Existence of one non-zero solution for a two point boundary value problem involving a fourth-order equation

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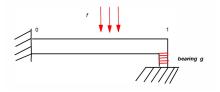
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$$\begin{cases} u^{(iv)}(x) + \lambda f(x, u(x)) = 0 \text{ in } [0, 1] \\ u(0) = u'(0) = 0 \\ u''(1) = 0 \quad u'''(1) = \mu g(u(1)) \end{cases}$$
(P<sub>\lambda, \mu)</sub>

- $f:[0,1]\times \mathbb{R} \to \mathbb{R}$  is an  $L^1-\text{Carathéodory}$  function
- $\bullet \ g: \mathbb{R} \to \mathbb{R}$  is a continuous function
- $\lambda, \mu$  are positive parameters

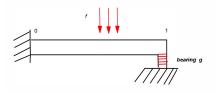
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#### variational methods

- L. YANG, H. CHEN, X. YANG,
  - Appl. Math. Letters (2011) existence of three solutions for problem  $(P_{\lambda,1})$
  - Appl. Math. and Computations (2011) existence of three solutions for problem  $(P_{\lambda,\lambda})$
- T.F. MA, *Appl. Math. and Computations* (2004) existence of at least two positive solutions for problem  $(P_{1,1})$
- A. CABADA, S. TERSIAN, Appl. Math. and Computations, (2013) existence of two non trivial solutions for problem  $(P_{\lambda,\lambda})$

#### iterative methods

• T.F. MA, J. DA SILVA, Appl. Math. Letters (2004) problem  $(P_{\lambda,\mu})$ .

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$$X := \{ u \in H^2([0,1]) : u(0) = u'(0) = 0 \}$$
$$\|u\| := \left( \int_0^1 (u''(t))^2 \, dt \right)^{\frac{1}{2}}$$

- the embedding  $X \hookrightarrow C^1([0,1])$  is compact
- $\|u\|_{C^1([0,1])} := \max\{\|u\|_{\infty}, \|u'\|_{\infty}\} \le \|u\|$  for each  $u \in X$

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# The functional $\Phi$

$$\forall x \in X \qquad \Phi(u) := \frac{1}{2} \|u\|^2$$

- $\Phi$  is Frechét differentiable,  $\Phi \in C^1(X)$
- $\langle \Phi'(u), v \rangle = \int_0^1 u''(x)v''(x) dx$ , for each  $u, v \in X$
- ullet  $\Phi$  is sequentially weakly lower semi-continuous and coercive
- $\Phi'$  admits a continuous inverse on  $X^*$

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# The functional $\Psi_{\lambda,\mu}$

 $\forall \lambda, \mu > 0, \forall u \in X$ 

$$\Psi_{\lambda,\mu}(u) := \int_0^1 F(x, u(x)) \, dx + \frac{\mu}{\lambda} G(u(1))$$

• 
$$F(x,\xi) := \int_0^{\xi} f(x,t) dt$$
 for each  $x \in [0,1], \xi \in \mathbb{R}$ 

• 
$$G(\xi) := \int_0^{\xi} g(t) dt$$
 for each  $\xi \in \mathbb{R}$ 

- $\Psi_{\lambda,\mu}$  is Frechét differentiable,  $\Psi_{\lambda,\mu} \in C^1(X)$
- $\langle \Psi'_{\lambda,\mu}(u), v \rangle = \int_0^1 f(x, u(x))v(x) dx + \frac{\mu}{\lambda}g(u(1))v(1)$ , for each  $u, v \in X$ •  $\Psi'_{\lambda,\mu}$  is compact

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$$I_{\lambda,\mu} := \Phi - \lambda \Psi_{\lambda,\mu},$$

### L. Yang, H. Chen, X. Yang (2011)

For each  $\lambda, \mu > 0$ , the critical points of  $I_{\lambda,\mu}$  are classical solutions for problem  $(P_{\lambda,\mu})$ .

