Some existence results for boundary value problems associated with singular equations

Stefano Biagi

Dipartimento di Ingegneria Industriale e Scienze Matematiche (DIISM)
Università Politecnica delle Marche - Ancona

Giornate di Equazioni Differenziali Ordinarie: metodi e prospettive

September 27 - 29, 2018 - Università Politecnica delle Marche (Ancona)

The results presented in this talk were obtained in a joint work with A. Calamai[1] and F. Papalini[2].

[1]Dipartimento di Ingegneria Civile, Edile e Architettura, Università Politecnica delle Marche
[2]Dipartimento di Ingegneria Industriale e Scienze Matematiche, Università Politecnica delle Marche

Main assumptions and notations

We are interested in **strongly nonlinear differential equations** of the type

$$(\bigstar) \quad \Big(\Phi\big(a(t,x(t))\,x'(t)\big)\Big)' = f(t,x(t),x'(t)), \quad \text{a.e. on } I := [0,T],$$

where Φ, a and f satisfy the following *structural assumptions*:

(A1) $\Phi : \mathbb{R} \to \mathbb{R}$ is a *strictly increasing homeomorphism*;

(A2) $a \in C(I \times \mathbb{R}, \mathbb{R})$ and there exists $h \in C(I, \mathbb{R})$ such that

$(A2)_1$ $h \geq 0$ on I and $1/h \in L^p(I)$ (for some $p > 1$);

$(A2)_2$ $a(t,x) \geq h(t)$ for every $t \in I$ and every $x \in \mathbb{R}$;

(A3) $f : I \times \mathbb{R}^2 \to \mathbb{R}$ is a *Carathéodory function*,

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The reasons of our interest

Ordinary differential equations of the form $(\bigstar)$ intervene in several models:

- non-Newtonian fluid theory;
- diffusion of flows in porous media;
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To prove the existence of a solution for several classes of **boundary value problems** associated with ODEs of the form $(\bigstar)$.

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Solution of an ODE of the form ($\bigstar$)

Definition

We say that $x \in W^{1,p}(I)$ is a **solution** of the ODE ($\bigstar$) if:

(1) $t \mapsto \Phi\big(a(t,x(t))\,x'(t)\big) \in W^{1,1}(I)$;

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An important remark

If $x \in W^{1,p}(I)$ is any solution of the ODE ($\bigstar$), we indicate by $\mathcal{A}_x$ the unique continuous function on I such that

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Some references

For the Φ-Laplacian operator: $\Phi(x')' = f(t,x,x')$

- Cabada and Pouso (1997, 1999); Cabada, O'Regan and Pouso (2008); El Khattabi, Frigon and Ayyadi (2013).

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- Bereanu and Mawhin (2008); Ferracuti and Papalini (2009); Calamai (2011).

For mixed differential operators: $\big(a(t,x)\,\Phi(x')\big)' = f(t,x,x')$ (with $a > 0$)

- Cupini, Marcelli and Papalini (2011); Marcelli (2012, 2013); Marcelli and Papalini (2017).

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- Liu (2012); Liu and Yang (2015); Calamai, Marcelli and Papalini (2018).

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$$\text{(DP)} \qquad \begin{cases} \Big(\Phi\big(a(t,x(t))\,x'(t)\big)\Big)' = f(t,x(t),x'(t)), & \text{a.e. on } I, \\ x(0) = \nu_1, \; x(T) = \nu_2. \end{cases}$$

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To prove the existence of (at least) one **solution of (DP)**, i.e., the existence of a solution $x \in W^{1,p}(I)$ of the ODE $(\bigstar)$ satisfying

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Our approach

A suitable combination of the method of lower/upper solutions with a fixed-point technique (applied to an auxiliary functional problem).

