Semilinear Elliptic Systems with Dependence on the Gradient

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Abstract. We provide results on the existence, non-existence, multiplicity, and localization of positive radial solutions for semilinear elliptic systems with Dirichlet or Robin boundary conditions on an annulus. Our approach is topological and relies on the classical fixed point index. We present an example to illustrate our theory.

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We study the existence of positive *radial* solutions for the system of BVPs

$$\begin{cases} -\Delta u = f_1(|x|, u, v, |\nabla u|, |\nabla v|) \text{ in } \Omega, \\ -\Delta v = f_2(|x|, u, v, |\nabla u|, |\nabla v|) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ v = 0 \text{ on } |x| = R_0 \text{ and } \dfrac{\partial v}{\partial r} = 0 \text{ on } |x| = R_1, \end{cases} \tag{1}$$

where $\Omega = \{x \in \mathbb{R}^n : R_0 < |x| < R_1\}$ is an annulus, the nonlinearities f_i are positive continuous functions and $\dfrac{\partial}{\partial r}$ denotes differentiation in the radial direction $r = |x|$.

When the nonlinearities are *indipendent* of the gradient, variational methods are frequently used to study the above problem and there are many papers in literature on this problem, especially on the problem involving the critical Sobolev exponent.

If the nonlinearities depend on the gradient of the solution, the problem is not variational and the critical point theory can not be applied directly.
In the case of special forms of nonlinearities, by changing variables, the problem (1) can be trasformed into a (BVP) indipendent of the gradient.
For example, when

$$f(x, u, |\nabla u|) = h(u) + \lambda |\nabla u|^2 + \eta,$$

Ghergu and Rădulescu, C.R. Acad. Sci. Paris, Ser. I (2004), use the above method to show the existence of positive classical solution under the assumption hat h is decreasing and unbounded at the origin.

The existence of positive radial solutions of elliptic equations with nonlinearities with dependence on gradient subject to Dirichlet or Robin boundary conditions has been investigated by several authors.
Topological methods:

1. Chen and Lu, Nonlinear Anal. (1999);
2. Lee, Nonlinear Anal. (2001);
3. Lee, J. Differential Equations (2001);
4. De Figueiredo and Ubilla, Nonlinear Anal. (2008);
5. De Figueiredo et al., Nonlinear Anal. (2009);
6. Bueno et al., Nonlinearity (2012);
7. Singh, Nonlinear Anal. (2015).

Other methods:

1. Faria et al., Math. Nachr. (2014);
2. Averna et al., Appl. Math. Lett. (2016).

Since we are looking for radial solutions, we obtain results to a wide class of nonlinearities by means of an approach based on fixed point index theory and invariance properties of a suitable cone, in which Harnack-type inequalities are used.

The use of cones and Harnack-type inequalities in the context of ordinary differential equations or systems depending on the first derivative has been studied in

1. Guo and Ge, J. Math. Anal. Appl. (2004);
2. Agarwal et al., J. Dyn. Control Syst. (2009);
3. De Figueiredo et al., Nonlinear Anal. (2009);
4. Yang and Kong, Nonlinear Anal. (2012);
5. Jankowski, Nonlinear Anal. (2013);
6. Zima, Bound. Value Probl. (2014);
7. Avery et al., Fixed Point Theory Appl. (2015);
8. Infante and Minhós, Mediterr. J. Math. (2017).

Consider in $\mathbb{R}^n$, $n \geq 2$, the equation

$$-\triangle w = f(|x|, w, |\nabla w|) \text{ in } \Omega, \tag{2}$$

subject to Dirichlet boundary conditions

$$w = 0 \text{ on } \partial\Omega,$$

or Robin boundary conditions

$$w = 0 \text{ on } |x| = R_0 \text{ and } \frac{\partial w}{\partial r} = 0 \text{ on } |x| = R_1.$$

Since we are looking for the existence of radial solutions $w = w(r),\ (r = |x|)$ of the system (1), we rewrite (2) in the form

$$-w''(r) - \frac{n-1}{r} w'(r) = f(r, w(r), |w'(r)|) \quad \text{in } [R_0, R_1]. \tag{3}$$