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G. BONANNO, Relations between the mountain pass theorem and ocal minima, *Adv. Nonlinear Anal.*, **1** (2012), 205–220.

# $\begin{cases} u^{(iv)}(x) + \lambda f(x, u(x)) = 0 \text{ in } [0, 1] \\ u(0) = u'(0) = 0 \\ u''(1) = 0 \quad u'''(1) = \mu g(u(1)) \end{cases}$ $(P_{\lambda, \mu})$

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# The main tool

• X real Banach space

• 
$$\Phi, \Psi: X \to \mathbb{R}, \ \Phi, \Psi \in C^1(X)$$

• 
$$\inf_{x \in X} \Phi(x) = \Phi(0) = \Psi(0) = 0$$

• 
$$\exists r > 0 \text{ and } \bar{x} \in X, r < \Phi(\bar{x}):$$
  
 $(a_1) \quad \frac{\sup_{\Phi(x) \le r} \Psi(x)}{r} < \frac{\Psi(\bar{x})}{\Phi(\bar{x})},$   
 $(a_2) \text{ for each } \lambda \in ]\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x) \le r} \Psi(x)}[ \text{ the functional } I_{\lambda} := \Phi - \lambda \Psi$   
satisfies  $(P.S.)^{[r]}$  condition.

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# The main tool

#### Theorem 1 (Bonanno(2012))

$$\begin{array}{l} \text{for each } \lambda \in \Lambda_r := ] \frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x) \leq r} \Psi(x)} [ \text{, there is } x_{0,\lambda} \in \Phi^{-1}(]0, r[) \text{ such } \\ \text{that } I'_{\lambda}(x_{0,\lambda}) \equiv \vartheta_{X^*} \text{ and } I_{\lambda}(x_{0,\lambda}) \leq I_{\lambda}(x) \text{ for all } x \in \Phi^{-1}(]0, r[). \end{array}$$

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# Notations

For  $\alpha > 0$ , we put

$$F^{\alpha} := \int_0^1 \max_{|\xi| \le \alpha} F(x,\xi) \ dx$$

and

$$G^{\alpha} := \max_{|\xi| \le \alpha} G(\xi)$$

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# The main result

 $(f_1)$  there exist  $\delta, \gamma \in \mathbb{R}$ , with  $0 < \delta < \gamma$ , such that

$$\frac{F^{\gamma}}{\gamma^2} < \frac{1}{8\pi^4} \left(\frac{3}{2}\right)^3 \frac{\int_{\frac{3}{4}}^1 F(x,\delta) \ dx}{\delta^2}$$

 $(f_2)$   $F(x,t) \ge 0$  for almost every  $x \in [0,1]$  and for all  $t \in [0,\delta]$ .

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## The main result

$$\Lambda_{\delta,\gamma} := \left] 4\pi^4 \left(\frac{2}{3}\right)^3 \frac{\delta^2}{\int_{\frac{3}{4}}^1 F(x,\delta) \, dx}, \frac{\gamma^2}{2F^{\gamma}} \right[$$

# Theorem 2 (Bonanno,C. , Tersian - *Electronic Journal of Qualitative Theory of Differential Equations* - (2015))

for each  $\lambda \in \Lambda_{\delta,\gamma}$  and for each  $g : \mathbb{R} \to \mathbb{R}$  continuous, there exists  $\eta_{\lambda,g} > 0$  such that for each  $\mu \in ]0, \eta_{\lambda,g}[$  the problem  $(P_{\lambda,\mu})$  admits at least one non-zero solution  $u_{\lambda}$  such that  $||u_{\lambda}||_{\infty}, ||u'_{\lambda}||_{\infty} < \gamma$ .