Definition: lower/upper solution

We say that $\alpha \in W^{1,p}(I)$ is a **lower** [resp. **upper**] **solution** of the ODE $(\bigstar)$ if

(1) $t \mapsto \Phi\big(a(t,\alpha(t))\,\alpha'(t)\big) \in W^{1,1}(I)$;

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Our main result

Theorem 1 (B., Calamai and Papalini)

We assume that, together with (A1)-to-(A3), the following hypotheses hold:

- there exists a pair of lower and upper solutions $\alpha, \beta \in W^{1,p}(I)$ of $(\bigstar)$ such that $\alpha(t) \le \beta(t)$ for every $t \in I$;
- for every $R > 0$ and every non-negative function $\gamma \in L^p(I)$ there exists a non-negative function $h = h_{R,\gamma} \in L^p(I)$ such that
$$|f(t,x,y(t))| \le h_{R,\gamma}(t) \tag{1}$$
for a.e. $t \in I$, every $|x| \le R$ and every $y \in L^p(I)$ with $|y(t)| \le \gamma(t)$ a.e. on I.
- there exist $H > 0$, a non-negative function $\mu \in L^q(I)$ (for some $1 < q \le \infty$), a non-negative function $l \in L^1(I)$ and a non-negative measurable function $\psi : (0,\infty) \to (0,\infty)$ such that
$$(\star)\ 1/\psi \in L^1_{\mathrm{loc}}(0,\infty) \quad \text{and} \quad \int_1^\infty \frac{1}{\psi(t)}\,\mathrm{d}t = \infty;$$
$$(\star)\ |f(t,x,y)| \le \psi\Big(|\Phi(a(t,x)\,y)|\Big)\cdot\Big(l(t) + \mu(t)\,|y|^{\frac{q-1}{q}}\Big); \tag{2}$$
for a.e. $t \in I$, every $x \in [\alpha(t),\beta(t)]$ and every $y \in \mathbb{R}$ with $|y| \ge H$.

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Theorem 1 (B., Calamai and Papalini) - continued

Then, for every $\nu_1 \in [\alpha(0), \beta(0)]$ and every $\nu_2 \in [\alpha(T), \beta(T)]$, the Dirichlet problem (DP) possesses at least one solution $x \in W^{1,p}(I)$ satisfying

$$\alpha(t) \leq x(t) \leq \beta(t) \quad \text{for every } t \in I.$$

Furthermore, if $M > 0$ is *any* real number such that $\sup_I |\alpha|, \sup_I |\beta| \leq M$, it is possible to find a real $L_0 > 0$, *only depending on* M, with the following property: *if* $L \geq L_0$ *is any real number such that* $\sup_I |\mathcal{A}_\alpha|, \sup_I |\mathcal{A}_\beta| \leq L$, *then*

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$$\max_{t \in I} \big|x(t)\big| \leq M \quad \textit{and} \quad \max_{t \in I} \big|\mathcal{A}_x(t)\big| \leq L.$$

A couple of examples

(1) Let us consider the Dirichlet problem

$$\begin{cases} \Big(\Phi\big(a(t,x(t))\,x'(t)\big)\Big)' = \sigma(t)(x(t)+\rho(t)) + g(x(t))\,x'(t) \\ x(0)=\nu_1,\ x(T)=\nu_2, \end{cases}$$

where Φ, a, σ, ρ and g satisfy the following assumptions:

($\star$) $\Phi : \mathbb{R} \to \mathbb{R}$ is a generic strictly increasing homeomorphism;

($\star$) $a \in C(I \times \mathbb{R}, \mathbb{R})$ satisfies assumption (A2);

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It is not difficult to recognize that:

(a) Setting $N := \max_I |\rho|$, the constant functions

$$\alpha(t) := -N \qquad \beta(t) := N \qquad (\text{for } t \in I)$$

are, respectively, a lower and a upper solution of $(\bigstar)$ s.t. $\alpha \leq \beta$ on I.

