TWO POSITIVE SOLUTIONS FOR NONLINEAR PROBLEMS

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On the Number of Solutions of Asymptotically Superlinear Two Point Boundary Value Problems

HERBERT AMANN

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30. IX. 1975, Volume 58, Issue 3, pp 207-218

Some Continuation and Variational Methods for Positive Solutions of Nonlinear Elliptic Eigenvalue Problems

MICHAEL G. CRANDALL & PAUL H. RABINOWITZ

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Combined Effects of Concave and Convex Nonlinearities in Some Elliptic Problems*

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$$(P_\lambda) \quad \begin{cases} -u'' = \lambda f(u) & \text{in} \quad]0,1[\\ u(0) = u(1) = 0 \end{cases}$$

$f : \mathbb{R} \to \mathbb{R}$ is a continuous and nonnegative function such that

$$f(0) > 0; \qquad (1)$$

$$\lim_{t \to +\infty} \frac{f(t)}{t} = +\infty. \qquad (2)$$

Then, there is $\lambda^* > 0$ such that the problem (P_λ) has at least two positive solutions for $0 < \lambda < \lambda^*$; at least one for $\lambda = \lambda^*$; none for $\lambda > \lambda^*$.

CRANDALL-RABINOWITZ

$$(P_\lambda) \quad \begin{cases} -\Delta u = \lambda f(u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

$f : \mathbb{R} \to \mathbb{R}$ *is a continuous, nonnegative and sub-critical function such that*

$$f(0) > 0; \qquad \textbf{1}$$

$$m > 2 \quad l > 0 \quad 0 < mF(\xi) \le \xi f(\xi) \quad \textit{for all } \xi \ge l \qquad \textbf{AR}$$

Then, there is $\lambda^* > 0$ *such that the problem* (P_λ) *has at least two positive solutions for* $0 < \lambda < \lambda^*$; *at least one for* $\lambda = \lambda^*$; *none for* $\lambda > \lambda^*$.

The (AR) condition

$$m > 2 \quad l > 0 \qquad 0 < mF(\xi) \leq \xi f(\xi) \quad \textit{for all } \xi \geq l$$

AR

is a bit more strong than

$$\lim_{t \to +\infty} \frac{f(t)}{t^q} = +\infty \qquad q>1$$

2'

that is, ***f is more than superlinear at infinity.***

The condition $f(0)>0$ *implies*

$$\lim_{t \to 0+} \frac{f(t)}{t} = +\infty$$

1'

that is, ***f is sublinear at zero.***

In both cases 0 is not a solution of the problem.

AMBROSETTI-BREZIS-CERAMI

$$(P_\mu) \quad \begin{cases} -\Delta u = \mu u^s + u^q & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

where $0<s<1<q$, with q subcritical (or critical).

Then, there is $\Lambda > 0$ such that the problem (P_μ) has at least two positive solutions for $0 < \mu < \Lambda$; at least one for $\mu = \Lambda$; none for $\mu > \Lambda$.

In this case

$$f(u) = \mu u^s + u^q$$

for which

0 is a solution of the problem

$$\lim_{t \to 0^+} \frac{f(t)}{t} = +\infty$$

*that is, **f is sublinear at zero***

$$\lim_{t \to +\infty} \frac{f(t)}{t^q} = +\infty \, , \; q>1$$

*that is, **f is more than superlinear at infinity***

The aim of this talk is to present an existence result of two positive solutions for the previous problems by requiring, besides the ***(AR) condition****, a condition which is more general than the sublinearity at zero. Precisely, in the ordinary case, we require:*

there are two positive constants c, d, *with* $d < c$, *such that*

$$\frac{F(c)}{c^2} < \frac{1}{4}\frac{F(d)}{d^2} \qquad \text{1''}$$

F is the primitive of f, with F(0)=0.

So, in particular, one has that

$$\lim_{t\to 0^+}\frac{f(t)}{t} = +\infty \quad \textit{implies} \quad \text{1''}$$

In addition, it may be satisfied also in some case where the functions f are superlinear (or linear) at zero.

A similar situation one has for elliptic case. In this case such a condition is a bit less simple.

The basic ingredients of such a result are:
*a **theorem of local minimum***
and
*the **Ambrosetti-Rabinowitz theorem**.*

1973

THE AMBROSETTI-RABINOWITZ THEOREM

Let X be a real Banach space, $I : X \to \mathbb{R}$ a continuously Gâteaux differentiable function which verifies (PS).

Assume that

(G) *there are $u_0, u_1 \in X$ and $r \in \mathbb{R}$, with $0 < r < \|u_1 - u_0\|$, such that*

$$\inf_{\|u-u_0\|=r} I(u) > \max\{I(u_0), I(u_1)\}.$$

Then, I admits a critical value c characterized by

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t))$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0; \gamma(1) = u_1\}.$$

A LOCAL MINIMUM THEOREM

BONANNO G., *A critical point theorem via the Ekeland variational principle*, Nonlinear Analysis, **75** (2012), 2992-3007.

Our aim is to present a local minimum theorem for functionals of the type:

Φ $\quad - \quad$ Ψ

$$\Phi - \Psi$$

A LOCAL MINIMUM THEOREM

Let X be a real Banach space and $\Phi, \Psi : X \to \mathbb{R}$ two continuously Gâteaux differentiable functions. Put

$$I = \Phi - \Psi$$

and assume that there are $x_0 \in X$ and $r_1, r_2 \in \mathbb{R}$, with $r_1 < \Phi(x_0) < r_2$, such that

$$\sup_{u \in \Phi^{-1}(]r_1, r_2[)} \Psi(u) \le r_2 - \Phi(x_0) + \Psi(x_0), \qquad (1)$$

$$\sup_{u \in \Phi^{-1}(]-\infty, r_1])} \Psi(u) \le r_1 - \Phi(x_0) + \Psi(x_0). \qquad (2)$$

Moreover, assume that I satisfies ${}^{[r_1]}(PS)^{[r_2]}$-condition.

Then, there is $u_0 \in \Phi^{-1}(]r_1, r_2[)$ such that $I(u_0) \le I(u)$ for all $u \in \Phi^{-1}(]r_1, r_2[)$ and $I'(u_0) = 0$.

A CONSEQUENCE:
A NON-ZERO LOCAL MINIMUM THEOREM

Let X *be a real Banach space and let* $\Phi, \Psi : X \to \mathbb{R}$ *be two continuously Gâteaux differentiable functionals such that* $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. *Assume that there are* $r \in \mathbb{R}$ *and* $\tilde{u} \in X$, *with* $0 < \Phi(\tilde{u}) < r$, *such that*

$$\frac{\displaystyle\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}$$

and, for each $\lambda \in \left] \dfrac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \dfrac{r}{\displaystyle\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)} \right[$, *the functional* $I_\lambda = \Phi - \lambda \Psi$ *satisfies* $(PS)^{[r]}$*-condition.*

Then, for each $\lambda \in \left] \dfrac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \dfrac{r}{\displaystyle\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)} \right[$, *there is* $u_0 \in \Phi^{-1}(]0, r[)$ *(hence,* $u_0 \neq 0$*) such that* $I_\lambda(u_0) \leq I_\lambda(u)$ *for all* $u \in \Phi^{-1}(]0, r[)$ *and* $I'_\lambda(u_0) = 0$.

A TWO NONZERO CRITICAL POINTS THEOREM

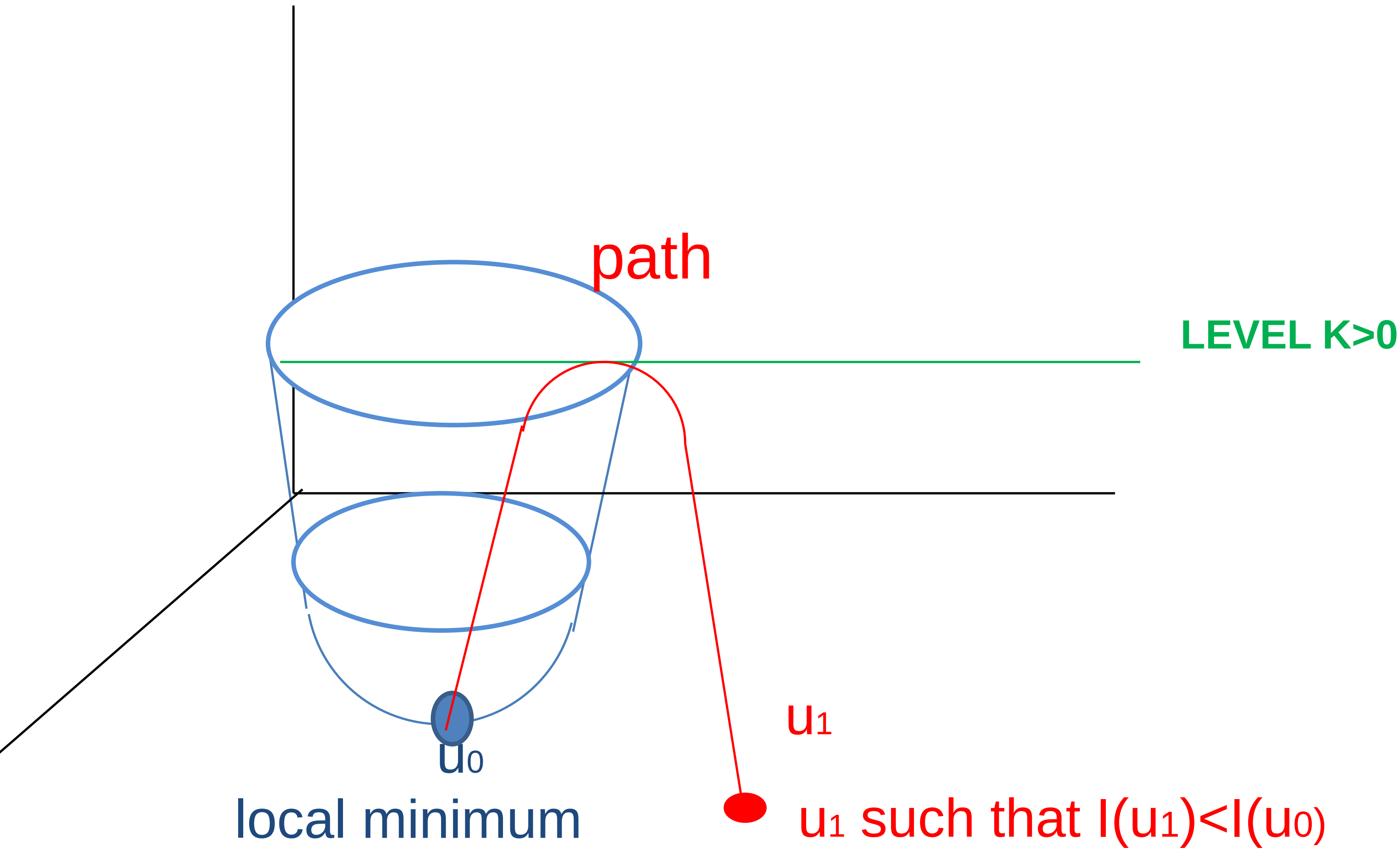
Theorem 2.1. *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that*

$$\frac{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \qquad \textbf{(G2)}$$

and, for each $\lambda \in \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}\right[$, the functional $I_\lambda = \Phi - \lambda \Psi$ satisfies (PS)-condition and it is unbounded from below.

Then, for each $\lambda \in \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}\right[$, the functional I_λ admits at least two non-zero critical points $u_{\lambda,1}$, $u_{\lambda,2}$ such that $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$.

BONANNO G., D'AGUI' G., *Two non-zero solutions for elliptic Dirichlet problems*, Zeitschrift für Analysis und ihre Anwendungen , **35** n.4 (2016), 449-464.



path
LEVEL K>0
u_1
u_0
local minimum
u_1 such that $I(u_1)<I(u_0)$

Consider the problem

$$(P_\lambda) \begin{cases} -\Delta u = \lambda f(u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a function which is nonnegative and continuous in $[0, +\infty[$.

Assume that

(h) *there exist* $s \in [1, 2[$, $q \in]2, 2N/(N-2)[$ *and two positive constants* a_s, a_q *such that*

$$f(t) \leq a_s|t|^{s-1} + a_q|t|^{q-1}$$

for all $t \geq 0$.