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$$\eta_{\lambda,g} = \begin{cases} \frac{\gamma^2 - 2\lambda F^{\gamma}}{2G^{\gamma}} & \text{if } G(\delta) \ge 0\\ \min\left\{\frac{\gamma^2 - 2\lambda F^{\gamma}}{2G^{\gamma}}, \frac{4\pi^4 \delta^2 - \lambda \left(\frac{3}{2}\right)^3 \int_{\frac{3}{4}}^1 F(x,\delta) \, dx}{\left(\frac{3}{2}\right)^3 G(\delta)} \right\} & \text{if } G(\delta) < 0, \end{cases}$$

#### Remark

We read 
$$\frac{\gamma^2-2\lambda F^\gamma}{2G^\gamma}=+\infty$$
 when  $G^\gamma=0$ 

#### Remark

For each  $\lambda \in \Lambda_{\delta,\gamma}$  and  $g : \mathbb{R} \to \mathbb{R}$  continuous it results  $\eta_{\lambda,q} > 0$ .

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• 
$$X := \{ u \in H^2([0,1]) : u(0) = u'(0) = 0 \}$$

• 
$$\lambda \in \Lambda_{\delta,\gamma}$$
,  $g : \mathbb{R} \to \mathbb{R}$  continuous,  $\mu \in ]0, \eta_{\lambda,g}[$ 

$$I_{\lambda,\mu}(u) := \underbrace{\frac{1}{2} \|u\|^2}_{\Phi(u)} - \lambda(\underbrace{\int_0^1 F(x, u(x)) \, dx + \frac{\mu}{\lambda} G(u(1))}_{\Psi_{\lambda,\mu}(u)})$$

• 
$$\inf_{x \in X} \Phi(x) = \Phi(0) = \Psi_{\lambda,\mu}(0) = 0$$

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 $I_{\lambda,\mu}$  verifies  $(a_2)$  of Theorem 1

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for all  $r \in ]-\infty, +\infty[$  the function  $I_{\lambda,\mu}$  satisfies the  $(P.S.)^{[r]}$ -condition.

 $I_{\lambda,\mu}$  verifies  $(a_2)$  of Theorem 1

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#### $I_{\lambda,\mu}$ verifies $(a_2)$ of Theorem 1

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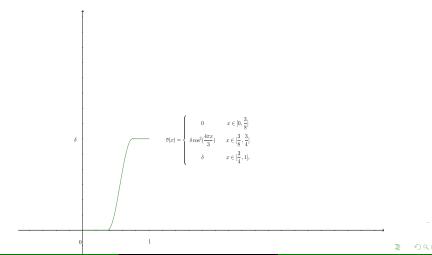
 $\bullet~\bar{v}$  depends on  $\delta$ 

$$\bar{v}(x) = \begin{cases} 0 & x \in [0, \frac{3}{8}] \\ \delta \cos^2(\frac{4\pi x}{3}) & x \in [\frac{3}{8}, \frac{3}{4}] \\ \delta & x \in [\frac{3}{4}, 1], \end{cases}$$

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$$\Phi(\bar{v}) = 4\pi^4 \delta^2 \left(\frac{2}{3}\right)^3 \quad \bar{v}(x) \in [0, \delta] \ \forall x \in [\frac{3}{8}, \frac{3}{4}] \quad + \quad (f_2)^{\text{s}}$$

$$\Psi_{\lambda,\mu}(\bar{v}) \ge \int_{\frac{3}{4}}^1 F(x, \delta) \ dx + \frac{\mu}{\lambda} G(\delta)$$

$$\Downarrow$$

$$\frac{\Psi_{\lambda,\mu}(\bar{v})}{\Phi(\bar{v})} \ge \frac{\int_{\frac{3}{4}}^{1} F(x,\delta) \, dx + \frac{\mu}{\lambda} G(\delta)}{4\pi^4 \delta^2 \left(\frac{2}{3}\right)^3}$$

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r

• 
$$r$$
 depends on  $\gamma$ ,  $r = \frac{\gamma^2}{2}$   
 $\forall u \in X : \Phi(u) \le r$ ,  
 $\Downarrow$   
 $\|u\|_{\infty} \le \gamma$   
 $\Downarrow$ 

$$\frac{1}{r} \sup_{u \in \Phi^{-1}(]-\infty,r])} \Psi_{\lambda,\mu}(u) \le \frac{2}{\gamma^2} F^{\gamma} + \frac{2}{\gamma^2} \frac{\mu}{\lambda} G^{\gamma}$$