(b) Setting $f(t,x,y) := \sigma(t)(x+\rho(t)) + g(x)y$, we have

- f is a Carathéodory function on $I \times \mathbb{R}^2$;
- f fulfills (1) with the choice

$$h_{R,\gamma}(t) := \sigma(t)\,\big(R + |\rho(t)|\big) + \big(\max_{[-R,R]} |g|\big) \cdot \gamma(t);$$

- f fulfills (2) with the choice

$$H := 1, \quad \psi \equiv 1, \quad l(t) := 2N\,\sigma(t), \quad \mu(t) := \max_{[-N,N]} |g| \quad \text{and} \quad q = \infty.$$

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(2) Let us consider the Dirichlet problem

$$\begin{cases} \Big(\Phi_r\big(a(t,x(t))\,x'(t)\big)\Big)' = \sigma(t)\cdot g(x(t))\cdot |x'(t)|^{\delta} \\ u(0) = \nu_1,\ u(T) = \nu_2, \end{cases}$$

where Φ_r, a, σ, g and the exponent δ satisfy the following assumptions:

($\star$) $\Phi_r : \mathbb{R} \to \mathbb{R},\ \Phi_r(\xi) := |\xi|^{r-2}\cdot\xi$ (for a suitable $r > 1$).

($\star$) $a \in C(I \times \mathbb{R}, \mathbb{R})$ satisfies assumption (A2);

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$$\frac{1}{\tau} + \frac{r-1}{p} < 1;$$

($\star$) $g \in C(\mathbb{R}, \mathbb{R})$ is a generic function;

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A couple of examples

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It is not difficult to recognize that:

(a) For every fixed $N \in \mathbb{R}$, the constant functions

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are, respectively, a lower and a upper solution of $(\bigstar)$ s.t. $\alpha \leq \beta$ on I.

(b) Setting $f(t,x,y) := \sigma(t) \cdot g(x) \cdot |y|^{\delta}$, we have

- f is a Carathéodory function on $I \times \mathbb{R}^2$;
- f fulfills (1) with the choice

$$h_{R,\gamma}(t) := \big(\max_{[-R,R]} |g| \big) \cdot |\sigma(t)| \cdot (\gamma(t))^{\delta};$$

- f fulfills (2) with the choice

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Idea of the proof

- Truncating operators: Let α, $\beta \in W^{1,p}(I)$ be, respectively, a lower and a upper solution of the equation $(\bigstar)$ such that $\alpha \leq \beta$ on I. We define

$$\mathcal{T} : W^{1,p}(I) \longrightarrow W^{1,p}(I), \qquad \mathcal{T}(x)(t) := \begin{cases} \alpha(t), & \text{if } x(t) < \alpha(t); \\ x(t), & \text{if } x(t) \in [\alpha(t), \beta(t)]; \\ \beta(t), & \text{if } x(t) > \beta(t); \end{cases}$$

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We also set

$$f^*(t,x,y) := \begin{cases} f\big(t,\beta(t),\beta'(t)\big) + \arctan\big(x(t)-\beta(t)\big), & \text{if } x > \beta(t); \\ f(t,x,y), & \text{if } x \in [\alpha(t),\beta(t)]; \\ f\big(t,\alpha(t),\alpha'(t)\big) + \arctan\big(x(t)-\alpha(t)\big), & \text{if } x < \alpha(t). \end{cases}$$

- Auxiliary problem: We consider the following functional problem

$$(\mathrm{DP})' \qquad \begin{cases} \Big(\Phi\big(A_x(t)\,x'(t)\big)\Big)' = F_x(t), & \text{a.e. on } I, \\ x(0) = \nu_1,\ x(T) = \nu_2. \end{cases}$$

where we have set

$$A : W^{1,p}(I) \longrightarrow C(I,\mathbb{R}), \qquad A_x(t) := a\big(t,\mathcal{T}(x)(t)\big),$$
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- Solving the auxiliary problem: We prove that (DP)' possesses (at least) one solution $x_0 \in W^{1,p}(I)$ by showing that the operator

$$\mathcal{P}_x(t) := \nu_1 + \int_0^t \frac{1}{A_x(s)}\, \Phi^{-1}\left(\xi_x + \int_0^s F_x(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s,$$

(from $W^{1,p}(I)$ into itself) has a fixed point. Here, ξ_x is the unique real constant (depending on $x \in W^{1,p}(I)$) such that $\mathcal{P}_x(T) = \nu_2$.