Set $w(t) = w(r(t))$, where, for $t \in [0, 1]$, (see do Ó et al., J. Math. Anal. Appl. (2007); De Figueiredo and Ubilla, Nonlinear Anal. (2008))

$$r(t) := \begin{cases} R_1^{1-t} R_0^t, & n = 2, \\ \left(\frac{A}{B-t}\right)^{\frac{1}{n-2}}, & n \geq 3, \end{cases}$$

$$A = \frac{(R_0 R_1)^{n-2}}{R_1^{n-2} - R_0^{n-2}} \quad \text{and} \quad B = \frac{R_1^{n-2}}{R_1^{n-2} - R_0^{n-2}}.$$

Take, for $t \in [0, 1]$,

$$p(t) := \begin{cases} r^2(t) \log^2(R_1/R_0), & n = 2, \\ \left(\frac{R_0 R_1 \left(R_1^{n-2} - R_0^{n-2}\right)}{n-2}\right)^2 \frac{1}{\left(R_1^{n-2} - (R_1^{n-2} - R_0^{n-2})t\right)^{\frac{2(n-1)}{n-2}}}, & n \geq 3. \end{cases} \tag{4}$$

Then the equation (3) becomes

$$-w''(t) = p(t)f\left(r(t), w(t), \left|\frac{w'(t)}{r'(t)}\right|\right) := g(t, w(t), |w'(t)|) \text{ on } [0,1].$$

If we set $u(t) = u(r(t))$ and $v(t) = v(r(t))$, thus to the system (1) we can associate the system of ODEs

$$\begin{cases} -u''(t) = g_1(t, u(t), v(t), |u'(t)|, |v'(t)|) & \text{in } [0,1], \\ -v''(t) = g_2(t, u(t), v(t), |u'(t)|, |v'(t)|) & \text{in } [0,1], \\ u(0) = u(1) = v(0) = v'(1) = 0. \end{cases} \tag{5}$$

We note that, for $i = 1, 2$, g_i is a positive continuous function in $[0,1] \times [0,+\infty)^4$.
By a *radial solution* of the system (1) we mean a solution of the system (5).

Let ω be a continuous function on $[0,1]$ and

$$C^1_\omega[0,1] = \{w \in C[0,1] : w \text{ is continuos differentiable on } (0,1)$$

$$\text{with } \sup_{t\in(0,1)} \omega(t)|w'(t)| < +\infty\}.$$

It can verify that $C^1_\omega[0,1]$ is a Banach space (see Agarwal et al., J. Dyn. Control Syst. (2009)) with the norm

$$||w|| := \max\left\{||w||_\infty, \|w'\|_\omega\right\},$$

where $||w||_\infty := \max_{t\in[0,1]} |w(t)|$ and $\|w'\|_\omega := \sup_{t\in(0,1)} \omega(t)|w'(t)|$.
Set $\omega_1(t) = t(1-t)$, $\omega_2(t) = t$, we study the existence of solutions of the system (5) by means of the fixed points of the suitable operator T on the space $C^1_{\omega_1}[0,1] \times C^1_{\omega_2}[0,1]$ equipped with the norm

$$||(u,v)|| := \max\{||u||,\ ||v||\}.$$

We define the integral operator
$T : C^1_{\omega_1}[0,1] \times C^1_{\omega_2}[0,1] \to C^1_{\omega_1}[0,1] \times C^1_{\omega_2}[0,1]$

$$T(u,v)(t) := \begin{pmatrix} T_1(u,v)(t) \\ T_2(u,v)(t) \end{pmatrix} \tag{6}$$

$$= \begin{pmatrix} \int_0^1 k_1(t,s) g_1(s, u(s), v(s), |u'(s)|, |v'(s)|)\, ds \\ \int_0^1 k_2(t,s) g_2(s, u(s), v(s), |u'(s)|, |v'(s)|) ds \end{pmatrix}$$

where the Green's functions k_i are given by

$$k_1(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1, \end{cases} \quad k_2(t,s) = \begin{cases} s, & 0 \le s \le t \le 1, \\ t, & 0 \le t \le s \le 1. \end{cases}$$

Fixed $[a_1, b_1] \subset (0,1)$, $c_1 = \min\{a_1, 1-b_1\}$, $[a_2, b_2] \subset (0,1]$ and $c_2 = a_2$, we consider the cones, for $i = 1, 2$,

$$K_i := \left\{ w \in C^1_{\omega_i}[0,\ 1] : w \geq 0,\ \min_{t\in[a_i,b_i]} w(t) \geq c_i ||w||_\infty,\ ||w||_\infty \geq \|w'\|_{\omega_i} \right\},$$

and the cone K in $C^1_{\omega_1}[0,1] \times C^1_{\omega_2}[0,1]$ defined by

$$K := \{(u,v) \in K_1 \times K_2\}.$$

Note that the functions in K_i are strictly positive on the sub-intervals $[a_i, b_i]$.