Moreover, put $R(x) = \sup\{\delta : B(x,\delta) \subseteq \Omega\}$ for all $x \in \Omega$, and $R = \sup_{x\in\Omega} R(x)$, for which there exists $x_0 \in \Omega$ such that $B(x_0, R) \subseteq \Omega$. Finally, put

$$K = \frac{R^2}{2(2^N-1)} \frac{1}{2T^2|\Omega|^{\frac{2}{N}}}$$

and

$$\Xi_\delta = \frac{1}{K} \frac{1}{2T^2|\Omega|^{\frac{2}{N}}} \frac{\delta^2}{F(\delta)}, \qquad \Lambda_\gamma = \frac{1}{2T^2|\Omega|^{\frac{2}{N}}} \frac{1}{\frac{a_s}{s}\gamma^{s-2} + \frac{a_q}{q}\gamma^{q-2}}$$

where γ, δ are positive constants.

$$T = \frac{1}{\sqrt{N(N-2)\pi}} \left(\frac{N!}{2\Gamma(1+N/2)}\right)^{1/N}$$

$$\|u\|_{L^{2^*}(\Omega)} \leq T\|u\| \quad \forall\, u \in H_0^1(\Omega)$$

Theorem 3.1. *Assume that* (h) *holds. Moreover, assume that there are two positive constants* γ *and* δ*, with* $\delta < \gamma$*, such that*

$$\frac{a_s}{s}\gamma^{s-2} + \frac{a_q}{q}\gamma^{q-2} < K\frac{F(\delta)}{\delta^2} \tag{3.1}$$

and there are two constants $m > 2$ *and* $l > 0$ *such that, for all* $\xi \geq l$*, one has*

$$0 < mF(\xi) \leq \xi f(\xi). \tag{AR}$$

Then, for each $\lambda \in]\Xi_\delta, \Lambda_\gamma[$*, problem* $\left(P_\lambda\right)$ *admits at least two positive weak solutions.*

$$\boxed{\bar{\lambda} = \frac{1}{2T^2|\Omega|^{\frac{2}{N}}}\left(\frac{s}{a_s}\right)^{\frac{q-2}{q-s}}\left(\frac{q}{a_q}\right)^{\frac{2-s}{q-s}}\left(\frac{2-s}{q-2}\right)^{\frac{2-s}{q-s}}\frac{q-2}{q-s}}$$

Corollary 3.1. *Assume* (h),

$$\limsup_{\xi\to 0^+}\frac{F(\xi)}{\xi^2} = +\infty \tag{3.1'}$$

and (AR).
Then, for each $\lambda \in]0, \bar{\lambda}[$*, problem* (P_λ) *admits at least two positive weak solutions.*

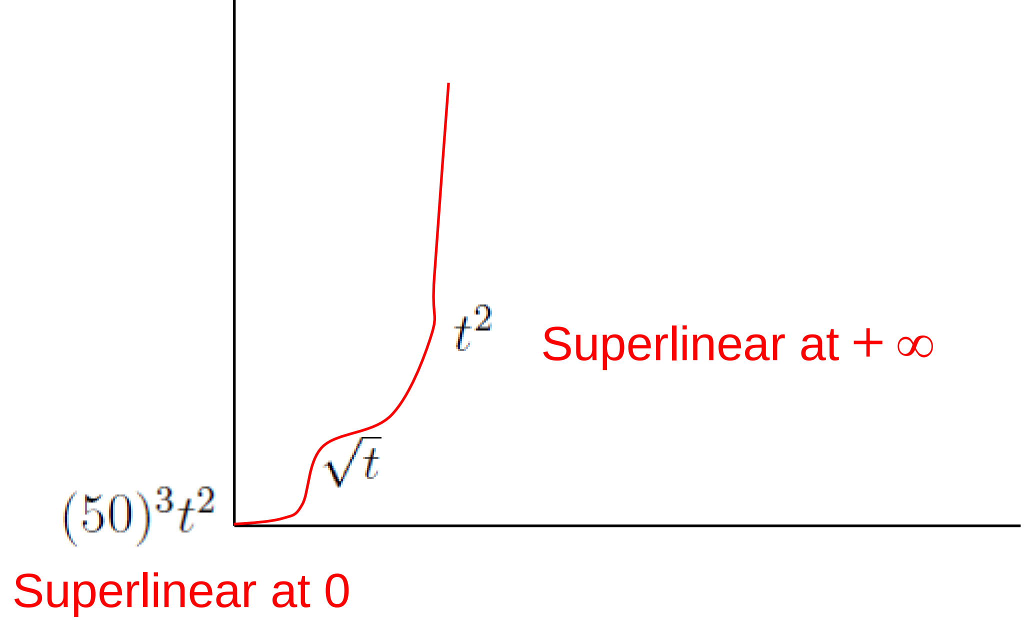
Example 3.1. Let $\Omega = \{x \in \mathbb{R}^3 : |x| < 1\}$ and let $f : \mathbb{R} \to \mathbb{R}$ be a function defined as follows

$$f(t) = \begin{cases} (50)^3 t^2 & \text{if} \quad \mathrm{t} \leq \left(\frac{1}{50}\right)^2, \\ \sqrt{t} & \text{if} \quad \left(\frac{1}{50}\right)^2 < \mathrm{t} \leq 1, \\ t^2 & \text{if} \quad \mathrm{t} \geq 1. \end{cases}$$

Owing to previous theorem, the problem

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

admits at least two positive weak solutions.



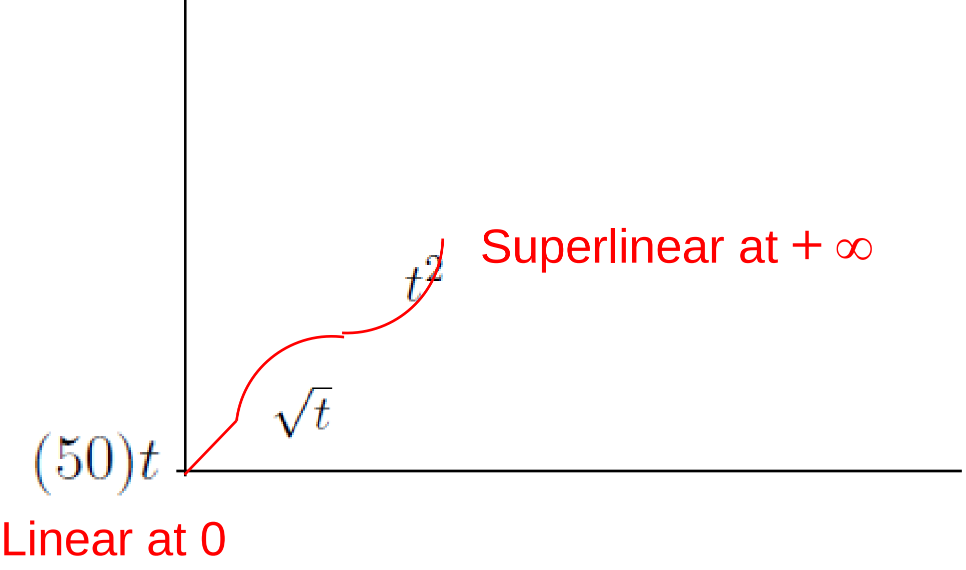
Let $\Omega = \{x \in \mathbb{R}^3 : |x| < 1\}$ and let $f : \mathbb{R} \to \mathbb{R}$ be a function defined as follows

$$f(t) = \begin{cases} (50)t & \text{if} \quad t \leq \left(\frac{1}{50}\right)^2, \\ \sqrt{t} & \text{if} \quad \left(\frac{1}{50}\right)^2 < t \leq 1, \\ t^2 & \text{if} \quad t \geq 1. \end{cases}$$

Owing to previous theorem, the problem

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

admits at least two positive weak solutions.



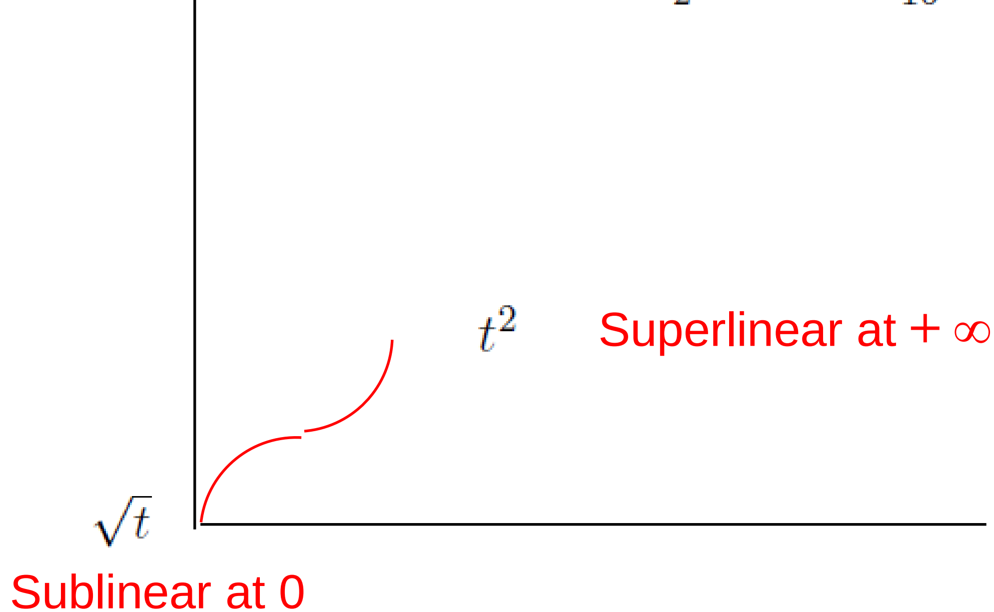
Example 3.2. Owing to Corollary 3.1, for each $\lambda \in \left]0, \frac{\sqrt{3}}{4T^2|\Omega|^{\frac{2}{N}}}\right[$ the problem

$$\begin{cases} -\Delta u = \lambda \max\{\sqrt{u}, u^2\} \quad \text{in} \quad \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

admits at least two positive weak solutions. Moreover, in particular if $\Omega = \{x \in \mathbb{R}^3 : |x| < 1\}$, the problem

$$\begin{cases} -\Delta u = \frac{1}{2} \max\{\sqrt{u}, u^2\} \quad \text{in} \quad \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

admits at least two positive weak solutions, since $\frac{1}{2} < \bar{\lambda} = \frac{9}{16} \sqrt[6]{3} \left(\frac{\pi}{2}\right)^{\frac{2}{3}}$.



We recall that in the Crandall-Rabinowitz theorem, besides (AR) condition, the key assumption is

$$f(0)>0$$

Hence, the Crandall-Rabinowitz theorem cannot applied to none of previous examples since, there, one has $f(0)=0$.
Moreover, we also observe that

$$f(0) > 0 \Rightarrow f \text{ sublinear at } 0^{+}$$

Now, put

$$\bar{\mu} = \left(\frac{1}{2T^2|\Omega|^{\frac{2}{N}}} \right)^{\frac{q-s}{q-2}} s(q-2)q^{\frac{2-s}{q-2}} \left(\frac{(2-s)^{(2-s)}}{(q-s)^{(q-s)}} \right)^{\frac{1}{q-2}}$$

Corollary 3.2. *Fix $1 \le s < 2 < q < 2^*$. Then, for each $\mu \in]0, \bar{\mu}[$ problem*

$$(D_\mu) \begin{cases} -\Delta u = \mu u^{s-1} + u^{q-1} & \text{in} \quad \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}$$

admits at least two positive weak solutions.

Indeed, it is enough to observe that

$$\bar{\lambda} = \frac{1}{2T^2|\Omega|^{\frac{2}{N}}} \left(\frac{s}{\mu} \right)^{\frac{q-2}{q-s}} (q)^{\frac{2-s}{q-s}} \left(\frac{2-s}{q-2} \right)^{\frac{2-s}{q-s}} \frac{q-2}{q-s} > 1.$$

So, from our result applied to $-\Delta u = \lambda \left(\mu u^{s-1} + u^{q-1} \right)$ *we obtain the conclusion.*

We observe that our results and the Ambosetti-Brezis-Cerami as well as Crandall-Rabinowitz are mutually independent. Indeed, on the hand, we can apply our results to problems where ABC and CR cannot be applied, as seen in the previous examples. On the other hand, when we can apply both CR (or ABC) and our results, the value $\boldsymbol{\lambda^*}$ *obtained in CR (or ABC), even if given in a theorical form, is the best. So, in this latest case we can use our results as a complement to CR in order to give a numerical lower bound of* $\boldsymbol{\lambda^*}$*, that is,*

$$\bar{\lambda} \leq \lambda^*.$$

The same remark also for ABC holds, that is a lower bound of $\boldsymbol{\Lambda}$ *is* $\bar{\boldsymbol{\mu}}$.

Previous results also for the ordinary case holds true.

Theorem 3.2. *Let $f : [0, +\infty[\to [0, +\infty[$ be a continuous function and assume that* (AR) *holds. Moreover, assume that there are two positive constants γ, δ, with $\delta < \gamma$ such that*

$$\frac{F(\gamma)}{\gamma^2} < \frac{1}{4}\frac{F(\delta)}{\delta^2}. \tag{3.1''}$$

Then, for each $\lambda \in \left]\frac{8\delta^2}{F(\delta)}, \frac{2\gamma^2}{F(\gamma)}\right[$, the problem

$$\begin{cases} -u'' = \lambda f(u) & \text{in} \quad]0,1[, \\ u(0) = u(1) = 0, \end{cases} \tag{P_λ^f}$$

admits at least two positive classical solutions.

If $\limsup_{\delta \to 0^+} \frac{F(\delta)}{\delta^2} = +\infty$ then $(3.1'')$ holds true and in this case the interval becames $\left]0, \bar{\lambda}\right[$, where $\quad \bar{\lambda} = \sup_{\gamma > 0} \frac{2\gamma^2}{F(\gamma)}$.

So, in particular, our result holds by assuming, besides the *(AR) – condition*, the following condition

$$\lim_{t \to 0^+} \frac{f(t)}{t} = +\infty.$$

Example 3.3. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined as follows

$$f(t) = \begin{cases} t^2 & \text{if} \quad t < 1, \\ \sqrt{t} & \text{if} \quad 1 \leq t < 10^2, \\ \frac{1}{10^3} t^2 & \text{if} \quad t \geq 10^2. \end{cases}$$

Owing to Theorem 3.2, the problem

$$\begin{cases} -u'' = 5^2 f(u) & \text{in} \quad]0, 1[, \\ u(0) = u(1) = 0, \end{cases}$$

admits at least two positive classical solutions. It is enough to to verify $\frac{1}{2}\frac{F(3)}{3^2} < \frac{1}{5^2} < \frac{1}{8}\frac{F(1)}{1^2}$. We observe that in this case, the nonlinearity f is not sublinear at zero.

