Existence of one non-zero solution for a two point boundary value problem involving a fourth-order equation

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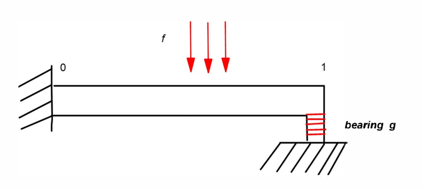
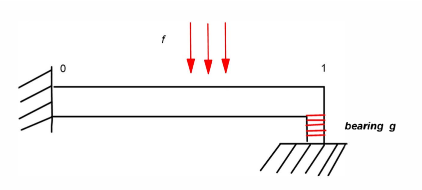
$$\begin{cases} u^{(iv)}(x) + \lambda f(x, u(x)) = 0 \text{ in } [0,1] \\ u(0) = u'(0) = 0 \\ u''(1) = 0 \quad u'''(1) = \mu g(u(1)) \end{cases} \qquad (P_{\lambda,\mu})$$

- $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function
- $g : \mathbb{R} \to \mathbb{R}$ is a continuous function
- λ, μ are positive parameters



- $u(0) = u'(0) = 0$
- $u''(1) = 0,\ u'''(1) = \mu g(u(1))$



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- **variational methods**
  - L. Yang, H. Chen, X. Yang,
    - *Appl. Math. Letters* (2011) existence of three solutions for problem $(P_{\lambda,1})$
    - *Appl. Math. and Computations* (2011) existence of three solutions for problem $(P_{\lambda,\lambda})$
  - T.F. Ma, *Appl. Math. and Computations* (2004) existence of at least two positive solutions for problem $(P_{1,1})$
  - A. Cabada, S. Tersian, *Appl. Math. and Computations*, (2013) existence of two non trivial solutions for problem $(P_{\lambda,\lambda})$
- **iterative methods**
  - T.F. Ma, J. Da Silva, *Appl. Math. Letters* (2004) problem $(P_{\lambda,\mu})$.

$$X := \{u \in H^2([0,1]) : u(0) = u'(0) = 0\}$$

$$\|u\| := \left(\int_0^1 (u''(t))^2 \, dt\right)^{\frac{1}{2}}$$

- the embedding $X \hookrightarrow C^1([0,1])$ is compact
- $\|u\|_{C^1([0,1])} := \max\{\|u\|_\infty, \|u'\|_\infty\} \le \|u\|$ for each $u \in X$

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The functional Φ

$$\forall x \in X \qquad \Phi(u) := \frac{1}{2}\|u\|^2$$

- Φ is Frechét differentiable, $\Phi \in C^1(X)$
- $\langle \Phi'(u), v\rangle = \int_0^1 u''(x)v''(x)\,dx$, for each $u, v \in X$
- Φ is sequentially weakly lower semi-continuous and coercive
- Φ' admits a continuous inverse on X^*

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The functional $\Psi_{\lambda,\mu}$

$\forall \lambda, \mu > 0, \forall u \in X$

$$\Psi_{\lambda,\mu}(u) := \int_0^1 F(x, u(x))\, dx + \frac{\mu}{\lambda} G(u(1))$$

- $F(x,\xi) := \int_0^\xi f(x,t)\, dt$ for each $x \in [0,1]$, $\xi \in \mathbb{R}$
- $G(\xi) := \int_0^\xi g(t)\, dt$ for each $\xi \in \mathbb{R}$

- $\Psi_{\lambda,\mu}$ is Frechét differentiable, $\Psi_{\lambda,\mu} \in C^1(X)$
- $\langle \Psi'_{\lambda,\mu}(u), v \rangle = \int_0^1 f(x,u(x))v(x)\, dx + \frac{\mu}{\lambda} g(u(1))v(1)$, for each $u, v \in X$
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$$I_{\lambda,\mu} := \Phi - \lambda\Psi_{\lambda,\mu},$$

L. Yang, H. Chen, X. Yang (2011)

For each $\lambda, \mu > 0$, the critical points of $I_{\lambda,\mu}$ are classical solutions for problem $(P_{\lambda,\mu})$.

