Giornate di Equazioni Differenziali Ordinarie: metodi e prospettive

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Esistenza di due o tre soluzioni per problemi differenziali nonlineari

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Aim of the talk

The talk is given in two parts:

PART I

I would like to give a survey on some critical point theorems involving the following class of functionals:

$$J_\lambda = \Phi - \lambda\Psi, \quad \Phi : X \to \mathbb{R}, \quad \Psi : X \to \mathbb{R}$$

where X is a Banach space and λ is a positive parameter.

A parameter local minimum problem

Find $\underline{\lambda}$ and $\overline{\lambda}$ $(0 < \underline{\lambda} < \overline{\lambda})$ such that for every $\lambda \in \Lambda := (\underline{\lambda}, \overline{\lambda})$, J_λ admits at least one local minimum (critical point) $u_\lambda \in X$.

Two lines of research:

- Existence and multiplicity of local minima (critical points)
- Estimate of the set Λ of parameters.

Some basic references

$$J_\lambda = \Phi - \lambda\Psi$$

- G. Bonanno, P. Candito, *Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities.* J. Differential Equations (2008).
- G. Bonanno, S. A. Marano, *On the structure of the critical set of non-differentiable functions with a weak compactness condition* App. Anal.(2010).
- G. Bonanno, P. Candito, G. D'Aguì, *Variational methods on finite dimensional Banach spaces and discrete problems.* Adv. Nonlinear Stud. (2014).
- G. Bonanno, G. D'Aguì, *Two non-zero solutions for elliptic Dirichlet problems.* Z. Anal. Anwend. (2016).

Some basic references

$$J : X \to \mathbf{R}$$

In other words, I would like to give a reading of the results for functionals $J_\lambda = \Phi - \lambda\Psi$ taking into account the relations between the Mountain Pass Theorem and local minima described in the following two papers:

- G. Bonanno, *A characterization of the mountain pass geometry for functionals bounded from below.* Differential Integral Equations (2012).
- G. Bonanno, *Relations between the mountain pass theorem and local minima.* Adv. Nonlinear Anal. (2012).

Ambrosetti-Rabinowitz, J. Funct. Anal. (1973)

Let X be a Banach space. If $J \in C^1(X)$, we say that J satisfies the "Palais-Smale condition" (the "PS-condition" for short), if the following holds: *"Every sequence* $\{x_n\}_{n\geq 1} \subseteq X$ *s.t.* $\{J(x_n)\}_{n\geq 1} \subseteq \mathbf{R}$ *is bounded and*

$$J'(x_n) \to 0 \text{ in } X^* \text{ as } n \to \infty,$$

admits a strongly convergent subsequence".

Mountain Pass Theorem

If $J \in C^1(X)$ *satisfies the PS-condition,* $x_0, x_1 \in X$*, with* $\|x_0 - x_1\| > \rho > 0$,

$$(MPG): \ \max\{J(x_0), J(x_1)\} < \inf\{J(x) : \|x - x_0\| = r\} = \eta_\rho,$$

$$c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \varphi(\gamma(t))$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = x_0, \gamma(1) = x_1\}$*, then* $c \geq \eta_\rho$ *and* c *is a critical value of* J.

Pucci-Serrin's Theorem

If J *admits two local minima, then* J *has a third critical point.*

Functionals unbounded from below

A first flash on the mountain pass geometry:

(a) *There exist* x_0, $x_1 \in X$ *and* $r \in (0, \|x_0 - x_1\|$ *such that*

$$\max\{J(x_0), J(x_1)\} \leq \inf\{J(x) : \|x - x_0\| = \rho\} = \eta_\rho$$

It is proved that if the functional J turns out to be **bounded from below on bounded subsets** of X and satisfies the PS-condition, then (a) holds true iff

(b) *J has at least one local minimum, which is not strictly global.*

G. Bonanno, ***Relations between the mountain pass theorem and local minima***. Adv. Nonlinear Anal. (2012).

Functionals bounded from below

A second flash on the mountain pass geometry:

(a) *There exist* $x_0, x_1 \in X$ *and* $r \in (0, \|x_0 - x_1\|$ *such that*

$$\max\{J(x_0), J(x_1)\} \leq \inf\{J(x) : \|x - x_0\| = \rho\} = \eta_\rho$$

It is proved that if the functional J turns out **bounded from below** and satisfies the PS-condition, then (a) holds true iff

(b) *J has at least two distinct local minima.*

G. Bonanno, ***A characterization of the mountain pass geometry for functionals bounded from below***. Differential Integral Equations (2012).