By a *positive solution* of the system (5) we mean a solution $(u,v) \in K$ of (5) such that $\|(u,v)\| > 0$.

The operator T defined in (6) leaves the cone K invariant and is compact.

The following theorem follows from classical results about fixed point index (more details can be seen, for example, Amann, SIAM. Rev. (2009), or Guo and Lakshmikantham, *Nonlinear Problems in Abstract Cones*, (1988)).

Theorem

Let K be a cone in an ordered Banach space X. Let Ω be an open bounded subset with $0 \in \Omega \cap K$ and $\overline{\Omega \cap K} \neq K$. Let Ω^1 be open in X with $\overline{\Omega^1} \subset \Omega \cap K$. Let $F : \overline{\Omega \cap K} \to K$ be a compact map. Suppose that

(1) *$Fx \neq \mu x$ for all $x \in \partial(\Omega \cap K)$ and for all $\mu \geq 1$.*

(2) *There exists $h \in K \setminus \{0\}$ such that $x \neq Fx + \lambda h$ for all $x \in \partial(\Omega^1 \cap K)$ and all $\lambda > 0$.*

Then F has at least one fixed point $x \in (\Omega \cap K) \setminus \overline{(\Omega^1 \cap K)}$.
Denoting by $i_K(F, U)$ the fixed point index of F in some $U \subset X$, we have

$$i_K(F, \Omega \cap K) = 1 \text{ and } i_K(F, \Omega^1 \cap K) = 0\,.$$

The same result holds if $i_K(F, \Omega \cap K) = 0$ and $i_K(F, \Omega^1 \cap K) = 1$.

For our index calculations, we utilize the open bounded sets (relative to K) , for $\rho_1, \rho_2 > 0$,

$$K_{\rho_1,\rho_2} := \{(u,v) \in K : ||u|| < \rho_1 \text{ and } ||v|| < \rho_2\}.$$

$$V_{\rho_1,\rho_2} := \{(u,v) \in K : \min_{t\in[a_1,b_1]} u(t) < \rho_1 \text{ and } \min_{t\in[a_2,b_2]} v(t) < \rho_2\}.$$

The use of different radii allows more freedom in the growth of the nonlinearities.

The sets defined above have the following properties:

(P_1) $K_{\rho_1,\rho_2} \subset V_{\rho_1,\rho_2} \subset K_{\rho_1/c_1,\rho_2/c_2}$.

(P_2) $(w_1,w_2) \in \partial K_{\rho_1,\rho_2}$ if and only if $(w_1,w_2) \in K$ and for some $i \in \{1,2\}$ $\|w_i\|_\infty = \rho_i$ and $c_i\rho_i \le w_i(t) \le \rho_i$ for $t \in [a_i,b_i]$.

(P_3) $(w_1,w_2) \in \partial V_{\rho_1,\rho_2}$ if and only if $(w_1,w_2) \in K$ and for some $i \in \{1,2\}$ $\min_{t\in[a_i,b_i]} w_i(t) = \rho_i$ and $\rho_i \le w_i(t) \le \rho_i/c_i$ for $t \in [a_i,b_i]$.