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General nonlinear boundary conditions

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We consider the following general boundary problem for $(\bigstar)$:

$$\text{(G)} \qquad \begin{cases} \Big(\Phi\big(a(t,x(t))\,x'(t)\big)\Big)' = f(t,x(t),x'(t)), & \text{a.e. on } I, \\ g(x(0),x(T),\mathcal{A}_x(0),\mathcal{A}_x(T)) = 0, \\ x(T) = h(x(0)). \end{cases}$$

Here, $h : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^4 \to \mathbb{R}$ satisfy the following general assumptions:

(G1) $h \in C(\mathbb{R},\mathbb{R})$ and is increasing on $\mathbb{R}$;

(G2) $g \in C(\mathbb{R}^4,\mathbb{R})$ and, for every fixed $u, v \in \mathbb{R}$, it holds that

- $(G2)_1$ $g(u,v,\cdot,z)$ is increasing for every fixed $z \in \mathbb{R}$;
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Our main aim

To prove the existence of (at least) one **solution of (G)**, i.e., the existence of a solution $x \in W^{1,p}(I)$ of the ODE $(\bigstar)$ satisfying

$$g(x(0), x(T), \mathcal{A}_x(0), \mathcal{A}_x(T)) = 0 \text{ and } x(T) = h(x(0)).$$

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We think of (G) as a superposition of Dirichlet problems and we use a compactness-type result for the solutions of $(\bigstar)$.

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Theorem 2 (B., Calamai and Papalini)

Let us assume that *all* the hypotheses of Theorem 1 are satisfied and that g and h fulfill (G1)-(G2). Moreover, if $\alpha, \beta \in W^{1,p}(I)$ are, resp., a lower and a upper solution of $(\bigstar)$ such that $\alpha \leq \beta$ on I, we suppose that

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Finally, let us assume that the function a satisfies the following condition:

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Our main result

Theorem 2 (B., Calamai and Papalini) - continued

Then the problem (G) possesses one solution $x \in W^{1,p}(I)$ such that

$$\alpha(t) \leq x(t) \leq \beta(t) \qquad \text{for every } t \in I.$$

Furthermore, if $M > 0$ is any real number such that $\sup_I |\alpha|, \sup_I |\beta| \leq M$ and $L_0 > 0$ *is as in Theorem 1*, the following fact holds true: *for every real number* $L \geq L_0$ *such that* $\sup_I |\mathcal{A}_\alpha|, \sup_I |\mathcal{A}_\beta| \leq L$, *we have*

$$\max_{t \in I} \big|x(t)\big| \leq M \quad \textit{and} \quad \max_{t \in I} \big|\mathcal{A}_x(t)\big| \leq L.$$

A particular case

By applying Theorem 2 in the particular case when

$$g(u,v,w,z) = w - z \quad \text{and} \quad h(r) = r,$$

we obtain the existence of one solution $x \in W^{1,p}(I)$ of the “periodic problem”

$$\begin{cases} \Big(\Phi\big(a(t,x(t))\,x'(t)\big)\Big)' = f(t,x(t),x'(t)), & \text{a.e. on } I, \\ \mathcal{A}_x(0) = \mathcal{A}_x(T), \\ x(0) = x(T). \end{cases}$$

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Sturm-Liouville and Neumann problems

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We consider the following boundary value problem for $(\star)$:

$$\text{(SL)} \qquad \begin{cases} \Big(\Phi\big(a(t,x(t))\,x'(t)\big)\Big)' = f(t,x(t),x'(t)), & \text{a.e. on } I, \\ p(x(0),\mathcal{A}_x(0)) = 0,\ \ q(x(T),\mathcal{A}_x(T)) = 0. \end{cases}$$

Here, the functions $p, q : \mathbb{R}^2 \longrightarrow \mathbb{R}$ satisfy the following general assumptions:

(S1) $p \in C(\mathbb{R}^2,\mathbb{R})$ and, for every $s \in \mathbb{R}$, the map $p(s,\cdot)$ is increasing on $\mathbb{R}$;

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To prove the existence of (at least) one **solution of (SL)**, i.e., the existence of a solution $x \in W^{1,p}(I)$ of the ODE $(\bigstar)$ satisfying

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To prove the existence of (at least) one **solution of (SL)**, i.e., the existence of a solution $x \in W^{1,p}(I)$ of the ODE $(\bigstar)$ satisfying

$$p(x(0), \mathcal{A}_x(0)) = 0 \text{ and } q(x(T), \mathcal{A}_x(T)) = 0.$$

Our approach

We think of (SL) as a superposition of general boundary problems of the type (G) and we use again a compactness-type result for the solutions of $(\bigstar)$.