G. Bonanno, Relations between the mountain pass theorem and local minima, *Adv. Nonlinear Anal.*, **1** (2012), 205–220.

$\Downarrow$

$$\begin{cases} u^{(iv)}(x) + \lambda f(x, u(x)) = 0 \ \text{ in } \ [0,1] \\ u(0) = u'(0) = 0 \\ u''(1) = 0 \quad u'''(1) = \mu g(u(1)) \end{cases} \qquad (P_{\lambda,\mu})$$

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The main tool

- X real Banach space
- $\Phi, \Psi : X \to \mathbb{R}$, $\Phi, \Psi \in C^1(X)$
- $\inf_{x\in X} \Phi(x) = \Phi(0) = \Psi(0) = 0$
- $\exists\, r > 0$ and $\bar{x} \in X$, $r < \Phi(\bar{x})$:

(a_1) $$\frac{\sup_{\Phi(x)\leq r} \Psi(x)}{r} < \frac{\Psi(\bar{x})}{\Phi(\bar{x})},$$

(a_2) for each $\lambda \in \left]\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\leq r} \Psi(x)}\right[$ the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies $(P.S.)^{[r]}$ condition. [3]

The main tool

Theorem 1 (Bonanno(2012))

for each $\lambda \in \Lambda_r := \left]\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup\limits_{\Phi(x)\leq r} \Psi(x)}\right[$, there is $x_{0,\lambda} \in \Phi^{-1}(]0,r[)$ such that $I'_\lambda(x_{0,\lambda}) \equiv \vartheta_{X^*}$ and $I_\lambda(x_{0,\lambda}) \leq I_\lambda(x)$ for all $x \in \Phi^{-1}(]0,r[)$.

Notations

For $\alpha > 0$, we put

$$F^{\alpha} := \int_0^1 \max_{|\xi|\leq\alpha} F(x,\xi)\, dx$$

and

$$G^{\alpha} := \max_{|\xi|\leq\alpha} G(\xi)$$

The main result

(f_1) there exist $\delta, \gamma \in \mathbb{R}$, with $0 < \delta < \gamma$, such that

$$\frac{F^{\gamma}}{\gamma^2} < \frac{1}{8\pi^4}\left(\frac{3}{2}\right)^3 \frac{\int_{\frac{3}{4}}^{1} F(x,\delta)\, dx}{\delta^2}$$ 7

(f_2) $F(x,t) \geq 0$ for almost every $x \in [0,1]$ and for all $t \in [0,\delta]$. 5 6

The main result

$$\Lambda_{\delta,\gamma} := \left]4\pi^4\left(\frac{2}{3}\right)^3\frac{\delta^2}{\int_{\frac{3}{4}}^1 F(x,\delta)\,dx}, \frac{\gamma^2}{2F^\gamma}\right[$$

$$\Downarrow$$

Theorem 2 (Bonanno,C. , Tersian - *Electronic Journal of Qualitative Theory of Differential Equations* - (2015))

for each $\lambda \in \Lambda_{\delta,\gamma}$ and for each $g : \mathbb{R} \to \mathbb{R}$ continuous, there exists $\eta_{\lambda,g} > 0$ such that for each $\mu \in]0, \eta_{\lambda,g}[$ the problem $(P_{\lambda,\mu})$ admits at least one non-zero solution u_λ such that $\|u_\lambda\|_\infty, \|u'_\lambda\|_\infty < \gamma$.

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$$\eta_{\lambda,g} = \begin{cases} \frac{\gamma^2 - 2\lambda F^\gamma}{2G^\gamma} & \text{if } G(\delta) \geq 0 \\ \min\left\{ \frac{\gamma^2 - 2\lambda F^\gamma}{2G^\gamma}, \frac{4\pi^4\delta^2 - \lambda\left(\frac{3}{2}\right)^3 \int_{\frac{3}{4}}^{1} F(x,\delta)\, dx}{\left(\frac{3}{2}\right)^3 G(\delta)} \right\} & \text{if } G(\delta) < 0, \end{cases}$$

Remark

We read $\frac{\gamma^2 - 2\lambda F^\gamma}{2G^\gamma} = +\infty$ when $G^\gamma = 0$

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For each $\lambda \in \Lambda_{\delta,\gamma}$ and $g : \mathbb{R} \to \mathbb{R}$ continuous it results $\eta_{\lambda,g} > 0$.