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Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities

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A first version of a local minimum theorem

Theorem 1 (Bo-Ca JDE, 2008, Remark 3.3.)

Let X be a reflexive Banach space, $\Phi : X \to \mathbf{R}$ and $\Psi : X \to \mathbf{R}$ two continuously Gâteaux differentiable functionals such that Φ is coercive and sequentially weakly lower semicontinuous, while Ψ is sequentially weakly upper semicontinuous.

Let $r > \inf_X \Phi$ and put

$$\varphi(r) := \inf_{v \in \Phi^{-1}(-\infty, r)} \frac{\sup_{u \in \Phi^{-1}(-\infty, r)} \Psi(u) - \Psi(v)}{r - \Phi(v)}.$$

Then, for every $\lambda \in \left(0, \frac{1}{\varphi(r)}\right)$ the functional $J_\lambda = \Phi - \lambda\Psi$ has a local minimum point $u_\lambda \in \Phi^{-1}(-\infty, r)$ such that $J_\lambda(u_\lambda) \leq J_\lambda(v)$ for every $v \in \Phi^{-1}(-\infty, r)$.

The energy functional J_λ of problem (D_λ) has the structure required on Theorem 1 with $X = W_0^{1,p}(\Omega)$, and

$$\Phi(u) := \frac{1}{p}\|u\|^p, \quad \Psi(u) := \int_\Omega F(u(x))\, dx, \quad -\Delta_p u = \lambda f(u).$$

Some notations

Provided $r, r_1, r_2 > \inf_X \Phi$, $r_2 > r_1$, we can define

$$\varphi^{(1)}(r) = \inf_{u \in \Phi^{-1}(]-\infty, r[)} \frac{\left(\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u) \right) - \Psi(u)}{r - \Phi(u)}$$

$$\varphi^{(2)}(r) = \inf_{u \in \Phi^{-1}(]-\infty, r[)} \sup_{v \in \Phi^{-1}([r, +\infty[)} \frac{\Psi(v) - \Psi(u)}{\Phi(v) - \Phi(u)}$$

$$\varphi_1(r_1, r_2) = \max \left\{ \varphi^{(1)}(r_1); \varphi^{(1)}(r_2) \right\}$$

$$\varphi_2(r_1, r_2) = \inf_{u \in \Phi^{-1}(]-\infty, r_1[)} \sup_{v \in \Phi^{-1}([r_1, r_2[)} \frac{\Psi(v) - \Psi(u)}{\Phi(v) - \Phi(u)}.$$

Moreover, $\varphi_1(r_1, r_2)$ could be 0; in this and similar cases, in the sequel, we agree to read $\frac{1}{\varphi_1(r_1, r_2)}$ as $+\infty$.

A first version of a two local minima theorem

Theorem 2 (Bo-Ca JDE, 2008, Theorem 3.1.)

Assume that there are $r_1, r_2 \in \mathbf{R}$, with $\inf_X \Phi < r_1 < r_2$, such that

$$(a_1) \qquad \varphi_1(r_1, r_2) < \varphi_2(r_1, r_2).$$

Then, for each

$$\lambda \in \Lambda_{r_1,r_2} := \left] \frac{1}{\varphi_2(r_1, r_2)}, \frac{1}{\varphi_1(r_1, r_2)} \right[$$

- the restriction of the functional J_λ to $\Phi^{-1}(]-\infty, r_1[)$ admits a global minimum u_1;
- the restriction of the functional J_λ to $\Phi^{-1}(]-\infty, r_2[)$ admits a global minimum $u_2 \notin \Phi^{-1}(]-\infty, r_1[)$.

Theorem 3 (Bo-Ca JDE, 2008, Theorem 3.2.)

Assume that there $r_1 \in \mathbf{R}$, with $\inf_X \Phi < r_1$, such that

$$(a_1') \qquad \varphi^{(1)}(r_1) < \varphi^{(2)}(r_1).$$

Assume also that for each

$$\lambda \in \Lambda^{r_1} := \left] \frac{1}{\varphi^{(2)}(r_1)}, \frac{1}{\varphi^{(1)}(r_1)} \right[$$

one has
(b_1) *the functional* $J_\lambda = \Phi - \lambda\Psi$ *is bounded below and fulfils* $(PS)_c$, $c \in \mathbf{R}$.