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r and  $\bar{v}$  verify  $(a_1)$  of Theorem 1

• 
$$G(\delta) \ge 0$$
,  $\eta_{\lambda,g} = \frac{\gamma^2 - 2\lambda F\gamma}{2G^{\gamma}}$ 

$$\underbrace{\frac{\frac{2}{\gamma^2}F^{\gamma} + \frac{2}{\gamma^2}\frac{\mu}{\lambda}G^{\gamma} < \frac{1}{\lambda}}{\frac{1}{\lambda} < \frac{\int_{\frac{3}{4}}^{1}F(x,\delta) dx}{4\pi^4\delta^2\left(\frac{2}{3}\right)^3} \le \frac{\int_{\frac{3}{4}}^{1}F(x,\delta) dx}{4\pi^4\delta^2\left(\frac{2}{3}\right)^3} + \frac{\mu}{\lambda}\frac{G(\delta)}{4\pi^4\delta^2\left(\frac{2}{3}\right)^3}}{\lambda \in \Lambda_{\delta,\gamma}}$$

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$$G(\delta) < 0$$
,  $\eta_{\lambda,g} = \min\left\{\frac{\gamma^2 - 2\lambda F^{\gamma}}{2G^{\gamma}}, \frac{4\pi^4 \delta^2 - \lambda\left(\frac{3}{2}\right)^3 \int_{\frac{3}{4}}^{\frac{1}{4}} F(x,\delta) \, dx}{\left(\frac{3}{2}\right)^3 G(\delta)}\right\}$ 

$$\underbrace{\frac{\frac{\mu < \eta_{\lambda,g}}{2}}{\frac{2}{\gamma^2}F^{\gamma} + \frac{2}{\gamma^2}\frac{\mu}{\lambda}G^{\gamma} < \frac{1}{\lambda}}}_{\substack{\mu < \eta_{\lambda,g}}} \underbrace{\frac{1}{\lambda} < \frac{\int_{\frac{3}{4}}^{\frac{1}{4}}F(x,\delta) \, dx}{4\pi^4\delta^2 \left(\frac{2}{3}\right)^3} + \frac{\mu}{\lambda}\frac{G(\delta)}{4\pi^4\delta^2 \left(\frac{2}{3}\right)^3}}_{\substack{\mu < \eta_{\lambda,g}}}$$

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r and  $\bar{v}$  verify  $(a_1)$  of Theorem 1

$$\frac{2}{\gamma^2}F^{\gamma} + \frac{2}{\gamma^2}\frac{\mu}{\lambda}G^{\gamma} < \frac{1}{\lambda} < \frac{\int_{\frac{3}{4}}^{\frac{1}{4}}F(x,\delta)\,dx}{4\pi^4\delta^2\left(\frac{2}{3}\right)^3} + \frac{\mu}{\lambda}\frac{G(\delta)}{4\pi^4\delta^2\left(\frac{2}{3}\right)^3}$$

$$\frac{1}{r} \sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi_{\lambda,\mu}(u) < \frac{1}{\lambda} < \frac{\Psi_{\lambda,\mu}(\bar{v})}{\Phi(\bar{v})}$$

$$\downarrow$$

$$(a_1) \text{ of Theorem 1 is verified}$$

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r and  $\bar{v}$  verify  $(a_1)$  of Theorem 1

$$\frac{2}{\gamma^2}F^{\gamma} + \frac{2}{\gamma^2}\frac{\mu}{\lambda}G^{\gamma} < \frac{1}{\lambda} < \frac{\int_{\frac{3}{4}}^{1}F(x,\delta)\,dx}{4\pi^4\delta^2\left(\frac{2}{3}\right)^3} + \frac{\mu}{\lambda}\frac{G(\delta)}{4\pi^4\delta^2\left(\frac{2}{3}\right)^3}$$