Example 3.4. For each $\lambda \in]0, 3[$ the problem

$$\begin{cases} -u'' = \lambda \max\{\sqrt{u}, u^2\} & \text{in} \quad]0, 1[, \\ u(0) = u(1) = 0, \end{cases}$$

admits at least two positive classical solutions.

We recall that in the Amann theorem, besides the condition

$$\lim_{t \to +\infty} \frac{f(t)}{t} = +\infty$$

the key assumption is

f(0)>0

Hence, the Amann theorem cannot applied to none of previous examples since, there, ***f(0)=0*** *is assumed.*
Moreover, we also observe that

$$f(0) > 0 \implies f \text{ sublinear at } 0^{+}$$

We observe that our results and the Amann theorem are mutually independent. Indeed, on the hand, we can apply our results to problems where the result of Amann cannot be applied, as seen in the previous examples. On the other hand, Amann requires only the superlinearity at infinity and, in addition, when we can apply both the results, the value λ^* obtained in Amann, even if given in a theorical form, is the best. So, in this latest case we can use our results as a complement to the result of Amann in order to give a numerical lower bound of λ^*, that is,

$$\bar{\lambda} \leq \lambda^*.$$

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BONANNO G., JEBELEAN P., SERBAN C., *Superlinear discrete problems,* Applied Mathematics Letters **52** (2016), 162-168.

BONANNO G., IANNIZZOTTO A., MARRAS M., *Two non-zero solutions for Neumann problems,* Journal of Convex Analysis, **25** n.2 (2018), 421-434.

BONANNO G., LIVREA R., SCHECHTER M., *Some notes on a superlinear second order Hamiltonian system*, Manuscripta Mathematica, **154** n.1-2 (2017), 59-77.

BONANNO G., D'AGUI' G., *Mixed elliptic problems involving the p-Laplacian with nonhomogeneous boundary conditions*, Discrete and Continuous Dynamical Systems - Series A, **37** n.11 (2017), 5797-5817.

D'AGUI' G., Di BELLA B., TERSIAN S, *Multiplicity results for superlinear boundary value problems with impulsive effects,* Mathematical Methods in the Applied Science, **39** (2016), 1060-1068.

D'AGUI' G., MAWHIN J., SCIAMMETTA A., *Positive solutions for a discrete two point nonlinear boundary value problem with p-laplacian,* Journal of Mathematical Analysis and Applications **447** (2017), 383-397.

We recall again the situation that we have obtained in order to the existence of two solutions. Our condition expresses a growth which is less than quadratic of the primitive in an appropriate interval, and it is clearly more general than the condition requested by Crandall-Rabinowitz, Ambrosetti-Brezis-Cerami, Amann, which is f(0)>0 or the sublinearity of f at 0.
Precisely, the condition

$$\frac{F(c)}{c^2} < \frac{1}{4}\frac{F(d)}{d^2} \qquad for \quad some \quad d < c$$

is more general than the condition

$$\lim_{t\to 0^+} \frac{f(t)}{t} = +\infty,$$

which is, in turn, more general than the condition

$$f(0) > 0$$

Nevertheless, when *f* is sublinear at 0 (or, also, f(0)>0) **we do not say** "*the results of CR, ABC, A can be also obtained by our results*", but instead of this, we say "*we can obtained a numerical estimate of the best value* λ*", precising that they are **mutually independent**. This because the **λ* obtained from them is the best value** for which the problem admits solutions while we do not know if our $\hat{\lambda}$, ensured by the local minimum theorem, is the best value, even if we know its numerical estimate.

Here, we explicit better such a situation. We recall the general expression of our $\hat{\lambda}$ obtained from the local minimum theorem. Precisely, one has

$$\hat{\lambda} = \sup_{r > \inf_X \Phi} \sup_{v \in \Phi^{-1}(]-\infty, r[)} \frac{r - \Phi(v)}{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u) - \Psi(v)}$$

$$\bar{\lambda}(r) = \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}$$

$$\bar{\lambda} = \sup_{r>0} \bar{\lambda}(r) = \sup_{r>0} \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}$$

$$\bar{\lambda} \leq \hat{\lambda}$$

Some notes on

λ^*

Who is

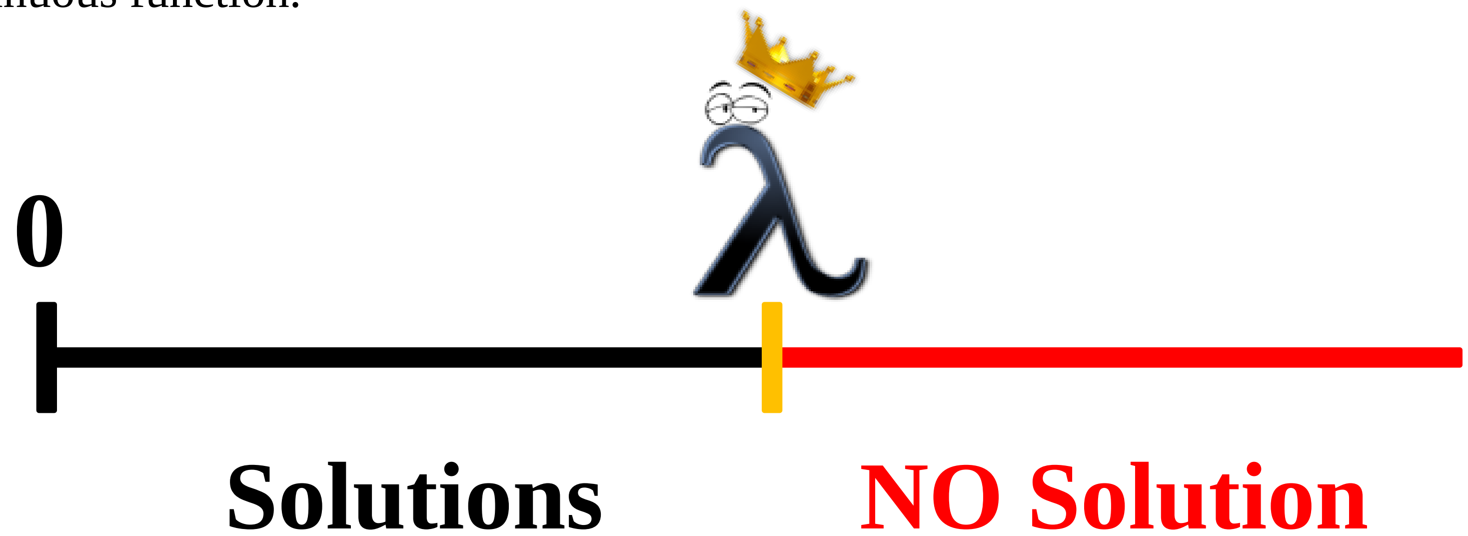
?

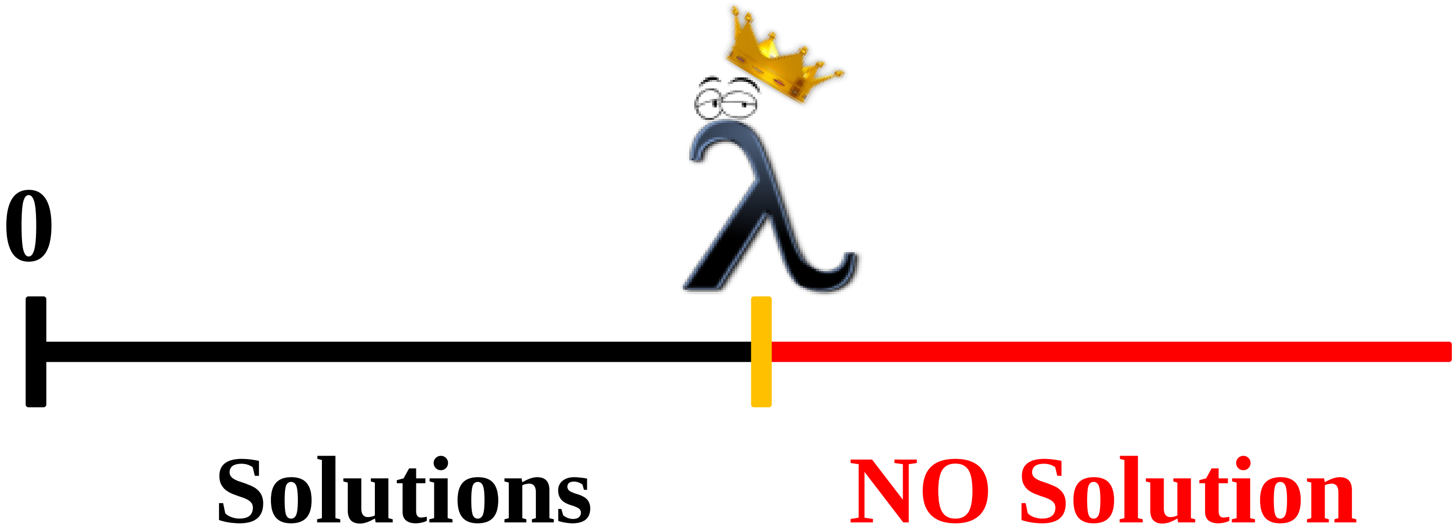
It is the **King** of *lambdas*

Given the problem

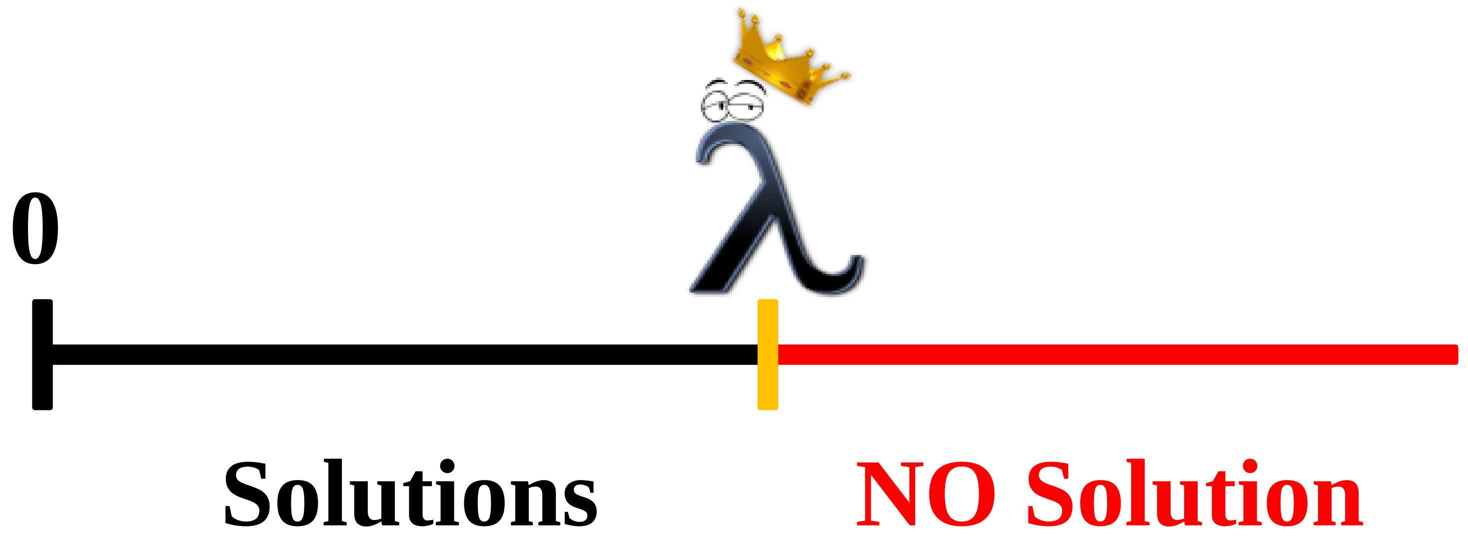
$$P_\lambda) \begin{cases} -\Delta u = \lambda f(u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a subcritical, **sublinear at zero and superlinear at infinity** continuous function.