Then, for each $\lambda \in \Lambda^{r_1}$, J_λ admits at least three distinct critical points.

G. Bonanno, S. A. Marano, *On the structure of the critical set of non-differentiable functions with a weak compactness condition* App. Anal.(2010).

G. Bonanno, *A critical point theorem via the Ekeland variational principle*. Nonlinear Anal. (2012).

Corollary 1. Let $f : \mathbf{R} \to \mathbf{R}$ be a positive, **bounded** and continuous function.
Assume that there exist two positive constants c and d, with $c < Kd$, such that

$$\frac{F(c)}{c^p} < \frac{R}{2}\frac{F(d)}{d^p}.$$

Then, for each

$$\lambda \in \left] \frac{2^{p+1}(2^N - 1)}{pD^p} \frac{d^p}{F(d)}, \frac{1}{m(\Omega)pK^p} \frac{c^p}{F(c)} \right[,$$

the problem

$$(D_\lambda) \begin{cases} -\Delta_p u = \lambda f(u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

admits at least three weak solutions.

The constants $R(K, D, \Omega)$, D and K are well estimate. $F(x) = \int_0^x f(t)\, dt$.
In particular, K and R involve the **Talenti's constant** in the case $N < p$.

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Two Non-Zero Solutions for Elliptic Dirichlet Problems

Gabriele Bonanno and Giuseppina D'Aguì

Abstract. The aim of this note is to point out a two non-zero critical points theorem for differentiable functionals and, as an application, to obtain existence results of two positive solutions for elliptic Dirichlet problems by requiring, in particular, a suitable condition on the nonlinearity which is more general than the sublinearity at zero.

Keywords. Variational methods, multiple critical points, elliptic nonlinear equations, boundary value problems

Mathematics Subject Classification (2010). Primary 49J35, 35J60, secondary 35J20, 34B15

Theorem *Let X be a real Banach space and let $\Phi, \Psi \colon X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that*

$$\frac{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}$$

and, for each $\lambda \in \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)} \right[$, the functional $I_\lambda = \Phi - \lambda \Psi$ satisfies (PS)-condition and it is unbounded from below.

Then, for each $\lambda \in \left] \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)} \right[$, the functional I_λ admits at least two non-zero critical points $u_{\lambda,1}$, $u_{\lambda,2}$ such that $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$.

G. Bonanno, G. D'Aguì, *Two non-zero solutions for elliptic Dirichlet problems*. Z. Anal. Anwend. (2016).

G. Bonanno, *A critical point theorem via the Ekeland variational principle*. Nonlinear Anal. (2012).

- H. Amann, *On the number of solutions of asymptotically superlinear two point boundary value problems.* Arch. Rational Mech. Anal. (1974)
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- F. Brock, L. Iturriaga, P. Ubilla, *A multiplicity result for the p-Laplacian involving a parameter.* Ann. Henri PoincarÃľ (2008).
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- H. Pan, R. Xing, *On the existence of positive solutions for some nonlinear boundary value problems and applications to MEMS models.* Discrete Contin. Dyn. Syst. (2015).

PART II

I would like to discuss some results about the existence of three solutions for a $p-$Laplacian problem obtained by combining variational methods, truncation techniques and sub-super solutions methods.
The Mountain Pass Geometry of the "energy functional" is performed in a different way white respect to the above mentioned papers.

Some basic references

- P. Candito, S. Carl and R. Livrea, *Multiple solutions for quasilinear elliptic problems via critical points in open sublevels and truncation principles*, J. Math. Anal. Appl. **395** (2012), 156–163. (When $p > N$)
- P. Candito, S. Carl and R. Livrea, *Critical points in open sublevels and multiple solutions for parameter-depending quasilinear elliptic equations*, Adv. Differential Equations **19** (2014), 1021–1042. (When $1 < p < \infty$)
- P. Candito, S. Carl and R. Livrea, *Variational versus pseudomonotone operator approach in parameter-dependent nonlinear elliptic problems* , Dynamic Systems and Applications **22** (2013). 397–410. (No variational methods).

