Strongly Exponentially Separated Linear Systems

GEDO. Ancona, Sept 29, 2018

F. B. Università Politecnica delle Marche, Ancona (I)
K. J. Palmer National Taiwan University, Taipei (Taiwan)

Outline

Outline

Outline

Outline

Outline

Outline

Exponential separation

Let $A(t)$, $t \in I$ be a (possibly unbounded) $n \times n$ continuous matrix.

Exponential separation

The linear system $\dot{x} = A(t)x$ is said to be exponentially separated of rank r on the interval I if there exist two non-trivial, invariant subspaces $V_s(t)$, $V_u(t)$ such that $\mathbb{R}^n = V_s(t) \oplus V_u(t)$, *rank* $V_1(t) = r$, and constants $k \geq 1$, $\alpha > 0$ such that for any pair of non zero solutions $x(t) \in V_s(t)$ and $y(t) \in V_u(t)$ it results

$$\frac{|x(t)|}{|x(s)|}\frac{|y(s)|}{|y(t)|} \leq ke^{-\alpha(t-s)}$$

Exponential separation

Let $A(t)$, $t \in I$ be a (possibly unbounded) $n \times n$ continuous matrix.

Exponential separation

The linear system $\dot{x} = A(t)x$ is said to be exponentially separated of rank r on the interval I if there exist two non-trivial, invariant subspaces $V_s(t)$, $V_u(t)$ such that $\mathbb{R}^n = V_s(t) \oplus V_u(t)$, *rank* $V_1(t) = r$, and constants $k \geq 1$, $\alpha > 0$ such that for any pair of non zero solutions $x(t) \in V_s(t)$ and $y(t) \in V_u(t)$ it results

$$\frac{|x(t)|}{|x(s)|}\frac{|y(s)|}{|y(t)|} \leq ke^{-\alpha(t-s)}$$

$V_s(t)$:stable space and $V_u(t)$:unstable space.

Exponential separation

Let $A(t)$, $t \in I$ be a (possibly unbounded) $n \times n$ continuous matrix.

Exponential separation

The linear system $\dot{x} = A(t)x$ is said to be exponentially separated of rank r on the interval I if there exist two non-trivial, invariant subspaces $V_s(t)$, $V_u(t)$ such that $\mathbb{R}^n = V_s(t) \oplus V_u(t)$, *rank* $V_1(t) = r$, and constants $k \geq 1$, $\alpha > 0$ such that for any pair of non zero solutions $x(t) \in V_s(t)$ and $y(t) \in V_u(t)$ it results

$$\frac{|x(t)|}{|x(s)|}\frac{|y(s)|}{|y(t)|} \leq ke^{-\alpha(t-s)}$$

$V_s(t)$:stable space and $V_u(t)$:unstable space. $P(t)$: proj. on $\mathbb{R}^n$: $\mathcal{R}P(t) = V_s(t)$, $\mathcal{N}P(t) = V_u(t) \Rightarrow$: $X(t,s) : V_{s,u}(s) \to V_{s,u}(t)$

Properties of ES

Gronwall Lemma $\Rightarrow$ on compact intervals a linear system is ES of any rank.

Properties of ES

Gronwall Lemma $\Rightarrow$ on compact intervals a linear system is ES of any rank. Take $I = \mathbb{R}_+ = [0, \infty), \mathbb{R}_- = (-\infty, 0], \mathbb{R} = (-\infty, \infty)$.

Properties of ES

Gronwall Lemma $\Rightarrow$ on compact intervals a linear system is ES of any rank. Take $I = \mathbb{R}_+ = [0, \infty), \mathbb{R}_- = (-\infty, 0], \mathbb{R} = (-\infty, \infty)$.

Proposition

An ES linear system on $[T, \infty)$, $T > 0$ is also ES on $[0, \infty)$ with the same rank. Similarly, an ES linear system on $(-\infty, T]$, $T < 0$ is also ES on $(-\infty, 0]$ with the same rank.

Properties of ES

Gronwall Lemma $\Rightarrow$ on compact intervals a linear system is ES of any rank. Take $I = \mathbb{R}_+ = [0, \infty), \mathbb{R}_- = (-\infty, 0], \mathbb{R} = (-\infty, \infty)$.

Proposition

An ES linear system on $[T, \infty)$, $T > 0$ is also ES on $[0, \infty)$ with the same rank. Similarly, an ES linear system on $(-\infty, T]$, $T < 0$ is also ES on $(-\infty, 0]$ with the same rank.

Proposition

For exponentially separated systems on $\mathbb{R}_+$ the stable subspace is uniquely defined (for a given dimension) and for exponentially separated systems on $\mathbb{R}_-$ the unstable subspace is uniquely defined. The other subspace can be any complement.

Proposition (Adrianova, 1995)

When the Lyapunov exponents are distinct, they vary continuously with respect to perturbations of the coefficient matrix if and only if the system is integrally separated

Proposition

A system is ES on $\mathbb{R}$ if and only

- if it is ES on $\mathbb{R}_+$ and $\mathbb{R}_-$,
- the respective ranks are the same
- the stable subspace on $\mathbb{R}_+$ and the unstable subspace on $\mathbb{R}_-$ intersect in $\{0\}$ at $t = 0$.

Proposition (Adrianova, 1995)

When the Lyapunov exponents are distinct, they vary continuously with respect to perturbations of the coefficient matrix if and only if the system is integrally separated

Proposition

A system is ES on $\mathbb{R}$ if and only

- if it is ES on $\mathbb{R}_+$ and $\mathbb{R}_-$,
- the respective ranks are the same
- the stable subspace on $\mathbb{R}_+$ and the unstable subspace on $\mathbb{R}_-$ intersect in $\{0\}$ at $t = 0$.

Since $X(t,s)P(s) \in \mathcal{R}P(t)$ and $X(t,s)(\mathbb{I} - P(s)) \in \mathcal{N}P(t)$ we get

$P(t)X(t,s)P(s) = X(t,s)P(s)$ and

Proposition (Adrianova, 1995)

When the Lyapunov exponents are distinct, they vary continuously with respect to perturbations of the coefficient matrix if and only if the system is integrally separated

Proposition

A system is ES on $\mathbb{R}$ if and only

- if it is ES on $\mathbb{R}_+$ and $\mathbb{R}_-$,
- the respective ranks are the same
- the stable subspace on $\mathbb{R}_+$ and the unstable subspace on $\mathbb{R}_-$ intersect in $\{0\}$ at $t = 0$.