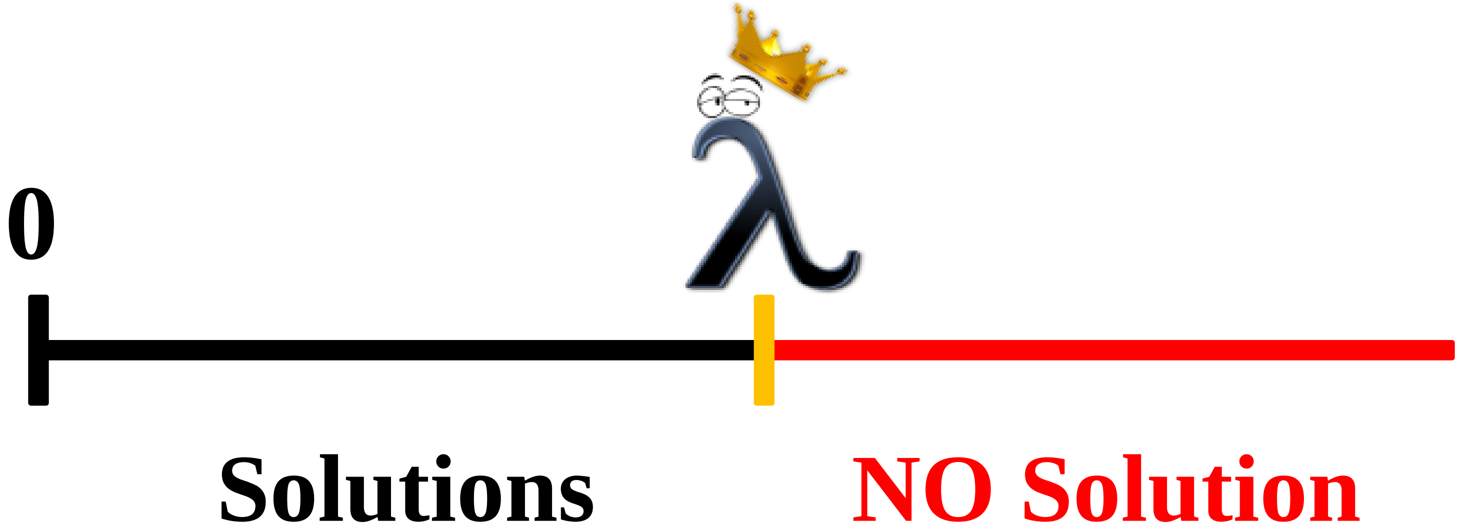




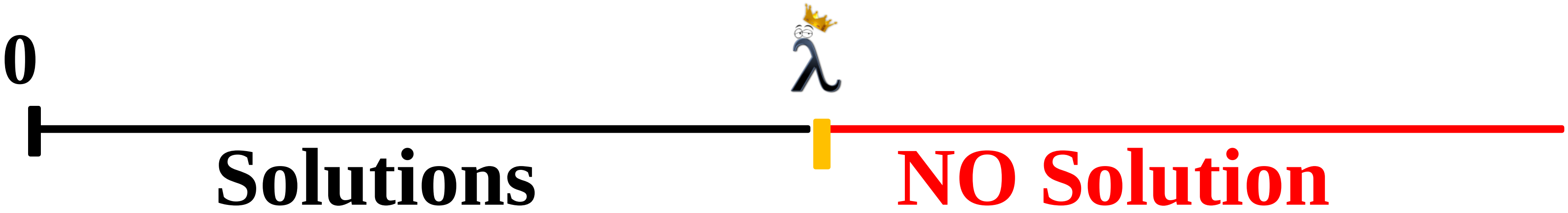
It is the best value for which problem $\boldsymbol{P_\lambda}$**)** admits solutions



Its existence has been proved by Crandall-Rabinowitz, Amann, Ambrosetti-Brezis-Cerami…



WE DO NOT KNOW ITS NUMERICAL VALUE!



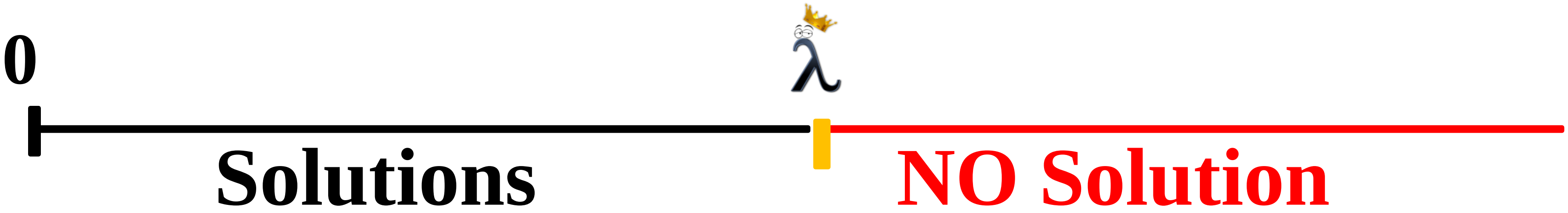
In literature we have some estimates of λ from above, that is, from some known results we can obtain

<u>UPPER BOUNDS of</u> λ

For instance, in the following works for general or specific problems:
Crandall-Rabinowitz,
De Coster-Habets,
Willem,
Brezis-Nirenberg,
Liu,
Laetsch,
…

<u>LOWER BOUNDS of</u> λ

On the contrary, in literature only few papers deals with lower bounds. We refer for instance to C. Bandle, Z. Liu,…



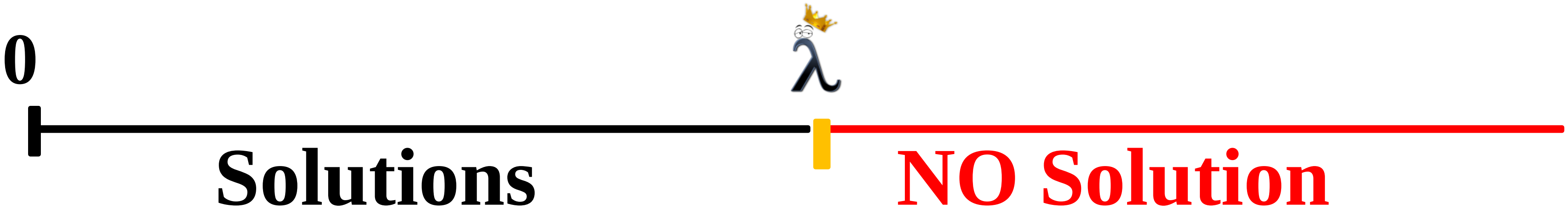
<u>UPPER BOUNDS of</u> λ^*

For instance, Crandall-Rabinowitz established that P_λ) has no solution ($f(0) \neq 0$) if

$$\boldsymbol{\lambda > \mu_1,}$$

where $\boldsymbol{\mu_1}$ is the first eigenvalue of the following problem

$$\begin{cases} -\Delta u = \mu f'(0) u & \text{in} \quad \Omega \\ \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$



UPPER BOUNDS of λ

For instance, De Coster-Habets established (in the ordinary case) that $P\lambda$) has no solution (f(0)$\neq$ 0) if

$$\lambda > \lambda_1 / K\ ,$$

where K is such that $f(u) \geq Ku$ for all $u \geq 0$.

For instance, in the book of Willem it is observed that the following problem

$$\begin{cases} -\Delta u + \lambda u = |u|^{p-2} u, \\ u \geq 0, u \in H_0^1(\Omega), \end{cases}$$

has no solution if

$$\lambda > \lambda_1.$$

0

λ

Solutions **NO Solution**

UPPER BOUNDS of λ

We focus on the constants of Crandall-Rabinowitz and De Coster –Habets, that we call **ΛCR** and **ΛDH**. So, we have:

$$\lambda \leq \Lambda_{CR} \qquad \qquad \lambda \leq \Lambda_{DH}$$

Which is the best constant among them?

To give an idea, we consider the following problem (it is known as Gelfand problem)

$$\begin{cases} -u'' = \lambda e^u & \text{in } \ ]0,1[\\ u(0) = u(1) = 0 \end{cases}$$

0 λ

Solutions **NO Solution**

$$\begin{cases} -u'' = \lambda e^u & \text{in } \ ]0,1[\\ u(0) = u(1) = 0 \end{cases}$$

From Amann, it admits

at least two solutions if $\lambda < \lambda$

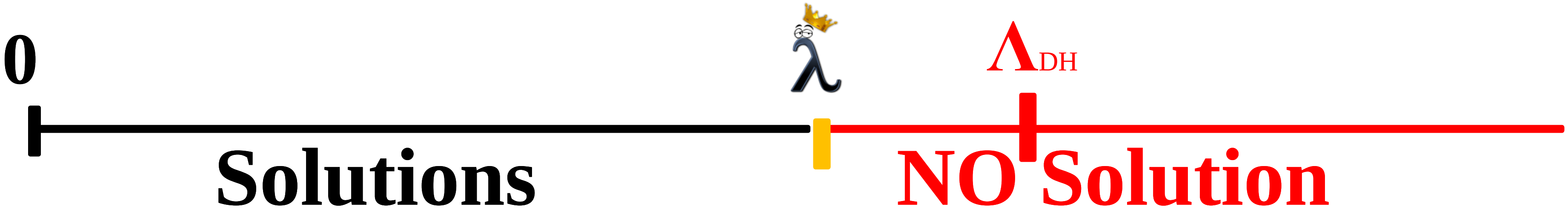
at least one solution if $\lambda = \lambda$

no solution if $\lambda > \lambda$

$$\Lambda_{CR} = \pi^2 \qquad \Lambda_{DH} = \frac{\pi^2}{e}$$

$$\lambda \leq \frac{\pi^2}{e}$$

So, a good upper bound of λ seems to be Λ_{DH}



$$\Lambda = \lambda_1 \sup_{r>0} \frac{r}{f(r)}$$

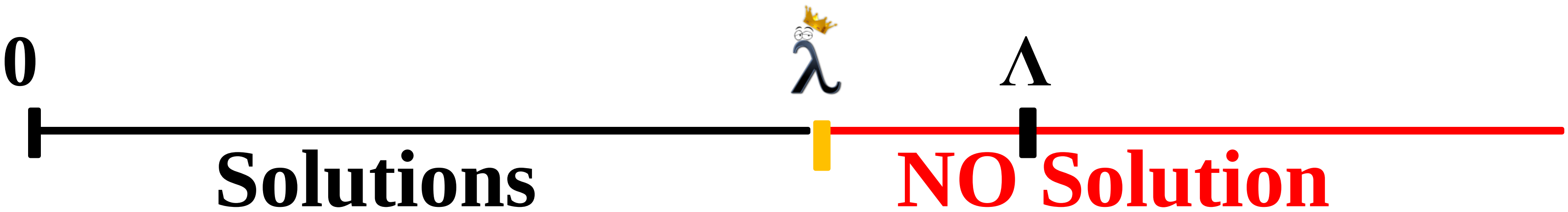
Proposition. *If $\lambda > \Lambda$, then P_λ) admits no solution.*

The proof is the same of De Coster –Habets, Willem, Liu, ABC,…

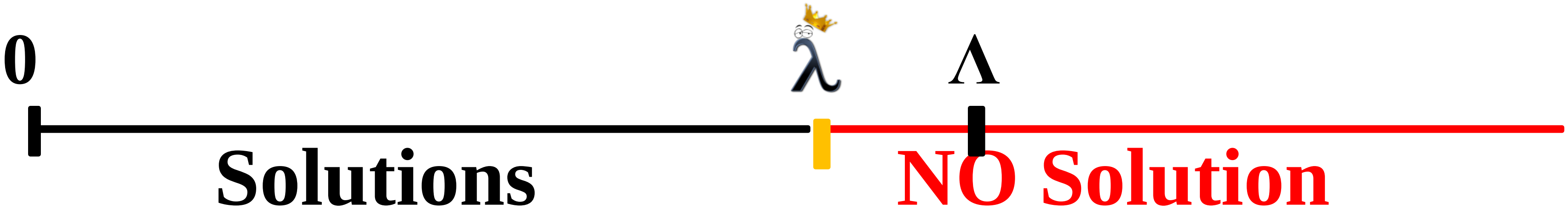
Clearly, $\Lambda = \Lambda_{DH}$ and it the same upper bound obtained by Liu for ABC problem in the ordinary case.

Moreover, as seen before, for the Gelfand problem one has:

$$\Lambda = \frac{\pi^2}{e}$$

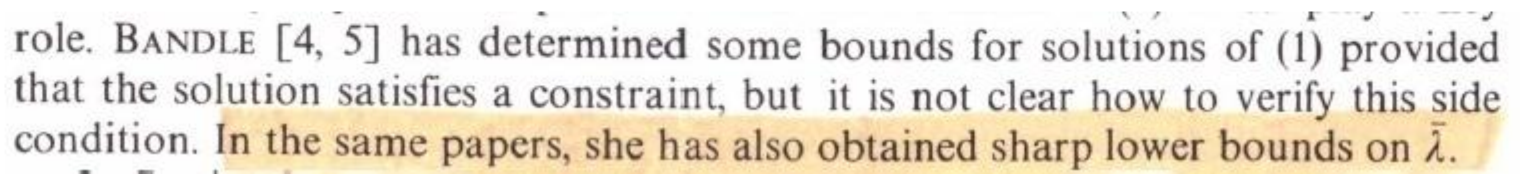


$$\Lambda = \lambda_1 \sup_{r > 0} \frac{r}{f(r)}$$



LOWER BOUNDS of λ

In literature only few papers deals with lower bounds. Crandall-Rabinowitz refer to C. Bandle (1973) (as you can see below) .


role. BANDLE [4, 5] has determined some bounds for solutions of (1) provided that the solution satisfies a constraint, but it is not clear how to verify this side condition. In the same papers, she has also obtained sharp lower bounds on $\bar{\lambda}$.

Indeed, Bandle investigated some specific problems for *n=2*.
Zhaoli Liu obtained a lower bound for the ABC problem for *n=1*.
At the best of our knowledgements, only in a my old paper a lower bound is established for general problem (even if when *f(0)≠0)*.

Proceedings of the Edinburgh Mathematical Society (1996) 39, 31–36 ©

EXISTENCE THEOREMS ON THE DIRICHLET PROBLEM FOR THE EQUATION $\Delta u + f(x,u) = 0$

by GABRIELE BONANNO

(Received 24th February 1994)

In this note we consider the Dirichlet problem $\Delta u + f(x,u) = 0$ in Ω, $u = 0$ on $\partial\Omega$; here Ω is a bounded domain in $\mathbb{R}^n$ $(n \geqq 3)$, with smooth boundary $\partial\Omega$. We prove the existence of strong solutions to the previous problem, which are positive if f satisfies a suitable condition. As a consequence we find that the problem with $f(x,u) = |u|^{((n+2)/(n-2))} + g(x,u)$, may have positive solutions even if g is not a lower-order perturbation of $|u|^{((n+2)/(n-2))}$. Next, we examine the case $f(x,u) = |u|^{((n+2)/(n-2))} + h(x)$.