In this later, an interesting comparison has been carried out between the intervals obtained by using a local minimum theorem and the analogous results obtained by using the pseudomonotone operators and Schauder's fixed point theorem.

A p-Laplacian model parameter-depending problem

$$(\mathrm{D}_\lambda) \quad \begin{cases} -\Delta_p u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where

- $\Omega \subset \mathbf{R}^N$ is a bounded domain, with a smooth boundary $\partial\Omega$,
- $f : \mathbf{R} \to \mathbf{R}$ is a continuous function,
- $-\Delta_p u := -\mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator $(1 < p < \infty)$, defined by

$$\langle -\Delta_p u, v \rangle := \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx \quad \forall\, u, v \in W_0^{1,p}(\Omega)$$

- $\lambda > 0$.

Growth condition

(f_*) *There exist two positive constant* M_1, M_2 *and* $q \in [1, p^*)$ *such that*

$$|f(s)| \leq M_1 + M_2|s|^{q-1} \quad \forall s \in \mathbf{R}.$$

Put

$$\lambda^* = \begin{cases} +\infty, & 1 \leq q < p; \\ \dfrac{1}{c_p^p M_2}, & q = p; \\ \dfrac{q^{\frac{p-1}{q-1}}}{p(q-1)} \left(\dfrac{q-p}{c_1 M_1}\right)^{\frac{q-p}{q-1}} \left(\dfrac{p-1}{c_q^q M_2}\right)^{\frac{p-1}{q-1}}, & p < q < p^*, \end{cases}$$

where c_1, c_p and c_q are the constants of the embedding of $W_0^{1,p}(\Omega)$ in the spaces $L^1(\Omega)$, $L^p(\Omega)$ and $L^q(\Omega)$ respectively, $1 < p < N$.

Theorem N– Positive, negative and nodal solution

Assume that f satisfy (f_*) and that

$$(f^{\lambda_2})' \quad \frac{\lambda_2}{\lambda^*} < L = \liminf_{s\to 0} \frac{f(s)}{|s|^{p-2}s} \le \limsup_{s\to 0} \frac{f(s)}{|s|^{p-2}s} < +\infty.$$

Then, for every $\lambda \in \left]\frac{\lambda_2}{L}, \lambda^*\right[$ *problem* (D_λ) *admits*

- *one positive solution, one negative solution*

$$u_+ = u_+(\lambda) \in \mathrm{int}(C_0^1(\bar\Omega)_+), \quad u_- = u_-(\lambda) \in -\mathrm{int}(C_0^1(\bar\Omega)_+);$$

- *and one third solution which is nodal*

$$u_0 = u_0(\lambda) \in C_0^1(\bar\Omega).$$

Being λ_2 the second eigenvalue of $-\Delta_p u$ in $W_0^{1,2}(\Omega)$.

Remarks on existence results

Theorem 2.1, CCL, ADE, 2014

Let $f : \mathbf{R} \to \mathbf{R}$ *be a continuous function with* $f(0) \neq 0$, *and assume that*

(f$_*$) *There exist two positive constants* M_1 *and* M_2 *and* $q \in [1, p^*)$ *such that*

$$|f(s)| \leq M_1 + M_2 |s|^{q-1}, \quad \forall s \in \mathbf{R}.$$

Then, for every $\lambda \in (0, \lambda^*)$, problem (1) admits a nontrivial solution $u_\lambda \in C_0^1(\overline{\Omega})$.

Remarks on the localization of a solution

Remark 1. The variational approach adopted in CCL, ADE, 2014, not only allows for the existence of at least one nontrivial solution u_λ, but also provides a localization of u_λ, namely

$$\|u_\lambda\| < (p\bar{r})^{1/p}, \tag{1}$$

where $\bar{r} = \bar{r}(\lambda)$ can a priori be estimated from below in the case $1 \leq q \leq p$, while in the case $p < q < p^*$ we have a uniform bound given by

$$\bar{r} = \frac{1}{p}\left[\frac{qc_1 M_1(p-1)}{c_q^q M_2(q-p)}\right]^{p/(q-1)}. \tag{2}$$