Since $X(t,s)P(s) \in \mathcal{R}P(t)$ and $X(t,s)(\mathbb{I} - P(s)) \in \mathcal{N}P(t)$ we get

$$P(t)X(t,s)P(s) = X(t,s)P(s) \quad \text{and} \quad P(t)X(t,s)P(s) = P(t)X(t,s)$$

Exponential dichotomy

so $X(t,s)P(s) = P(t)X(t,s)$.

Exponential dichotomy

so $X(t,s)P(s) = P(t)X(t,s)$.

Exponential dichotomy

The linear system $\dot{x} = A(t)x$ is said to have an exponential dichotomy of rank r on the interval I if there exist a continuous projection $P(t)$ such that *rank* $P(t) = r$ and the fundamental matrix $X(t,s)$ of the system with $X(s,s) = \mathbb{I}$, satisfies the following:

i) $X(t,s)P(s) = P(t)X(t,s)$, for all $s, t \in I$;

ii) $\|X(t,s)P(s)\| \leq ke^{-\alpha(t-s)}$, for all $s \leq t \in I$;

iii) $\|X(s,t)[\mathbb{I} - P(t)]\| \leq ke^{-\alpha(t-s)}$, for all $s \leq t \in I$;

Exponential dichotomy

so $X(t,s)P(s) = P(t)X(t,s)$.

Exponential dichotomy

The linear system $\dot{x} = A(t)x$ is said to have an exponential dichotomy of rank r on the interval I if there exist a continuous projection $P(t)$ such that *rank* $P(t) = r$ and the fundamental matrix $X(t,s)$ of the system with $X(s,s) = \mathbb{I}$, satisfies the following:

i) $X(t,s)P(s) = P(t)X(t,s)$, for all $s, t \in I$;

ii) $\|X(t,s)P(s)\| \leq ke^{-\alpha(t-s)}$, for all $s \leq t \in I$;

iii) $\|X(s,t)[\mathbb{I} - P(t)]\| \leq ke^{-\alpha(t-s)}$, for all $s \leq t \in I$;

Perron, 1930; Massera and Shaffer, 1966; Coppel 1978...

ED implies that $|X(t,s)P(s)||X(s,t)(I-P(t))| \leq Ke^{-2\alpha(t-s)}$.

ED implies that $|X(t,s)P(s)||X(s,t)(I-P(t))| \leq Ke^{-2\alpha(t-s)}$.

$\dot{x} = A(t)x$ is ES with projection $P(t) : \mathbb{R}^n \to \mathbb{R}^n$ if and only if, for any $x \in \mathcal{R}P(s)$ and $y \in \mathcal{N}P(t)$, we have

$$|X(t,s)x||X(s,t)y| \leq Ke^{-\alpha(t-s)}|x|\,|y|.$$

ED implies that $|X(t,s)P(s)||X(s,t)(I-P(t))| \leq Ke^{-2\alpha(t-s)}$.

$\dot{x} = A(t)x$ is ES with projection $P(t) : \mathbb{R}^n \to \mathbb{R}^n$ if and only if, for any $x \in \mathcal{R}P(s)$ and $y \in \mathcal{N}P(t)$, we have

$$|X(t,s)x||X(s,t)y| \leq Ke^{-\alpha(t-s)}|x|\,|y|.$$

Indeed, write $x := X(s,0)P(0)\xi$, $y := X(t,0)(\mathbb{I}-P(0))\eta$ and set $x(t) = X(t,0)P(0)\xi$, $y(t) = X(t,0)(\mathbb{I}-P(0))\eta$:

$$\frac{|x(t)|}{|x(s)|} = \frac{|X(t,s)X(s,0)P(0)\xi|}{|X(s,0)P(0)\xi|} := \frac{|X(t,s)x|}{|x|},$$

$$\frac{|y(s)|}{|y(t)|} = \frac{|X(s,t)X(t,0)(\mathbb{I}-P(0))\eta|}{|X(t,0)(\mathbb{I}-P(0))\eta|} = \frac{|X(s,t)y|}{|y|}$$

and the inequality follows.

ED implies that $|X(t,s)P(s)||X(s,t)(I-P(t))| \leq Ke^{-2\alpha(t-s)}$.

$\dot{x} = A(t)x$ is ES with projection $P(t) : \mathbb{R}^n \to \mathbb{R}^n$ if and only if, for any $x \in \mathcal{R}P(s)$ and $y \in \mathcal{N}P(t)$, we have

$$|X(t,s)x||X(s,t)y| \leq Ke^{-\alpha(t-s)}|x|\,|y|.$$

Indeed, write $x := X(s,0)P(0)\xi$, $y := X(t,0)(\mathbb{I} - P(0))\eta$ and set $x(t) = X(t,0)P(0)\xi$, $y(t) = X(t,0)(\mathbb{I} - P(0))\eta$:

$$\frac{|x(t)|}{|x(s)|} = \frac{|X(t,s)X(s,0)P(0)\xi|}{|X(s,0)P(0)\xi|} := \frac{|X(t,s)x|}{|x|},$$

$$\frac{|y(s)|}{|y(t)|} = \frac{|X(s,t)X(t,0)(\mathbb{I}-P(0))\eta|}{|X(t,0)(\mathbb{I}-P(0))\eta|} = \frac{|X(s,t)y|}{|y|}$$

and the inequality follows. Viceversa, replace x and y with $x(s) := X(s,0)x \in \mathcal{R}P(s)$ and $y(t) := X(t,0)y \in \mathcal{N}P(t)$.

ED implies that $|X(t,s)P(s)||X(s,t)(I-P(t))| \leq Ke^{-2\alpha(t-s)}$.

