtained as a local minimum of energy functional and the second one is obtained as a mountain pass critical point.

Outline

R. P. Agarwal, D. O'Regan, *Boundary value problems for discrete equations*, Appl. Math. Lett. **10** (1997), no. 4, 83–89.

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A. Ambrosetti, P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973) 349–381.

C. Bereanu, P. Jebelean, C. Şerban, *Periodic and Neumann problems for discrete $p(\cdot)-$Laplacian*, J. Math. Anal. Appl. 399 (2013), no. 1, 75–87.

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Thanks for your kind attention