1991 *Mathematics subject classification*: 35J65, 35J60.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geqq 3$, with a $C^{1,1}$-boundary $\partial\Omega$, and let f be

This paper is based on two powerfull results:

1. A fixed point theorem due to **Arino-Gautier-Penot**.
2. A regularity result due to **G. Talenti**;

Funkcialaj Ekvacioj, **27** (1984), 273–279

A Fixed Point Theorem for Sequentially Continuous Mappings with Application to Ordinary Differential Equations

By

O. Arino, S. Gautier, J. P. Penot

(Université de Pau et des Pays de l'Adour, France)

As everybody knows, the road to Hell is paved with good intentions... and faults. The story of existence results for ordinary differential equations is an illustration of this saying. Not only the common belief that continuous vector fields

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GIORGIO TALENTI

Elliptic equations and rearrangements

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 3, nº 4 (1976), p. 697-718

Elliptic Equations and Rearrangements.

GIORGIO TALENTI (*)

dedicated to Jean Leray

1. – Introduction.

We are concerned with linear elliptic second-order partial differential equations in divergence form. The equations we deal with are uniformly

Fix a bounded open set $\Omega \subseteq \mathbb{R}^n$, $n \geq 3$, with a $C^{1,1}-$ boundary $\partial\Omega$ and $v \in L^\infty(\Omega)$. Moreover, consider the problem

$$\begin{cases} -\Delta u = v(x) & \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P}$$

It is well known that (P) admits a unique weak solution $u \in W_0^{1,2}(\Omega)$ (see, for instance, [24, Theorem 9.15]). Moreover, owing to the regularity result of Talenti [34, Theorem 2] one has $u \in L^\infty(\Omega)$ and

$$\|u\|_\infty \leq B\|v\|_\infty \tag{2.1}$$

where

$$B = \frac{1}{2n\pi}\left(\Gamma\left(1+\frac{n}{2}\right)|\Omega|\right)^{\frac{2}{n}} \tag{2.2}$$

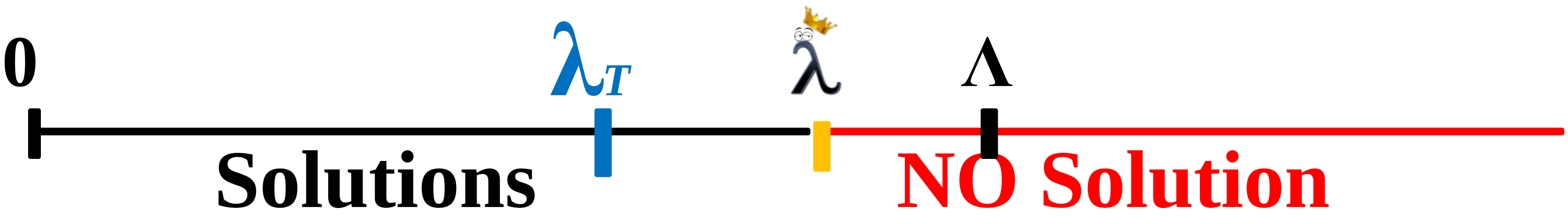
(see [34, Remark 1]).

$$\lambda_T = \frac{1}{B} \sup_{s>0} \frac{s}{\max\limits_{t\in[0,s]} f(t)}$$

$$B = \frac{1}{2n\pi} \left(\Gamma\left(1 + \frac{n}{2}\right) |\Omega| \right)^{\frac{2}{n}}$$

$$\lambda_T = 8 \sup_{s>0} \frac{s}{\max\limits_{t\in[0,s]} f(t)}.$$

When *f(0) ≠ 0* one has:

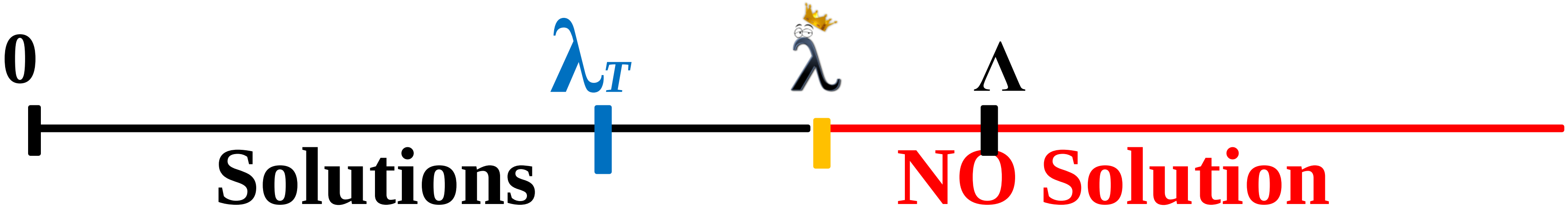


Indeed, by using a variational approach, it is possible to prove that λ_T *holds true even in the case* ***f(0)=0****. Hence, we have*

$$\lambda_T = \frac{1}{B} \sup_{s>0} \frac{s}{\max_{t\in[0,s]} f(t)}$$

By using in a suitable way the direct methods theorem instead of Arino-Gauthier-Penot fixed point theorem.

BONANNO G., *Dirichlet problems without asymptotic conditions on the nonlinear term,* Rendiconti dell'Istituto di Matematica dell'Università di Trieste, **49** (2017), 319-333.



LOWER BOUNDS of λ

$$\hat{\lambda} = \sup_{r > \inf_X \Phi} \; \sup_{v \in \Phi^{-1}(]-\infty, r[)} \frac{r - \Phi(v)}{\displaystyle\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u) - \Psi(v)}$$

$$\bar{\lambda}(r) = \frac{r}{\displaystyle\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}$$

$$\bar{\lambda} = \sup_{r>0} \bar{\lambda}(r) = \sup_{r>0} \frac{r}{\displaystyle\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}$$

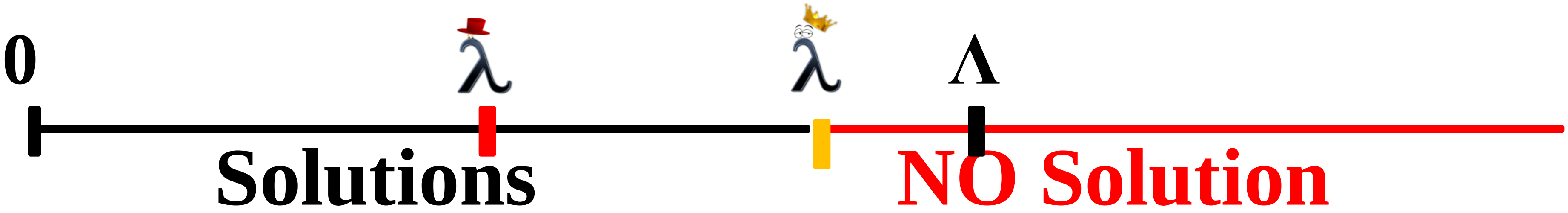
$$\bar{\lambda} \leq \hat{\lambda}$$

$$\hat{\lambda}$$

Where is the “position” of

?

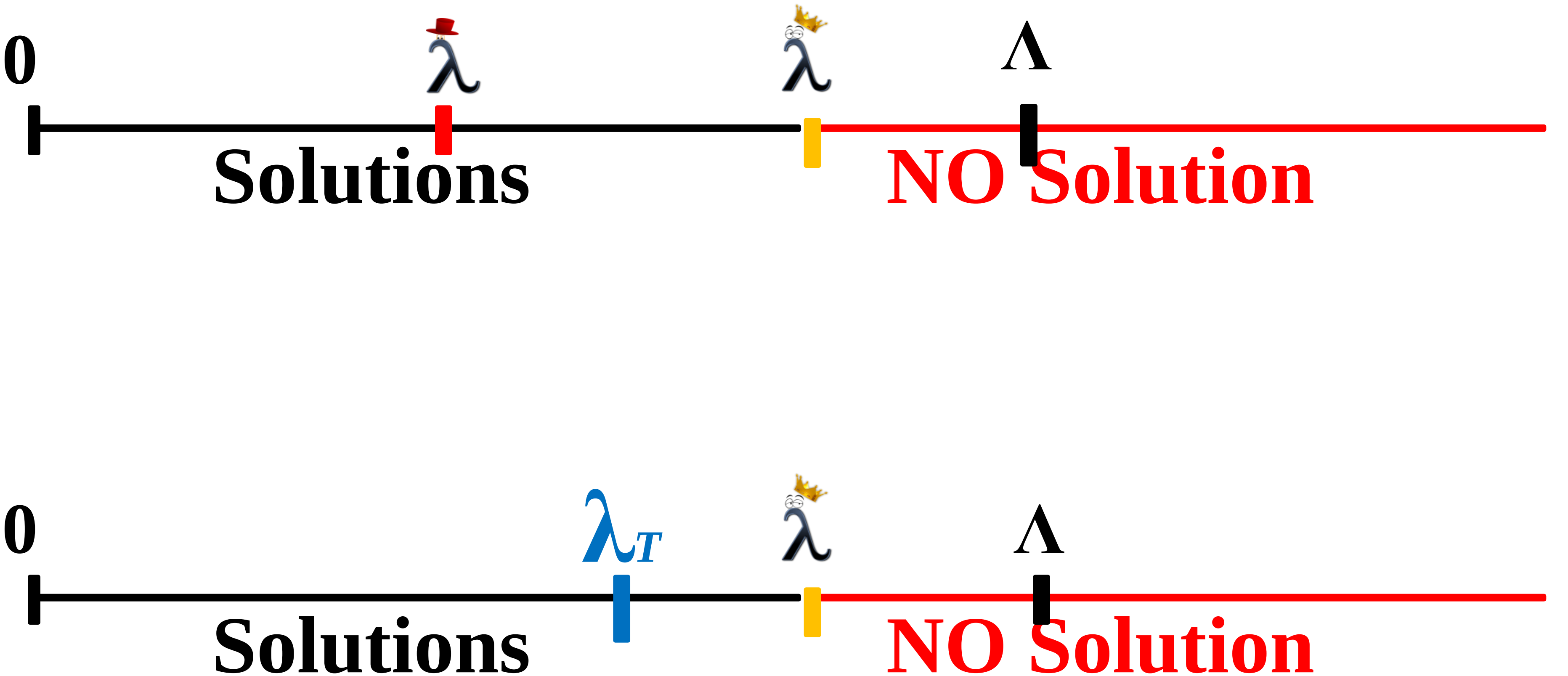
$\leq$



$$\hat{\lambda} = \sup_{r > \inf_X \Phi} \; \sup_{v \in \Phi^{-1}(]-\infty, r[)} \frac{r - \Phi(v)}{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u) - \Psi(v)}$$

$$\bar{\lambda} \leq \hat{\lambda}$$

$$\bar{\lambda} = \sup_{r>0} \bar{\lambda}(r) = \sup_{r>0} \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}$$


0
λ
λ
Λ
Solutions
NO Solution
0
λ_T
λ
Λ
Solutions
NO Solution

Now, we summarise the situation!

$$\lambda = \sup_{r > \inf_X \Phi} \; \sup_{v \in \Phi^{-1}(]-\infty, r[)} \frac{r - \Phi(v)}{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u) - \Psi(v)}$$

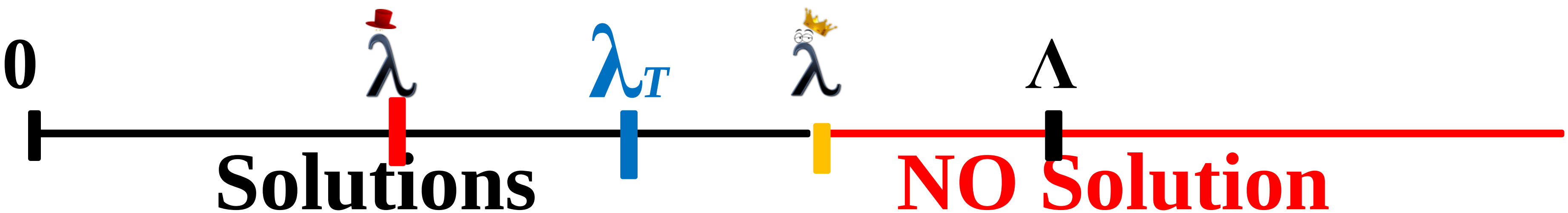
$$\lambda_{\mathrm{T}} = \frac{1}{B} \sup_{s > 0} \frac{s}{\max_{t \in [0,s]} f(t)}$$

They are both lower bounds of λ

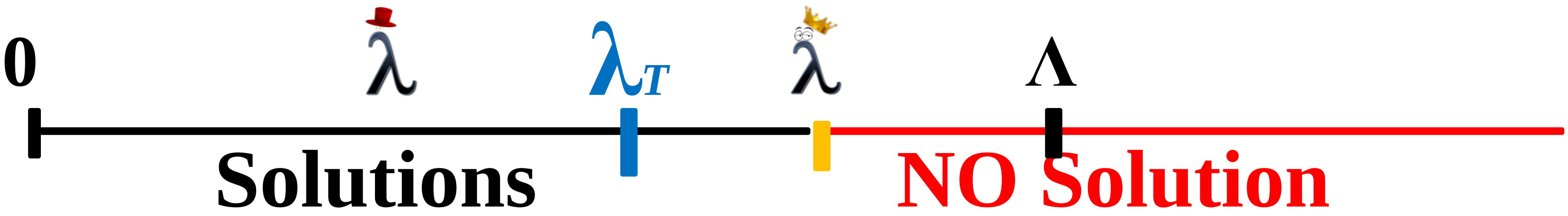
$$\begin{cases} -u'' = \lambda e^u & \text{in } \;]0,1[\\ u(0) = u(1) = 0 \end{cases}$$

$$\lambda \;\geq \sup_{r>0} \frac{2r^2}{F(r)} = \sup_{r>0} \frac{2r^2}{e^r - 1} \geq \frac{2}{e-1} = 1,163...$$

$$\lambda_T \;= 8 \sup_{r>0} \frac{r}{\sup\limits_{t\in[0,r]} f(t)} = 8 \sup_{r>0} \frac{r}{e^r} = \frac{8}{e} = 2,943...$$



So, it seems that we have the following situation:



In particular, we have

$$\lambda_T \leq \lambda \leq \Lambda$$

and, for the Gelfand problem, one has: $\lambda \in \left[\frac{8}{e}, \frac{\pi^2}{e}\right]$

In general, we have

$$\lambda_T = \frac{1}{B} \sup_{r>0} \frac{r}{\sup f(t)} \qquad \Lambda = \lambda_1 \sup_{r>0} \frac{r}{f(r)}$$

$$\lambda \in \left[\frac{1}{B} \sup_{r>0} \frac{r}{\sup\limits_{t\in[0,r]} f(t)}, \lambda_1 \sup_{r>0} \frac{r}{f(r)} \right]$$

Taking into account of the expressions of λ_T and $\boldsymbol{\Lambda}$**,** by using a suitable combination among the “costant function” and eigenfunction, and arguing as in the proof used for obtain λ_T it is possible to establish the following result.