$\dot{x} = A(t)x$ is ES with projection $P(t) : \mathbb{R}^n \to \mathbb{R}^n$ if and only if, for any $x \in \mathcal{R}P(s)$ and $y \in \mathcal{N}P(t)$, we have

$$|X(t,s)x||X(s,t)y| \leq Ke^{-\alpha(t-s)}|x|\,|y|.$$

Indeed, write $x := X(s,0)P(0)\xi$, $y := X(t,0)(\mathbb{I}-P(0))\eta$ and set $x(t) = X(t,0)P(0)\xi$, $y(t) = X(t,0)(\mathbb{I}-P(0))\eta$:

$$\frac{|x(t)|}{|x(s)|} = \frac{|X(t,s)X(s,0)P(0)\xi|}{|X(s,0)P(0)\xi|} := \frac{|X(t,s)x|}{|x|},$$

$$\frac{|y(s)|}{|y(t)|} = \frac{|X(s,t)X(t,0)(\mathbb{I}-P(0))\eta|}{|X(t,0)(\mathbb{I}-P(0))\eta|} = \frac{|X(s,t)y|}{|y|}$$

and the inequality follows. Viceversa, replace x and y with $x(s) := X(s,0)x \in \mathcal{R}P(s)$ and $y(t) := X(t,0)y \in \mathcal{N}P(t)$.
ED implies ES with the same projection (and hence rank)

Strong exponential separation

Writing $P(s)x$ and $(\mathbb{I} - P(s))y$ instead of x, y resp. we get:
$|X(t,s)P(s)x||X(s,t)(\mathbb{I} - P(t)y| \leq Ke^{-\alpha(t-s)}|P(s)x|\,|(\mathbb{I} - P(t))y|$.

Strong exponential separation

Writing $P(s)x$ and $(\mathbb{I} - P(s))y$ instead of x, y resp. we get:
$|X(t,s)P(s)x||X(s,t)(\mathbb{I} - P(t)y| \leq Ke^{-\alpha(t-s)}|P(s)x|\,|(\mathbb{I} - P(t))y|$.

Strong exponential separation

$\dot{x} = A(t)x$ is Strongly Exponentially Separated with projection $P(t) : \mathbb{R}^n \to \mathbb{R}^n$, if it is exponentially separated with projection $P(t)$ and $P(t)$ is bounded.

Strong exponential separation

Writing $P(s)x$ and $(\mathbb{I} - P(s))y$ instead of x, y resp. we get:
$|X(t,s)P(s)x||X(s,t)(\mathbb{I} - P(t)y| \leq Ke^{-\alpha(t-s)}|P(s)x|\,|(\mathbb{I} - P(t))y|$.

Strong exponential separation

$\dot{x} = A(t)x$ is Strongly Exponentially Separated with projection $P(t) : \mathbb{R}^n \to \mathbb{R}^n$, if it is exponentially separated with projection $P(t)$ and $P(t)$ is bounded.

Strong exponential separation condition

$\dot{x} = A(t)x$ is Strongly Exponentially Separated with projection $P(t) : \mathbb{R}^n \to \mathbb{R}^n$ if and only if

$$|X(t,s)P(s)||X(s,t)(\mathbb{I} - P(s))| \leq Ke^{-\alpha(t-s)}.$$

Proposition

If $\dot{x} = A(t)x$ is ES (resp. SES) with projection $P(t)$ and $Q(t)$ is another projection with $\mathcal{R}Q(t) = \mathcal{R}P(t)$, then $\dot{x} = A(t)x$ is ES (resp. SES) with projection $Q(t)$

Proposition

If $\dot{x} = A(t)x$ is ES (resp. SES) with projection $P(t)$ and $Q(t)$ is another projection with $\mathcal{R}Q(t) = \mathcal{R}P(t)$, then $\dot{x} = A(t)x$ is ES (resp. SES) with projection $Q(t)$

Example: $\dot{u} = u + e^t v$, $\dot{v} = 2v$ has the two solutions $x(t) = (e^t, 0)$, $y(t) = (e^{3t}, 2e^{2t})$ and $\frac{|x(t)|}{|x(s)|}\frac{|y(s)|}{|y(t)|} \le Ke^{-2(t-s)}$, for $0 \le s \le t$. So it is ES with projection $P(t) = \begin{pmatrix} 1 & -\frac{1}{2}e^t \\ 0 & 0 \end{pmatrix}$. But $P(t)$ is not bounded hence the system is not SES.

Proposition

If $\dot{x} = A(t)x$ is ES (resp. SES) with projection $P(t)$ and $Q(t)$ is another projection with $\mathcal{R}Q(t) = \mathcal{R}P(t)$, then $\dot{x} = A(t)x$ is ES (resp. SES) with projection $Q(t)$

Example: $\dot{u} = u + e^t v$, $\dot{v} = 2v$ has the two solutions $x(t) = (e^t, 0)$, $y(t) = (e^{3t}, 2e^{2t})$ and $\frac{|x(t)|}{|x(s)|}\frac{|y(s)|}{|y(t)|} \leq Ke^{-2(t-s)}$, for $0 \leq s \leq t$. So it is ES with projection $P(t) = \begin{pmatrix} 1 & -\frac{1}{2}e^t \\ 0 & 0 \end{pmatrix}$. But $P(t)$ is not bounded hence the system is not SES.

Proposition

If $\dot{x} = A(t)x$ is SES on I with projection $P(t)$ then its adjoint system $\dot{x} = -A(t)^* x$ is SES on I with projection $\mathbb{I} - P(t)^*$.

Proof. A fundamental system of $\dot{x} = -A(t)^* x$ is $Y(t,s) = X(s,t)^*$ and

$$\begin{aligned} Y(t,s)[\mathbb{I} - P(s)^*] &= X(s,t)^*[\mathbb{I} - P(s)^*] = [(\mathbb{I} - P(s))X(s,t)]^* \\ &= [X(s,t)(\mathbb{I} - P(t))]^* \\ Y(s,t)P(t)^* &= X(t,s)^* P(t)^* = [P(t)X(t,s)]^* = [X(t,s)P(s)]^* \end{aligned}$$

Proof. A fundamental system of $\dot{x} = -A(t)^* x$ is $Y(t,s) = X(s,t)^*$ and

$$\begin{aligned} Y(t,s)[\mathbb{I} - P(s)^*] &= X(s,t)^*[\mathbb{I} - P(s)^*] = [(\mathbb{I} - P(s))X(s,t)]^* \\ &= [X(s,t)(\mathbb{I} - P(t))]^* \\ Y(s,t)P(t)^* &= X(t,s)^* P(t)^* = [P(t)X(t,s)]^* = [X(t,s)P(s)]^* \end{aligned}$$

Conclusion is obvious.

