UNIVERSITÀ
DEGLI STUDI DI TRIESTE

DEAMS

Solvability of a MEMS model driven by capillarity and pressure effects

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Giornate di Equazioni Differenziali Ordinarie: metodi e prospettive
Ancona
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Capillarity phenomena






✎ R. Finn, *Equilibrium capillary surfaces*, Springer-Verlag, New York, 1986.

Outline

1. **Motivations**
   - A Micro-Electro Mechanical System model
2. **Solvability of the one-dimensional problem**
   - Properties of solutions
   - Non-existence/Existence of positive solutions
   - Multiplicity of positive solutions
3. **Conclusions**
   - Highlights and additional results

Basic features...

...of a Micro-Electro Mechanical System

Materials

A liquid and a solid in contact (small scales)

Forces

- Capillary forces
- Electrostatic force
- Gravitational force or external pressure

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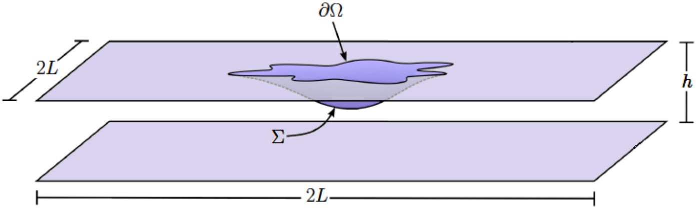
A MEMS physical model



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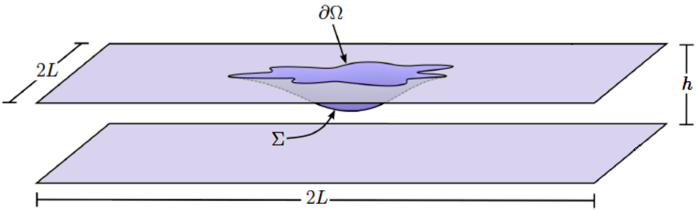

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- Parallel metallic plates
- Soap membrane suspended

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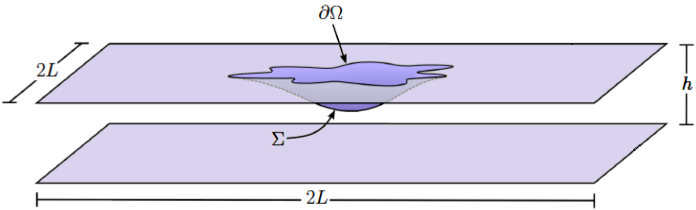

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Interplay of

- capillary
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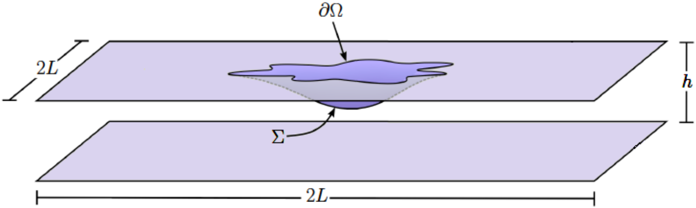

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Which is the form of the soap drop at equilibrium?

The MEMS mathematical model

$$\begin{cases} -(1-bu)\,\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \dfrac{a}{(u-R)^2} + \dfrac{b}{\sqrt{1+|\nabla u|^2}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

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Ω	bounded domain in $\mathbb{R}^N$
R	distance of the plates vs diameter of the membrane
$a = a(R)$	electrostatic force vs capillary forces
$b = b(R)$	gravity force vs capillary forces

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Parameters conditions

$$R > 0, \quad a, b \in \mathbb{R}$$

Aim ... and troubles

Goals

- To study the qualitative and quantitative properties of the solutions of the problem.
- To investigate the existence and regularity of positive solutions of the problem.
- To analyse the multiplicity matter.

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One-dimensional problem.

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Focus

One-dimensional problem.

Difficulty

The Euclidean curvature operator produces possible occurrence of blow-up gradient phenomena, in any dimension $N \geq 1$.