We control the growth of the f_i on the following stripes:

$$\begin{aligned}
\Omega^{\rho_1,\rho_2} &= [R_0, R_1] \times [0, \rho_1] \times [0, \rho_2] \times [0, +\infty) \times [0, +\infty)\,, \\
A_1^{s_1,s_2} &= [\min\{r(a_1), r(b_1)\}, \max\{r(a_1), r(b_1)\}] \times \left[s_1, \frac{s_1}{c_1}\right] \times \left[0, \frac{s_2}{c_2}\right] \\
&\quad \times [0, +\infty) \times [0, +\infty)\,, \\
A_2^{s_1,s_2} &= [\min\{r(a_2), r(b_2)\}, \max\{r(a_2), r(b_2)\}] \times \left[0, \frac{s_2}{c_2}\right] \times \left[s_2, \frac{s_2}{c_2}\right] \times \\
&\quad [0, +\infty) \times [0, +\infty)\,,
\end{aligned}$$

and we use the following real constants

$$m_i := \left(\sup_{t\in[0,1]} \int_0^1 k_i(t,s)\, ds \right)^{-1}, \tag{7}$$

$$M_i := \left(\inf_{t\in[a_i,b_i]} \int_{a_i}^{b_i} k_i(t,s) ds \right)^{-1}. \tag{8}$$

Theorem

Suppose that there exist $\rho_1, \rho_2, s_1, s_2 \in (0, +\infty)$, *with* $\rho_i < c_i\, s_i\,,\ \ i = 1, 2$, *such that the following conditions hold*

$$\sup_{\Omega^{\rho_1,\rho_2}} f_i(r, w_1, w_2, z_1, z_2) < \frac{m_i}{\sup\limits_{t\in[0,1]} p(t)}\, \rho_i \tag{9}$$

and

$$\inf_{A_i^{s_1,s_2}} f_i(r, w_1, w_2, z_1, z_2) > \frac{M_i}{\inf\limits_{t\in[a_i,b_i]} p(t)}\, s_i. \tag{10}$$

Then the system (1) *has at least one positive radial solution.*

The conditions (9) and (10) are nothing but growth estimates on two stripes.

Proof's Sketch:

We prove that $i_K(T, K_{\rho_1,\rho_2}) = 1$ and $i_K(T, V_{s_1,s_2}) = 0$. From index properties, it follows that the compact operator T has a fixed point in $V_{s_1,s_2} \setminus \overline{K}_{\rho_1,\rho_2}$. Then the system (1) admits a positive radial solution.

Examples

The Theorem can be apply when the nonlinearities f_i are of the type

$$f_i(|x|, u, v, |\nabla u|, |\nabla v|) = (\delta_i u^{\alpha_i} + \gamma_i v^{\beta_i}) q_i(|x|, u, v, |\nabla u|, |\nabla v|)$$

with q_i continuous functions bounded by a strictly positive constant, $\alpha_i, \beta_i > 1$ and suitable $\delta_i, \gamma_i \geq 0$.
For example we can consider in $\mathbb{R}^3$ the system of BVPs

$$\begin{cases} -\Delta u = \frac{e^{-|x|^2}}{6} (2 - \sin(|\nabla u|^2 + |\nabla v|^2) \, u^5 \text{ in } \Omega, \\ -\Delta v = \frac{1}{\pi} e^{-|x|^2} \arctan\left(1 + |\nabla u|^2 + |\nabla v|^2\right) \, v^5 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ v = 0 \text{ on } |x| = 1 \text{ and } \dfrac{\partial v}{\partial r} = 0 \text{ on } |x| = e, \end{cases} \tag{11}$$

where $\Omega = \{x \in \mathbb{R}^3 : 1 < |x| < e\}$.

By means of index properties and the aboveTheorem, it is possible to obtain results about the existence of *multiple* positive solutions of the system (1). For brevity, we state a result on the existence of two positive solutions.

Theorem

Suppose that there exist $\rho_i, s_i, \theta_i \in (0,\infty)$ *with* $\rho_i/c_i < s_i < \theta_i$ *such that*

$$\inf_{A_i^{\rho_1,\rho_2}} f_i(r, w_1, w_2, z_1, z_2) > \frac{M_i}{\inf_{t\in[a_i,b_i]} p(t)} \rho_i,$$

$$\sup_{\Omega^{s_1,s_2}} f_i(r, w_1, w_2, z_1, z_2) < \frac{m_i}{\sup_{t\in[0,1]} p(t)} s_i$$

$$\inf_{A_i^{\theta_1,\theta_2}} f_i(r, w_1, w_2, z_1, z_2) > \frac{M_i}{\inf_{t\in[a_i,b_i]} p(t)} \theta_i.$$

Then the system (1) *has at least two positive radial solutions.*

Thank you very much for your attention!