Example

Consider the system

$$\dot{x} = \begin{pmatrix} 3 & 0 & -e^t \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix} x$$

A fundamental matrix is

$$X(t) = \begin{pmatrix} e^{3t} & 0 & e^{2t} \\ 0 & e^{2t} & e^t \\ 0 & 0 & e^t \end{pmatrix} := (x(t), y_1(t), y_2(t))$$

Take $V_s(t) = \mathrm{span}\{y_1(t), y_2(t)\}$, $V_u(t) = \mathrm{span}\{x(t)\}$. Let $y(t) = \alpha y_1(t) + \beta y_2(t)$, $\alpha^2 + \beta^2 \neq 0$. We have (take the ℓ^1-norm)

$$|\alpha| + |\beta| \leq |y(t)e^{-2t}| = |\beta| + |\alpha + \beta e^{-t}| + |\beta e^{-t}| \leq |\alpha| + 3|\beta|$$
$$|x(t)| = e^{3t}$$

Hence

$$e^{-(t-s)} \leq \frac{|y(t)|}{|y(s)|}\frac{|x(s)|}{|x(t)|} \leq \frac{|\alpha| + 3|\beta|}{|\alpha| + |\beta|} e^{-(t-s)} \leq 3e^{-(t-s)}$$

Hence

$$e^{-(t-s)} \leq \frac{|y(t)|}{|y(s)|}\frac{|x(s)|}{|x(t)|} \leq \frac{|\alpha| + 3|\beta|}{|\alpha| + |\beta|}e^{-(t-s)} \leq 3e^{-(t-s)}$$

The projection is

$$P(t) = \begin{pmatrix} 0 & 0 & e^{t} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence

$$e^{-(t-s)} \le \frac{|y(t)|}{|y(s)|}\frac{|x(s)|}{|x(t)|} \le \frac{|\alpha| + 3|\beta|}{|\alpha| + |\beta|} e^{-(t-s)} \le 3e^{-(t-s)}$$

The projection is

$$P(t) = \begin{pmatrix} 0 & 0 & e^t \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The system is ES but not SES. A fundamental matrix of the adjoint system is

$$Y(t) = X(t)^{-1,*}X(0)^* = \begin{pmatrix} e^{-3t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} - e^{-2t} & e^{-t} \end{pmatrix}$$

and we have

$$Y(t)(\mathbb{I} - P(0)^*) = \begin{pmatrix} e^{-3t} & 0 & 0 \\ 0 & 0 & 0 \\ -e^{-2t} & 0 & 0 \end{pmatrix}$$

$$Y(t)P(0)^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e^{-2t} & 0 \\ e^{-t} & e^{-t} - e^{-2t} & e^{-t} \end{pmatrix}$$

so $V_s^*(t) = \mathrm{span}\{u(t)\}$, $V_u^*(t) = \mathrm{span}\{v_1(t), v_2(t)\}$ with

$$u(t) := \begin{pmatrix} e^{-3t} \\ 0 \\ -e^{-2t} \end{pmatrix},\ v_1(t) := \begin{pmatrix} 0 \\ 0 \\ e^{-t} \end{pmatrix}, v_2(t) := \begin{pmatrix} 0 \\ e^{-2t} \\ -e^{-2t} \end{pmatrix}$$

$$Y(t)(\mathbb{I} - P(0)^*) = \begin{pmatrix} e^{-3t} & 0 & 0 \\ 0 & 0 & 0 \\ -e^{-2t} & 0 & 0 \end{pmatrix}$$

$$Y(t)P(0)^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e^{-2t} & 0 \\ e^{-t} & e^{-t} - e^{-2t} & e^{-t} \end{pmatrix}$$

so $V_s^*(t) = \mathrm{span}\{u(t)\}$, $V_u^*(t) = \mathrm{span}\{v_1(t), v_2(t)\}$ with

$$u(t) := \begin{pmatrix} e^{-3t} \\ 0 \\ -e^{-2t} \end{pmatrix}, \; v_1(t) := \begin{pmatrix} 0 \\ 0 \\ e^{-t} \end{pmatrix}, v_2(t) := \begin{pmatrix} 0 \\ e^{-2t} \\ -e^{-2t} \end{pmatrix}$$

But (with the ℓ^1-norm)

$$e^{-2t} \leq |u(t)| = e^{-2t}(1 + e^{-t}) \leq 2e^{-2t}, \quad |v_2(t)| = 2e^{-2t}.$$

$$Y(t)(\mathbb{I} - P(0)^*) = \begin{pmatrix} e^{-3t} & 0 & 0 \\ 0 & 0 & 0 \\ -e^{-2t} & 0 & 0 \end{pmatrix}$$

$$Y(t)P(0)^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e^{-2t} & 0 \\ e^{-t} & e^{-t} - e^{-2t} & e^{-t} \end{pmatrix}$$

so $V_s^*(t) = \mathrm{span}\{u(t)\}$, $V_u^*(t) = \mathrm{span}\{v_1(t), v_2(t)\}$ with

$$u(t) := \begin{pmatrix} e^{-3t} \\ 0 \\ -e^{-2t} \end{pmatrix}, \; v_1(t) := \begin{pmatrix} 0 \\ 0 \\ e^{-t} \end{pmatrix}, v_2(t) := \begin{pmatrix} 0 \\ e^{-2t} \\ -e^{-2t} \end{pmatrix}$$

But (with the ℓ^1-norm)

$$e^{-2t} \leq |u(t)| = e^{-2t}(1 + e^{-t}) \leq 2e^{-2t}, \quad |v_2(t)| = 2e^{-2t}.$$

The adjoint system is not ES with projection $(\mathbb{I} - P(t)^*)$.

Kinematic similarity and reducibility

1) Systems $\dot{x} = A(t)x$ and $\dot{y} = B(t)y$ are kinematically similar if there exists a bounded, invertible continuously differentiable matrix function $S(t)$ with bounded inverse such that the transformation $x = S(t)y$ takes $\dot{x} = A(t)x$ and $\dot{y} = B(t)y$.
2) A system is reducible if it is kinematically similar to a block diagonal system

$$\dot{y} = \begin{pmatrix} A_1(t) & 0 \\ 0 & A_2(t) \end{pmatrix} y.$$

3) A system is reducible if and only if it has a fundamental matrix $X(t)$ such that $P(t) = X(t)PX^{-1}(t)$ is bounded, where P is a projection of rank $\neq 0, n$.

Proposition

If $\dot{x} = A(t)x$ is ES with projection $P(t)$ and $A(t)$ is bounded, then $P(t)$ is bounded. Thus $\dot{x} = A(t)x$ is reducible and SES.