#### ₩

$$\frac{1}{r} \sup_{u \in \Phi^{-1}(]-\infty,r])} \Psi_{\lambda,\mu}(u) < \frac{1}{\lambda} < \frac{\Psi_{\lambda,\mu}(\bar{v})}{\Phi(\bar{v})}$$

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 $\Phi(\bar{v}) < r$ 

• 
$$\delta < \gamma$$
 +  $(f_1)^{\circ} \Longrightarrow \sqrt{8\pi^4 \left(\frac{2}{3}\right)^3} \delta < \gamma$   
 $\Downarrow$   
 $\Phi(\bar{v}) = 4\pi^4 \delta^2 \left(\frac{2}{3}\right)^3 < \frac{\gamma^2}{2} = r$   
•  $\lambda \in \Lambda_{\delta,\gamma} \subseteq \left[\frac{\Phi(\bar{v})}{\Psi_{\lambda,\mu}(\bar{v})}, \frac{r}{\sup_{\Phi(u) \le r} \Psi_{\lambda,\mu}(u)}\right[$   
 $\Downarrow$ 

Theorem 1 guarantees the existence of a non-trivial classical solution of problem  $(P_{\lambda,\mu})$ ,  $u_{\lambda}$  such that  $||u_{\lambda}||_{\infty}, ||u'_{\lambda}||_{\infty} < \gamma$ 

 $\begin{array}{c} & \text{The problem} \\ & \text{Physical meaning} \\ \text{Variational structure of problem } (P_{\lambda,\mu}) \\ & \text{The main tool} \\ & \text{The main result} \\ & \text{Proof of Theorem 2} \\ & \text{Consequences} \end{array}$ 

Consequence

# An example of application of Theorem 2

• 
$$f: \mathbb{R} \to \mathbb{R}$$
  

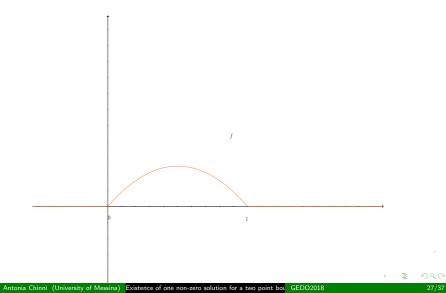
$$f(u) := \begin{cases} 0, & u < 0, \\ u - u^2, & 0 \le u \le 1, \\ 0 & u > 1. \end{cases}$$
•  $\delta = \frac{1}{2}, \gamma = 22$ 

$$\Downarrow$$

for each  $\lambda \in ]1385.4, 1452[$  and each  $g: \mathbb{R} \to \mathbb{R}$  continuous there exists  $\eta_{\lambda,g} > 0$  such that for each  $\mu \in ]0, \eta_{\lambda,g}[$ , the problem  $(P_{\lambda,\mu})$  admits at least one non-zero solution  $u_{\lambda}$  with  $||u||_{\infty}, ||u'||_{\infty} < 22$ .

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Consequence

A particular case of Theorem 2

 $(f_1'') \ f: \mathbb{R} \to [0, +\infty[ \text{ continuous, } \limsup_{t \to 0^+} \frac{F(t)}{t^2} = +\infty$ 

∜

### Theorem 3 (Bonanno, C., Tersian - EJQTDE - (2015))

$$\begin{split} &\forall \gamma > 0, \, \forall \lambda \in ]0, \frac{\gamma^2}{2F(\gamma)} [, \, \forall g : \mathbb{R} \to [0, +\infty[ \text{ continuous and} \\ &\forall \mu \in ]0, \frac{\gamma^2 - 2F(\gamma)\lambda}{2G(\gamma)} [, \, \text{the problem} \end{split}$$

$$\begin{cases} u^{(iv)}(x) + \lambda f(u(x)) = 0 \text{ in } [0,1] \\ u(0) = u'(0) = 0 \\ u''(1) = 0 \quad u'''(1) = \mu g(u(1)) \end{cases}$$
 ( $\tilde{P}_{\lambda,\mu}$ 

admits at least one non-zero classical solution u such that  $\|u\|_{\infty}, \|u'\|_{\infty} < \gamma$ 