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- $X := \{u \in H^2([0,1]) : u(0) = u'(0) = 0\}$
- $\lambda \in \Lambda_{\delta,\gamma}$, $g : \mathbb{R} \to \mathbb{R}$ continuous, $\mu \in]0, \eta_{\lambda,g}[$

$$I_{\lambda,\mu}(u) := \underbrace{\frac{1}{2}\|u\|^2}_{\Phi(u)} - \underbrace{\lambda(\int_0^1 F(x,u(x))\,dx + \frac{\mu}{\lambda}G(u(1)))}_{\Psi_{\lambda,\mu}(u)}$$

- $\inf_{x\in X} \Phi(x) = \Phi(0) = \Psi_{\lambda,\mu}(0) = 0$

$I_{\lambda,\mu}$ verifies (a_2) of Theorem 1

4

for all $r \in]-\infty, +\infty[$ the function $I_{\lambda,\mu}$ satisfies the$(P.S.)^{[r]}$-condition.

$$\Downarrow$$

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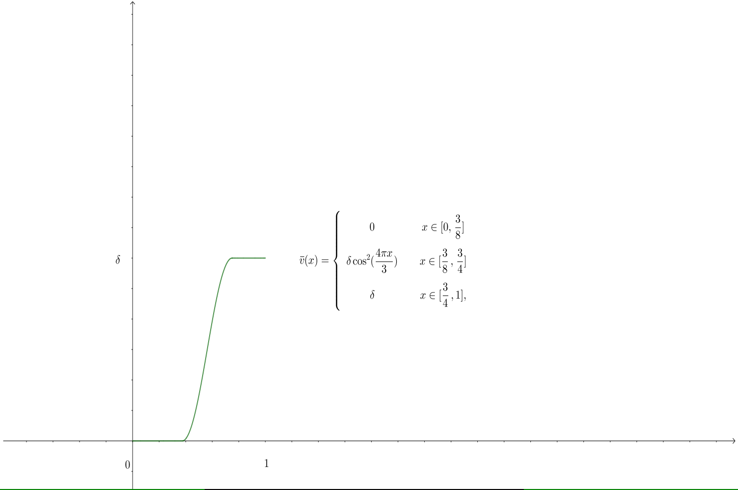
$\bar{v}$

- $\bar{v}$ depends on δ

$$\bar{v}(x) = \begin{cases} 0 & x \in [0, \frac{3}{8}] \\ \delta \cos^2(\frac{4\pi x}{3}) & x \in [\frac{3}{8}, \frac{3}{4}] \\ \delta & x \in [\frac{3}{4}, 1], \end{cases}$$

$\bar{v}$



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$\bar{v}$

$$\Phi(\bar{v}) = 4\pi^4\delta^2\left(\frac{2}{3}\right)^3 \quad \bar{v}(x)\in[0,\delta]\ \forall x\in\left[\frac{3}{8},\frac{3}{4}\right] \quad + \quad (f_2)$$ 5

$$\Downarrow$$

$$\Psi_{\lambda,\mu}(\bar{v}) \geq \int_{\frac{3}{4}}^{1} F(x,\delta)\,dx + \frac{\mu}{\lambda}G(\delta)$$

$$\Downarrow$$

$$\frac{\Psi_{\lambda,\mu}(\bar{v})}{\Phi(\bar{v})} \geq \frac{\int_{\frac{3}{4}}^{1} F(x,\delta)\,dx + \frac{\mu}{\lambda}G(\delta)}{4\pi^4\delta^2\left(\frac{2}{3}\right)^3}$$

r

- r depends on γ, $r = \frac{\gamma^2}{2}$

$$\forall u \in X : \Phi(u) \leq r,$$

$$\Downarrow$$

$$\|u\|_\infty \leq \gamma$$

$$\Downarrow$$

$$\frac{1}{r} \sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi_{\lambda,\mu}(u) \leq \frac{2}{\gamma^2} F^\gamma + \frac{2}{\gamma^2} \frac{\mu}{\lambda} G^\gamma$$