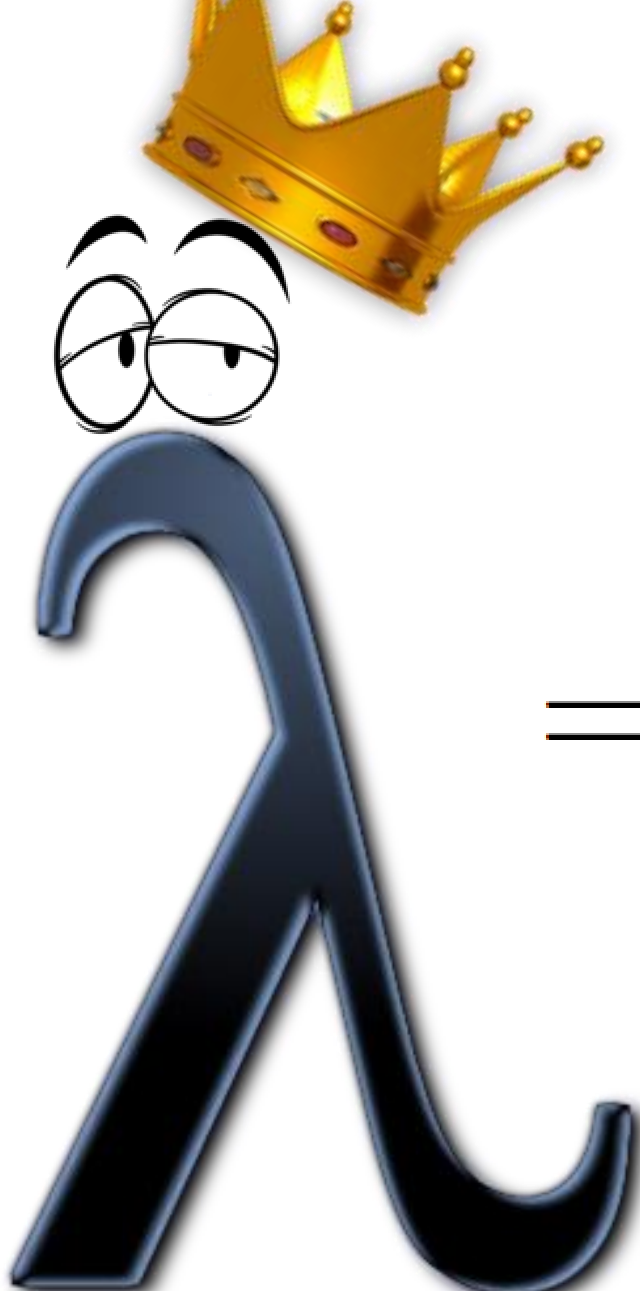
Put

$$\tilde{\lambda} = \lambda_1 \sup_{r>0} \frac{r}{\sup\limits_{t\in[0,r]} f(t)}.$$

Then, for each $\lambda \in [0, \tilde{\lambda}[$, *problem* (P_λ) *admits at least one positive solution.*

So, when f is monotone, we have

$$\tilde{\lambda} = \Lambda$$



$$= \lambda_1 \sup_{r>0} \frac{r}{f(r)}$$

So, in particular for the Gelfand problem we have

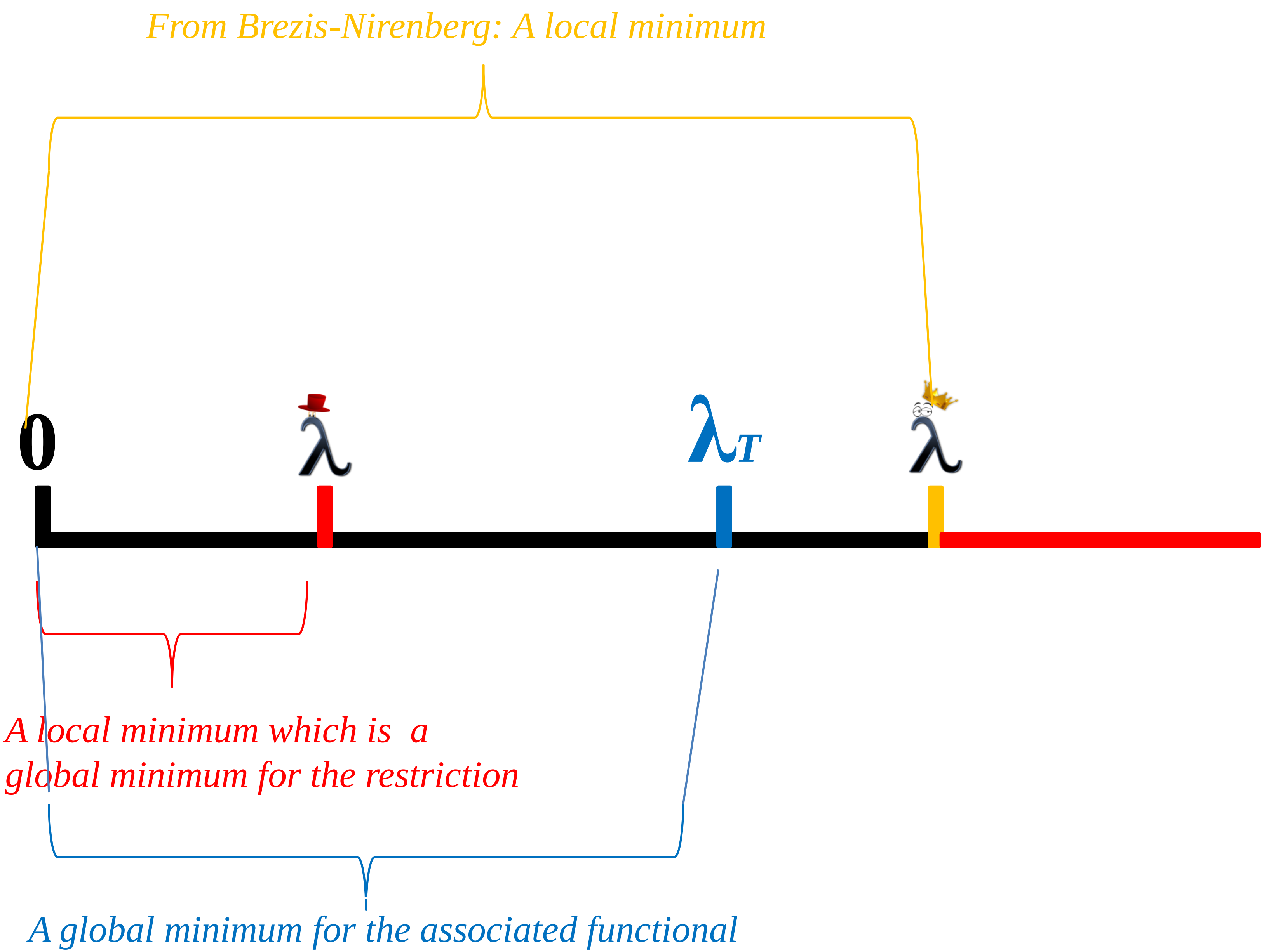
$$= \frac{\pi^2}{e}$$

Hence, from such a result and the Amann result, we obtain that the problem

$$\begin{cases} -u'' = \lambda e^u & \text{in } \ ]0,1[\\ u(0) = u(1) = 0 \end{cases}$$

admits at least two positive solutions if $\lambda < \frac{\pi^2}{e}$, , at least one positive solution if $\lambda = \frac{\pi^2}{e}$, no solution if $\lambda > \frac{\pi^2}{e}$.

0
λ
λ_T
λ
Λ


From Brezis-Nirenberg: A local minimum
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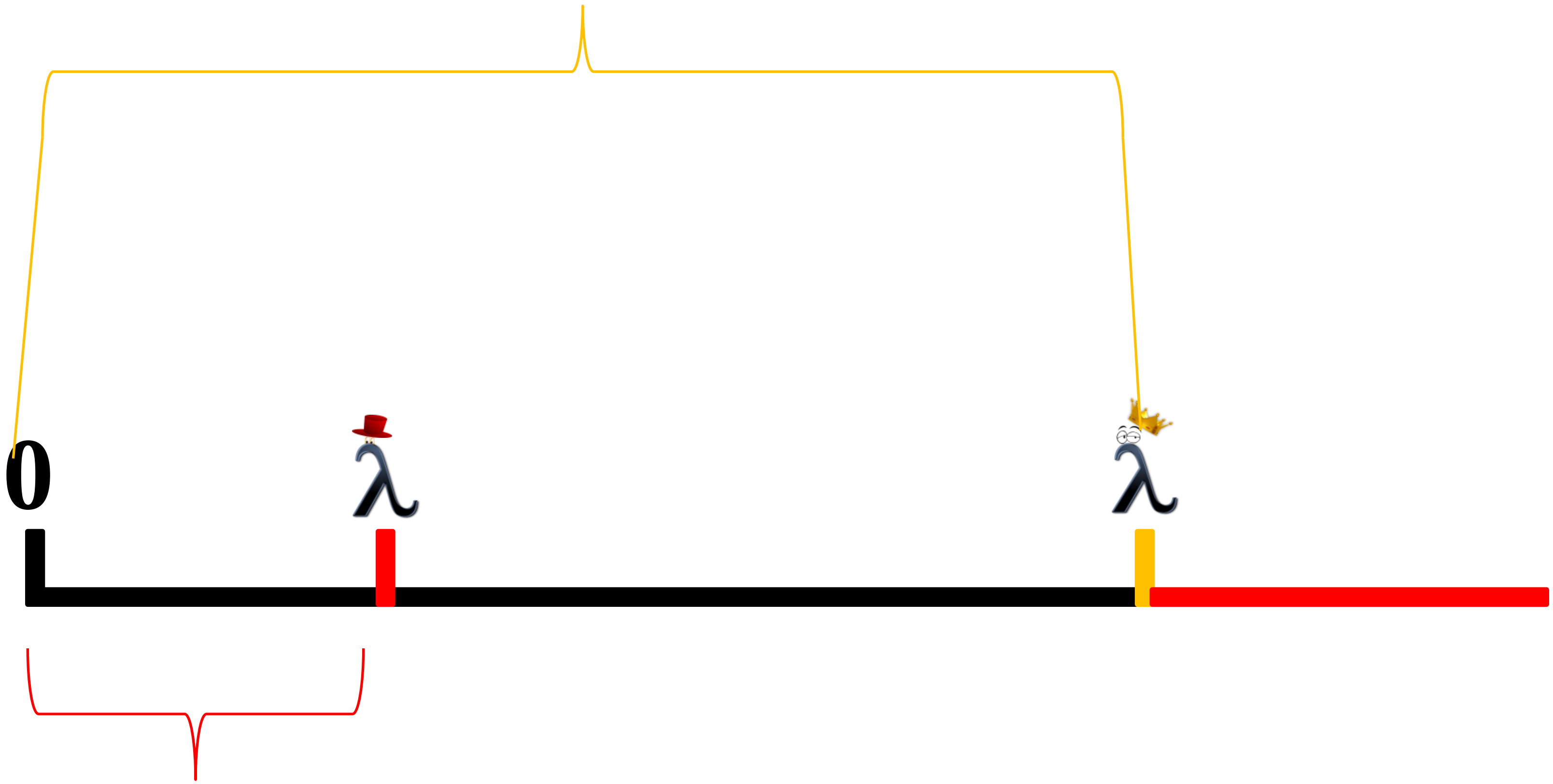
A local minimum which is a
global minimum for the restriction
A global minimum for the associated functional

From Brezis-Nirenberg: A local minimum



A local minimum which is a
global minimum for the restriction

Journal of Global Optimization **12:** 101–104, 1998.

On a Class of Functionals Whose Local Minima are Global

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(Received: 12 December 1996; accepted: 20 June 1997)

Abstract. In this note we introduce a suitable class of functionals, including the class of integral functionals, and prove that any (strict) local minimum of a functional of this class, defined on a

References

Giner, E.: Local minimizers of integral functionals are global minimizers, *Proc. Amer. Math. Soc.* (*3*), **123** (1995), 755–757.

[5] SAINT RAYMOND J., *Connexité des sous-niveaux des fonctionnelles intégrales*, Rend. Circ. Mat. Palermo, **44** (1995), 162-168.