Special cases

No gravity: $b = 0$

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \frac{a}{(u-R)^2} \quad \text{in } \Omega.$$

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No electric field: $a = 0$

$$-(1-bu)\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \frac{b}{\sqrt{1+|\nabla u|^2}} \quad \text{in } \Omega.$$

To be studied!

The one-dimensional case

1D problem:

$$\begin{cases} -(1-bu)\left(\dfrac{u'}{\sqrt{1+|u'|^2}}\right)' = \dfrac{a}{(u-R)^2} + \dfrac{b}{\sqrt{1+|u'|^2}} & \text{in }]-r,r[, \\ u(-r)=0,\ u(r)=0. \end{cases}$$

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Notion of (generalised) solution

$u \in C^2(]-r,r[) \cap C([-r,r])$

(a) $\dfrac{u'}{\sqrt{1+|u'|^2}} \in C([-r,r])$;

(b) $u(t) < R$ and $bu(t) < 1$ for all $t \in]-r,r[$;

(c) u satisfies the equation pointwise;

(d) either $u(-r)=0$, or $u(-r)>0$ and $u'(-r^+)=+\infty$, or $u(-r)<0$ and $u'(-r^+)=-\infty$;

(e) either $u(r)=0$, or $u(r)>0$ and $u'(r^-)=-\infty$, or $u(r)<0$ and $u'(r^-)=+\infty$.

If moreover $u(-r)=u(r)=0$, then u is a classical solution.

✎ A. Lichnewsky, R. Temam, *Pseudosolutions of the time-dependent minimal surface problem*, J. Differential Equation 30 (1978), 340-364.

✎ C. Corsato, C. De Coster, P. Omari, *The Dirichlet problem for a prescribed anisotropic mean curvature equation: existence, uniqueness and regularity of solutions*, J. Differential Equations 260 (2016), no. 5, 4572-4618.

Qualitative and quantitative results

Let u be a solution of the one-dimensional problem. Then

Properties

- Finite number of zeroes
- Symmetry w.r.t. the mid point of a maximal sign-unchanged interval
- At least one zero $\Rightarrow$ u is a classical solution
- Even number of zeroes $\Rightarrow$ u is even

Non existence of solutions

- **Absence of gravity** $(b = 0)$

  Semilinear problem:

  ✎ J.A. Pelesko, D.H. Bernstein, *Modeling MEMS and NEMS*, Chapman and Hall/CRC, Boca Raton, FL, 2003.

  General problem:

  ✎ N.D. Brubaker, *Mathematical Theory of Electro-Capillary Surfaces*, Ph.D. Thesis, University of Delaware, ProQuest Dissertations Publishing, 2013.

Existence of a threshold $\hat{a} \in \mathbb{R}^+$ discriminating between existence (for all $0 \leq a < \hat{a}$) and non-existence (for all $a > \hat{a}$).

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Existence of a threshold $\hat{a} \in \mathbb{R}^+$ discriminating between existence (for all $0 \leq a < \hat{a}$) and non-existence (for all $a > \hat{a}$).

- Possible **presence of gravity** $(b \in \mathbb{R})$

Theorem

For every $b \in \mathbb{R}$, there exists $\hat{a}(b) \in \mathbb{R}^+$ such that the (N-dimensional) problem has no solutions for all $a \geq \hat{a}(b)$.

An estimate of interest in physics

For any $b \in \mathbb{R}$, it is possible to upper estimate the *pull-in voltage* $\hat{a}(b)$:

$$\hat{a}(b) \leq \begin{cases} \max\left\{ \min\left\{ \frac{(|\partial\Omega|+|b|)(1+R|b|)}{|\Omega|} R^2, \frac{\lambda_1}{2} R^3 (1+R|b|) \right\}, \frac{(1+R|b|)^2}{|b|} \right\} & \text{if } b < 0, \\ \min\left\{ \frac{|\partial\Omega|}{|\Omega|} R^2, \frac{\lambda_1}{2} R^3 \right\} & \text{if } b \geq 0. \end{cases}$$

...what about the existence of solutions?