Proposition

If $\dot{x} = A(t)x$ is ES with projection $P(t)$ and $A(t)$ is bounded, then $P(t)$ is bounded. Thus $\dot{x} = A(t)x$ is reducible and SES.

If $\dot{x} = A(t)x$ is SES with projection $P(t)$ then $P(t)$ is bounded. Then there is a kinematic similarity $S(t)$ such that $P(t) = S(t)PS^{-1}(t)$, where $P = \begin{pmatrix} \mathbf{I}_r & 0 \\ 0 & 0 \end{pmatrix}$ with $r =$ the rank of $P(t)$. Then the transformation $x = S(t)y$ takes $\dot{x} = A(t)x$ into a system for which a fundamental matrix is $Y(t) = S^{-1}(t)X(t)$ which commutes with P. So the transformed system has the form

$$\dot{y}_1 = A_1(t)y_1$$
$$\dot{y}_2 = A_2(t)y_2$$

and hence $\dot{x} = A(t)x$ is reducible.

Roughness

Preliminary Lemma

Let $X_1(t,s)$, $X_2(t,s)$ be the transition matrices of the systems $\dot{x}_1 = A_1(t)x_1$, $\dot{x}_2 = A_2(t)x_2$. Suppose there exist positive constants K and α such that $|X_1(t,s)||X_2(s,t)| \leq Ke^{-\alpha(t-s)}$, $s \leq t \in J$. Then there exists $\delta > 0$ such that if $|C_i(t)| \leq \delta$, $i = 1,2$, the fundamental matrices $\tilde{X}_1(t,s)$, $\tilde{X}_2(t,s)$ of the perturbed linear systems $\dot{x}_1 = [A_1(t) + C_1(t)]x_1$ and $\dot{x}_2 = [A_2(t) + C_2(t)]x_2$ satisfy:

$$|\tilde{X}_1(t,s)||\tilde{X}_2(t,s)| \leq Ke^{-(\alpha-2K\delta)(t-s)}$$

$s \leq t \in J$.

Sketch of proof

Let $\dot{x}_2(t) = A_2(t)x_2(t)$, $x_2(t) \neq 0$. Set $p(t) = \frac{d}{dt}\log|x_2(t)|$. Then

$$|X_1(t)|\,|x_2(s)| = |X_1(t)|\,|X_2(s,t)x_2(t)| \leq Ke^{-\alpha(t-s)}|x_2(t)|$$

so $|X_1(t)| \leq Ke^{\int_s^t p(u)-\alpha du}$.

Sketch of proof

Let $\dot{x}_2(t) = A_2(t)x_2(t)$, $x_2(t) \neq 0$. Set $p(t) = \frac{d}{dt}\log|x_2(t)|$. Then

$$|X_1(t)|\,|x_2(s)| = |X_1(t)|\,|X_2(s,t)x_2(t)| \leq Ke^{-\alpha(t-s)}|x_2(t)|$$

so $|X_1(t)| \leq Ke^{\int_s^t p(u)-\alpha du}$. From the variation of constants formula and Gronwall inequality we get:

$$|\tilde{X}_1(t)| \leq Ke^{-(\alpha-K\delta)(t-s)}\frac{|x_2(t)|}{|x_2(s)|} \Leftrightarrow$$
$$|\tilde{X}_1(t)||X_2(s,t)x_2(t)| \leq Ke^{-(\alpha-K\delta)(t-s)}|x_2(t)|$$

Sketch of proof

Let $\dot{x}_2(t) = A_2(t)x_2(t)$, $x_2(t) \neq 0$. Set $p(t) = \frac{d}{dt}\log|x_2(t)|$. Then

$$|X_1(t)|\,|x_2(s)| = |X_1(t)|\,|X_2(s,t)x_2(t)| \leq Ke^{-\alpha(t-s)}|x_2(t)|$$

so $|X_1(t)| \leq Ke^{\int_s^t p(u)-\alpha du}$. From the variation of constants formula and Gronwall inequality we get:

$$|\tilde{X}_1(t)| \leq Ke^{-(\alpha-K\delta)(t-s)}\frac{|x_2(t)|}{|x_2(s)|} \Leftrightarrow$$
$$|\tilde{X}_1(t)||X_2(s,t)x_2(t)| \leq Ke^{-(\alpha-K\delta)(t-s)}|x_2(t)|$$

Hence $|\tilde{X}_1(t)||X_2(s,t)| \leq Ke^{-(\alpha-K\delta)(t-s)}$. Now we take $x_1(t) \neq 0$ such that $\dot{x}_1(t) = [A_1(t) + C_1(t)]x_1(t)$ and apply a similar argument as the above to get the result.

Roughness

R-Theorem

Suppose $\dot{x} = A(t)x$ is SES on an interval J with projection $P(t)$. Then $\exists \delta > 0$ such that if $|B(t)| \leq \delta$, the perturbed system $\dot{x} = [A(t) + B(t)]x$ is also strongly exponentially separated with projection $Q(t)$ of the same rank and there exists a constant N such that $|Q(t) - P(t)| \leq N\delta$, $\forall t \in J$.

Roughness

R-Theorem

Suppose $\dot{x} = A(t)x$ is SES on an interval J with projection $P(t)$. Then $\exists \delta > 0$ such that if $|B(t)| \leq \delta$, the perturbed system $\dot{x} = [A(t) + B(t)]x$ is also strongly exponentially separated with projection $Q(t)$ of the same rank and there exists a constant N such that $|Q(t) - P(t)| \leq N\delta$, $\forall t \in J$.

Sketch of proof. Let $T(t)$ be a kinematic similarity that takes $\dot{x} = A(t)x$ into $\dot{x}_1 = A_1(t)x_1$, $\dot{x}_2 = A_1(t)x_2$.

Roughness

R-Theorem

Suppose $\dot{x} = A(t)x$ is SES on an interval J with projection $P(t)$. Then $\exists\delta > 0$ such that if $|B(t)| \leq \delta$, the perturbed system $\dot{x} = [A(t) + B(t)]x$ is also strongly exponentially separated with projection $Q(t)$ of the same rank and there exists a constant N such that $|Q(t) - P(t)| \leq N\delta$, $\forall t \in J$.