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## variational structure in Theorem 2

$$I_{\lambda,\mu}(u) := \underbrace{\frac{1}{2} \|u\|^2}_{\Phi(u)} -\lambda(\underbrace{\int_0^1 F(x, u(x)) \, dx + \frac{\mu}{\lambda} G(u(1))}_{\Psi_{\lambda,\mu}(u)})$$

$$I_{\lambda,\mu}(u) := \underbrace{\frac{1}{2} \|u\|^2}_{\Phi(u)} - \mu(\underbrace{\frac{\lambda}{\mu} \int_0^1 F(x, u(x)) \, dx + G(u(1))}_{\tilde{\Psi}_{\lambda,\mu}(u)})$$

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## variational structure in Theorem 2

$$I_{\lambda,\mu}(u) := \underbrace{\frac{1}{2} \|u\|^2}_{\Phi(u)} -\lambda(\underbrace{\int_0^1 F(x, u(x)) \, dx + \frac{\mu}{\lambda} G(u(1))}_{\Psi_{\lambda,\mu}(u)})$$

$$I_{\lambda,\mu}(u) := \underbrace{\frac{1}{2} \|u\|^2}_{\Phi(u)} - \mu(\underbrace{\frac{\lambda}{\mu} \int_0^1 F(x, u(x)) \, dx + G(u(1))}_{\tilde{\Psi}_{\lambda,\mu}(u)})$$

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$$(g_1)$$
 there exist  $\delta, \gamma \in \mathbb{R}$  with  $0 < \delta < \gamma$ :

$$\frac{G^{\gamma}}{\gamma^2} < \frac{1}{8\pi^4} \left(\frac{3}{2}\right)^3 \frac{G(\delta)}{\delta^2}$$

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$$\Gamma_{\delta,\gamma} := \left] 4\pi^4 \left(\frac{2}{3}\right)^3 \frac{\delta^2}{G(\delta)}, \frac{\gamma^2}{2G^{\gamma}} \right[$$

$$\Downarrow$$

### Theorem 2'

for each  $\mu \in \Gamma_{\delta,\gamma}$  and for each  $f:[0,1] \times \mathbb{R} \to \mathbb{R} L^1$ -Carathéodory function verifying condition  $(f_2)$  of Theorem 2<sup>(6)</sup>, there exists  $\theta_{\mu,f} := \frac{\gamma^2 - 2\mu G^{\gamma}}{2F^{\gamma}} > 0$  such that for each  $\lambda \in ]0, \theta_{\mu,f}[$  the problem  $(P_{\lambda,\mu})$  admits at least one non-zero solution u such that  $||u||_{\infty}, ||u'||_{\infty} < \gamma$ 

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#### Consequence

A particular case of Theorem 2'

$$(g_1'') \ g: \mathbb{R} \to [0, +\infty[ \text{ continuous, } \limsup_{t \to 0^+} \frac{g(t)}{t} = +\infty$$

∜

### Theorem 3' (Bonanno, C., Tersian - EJQTDE - (2015))

$$\begin{split} &\forall \gamma > 0, \, \forall \mu \in ]0, \frac{\gamma^2}{2G(\gamma)} [, \, \forall f : \mathbb{R} \to [0, +\infty[ \text{ continuous and} \\ &\forall \lambda \in ]0, \frac{\gamma^2 - 2\mu G(\gamma)}{2F(\gamma)} [, \, \text{the problem} \end{split}$$

$$\begin{cases} u^{(iv)}(x) + \lambda f(u(x)) = 0 \text{ in } [0,1] \\ u(0) = u'(0) = 0 \\ u''(1) = 0 \quad u'''(1) = \mu g(u(1)) \end{cases}$$
 ( $\tilde{P}_{\lambda,\mu}$ 

admits at least one non-zero classical solution u such that  $\|u\|_{\infty}, \|u'\|_{\infty} < \gamma$ 