In absence of the electric field ($a = 0$)

1D-no-electric-field problem:

$$\begin{cases} -(1 - bu)\left(\dfrac{u'}{\sqrt{1 + |u'|^2}}\right)' = \dfrac{b}{\sqrt{1 + |u'|^2}}, & \text{in }]-r, r[, \\ u(-r) = 0,\ u(r) = 0. \end{cases} \tag{1}$$

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Proposition (Existence and exact multiplicity of solutions)

There exists $b^{\#} = b^{\#}(r) \in \mathbb{R}^+$ such that

	$b < 0$	$b > 0$
$\|b\| < b^{\#}$	$u_1, u_2 \ll 0$	$u_1, u_2 \gg 0$
$\|b\| = b^{\#}$	$u \ll 0$	$u \gg 0$
$\|b\| > b^{\#}$	–	–

Intermezzo

$$\begin{cases} -(1-bu)\left(\dfrac{u'}{\sqrt{1+|u'|^2}}\right)' = \dfrac{a}{(u-R)^2} + \dfrac{b}{\sqrt{1+|u'|^2}} & \text{in }]-r,r[, \\ u(-r)=0,\ u(r)=0. \end{cases}$$

If

$$\frac{a}{R^2} + b = 0$$

then 0 is a solution of the 1D problem.

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What happens if

$$\frac{a}{R^2} + b < 0, \quad \frac{a}{R^2} + b > 0?$$

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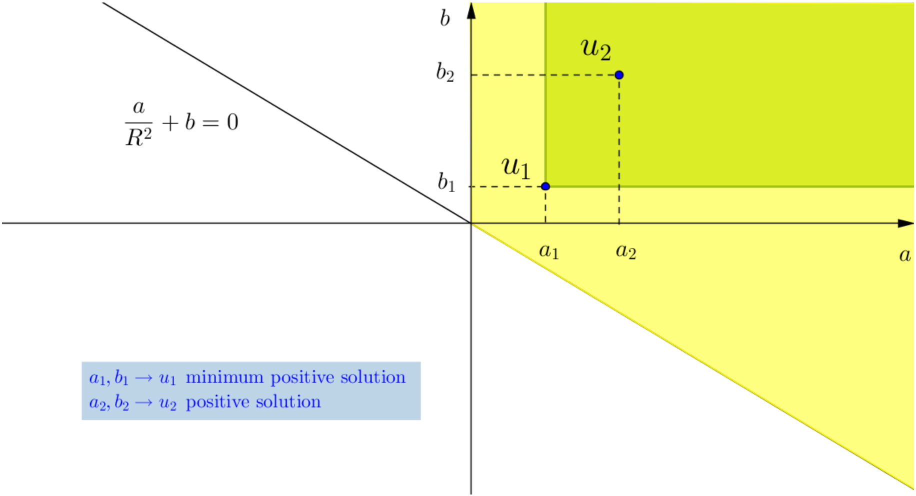
We will concentrate on

$$a \geq 0, \quad \frac{a}{R^2} + b > 0.$$

A comparison result



$u_1 \ll u_2$

A comparison result, formalisation

Theorem (Ordering between positive solutions)

Let us assume

$$a_1 \geq 0, \qquad \frac{a_1}{R^2} + b_1 > 0,$$
$$a_2 \geq a_1, \qquad b_2 \geq b_1.$$

Let

$$u_1 = u_1(a_1, b_1) \qquad \text{be the minimum positive solution,}$$
$$u_2 = u_2(a_2, b_2) \qquad \text{be a positive solution}$$