r and $\bar{v}$ verify (a_1) of Theorem 1

- $G(\delta) \geq 0$, $\eta_{\lambda,g} = \frac{\gamma^2 - 2\lambda F^\gamma}{2G^\gamma}$

$$\overbrace{\frac{2}{\gamma^2}F^\gamma + \frac{2}{\gamma^2}\frac{\mu}{\lambda}G^\gamma < \frac{1}{\lambda}}^{\mu<\eta_{\lambda,g}}$$

$$\underbrace{\frac{1}{\lambda} < \frac{\int_{\frac{3}{4}}^{1} F(x,\delta)\,dx}{4\pi^4\delta^2\left(\frac{2}{3}\right)^3}}_{\lambda\in\Lambda_{\delta,\gamma}} \leq \frac{\int_{\frac{3}{4}}^{1} F(x,\delta)\,dx}{4\pi^4\delta^2\left(\frac{2}{3}\right)^3} + \frac{\mu}{\lambda}\frac{G(\delta)}{4\pi^4\delta^2\left(\frac{2}{3}\right)^3}$$

r and $\bar{v}$ verify (a_1) of Theorem 1

- $G(\delta) < 0$, $\eta_{\lambda,g} = \min\left\{ \frac{\gamma^2 - 2\lambda F^\gamma}{2G^\gamma}, \frac{4\pi^4\delta^2 - \lambda\left(\frac{3}{2}\right)^3 \int_{\frac{3}{4}}^1 F(x,\delta)\,dx}{\left(\frac{3}{2}\right)^3 G(\delta)} \right\}$

$$\overbrace{\frac{2}{\gamma^2}F^\gamma + \frac{2}{\gamma^2}\frac{\mu}{\lambda}G^\gamma < \frac{1}{\lambda}}^{\mu<\eta_{\lambda,g}}$$

$$\underbrace{\frac{1}{\lambda} < \frac{\int_{\frac{3}{4}}^1 F(x,\delta)\,dx}{4\pi^4\delta^2\left(\frac{2}{3}\right)^3} + \frac{\mu}{\lambda}\frac{G(\delta)}{4\pi^4\delta^2\left(\frac{2}{3}\right)^3}}_{\mu<\eta_{\lambda,g}}$$

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$$\Downarrow$$

$$\frac{1}{r}\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi_{\lambda,\mu}(u) < \frac{1}{\lambda} < \frac{\Psi_{\lambda,\mu}(\bar{v})}{\Phi(\bar{v})}$$

$$\Downarrow$$

(a_1) of Theorem 1 is verified

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$$\frac{2}{\gamma^2}F^\gamma + \frac{2}{\gamma^2}\frac{\mu}{\lambda}G^\gamma < \frac{1}{\lambda} < \frac{\int_{\frac{3}{4}}^{1} F(x,\delta)\,dx}{4\pi^4\delta^2\left(\frac{2}{3}\right)^3} + \frac{\mu}{\lambda}\frac{G(\delta)}{4\pi^4\delta^2\left(\frac{2}{3}\right)^3}$$

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$$\frac{1}{r}\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi_{\lambda,\mu}(u) < \frac{1}{\lambda} < \frac{\Psi_{\lambda,\mu}(\bar{v})}{\Phi(\bar{v})}$$

$$\Downarrow$$

(a_1) of Theorem 1 is verified

$\Phi(\bar{v}) < r$

- $\delta < \gamma \quad + \quad (f_1)$ 7 $\Longrightarrow \sqrt{8\pi^4 \left(\frac{2}{3}\right)^3}\delta < \gamma$

$$\Downarrow$$

$$\Phi(\bar{v}) = 4\pi^4\delta^2 \left(\frac{2}{3}\right)^3 < \frac{\gamma^2}{2} = r$$