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#### Consequence

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# Corollary of Theorem 3'

$$f: \mathbb{R} \to [0, +\infty[, f \text{ continuous},$$

### Corollary of Theorem 3'

for each 
$$\lambda \in \left]0, \frac{1}{10\int_0^2 f(t)dt}\right[$$
 the problem  
$$\begin{cases} u^{(iv)}(x) + \lambda f(u(x)) = 0 \text{ in } [0,1]\\ u(0) = u'(0) = 0\\ u''(1) = 0 \quad u'''(1) = \sqrt{|u(1)|} \end{cases}$$

admits at least one non-zero classical solution

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- G. BONANNO, A critical point theorem via the Ekeland variational principle, *Nonlinear Analysis* **75**(2012), 2992–3007.
- G. BONANNO, Relations between the mountain pass theorem and local minima, *Adv. Nonlinear Anal.*, **1** (2012), 205–220.
- G.BONANNO, A.CHINNÌ, S.A.TERSIAN, Existence results for a two point boundary value problem involving a fourth-order equation, *Electronic Journal of Qualitative Theory of Differential Equations*, **33** (2015), 1–9.

G. BONANNO, B. DI BELLA, A boundary value problem for fourth-order elastic beam equations, *J. Math. Anal. Appl.*, **343** (2008), 1166–1176.

A. CABADA, S. TERSIAN, Multiplicity of solutions of a two point boundary value problem for a fourth-order equation, *Appl. Math. and Computations*, **219** (2013), n. 10, 5261–5267.

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- M.R. GROSSINHO M.R., S. TERSIAN, The dual variational principle and equilibria for a beam resting on a discontinuous nonlinear elastic foundation, *Nonlinear Anal.*, **41** (2000), 417–431.
- L. YANG, H. CHEN, X. YANG, The multiplicity of solutions for fourth-order equations generated from a boundary condition, *Appl. Math. Letters*, **24** (2011), 1599–1603.
- T.F. MA, J. DA SILVA, Iterative solutions for a beam equation with nonlinear boundary conditions of third order, *Appl. Math. Comput.*, **159** (2004), 11–18.
- **T**.F. MA, Positive solutions for a beam equation on a nonlinear elastic foundation, *Appl. Math. Comput.*, **159** (2004), 11–18.

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- $\bullet~X$  real Banach space
- $\Phi, \Psi: X \to \mathbb{R}, \ \Phi, \Psi \in C^1(X)$

•  $r \in \mathbb{R}$ ,

 $I(\cdot)=\Phi(\cdot)-\Psi(\cdot)$  verifies  $(P.S.)^{[r]}$  condition if any sequence  $\{u_n\}_{n\in\mathbb{N}}$  in X such that

- $(\alpha) \ \{I(u_n)\}$  is bounded;
- ( $\beta$ )  $\lim_{n \to +\infty} \|I'(u_n)\|_{X^*} = 0;$
- $(\gamma) \ \Phi(u_n) < r \text{ for each } n \in \mathbb{N};$

has a convergent subsequence.

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 $(P.S.)^{[r]}$ 

## (Bonanno(2012))

 $\begin{array}{l} X \mbox{ reflexive} \\ \Phi \in C^1(X) \\ \Phi \mbox{ s.w.l. semicontinuous} \\ \Phi \mbox{ coercive} \\ \Phi' \mbox{ admits a continuous inverse} \end{array}$ 

 $\Psi \in C^1(X)$  $\Psi' \text{ compact}$ 

## for all $r\in ]-\infty,+\infty[$ the function $\Phi-\Psi$ satisfies the $(P.S.)^{[r]}\text{-condition}.$

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