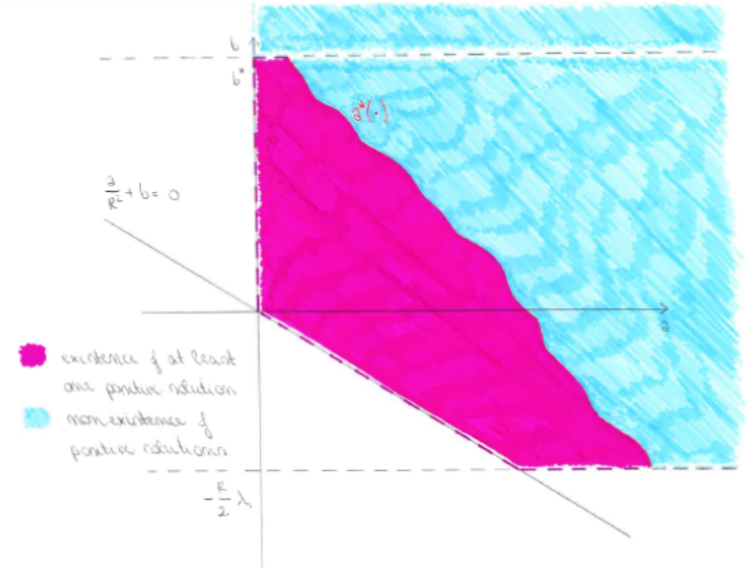
of the associated problems.

If $u_1 \neq u_2$, then it holds

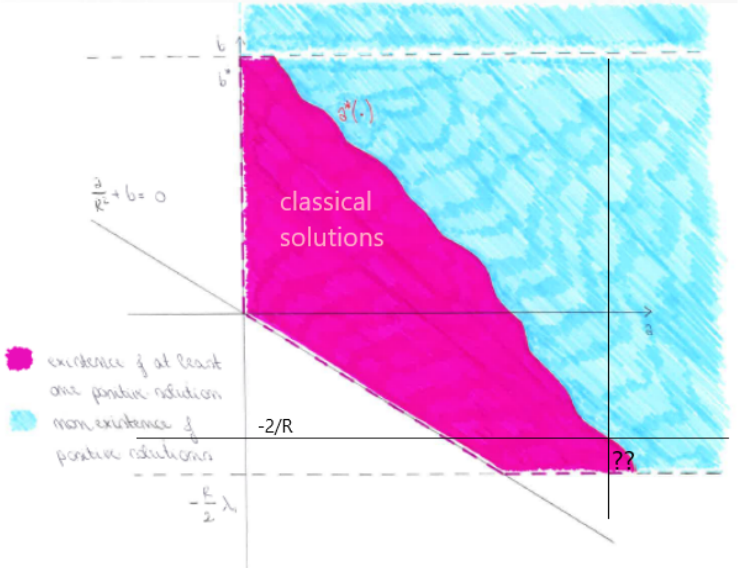
$$u_1 \ll u_2.$$

An existence result

By the comparison theorem, an existence and regularity result of positive solutions:



A regularity result



An existence result, formalisation

Theorem (Existence and regularity of positive solutions)

Let us set

$$b^* = \sup\{b \mid \text{the 1D-no-electric-field problem has a solution } u \text{ with } u(0) < R\}.$$

Then we have $0 < b^* < +\infty$ and there exists a decreasing function $a^* :]-\frac{R}{2}\lambda_1, b^*[\to]0, +\infty[$ such that

(a) for all b, $\frac{a^*(b)}{R^2} + b > 0$;

(b) the 1D problem has at least one positive solution u_1 in the following cases:

- $0 \leq b < b^*$ and $0 < a \leq a^*(b)$,
- $-\frac{R}{2}\lambda_1 < b < 0$ and $-R^2 b < a \leq a^*(b)$;

(c) the 1D problem has no positive solution in the following cases:

- $b > b^*$ and $a > 0$,
- $-\frac{R}{2}\lambda_1 < b < b^*$ and $a > a^*(b)$.

A regularity result, formalisation

Theorem (continued)

The solution u_1 of the 1D problem found is classical and, in particular, $u_1 \in C^2([-r, r])$, in the following cases:

- $0 \leq b < b^*$ and $0 < a < a^*(b)$,
- $\pi R \leq 4r$, $-\frac{R}{2}\lambda_1 < b < 0$ and $-R^2 b < a < a^*(b)$,
- $\pi R > 4r$, $-\frac{2}{R} \leq b < 0$ and $-R^2 b < a < a^*(b)$,
- $\pi R > 4r$, $-\frac{R}{2}\lambda_1 < b < -\frac{2}{R}$ and $-R^2 b < a < a^*(-\frac{2}{R})$.