Our main result

Theorem 3 (B., Calamai and Papalini)

Let us assume that *all* the hypotheses of Theorem 1 are satisfied and that p and q fulfill (S1)-(S2). Moreover, if $\alpha, \beta \in W^{1,p}(I)$ are, resp., a lower and a upper solution of ($\star$) such that $\alpha \leq \beta$ on I, we suppose that

$$\begin{cases} p(\alpha(0), \mathcal{A}_\alpha(0)) \geq 0, \\ q(\alpha(T), \mathcal{A}_\alpha(T)) \geq 0; \end{cases} \qquad \begin{cases} p(\beta(0), \mathcal{A}_\beta(0)) \leq 0, \\ q(\beta(T), \mathcal{A}_\beta(T)) \leq 0. \end{cases}$$

Finally, let us assume that the function a satisfies the following condition:

$$a(0,x) \neq 0 \quad \text{and} \quad a(T,x) \neq 0 \qquad \text{for every } x \in \mathbb{R}.$$

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Our main result

Theorem 3 (B., Calamai and Papalini) - continued

Then the problem (SL) possesses one solution $x \in W^{1,p}(I)$ such that

$$\alpha(t) \leq x(t) \leq \beta(t) \qquad \text{for every } t \in I.$$

Furthermore, if $M > 0$ is any real number such that $\sup_I |\alpha|, \sup_I |\beta| \leq M$ and $L_0 > 0$ *is as in Theorem 1*, the following fact holds true: *for every real number* $L \geq L_0$ *such that* $\sup_I |\mathcal{A}_\alpha|, \sup_I |\mathcal{A}_\beta| \leq L$*, we have*

$$\max_{t \in I} \big|x(t)\big| \leq M \quad \textit{and} \quad \max_{t \in I} \big|\mathcal{A}_x(t)\big| \leq L.$$

Two particular cases

(1) By applying Theorem 3 in the particular case when

$$p(s,t) := \ell_1\, s + m_1\, t - \nu_1 \qquad \text{and} \qquad q(s,t) := \ell_2\, s - m_2\, t - \nu_2$$

(for some $m_1,\ m_2 \geq 0$ and $\ell_1, \ell_2\, \nu_1,\ \nu_2 \in \mathbb{R}$) we obtain the existence of one solution $x \in W^{1,p}(I)$ of the "Sturm-Liouville problem"

$$\begin{cases} \Big(\Phi\big(a(t,x(t))\, x'(t)\big)\Big)' = f(t,x(t),x'(t)), & \text{a.e. on } I, \\ \ell_1\, x(0) + m_1\, \mathcal{A}_x(0) = \nu_1, \\ \ell_2\, x(T) - m_2\, \mathcal{A}_x(T) = \nu_2. \end{cases}$$

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(2) By applying Theorem 3 in the particular case when

$$p(s,t) := t - \nu_1 \qquad \text{and} \qquad q(s,t) := \nu_2 - t$$

(for some fixed constants $\nu_1, \nu_2 \in \mathbb{R}$) we obtain the existence of one solution $x \in W^{1,p}(I)$ of the "Neumann problem"

$$\begin{cases} \Big(\Phi\big(a(t,x(t))\,x'(t)\big)\Big)' = f(t,x(t),x'(t)), & \text{a.e. on } I, \\ \mathcal{A}_x(0) = \nu_1, \\ \mathcal{A}_x(T) = \nu_2. \end{cases}$$

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Thank you for your attention!