THEOREM 1. *Let* $J : M \longrightarrow \overline{\mathbb{R}}$ *be a strictly increasing functional. Then, any local minimum* u *of* J_f *in* X*, such that* $J_f(u) \in \mathbb{R}$ *is a global minimum. Moreover, for every* $v \in X$*, one has* $f(t, u(t)) \leq f(t, v(t))$ μ*-a.e. in* T .

A classical example of a functional of this type is the integral functional, namely

$$I_f(u) = \int_T^* f(t, u(t))d\mu,$$

We assume that X is decomposable. This means that if $A \in \mathcal{F}$ and $u, v \in X$, then $1_A u + 1_{T\setminus A} v \in X$, where 1_A is the characteristic function of A.

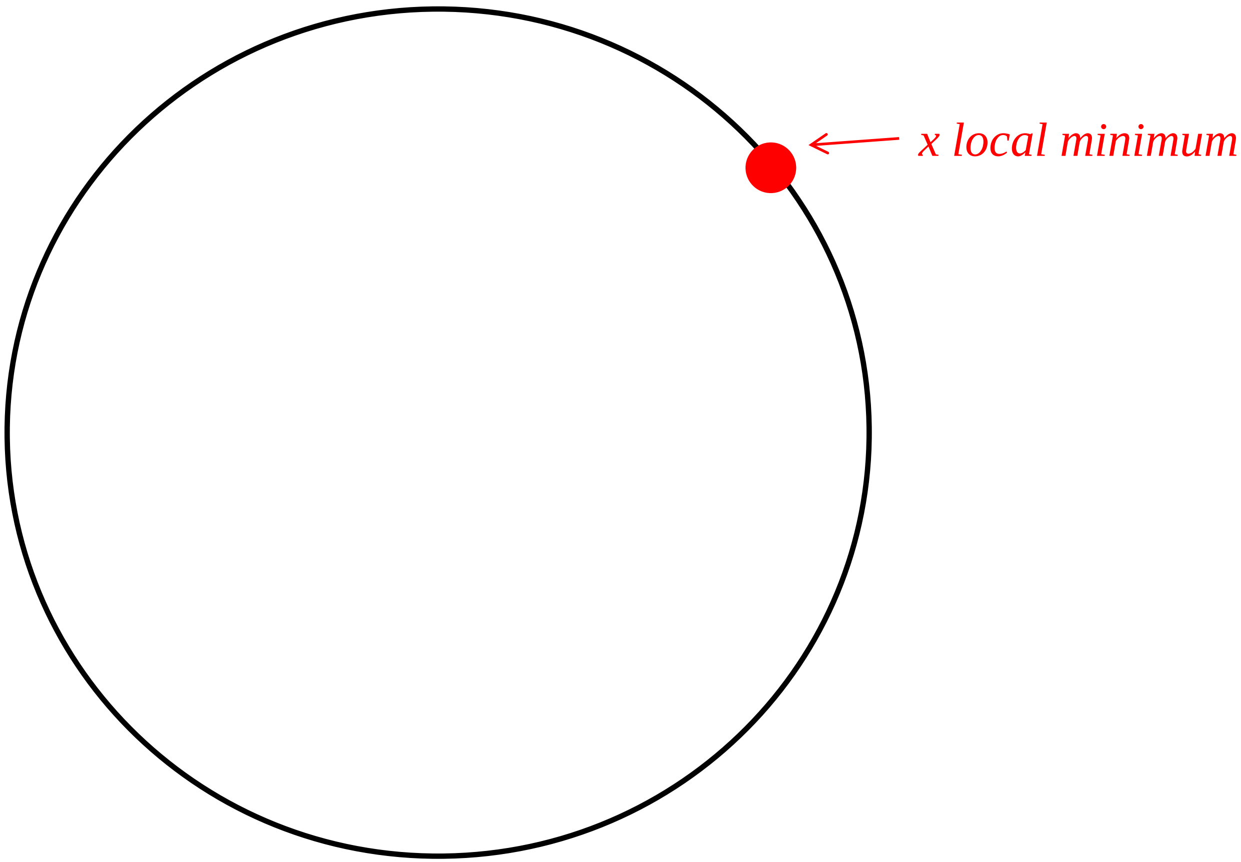
To give an idea, taking into account of our usual notations:

Any local minimum of

$$\Psi(u)=\int_\Omega f(u(x))dx, \quad u \text{ in } L^p(\Omega)$$

is a global minimum.

$\Phi^{-1}(]-\infty,r])$

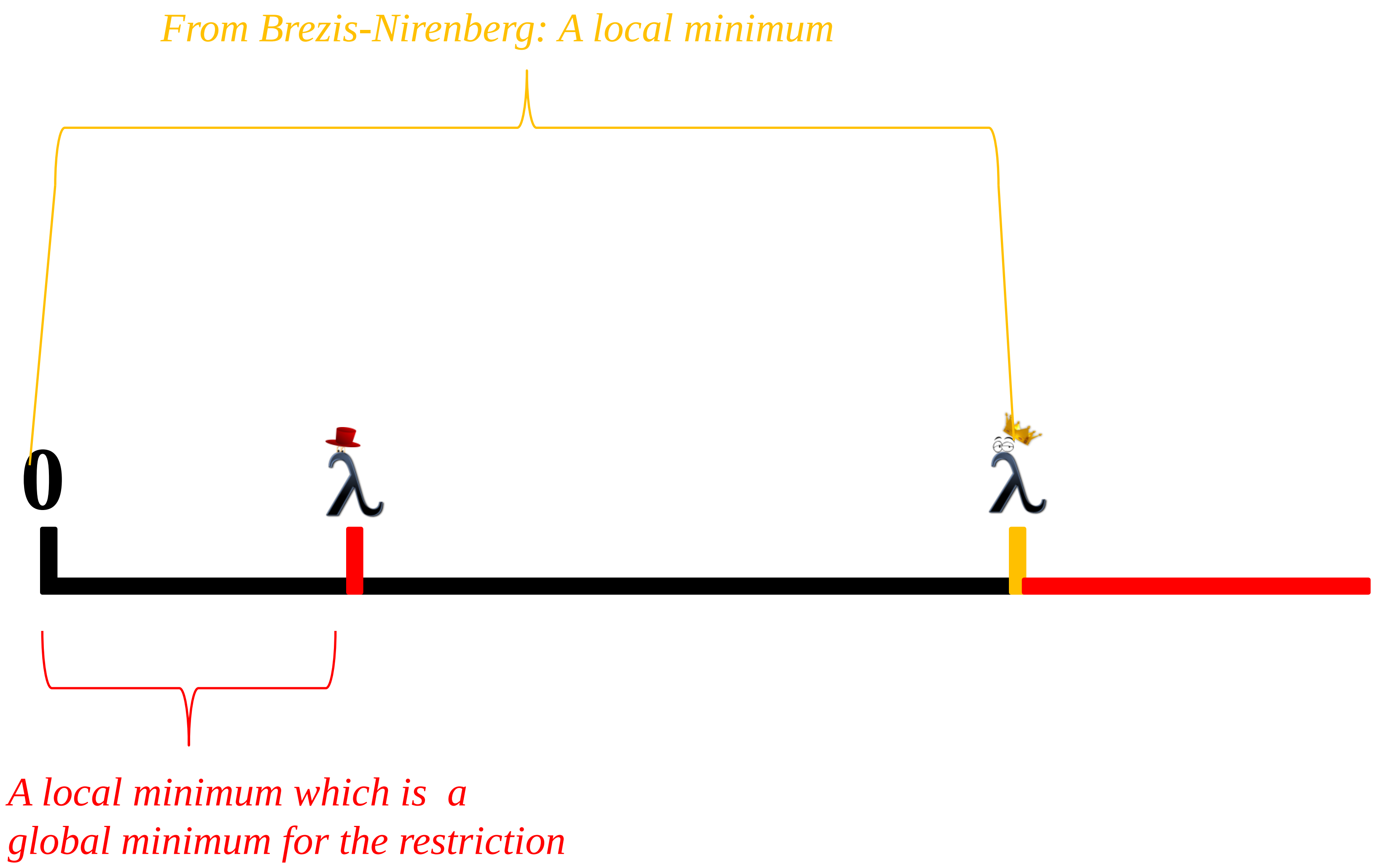


If x is a local minimum for $\boldsymbol{\Phi}$**-**$\boldsymbol{\Psi}$ in X, it is a local minimum in $\mathbf{\Phi^{-1}}(]-\infty,r])$, then it is a local minimum at its boundary, then it is a local minimum for the functional $\boldsymbol{r}$**-**$\boldsymbol{\Psi}$.

So, we can obtain that:

If u_0 is a local minimum for Φ-Ψ in the Sobolev space X, then there is $r>\Phi(u_0)$ such that Φ-Ψ admits a global minimum in $\Phi^{-1}(]-\infty,r])$.