Sketch of proof. Let $T(t)$ be a kinematic similarity that takes $\dot{x} = A(t)x$ into $\dot{x}_1 = A_1(t)x_1$, $\dot{x}_2 = A_1(t)x_2$. If we apply the same transformation to the perturbed system we get

$$\begin{aligned} \dot{x}_1 &= [A_1(t) + C_{11}(t)]x_1 + C_{12}(t)x_2, \quad |C_{11}(t)|, |C_{12}(t)| \leq \tilde{\delta} \\ \dot{x}_2 &= C_{21}(t)]x_1 + [A_2(t) + C_{22}(t)]x_2, \quad |C_{21}(t)|, |C_{21}(t)| \leq \tilde{\delta} \end{aligned}$$

Now we apply the transformation $S(t) = \begin{pmatrix} I & H_{12}(t) \\ H_{21}(t) & I \end{pmatrix}$.
Need $|H_{12}(t)|, |H_{21}(t)| \ll 1$ to have that $S(t)$ is invertible.

Now we apply the transformation $S(t) = \begin{pmatrix} I & H_{12}(t) \\ H_{21}(t) & I \end{pmatrix}$.

Need $|H_{12}(t)|, |H_{21}(t)| \ll 1$ to have that $S(t)$ is invertible. We obtain the block diagonal system:

$$\begin{aligned} \dot{x}_1 &= [A_1(t) + C_{11}(t) + C_{12}(t)H_{21}(t)]x_1 \\ \dot{x}_2 &= [A_2(t) + C_{22}(t) + C_{21}(t)H_{12}(t)]x_2 \end{aligned}$$

provided $H_{12}(t)$ and $H_{21}(t)$ satisfy certain differential equations.

Now we apply the transformation $S(t) = \begin{pmatrix} I & H_{12}(t) \\ H_{21}(t) & I \end{pmatrix}$.

Need $|H_{12}(t)|, |H_{21}(t)| \ll 1$ to have that $S(t)$ is invertible. We obtain the block diagonal system:

$$\dot{x}_1 = [A_1(t) + C_{11}(t) + C_{12}(t)H_{21}(t)]x_1$$
$$\dot{x}_2 = [A_2(t) + C_{22}(t) + C_{21}(t)H_{12}(t)]x_2$$

provided $H_{12}(t)$ and $H_{21}(t)$ satisfy certain differential equations. We prove these equation have bounded solution of small norm. Then we apply the preliminary Lemma.

Now we apply the transformation $S(t) = \begin{pmatrix} I & H_{12}(t) \\ H_{21}(t) & I \end{pmatrix}$.
Need $|H_{12}(t)|, |H_{21}(t)| \ll 1$ to have that $S(t)$ is invertible. We obtain the block diagonal system:

$$\begin{aligned} \dot{x}_1 &= [A_1(t) + C_{11}(t) + C_{12}(t)H_{21}(t)]x_1 \\ \dot{x}_2 &= [A_2(t) + C_{22}(t) + C_{21}(t)H_{12}(t)]x_2 \end{aligned}$$

provided $H_{12}(t)$ and $H_{21}(t)$ satisfy certain differential equations. We prove these equation have bounded solution of small norm. Then we apply the preliminary Lemma. $|Q(t) - P(t)| \leq N\delta$, $\forall t \in J$ also follows from the proof.

Exp Separation in upper triangular systems

We write

$$\mathrm{triang}(A_{ij}) = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \dots & A_{1k} \\ 0 & A_{22} & A_{23} & \dots & A_{2k} \\ 0 & 0 & A_{33} & \dots & A_{3k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_{kk} \end{pmatrix}$$

and

$$\mathrm{diag}(A_i) = \begin{pmatrix} A_1 & 0 & 0 & \dots & 0 \\ 0 & A_2 & 0 & \dots & 0 \\ 0 & 0 & A_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_k \end{pmatrix}$$

A Lemma

Lemma

i) If a upper-triangular system $\dot{x} = \mathrm{triang}(A_{ij})x$ is SES on J the projection at $t = 0$ can be taken as:

$$P(0) = \mathrm{triang}(P_{ij})x$$

where P_{ii} are projections.

A Lemma

Lemma

i) If a upper-triangular system $\dot{x} = \mathrm{triang}(A_{ij})x$ is SES on J the projection at $t = 0$ can be taken as:

$$P(0) = \mathrm{triang}(P_{ij})x$$

where P_{ii} are projections.

ii) If a diagonal system $\dot{x} = \mathrm{diag}(A_i)x$ is SES on J the projection at $t = 0$ can be taken as:

$$P(0) = \mathrm{diag}(P_i)x$$

where P_i are projections.

Proposition

Suppose $A_{ij}(t)$ is bounded for $i < j$. If $\dot{x} = \mathrm{diag}(A_{ii}(t))x$ is SES then also $\dot{x} = \mathrm{triang}(A_{ij}(t))x$ is SES.

Proof. Let $S = \mathrm{diag}(\beta^{i-1} I_{n_i})$ and $Sy = x$. We get

$$\dot{y} = \mathrm{triang}(\beta^{j-i} A_{ij}(t))y. \tag{1}$$

Proposition

Suppose $A_{ij}(t)$ is bounded for $i < j$. If $\dot{x} = \mathrm{diag}(A_{ii}(t))x$ is SES then also $\dot{x} = \mathrm{triang}(A_{ij}(t))x$ is SES.

Proof. Let $S = \mathrm{diag}(\beta^{i-1} I_{n_i})$ and $Sy = x$. We get

$$\dot{y} = \mathrm{triang}(\beta^{j-i} A_{ij}(t))y. \tag{1}$$

$A_{ij}(t)$ is bounded for $i < j$, $\Rightarrow$ (1) is a small perturbation of the diagonal system $\dot{x} = \mathrm{diag}(A_{ii}(t))x$.

Proposition

Suppose $A_{ij}(t)$ is bounded for $i < j$. If $\dot{x} = \mathrm{diag}(A_{ii}(t))x$ is SES then also $\dot{x} = \mathrm{triang}(A_{ij}(t))x$ is SES.

Proof. Let $S = \mathrm{diag}(\beta^{i-1} I_{n_i})$ and $Sy = x$. We get

$$\dot{y} = \mathrm{triang}(\beta^{j-i} A_{ij}(t))y. \tag{1}$$

$A_{ij}(t)$ is bounded for $i < j$, $\Rightarrow$ (1) is a small perturbation of the diagonal system $\dot{x} = \mathrm{diag}(A_{ii}(t))x$. Conclusion $\Leftarrow$ roughness.