- $\lambda \in \Lambda_{\delta,\gamma} \subseteq \left] \frac{\Phi(\bar{v})}{\Psi_{\lambda,\mu}(\bar{v})}, \frac{r}{\sup_{\Phi(u)\leq r} \Psi_{\lambda,\mu}(u)} \right[$

$$\Downarrow$$

Theorem 1 guarantees the existence of a non-trivial classical solution of problem $(P_{\lambda,\mu})$, u_λ such that $\|u_\lambda\|_\infty, \|u'_\lambda\|_\infty < \gamma$

An example of application of Theorem 2

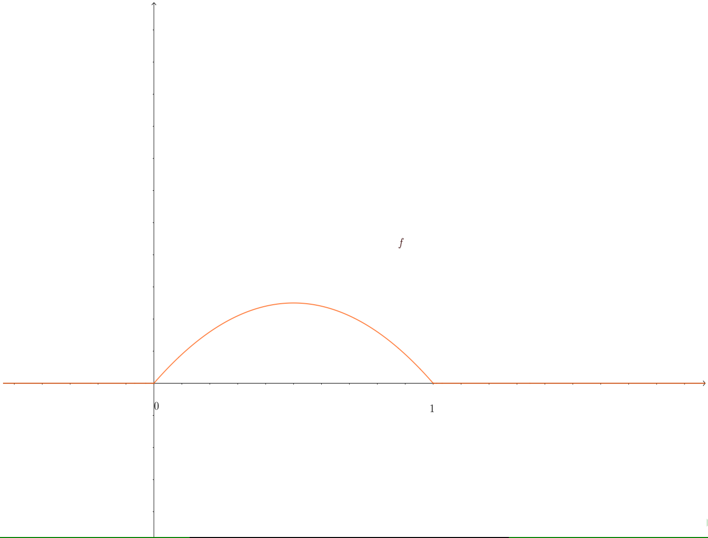
- $f : \mathbb{R} \to \mathbb{R}$

$$f(u) := \begin{cases} 0, & u < 0, \\ u - u^2, & 0 \leq u \leq 1, \\ 0 & u > 1. \end{cases}$$

- $\delta = \frac{1}{2}$, $\gamma = 22$

$$\Downarrow$$

for each $\lambda \in]1385.4, 1452[$ and each $g : \mathbb{R} \to \mathbb{R}$ continuous there exists $\eta_{\lambda,g} > 0$ such that for each $\mu \in]0, \eta_{\lambda,g}[$, the problem $(P_{\lambda,\mu})$ admits at least one non-zero solution u_λ with $||u||_\infty, ||u'||_\infty < 22$.


f
0
1

A particular case of Theorem 2

(f_1'') $f : \mathbb{R} \to [0, +\infty[$ continuous, $\limsup_{t\to 0^+} \frac{F(t)}{t^2} = +\infty$

$$\Downarrow$$

Theorem 3 (Bonanno,C. , Tersian - *EJQTDE* - (2015))

$\forall \gamma > 0$, $\forall \lambda \in]0, \frac{\gamma^2}{2F(\gamma)}[$, $\forall g : \mathbb{R} \to [0, +\infty[$ continuous and $\forall \mu \in]0, \frac{\gamma^2 - 2F(\gamma)\lambda}{2G(\gamma)}[$, the problem

$$\begin{cases} u^{(iv)}(x) + \lambda f(u(x)) = 0 \text{ in } [0,1] \\ u(0) = u'(0) = 0 \\ u''(1) = 0 \quad u'''(1) = \mu g(u(1)) \end{cases} \qquad (\tilde{P}_{\lambda,\mu})$$

admits at least one non-zero classical solution u such that $\|u\|_\infty, \|u'\|_\infty < \gamma$

variational structure in Theorem 2

$$I_{\lambda,\mu}(u) := \underbrace{\frac{1}{2}\|u\|^2}_{\Phi(u)} - \lambda\underbrace{\left(\int_0^1 F(x,u(x))\,dx + \frac{\mu}{\lambda}G(u(1))\right)}_{\Psi_{\lambda,\mu}(u)}$$

$$I_{\lambda,\mu}(u) := \underbrace{\frac{1}{2}\|u\|^2}_{\Phi(u)} - \mu\underbrace{\left(\frac{\lambda}{\mu}\int_0^1 F(x,u(x))\,dx + G(u(1))\right)}_{\tilde{\Psi}_{\lambda,\mu}(u)}$$