Multiplicity of positive solutions

- **Absence of gravity** $(b = 0)$

Existence of a positive solution for $\hat{a} \in \mathbb{R}^+$

$\Downarrow$

Existence of (at least) two positive solutions for all $0 < a < \hat{a}$,
$0 \ll u_1 \ll u_2$, and u_1 is the minimum among all the positive solutions.

✎ N.D. Brubaker, J.A. Pelesko, *Analysis of a one-dimensional prescribed mean curvature equation with singular nonlinearity*, Nonlinear Analysis 75 (2012), 5086-5102.

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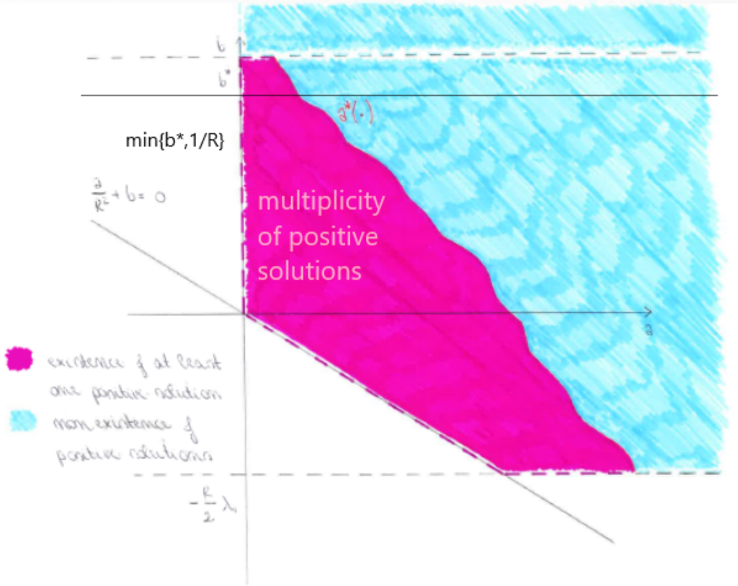
- **General one-dimensional case** $\left(a \geq 0, \dfrac{a}{R^2} + b > 0\right)$

Generalisation obtained (lower and upper solution method).

A multiplicity result, qualitative graph



A multiplicity result, formalisation

Theorem (Existence and multiplicity of positive solutions)

Let us set b^* as in the previous Theorem. Then the 1D problem has at least two positive solutions u_1 and u_2, satisfying $0 \ll u_1 \ll u_2$, where u_1 is its minimum positive solution, in the following cases:

- $0 \leq b < \min\{b^*, \frac{1}{R}\}$ and $0 < a < a^*(b)$,
- $-\frac{R}{2}\lambda_1 < b < 0$ and $-R^2 b < a < a^*(b)$.

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- $0 \leq b < \min\{b^*, \frac{1}{R}\}$ and $0 < a < a^*(b)$,
- $-\frac{R}{2}\lambda_1 < b < 0$ and $-R^2 b < a < a^*(b)$.

Moreover, the solution u_1 is classical.

Highlights

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- We have focused the attention on the one-dimensional version of the problem.
- We have analysed several qualitative and quantitative properties of its solutions and stated existence, multiplicity and regularity result of its positive solutions.

Additional results obtained

$$\begin{cases} -(1-bu)\left(\dfrac{u'}{\sqrt{1+|u'|^2}}\right)' = \dfrac{a}{(u-R)^2} + \dfrac{b}{\sqrt{1+|u'|^2}} & \text{in }]-r,r[, \\ u(-r)=0, \; u(r)=0. \end{cases}$$

- Existence and regularity of possibly sign-changing solutions, for suitable choices of a, b, R, r.

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- Existence and regularity of possibly sign-changing solutions, for suitable choices of a, b, R, r.
- Bifurcation of nodal solutions.

What about the N-dimensional problem?

$$\begin{cases} -(1-bu)\,\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \dfrac{a}{(u-R)^2} + \dfrac{b}{\sqrt{1+|\nabla u|^2}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

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Follow-up: loading...

Investigation of qualitative properties, existence and regularity of solutions of the general problem (2).

Thank you for your attention!