Proposition

Suppose $A_{ij}(t)$ is bounded for $i < j$. If $\dot{x} = \mathrm{diag}(A_{ii}(t))x$ is SES then also $\dot{x} = \mathrm{triang}(A_{ij}(t))x$ is SES.

Proof. Let $S = \mathrm{diag}(\beta^{i-1} I_{n_i})$ and $Sy = x$. We get

$$\dot{y} = \mathrm{triang}(\beta^{j-i} A_{ij}(t))y. \tag{1}$$

$A_{ij}(t)$ is bounded for $i < j$, $\Rightarrow$ (1) is a small perturbation of the diagonal system $\dot{x} = \mathrm{diag}(A_{ii}(t))x$. Conclusion $\Leftarrow$ roughness.

Boundedness and SES assumption cannot be easily removed.

Examples

$$\dot{u} = u + \beta e^t v, \quad \dot{v} = 0$$

Examples

$\dot{u} = u + \beta e^t v, \quad \dot{v} = 0$ The diagonal system is SES, with projection $P(t) = \mathrm{diag}(0, 1)$. If the β-triangular system is SES the projection $Q(t)$ is close to $P(t) = \mathrm{diag}(0, 1)$. So we can assume $\ker Q(0) = \ker P(0) = \langle e_1 \rangle$. That is $Q(0) = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}$ and $b^2 = 1$, $ab = a$. The only choices are $(a, b) = (0 - 1)$ or $a \in \mathbb{R}$ and $b = 1$. The fundamental matrix of the β-triangular system is

$$Y_\beta(t) = \begin{pmatrix} e^t & \beta t e^t \\ 0 & 1 \end{pmatrix}$$

then either $Y_\beta(t)Q(0)Y_\beta^{-1}(t) = \begin{pmatrix} 0 & -\beta t e^t \\ 0 & -1 \end{pmatrix}$ or $= \begin{pmatrix} 0 & (a+\beta t)e^t \\ 0 & 1 \end{pmatrix}$. In both cases $Q(t)$ is not bounded.

Proposition

Suppose $\dot{x} = \mathrm{triang}(A_{ij}(t))x$ is SES. Then $\dot{x} = \mathrm{diag}(A_{ii}(t))x$ is SES.

The transition matrices $X(t,s)$ of the diagonal and $\tilde{X}(t,s)$ of the upper triangular, system are:

$$\begin{pmatrix} X_1(t,s) & * & \dots & * \\ 0 & X_2(t,s) & \dots & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & X_m(t,s) \end{pmatrix}$$

where $* = 0$ for $X(t,s)$ and $* = W_{ij}(t,s)$, $i < j$, for $\tilde{X}(t,s)$.

Similarly the projections $P(t)$, $\tilde{P}(t)$ of the ES are like

$$\begin{pmatrix} P_1(t) & * & \dots & * \\ 0 & P_2(t) & \dots & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & P_m(t) \end{pmatrix}$$

where $* = P_{ij}(t)$ for $\tilde{P}(t)$ and $* = 0$ for $P(t)$.

Similarly the projections $P(t)$, $\tilde{P}(t)$ of the ES are like

$$\begin{pmatrix} P_1(t) & * & \dots & * \\ 0 & P_2(t) & \dots & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & P_m(t) \end{pmatrix}$$

where $* = P_{ij}(t)$ for $\tilde{P}(t)$ and $* = 0$ for $P(t)$. We check that the diagonal terms of $\tilde{X}(t,s)\tilde{P}(s)$ are the same as those of $X(t,s)P(s)$, and the out-of-diagonal term of $X(t,s)P(s)$ are zero.

Similarly the projections $P(t)$, $\tilde{P}(t)$ of the ES are like

$$\begin{pmatrix} P_1(t) & * & \dots & * \\ 0 & P_2(t) & \dots & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & P_m(t) \end{pmatrix}$$

where $* = P_{ij}(t)$ for $\tilde{P}(t)$ and $* = 0$ for $P(t)$. We check that the diagonal terms of $\tilde{X}(t,s)\tilde{P}(s)$ are the same as those of $X(t,s)P(s)$, and the out-of-diagonal term of $X(t,s)P(s)$ are zero. Similarly for $\tilde{X}(t,s)[\mathbb{I} - \tilde{P}(s)]$ and $X(t,s)[\mathbb{I} - P(s)]$.

Similarly the projections $P(t)$, $\tilde{P}(t)$ of the ES are like

$$\begin{pmatrix} P_1(t) & * & \dots & * \\ 0 & P_2(t) & \dots & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & P_m(t) \end{pmatrix}$$

where $* = P_{ij}(t)$ for $\tilde{P}(t)$ and $* = 0$ for $P(t)$. We check that the diagonal terms of $\tilde{X}(t,s)\tilde{P}(s)$ are the same as those of $X(t,s)P(s)$, and the out-of-diagonal term of $X(t,s)P(s)$ are zero. Similarly for $\tilde{X}(t,s)[\mathbb{I}-\tilde{P}(s)]$ and $X(t,s)[\mathbb{I}-P(s)]$. So

$$\begin{aligned} &|X(t,s)P(s)|\,|X(t,s)[\mathbb{I}-P(s)]| \\ &\le |\tilde{X}(t,s)\tilde{P}(s)|\,|\tilde{X}(t,s)[\mathbb{I}-\tilde{P}(s)]| \le Ke^{-\alpha(t-s)}. \end{aligned}$$

Corollary

Suppose $a_{ij}(t)$, $1 \leq i < j \leq n$ are bounded. Then $\dot{x} = \mathrm{triang}(a_{ij}(t))x$ is ES if and only if its diagonal part $\dot{x}_i = a_{ii}(t)x_i$, $1 \leq i \leq n$ is ES. Moreover $\dot{x}_i = a_{ii}(t)x_i$ is ES if and only if the set $\mathcal{I} = \{1, \ldots, n\}$ can be split into the disjoint union $\mathcal{I} = \mathcal{I}_1 \dot{\cup} \mathcal{I}_2$ and there exist $K \geq 1$ and $\alpha > 0$ such that

$$\int_s^t a_i(u) - a_j(u) du \leq K - \alpha(t - s)$$

for any $i \in \mathcal{I}_1$ and $j \in \mathcal{I}_2$ and for all $s \leq t$, $s, t \in I$.