Remark 2. The following variational characterization holds: u_λ *is a local minimum of the energy functional* J_λ *related to problem* (D_λ).
It is worth noticing that this last variational property of u_λ plays a crucial role whenever the existence and multiplicity of solutions are investigated in the case $f(0) = 0$

From “ bounded to unbounded” functionals

Remark 3.
We note that for every $\lambda \in [0, \lambda^*)$, the functional $J_\lambda = \Phi - \lambda\Psi$ is coercive in X provided that (f_*) holds and $1 \leq q \leq p$.
Indeed, for every $u \in X$, we have

$$J_\lambda(u) = \frac{1}{p}\|u\|^p - \lambda \int_\Omega F(u(x))\ dx \geq \frac{1}{p}\|u\|^p - \lambda c_1 M_1 \|u\| - \lambda \frac{M_2}{q} c_q^q \|u\|^q$$

$$= \left(\frac{1}{p}\|u\|^{p-q} - \lambda c_q^q \frac{M_2}{q} \right) \|u\|^q - \lambda c_1 M_1 \|u\|.$$

from which it is clear that our claim holds true.

Consequently, the advantage of using the LMT, instead of the classical direct method in calculus of variations, is more obvious in case that the functional J_λ is unbounded from below, which may occur if $p < q < p^*$.

No Variational Methods

Theorem CCL-DSA, 2013 – Existence of one nontrivial solution

Assume that f satisfy (f_*) and

$$f(0) \neq 0.$$

Then, for every $0 < |\lambda| < \mu^*$ problem (D_λ) admits at least one nontrivial solution $u \in C_0^1(\bar{\Omega})$, where

$$\mu^* := \begin{cases} +\infty, & 1 \leq q < p; \\ \dfrac{1}{c_p^p M_2}, & q = p; \\ \dfrac{1}{q-1} \left(\dfrac{q-p}{c_1 M_1} \right)^{\frac{q-p}{q-1}} \left(\dfrac{p-1}{c_q^q M_2} \right)^{\frac{p-1}{q-1}}, & p < q < p^*, \end{cases}$$

where c_1, c_p and c_q are the constant of the embedding of $W_0^{1,p}(\Omega)$ in $L^1(\Omega)$, $L^p(\Omega)$ and $L^q(\Omega)$ respectively.

Comparing λ^* and μ^*

$$\lambda^* = \mu^* \quad \text{if } 1 \leq q \leq p.$$

Otherwise, for $p < q < p^*$ one has

$$\lambda^* = \frac{q^{\frac{p-1}{q-1}}}{p(q-1)} \left(\frac{q-p}{c_1 M_1} \right)^{\frac{q-p}{q-1}} \left(\frac{p-1}{c_q^q M_2} \right)^{\frac{p-1}{q-1}},$$

$$\mu^* = \frac{1}{q-1} \left(\frac{q-p}{c_1 M_1} \right)^{\frac{q-p}{q-1}} \left(\frac{p-1}{c_q^q M_2} \right)^{\frac{p-1}{q-1}}.$$

It is easy to show that

$$\lambda^* < \mu^*.$$

Hence, in any case the intervals of parameters obtained are larger than those pointed out in the previous results.

A two-point boundary value problem with the prescribed mean curvature equation

$$(\mathrm{P}_\lambda) \quad \begin{cases} -\left(\frac{u'}{\sqrt{1+|u'|^2}}\right)' = \lambda\theta(t)f(u) & \text{in }]0,1[\\ u(0)=u(1)=0, \end{cases}$$

where

- $\theta : [0,1] \to \mathbf{R}$ is a continuous positive function with

$$\int_0^1 \theta(t)\, dt = 1,$$

- $f : \mathbf{R} \to \mathbf{R}$ is a continuous function,
- λ is a positive parameter.

P. Candito, R. Livrea, J. Mawhin, *Three solutions for a two-point boundary value problem with the prescribed mean curvature equation.* Differential Integral Equations **28** (2015).

References

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- F. Obersnel and P. Omari, *Positive solutions of the Dirichlet problem for the prescribed mean curvature equation*, J. Differential Equation, (2010).
- F. Obersnel and P. Omari, *Multiple non-trivial solutions of the Dirichlet problem for the prescribed mean curvature equation*, Contemp. Math., (2011).
- C. Corsato, C. De Coster, F. Obersnel, P. Omari, A. Soranzo, *A prescribed anisotropic mean curvature equation modeling the corneal shape: a paradigm of nonlinear analysis*. Submitted.