Hence, our target has been obtained. Indeed,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A local minimum which is a
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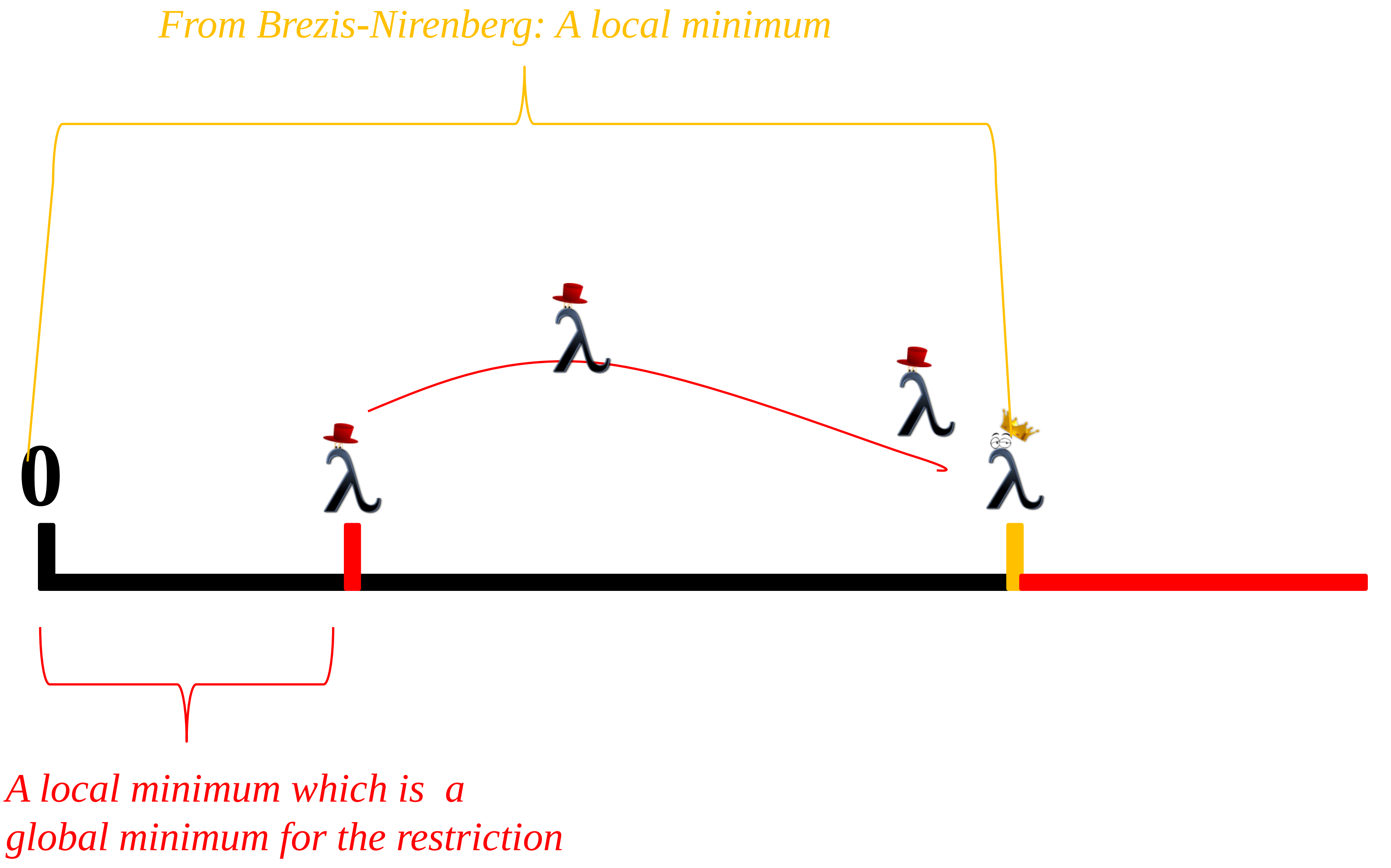
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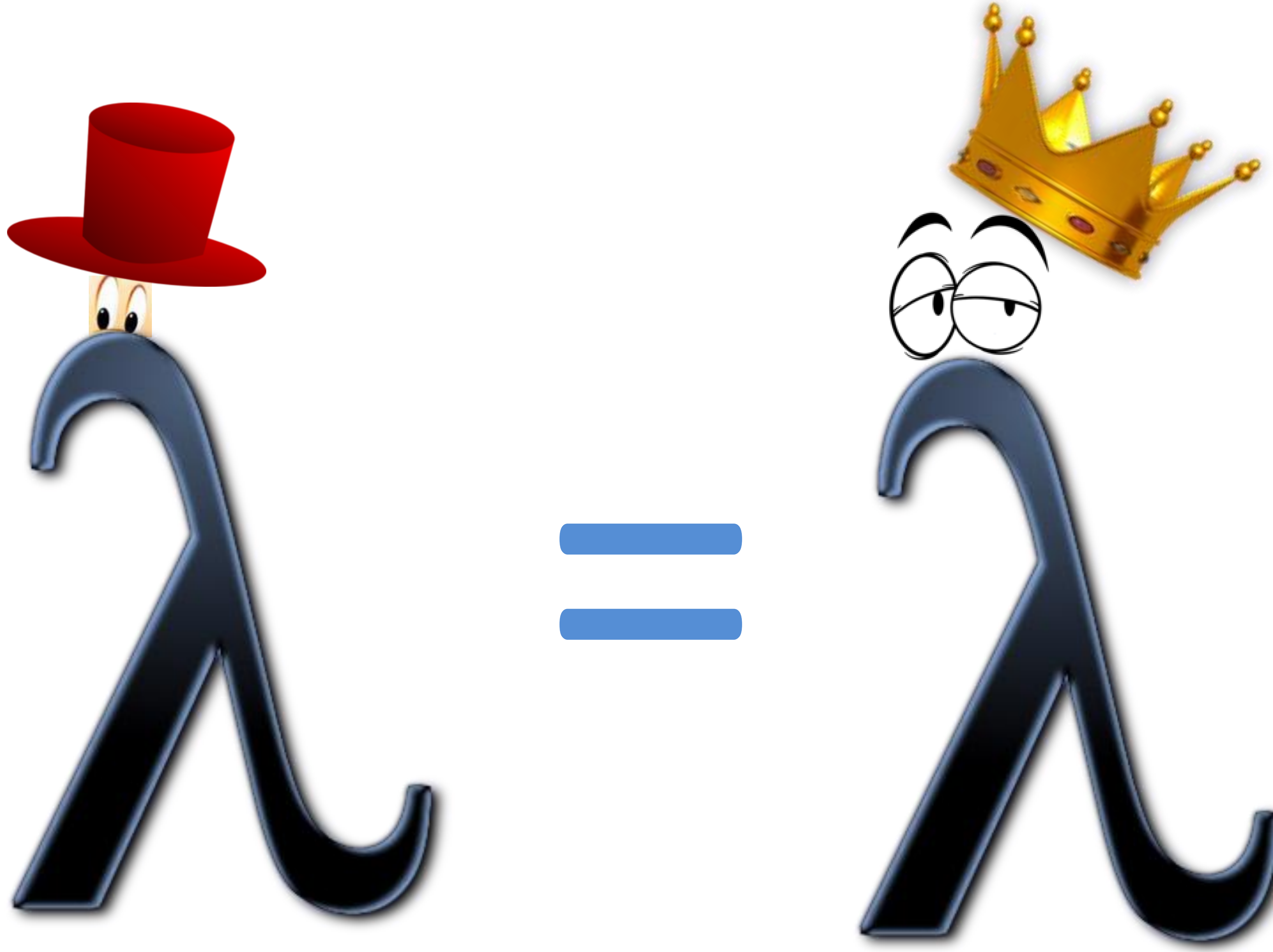
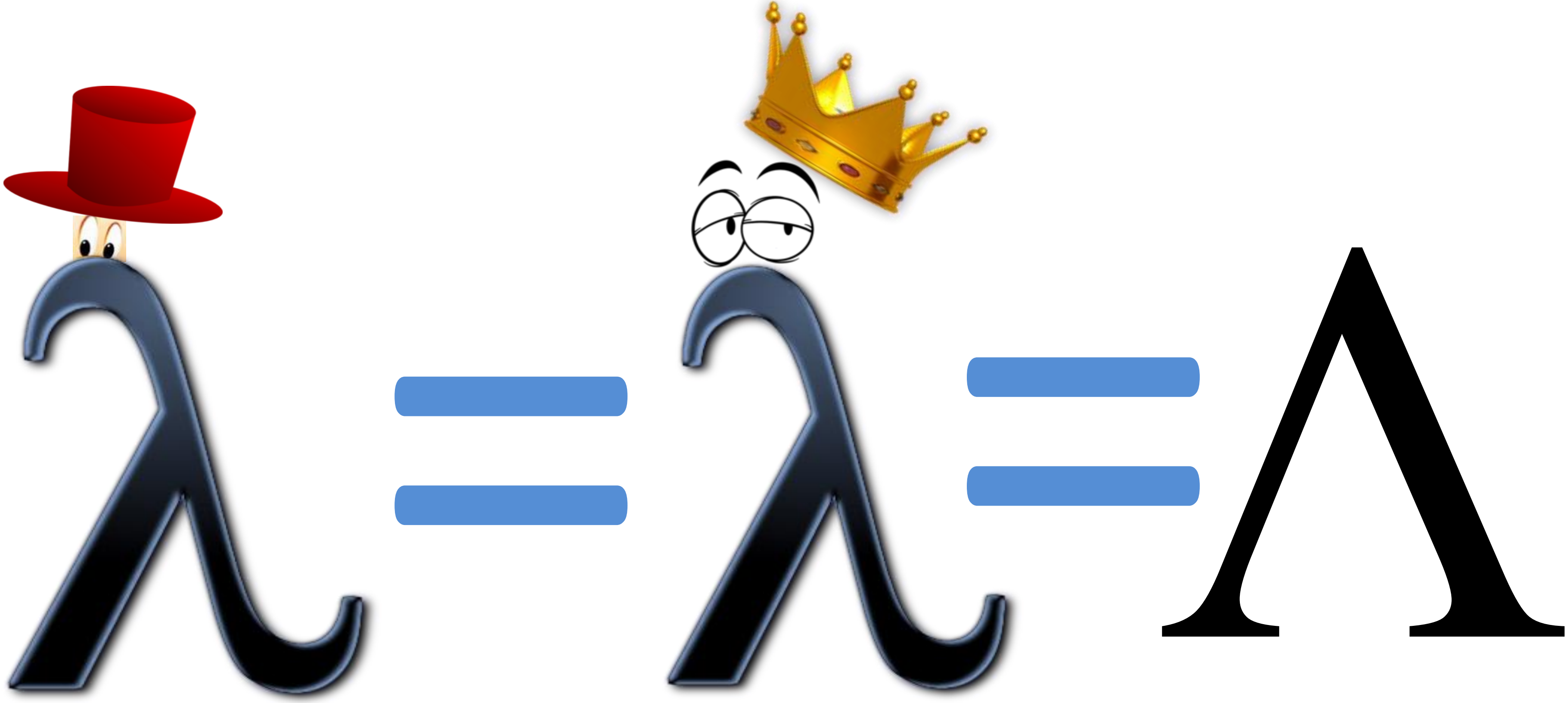
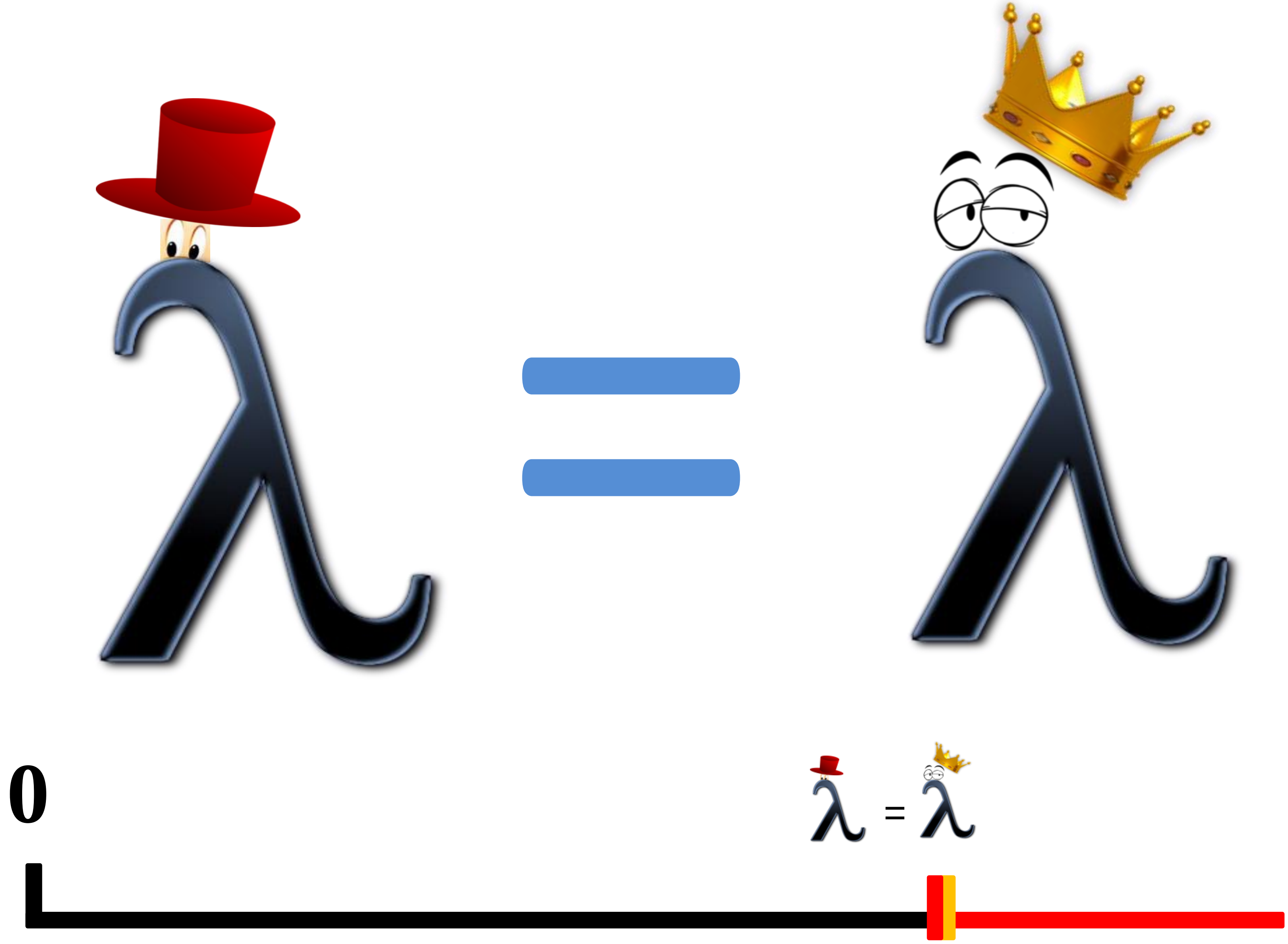




When f is monotone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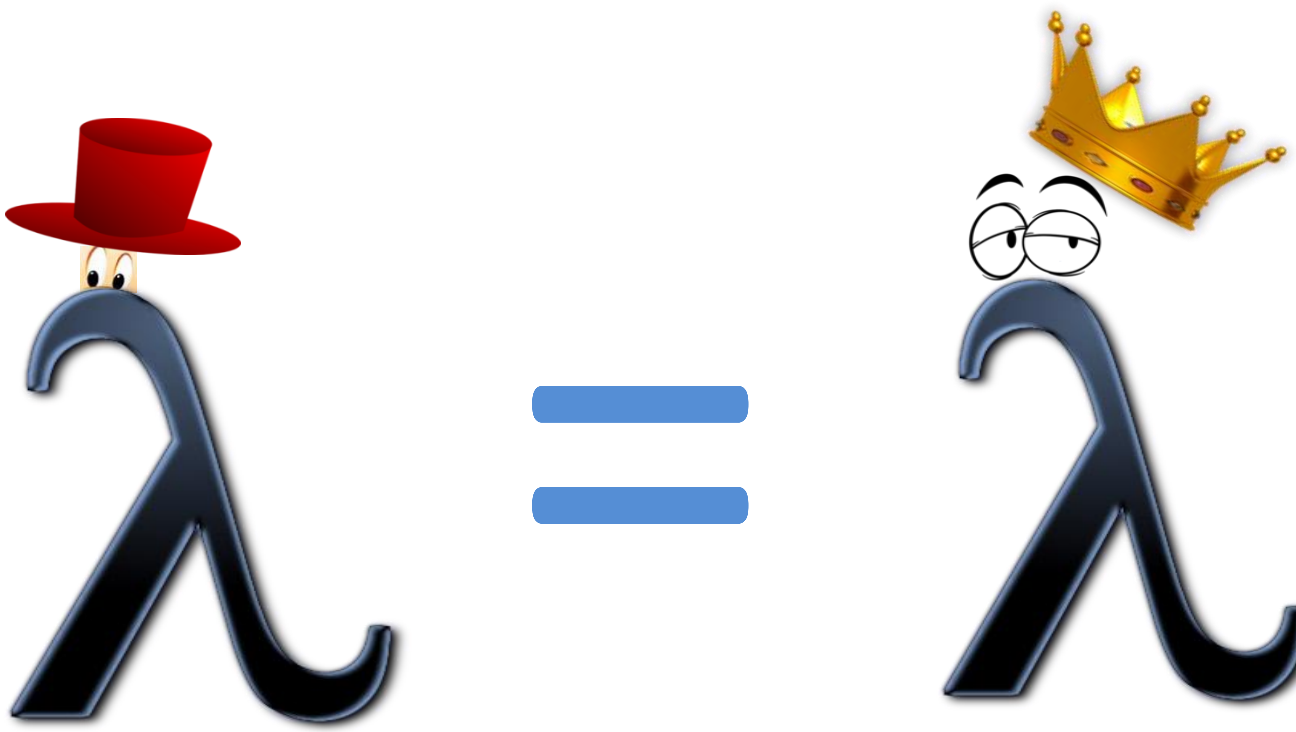
A TWO NONZERO CRITICAL POINTS THEOREM

So, now we can say that we can apply our two non-zero critical points theorem even when f is not sublinear at zero and, on the contrary, when f is sublinear at zero, we obtain the results as established by ABC and CR.




Thank you very much for your kind attention