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(g_1) there exist $\delta, \gamma \in \mathbb{R}$ with $0 < \delta < \gamma$:

$$\frac{G^{\gamma}}{\gamma^2} < \frac{1}{8\pi^4}\left(\frac{3}{2}\right)^3 \frac{G(\delta)}{\delta^2}$$

$$\Gamma_{\delta,\gamma} := \left]4\pi^4 \left(\frac{2}{3}\right)^3 \frac{\delta^2}{G(\delta)}, \frac{\gamma^2}{2G^\gamma}\right[$$

$$\Downarrow$$

Theorem 2'

for each $\mu \in \Gamma_{\delta,\gamma}$ and for each $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ L^1-Carathéodory function verifying condition (f_2) of Theorem 2 [6], there exists $\theta_{\mu,f} := \dfrac{\gamma^2 - 2\mu G^\gamma}{2F^\gamma} > 0$ such that for each $\lambda \in]0, \theta_{\mu,f}[$ the problem $(P_{\lambda,\mu})$ admits at least one non-zero solution u such that $\|u\|_\infty, \|u'\|_\infty < \gamma$

A particular case of Theorem 2'

(g_1'') $g : \mathbb{R} \to [0, +\infty[$ continuous, $\limsup_{t\to 0^+} \frac{g(t)}{t} = +\infty$

$$\Downarrow$$

Theorem 3' (Bonanno,C. , Tersian - *EJQTDE* - (2015))

$\forall \gamma > 0$, $\forall \mu \in]0, \frac{\gamma^2}{2G(\gamma)}[$, $\forall f : \mathbb{R} \to [0, +\infty[$ continuous and $\forall \lambda \in]0, \frac{\gamma^2 - 2\mu G(\gamma)}{2F(\gamma)}[$, the problem

$$\begin{cases} u^{(iv)}(x) + \lambda f(u(x)) = 0 \ \text{ in } \ [0,1] \\ u(0) = u'(0) = 0 \\ u''(1) = 0 \quad u'''(1) = \mu g(u(1)) \end{cases} \qquad (\tilde{P}_{\lambda,\mu})$$

admits at least one non-zero classical solution u such that $\|u\|_\infty, \|u'\|_\infty < \gamma$

Corollary of Theorem 3'

$f : \mathbb{R} \to [0, +\infty[$, f continuous,

$$\Downarrow$$

Corollary of Theorem 3'

for each $\lambda \in \left]0, \frac{1}{10\int_0^2 f(t)dt}\right[$ the problem

$$\begin{cases} u^{(iv)}(x) + \lambda f(u(x)) = 0 \text{ in } [0,1] \\ u(0) = u'(0) = 0 \\ u''(1) = 0 \quad u'''(1) = \sqrt{|u(1)|} \end{cases}$$

admits at least one non-zero classical solution

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T.F. Ma, Positive solutions for a beam equation on a nonlinear elastic foundation, *Appl. Math. Comput.*, **159** (2004), 11–18.

$(P.S.)^{[r]}$

- X real Banach space
- $\Phi, \Psi : X \to \mathbb{R}$, $\Phi, \Psi \in C^1(X)$
- $r \in \mathbb{R}$,

$I(\cdot) = \Phi(\cdot) - \Psi(\cdot)$ verifies $(P.S.)^{[r]}$ condition if any sequence $\{u_n\}_{n\in\mathbb{N}}$ in X such that

(α) $\{I(u_n)\}$ is bounded;

(β) $\lim_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0$;

(γ) $\Phi(u_n) < r$ for each $n \in \mathbb{N}$;

has a convergent subsequence. 3

$(P.S.)^{[r]}$

(Bonanno(2012))

X reflexive

$\Phi \in C^1(X)$ $\quad\quad$ $\Psi \in C^1(X)$

Φ s.w.l. semicontinuous $\quad\quad$ Ψ' compact

Φ coercive

Φ' admits a continuous inverse

$$\Downarrow$$

for all $r \in]-\infty, +\infty[$ the function $\Phi - \Psi$ satisfies the $(P.S.)^{[r]}$-condition.

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