Theorem 5

Let $f : \mathbf{R} \to \mathbf{R}$ *be a continuous function,* $F(s) = \int_0^s f(t)dt$ *for all* $s \in \mathbf{R}$*, and assume that*

$$\lim_{t\to 0} \frac{f(t)}{t} = +\infty.$$

Then, for every $c > 0$*, if we put*

$$\lambda^*(c) = 2\left[\left(\frac{\max_{|s|\le c} F(s)}{c^2}\right)^2 + \left(2\max_{|t|\le c}|f(t)|\right)^2\right]^{-1/2},$$

for every $\lambda \in]0, \lambda^*(c)[$ *problem*

$$-\left(\frac{u'}{\sqrt{1+|u'|^2}}\right)' = \lambda\theta(t)f(u) \quad in\ ]0,1[, \quad u(0) = u(1) = 0,$$

admits at least three nontrivial classical solutions $u_i = u_{i,\lambda} \in C^2([0,1])$*, such that*

$u_1 \in \mathrm{int}(C_0^1([0,1])_+)$, $u_2 \in -\mathrm{int}(C_0^1([0,1])_+)$, $\|u_{i,\lambda}\|_\infty \le c$, $i = 1, 2, 3.$

Example

If θ is a continuous and positive function satisfying $\int_0^1 \theta(t)\, dt = 1$, the following problem

$$\begin{cases} -\left(\frac{u'}{\sqrt{1+|u'|^2}}\right)' = \theta(t)\left[\frac{u}{\sqrt{|u|}} + u^2 e^{u^3}\right] & \text{in } [0,1] \\ u(0) = u(1) = 0 \end{cases}$$

admits at least three nontrivial classical solutions $u_i \in C^2([0,1])$, $i = 1, 2, 3$.

- $u_1 \in \mathrm{int}(C_0^1([0,1])_+)$
- $u_2 \in -\mathrm{int}(C_0^1([0,1])_+)$
- $\|u_i\|_\infty \leq 1/4$, $i = 1, 2, 3$.

Apply Theorem 4 with $f(x) = \frac{x}{\sqrt{|x|}} + x^2 e^{x^3}$ for all $x \in \mathbf{R}$ and $c = 1/4$, being $\lambda^*(c) > 1$.

Theorem 6

Let $f : \mathbf{R} \to \mathbf{R}$ *be a continuous function,* $F(s) = \int_0^s f(t)dt$ *for all* $s \in \mathbf{R}$*, and assume that*

$$\lim_{s \to 0} \frac{F(s)}{s^2} = +\infty$$

in addition to

$$tf(t) > 0 \quad \forall t \in]-c, c[\setminus\{0\},$$

for some $c > 0$.
Then, put

$$\lambda^*(c) = 2\left[\left(\frac{\max_{|s|\leq c} F(s)}{c^2}\right)^2 + \left(2\max_{|t|\leq c}|f(t)|\right)^2\right]^{-1/2},$$

for every $\lambda \in]0, \lambda^*(c)[$ *problem* (D_λ) *admits at least three nontrivial classical solutions* $u_1(\lambda), u_2(\lambda), u_3(\lambda) \in C^2([0,1])$ *such that:*

$u_1 \in \mathrm{int}(C_0^1([0,1])_+)$, $u_2 \in -\mathrm{int}(C_0^1([0,1])_+)$ *and* $\|u_{i,\lambda}\|_\infty \leq c$.

Example

If $a \in [0,1[$, the following problem

$$\begin{cases} \left(\frac{u'}{\sqrt{1+|u'|^2}}\right)' = (1 + a\sin 2\pi t)\, u \log|u| & \text{in } [0,1] \\ u(0) = u(1) = 0 \end{cases}$$

admits at least three nontrivial classical solutions $u_i \in C^2([0,1])$, $i = 1,2,3$

- $u_1 \in \mathrm{int}(C_0^1([0,1])_+)$,
- $u_2 \in -\mathrm{int}(C_0^1([0,1])_+)$,
- $\|u_i\|_\infty \leq 1$, $i = 1,2,3$.

Grazie a tutti voi per l'attenzione.

La "Rokka du Dragu" nella Calabria Greca