Hamiltonian systems

If $\dot{x} = A(t)x$, $x \in \mathbb{R}^{2n}$, is Hamiltonian then its fundamental matrix $X(t)$ is symplectic, that is

$$X(t)^* J X(t) = J, \quad \text{where} \quad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

Hamiltonian systems

If $\dot{x} = A(t)x$, $x \in \mathbb{R}^{2n}$, is Hamiltonian then its fundamental matrix $X(t)$ is symplectic, that is

$$X(t)^* J X(t) = J, \quad \text{where} \quad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

Theorem

Let $\dot{x} = A(t)x$ be a linear Hamiltonian system, where $A(t)$ is bounded and piecewise continuous. If $\dot{x} = A(t)x$ is SES on an interval $I = \mathbb{R}, \mathbb{R}_+, \mathbb{R}_-$ and the stable and unstable subspaces have the same dimension $(= n)$. Then it has an exponential dichotomy on I with stable subspace of dimension n.

Hamiltonian systems

If $\dot{x} = A(t)x$, $x \in \mathbb{R}^{2n}$, is Hamiltonian then its fundamental matrix $X(t)$ is symplectic, that is

$$X(t)^* J X(t) = J, \quad \text{where} \quad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

Theorem

Let $\dot{x} = A(t)x$ be a linear Hamiltonian system, where $A(t)$ is bounded and piecewise continuous. If $\dot{x} = A(t)x$ is SES on an interval $I = \mathbb{R}, \mathbb{R}_+, \mathbb{R}_-$ and the stable and unstable subspaces have the same dimension $(= n)$. Then it has an exponential dichotomy on I with stable subspace of dimension n.

Proof on $\mathbb{R}_+$. Write

$$X(t) = \begin{pmatrix} X_{11}(t) & X_{12}(t) \\ X_{21}(t) & X_{22}(t) \end{pmatrix}$$

$$X(t) = \begin{pmatrix} X_{11}(t) & X_{12}(t) \\ X_{21}(t) & X_{22}(t) \end{pmatrix}$$

Iwasawa decomposition (Gram-Schmidt):

$$\begin{pmatrix} X_{11}(t) \\ X_{21}(t) \end{pmatrix} = Q(t)R(t) = \begin{pmatrix} Q_{11}(t) \\ Q_{21}(t) \end{pmatrix} R_{11}(t).$$

where the columns of $Q(t)$ are orthonormal and $R(t)$ is upper triangular with positive diagonal entries.

$$X(t) = \begin{pmatrix} X_{11}(t) & X_{12}(t) \\ X_{21}(t) & X_{22}(t) \end{pmatrix}$$

Iwasawa decomposition (Gram-Schmidt):

$$\begin{pmatrix} X_{11}(t) \\ X_{21}(t) \end{pmatrix} = Q(t)R(t) = \begin{pmatrix} Q_{11}(t) \\ Q_{21}(t) \end{pmatrix} R_{11}(t).$$

where the columns of $Q(t)$ are orthonormal and $R(t)$ is upper triangular with positive diagonal entries. Setting

$$G(t) = \begin{pmatrix} Q_{11}(t) & -Q_{21}(t) \\ Q_{21}(t) & Q_{11}(t) \end{pmatrix}$$

we find $G^*G = \mathbb{I}$, so $G^* = G^{-1}$.

$$X(t) = \begin{pmatrix} X_{11}(t) & X_{12}(t) \\ X_{21}(t) & X_{22}(t) \end{pmatrix}$$

Iwasawa decomposition (Gram-Schmidt):

$$\begin{pmatrix} X_{11}(t) \\ X_{21}(t) \end{pmatrix} = Q(t)R(t) = \begin{pmatrix} Q_{11}(t) \\ Q_{21}(t) \end{pmatrix} R_{11}(t).$$

where the columns of $Q(t)$ are orthonormal and $R(t)$ is upper triangular with positive diagonal entries. Setting

$$G(t) = \begin{pmatrix} Q_{11}(t) & -Q_{21}(t) \\ Q_{21}(t) & Q_{11}(t) \end{pmatrix}$$

we find $G^* G = \mathbb{I}$, so $G^* = G^{-1}$. We take

$$R(t) = \begin{pmatrix} R_{11}(t) & R_{21}(t) \\ 0 & R_{22}(t) \end{pmatrix}, \quad \begin{pmatrix} R_{21}(t) \\ R_{22}(t) \end{pmatrix} = G^* \begin{pmatrix} X_{12}(t) \\ X_{22}(t) \end{pmatrix}$$

Then $X(t) = G(t)R(t)$.

Then $X(t) = G(t)R(t)$. Note: $G(t)$ is orthogonal and $R(t)$ is block-upper triangular.

Then $X(t) = G(t)R(t)$. Note: $G(t)$ is orthogonal and $R(t)$ is block-upper triangular. $X(t)$ symplectic $\Rightarrow R_{22}(t) = [R_{11}(t)^*]^{-1}$.

Then $X(t) = G(t)R(t)$. Note: $G(t)$ is orthogonal and $R(t)$ is block-upper triangular. $X(t)$ symplectic $\Rightarrow R_{22}(t) = [R_{11}(t)^*]^{-1}$. All matrices are C^1, and the transformation $x = G(t)y$ takes $\dot{x} = A(t)x$ to:

$$\dot{y} = B(t)y, \quad B(t) = G(t)^{-1}[A(t)G(t) - \dot{G}(t)]$$

Then $X(t) = G(t)R(t)$. Note: $G(t)$ is orthogonal and $R(t)$ is block-upper triangular. $X(t)$ symplectic $\Rightarrow R_{22}(t) = [R_{11}(t)^*]^{-1}$. All matrices are C^1, and the transformation $x = G(t)y$ takes $\dot{x} = A(t)x$ to:

$$\dot{y} = B(t)y, \quad B(t) = G(t)^{-1}[A(t)G(t) - \dot{G}(t)]$$

i) $B(t)$ is bounded.

Then $X(t) = G(t)R(t)$. Note: $G(t)$ is orthogonal and $R(t)$ is block-upper triangular. $X(t)$ symplectic $\Rightarrow R_{22}(t) = [R_{11}(t)^*]^{-1}$. All matrices are C^1, and the transformation $x = G(t)y$ takes $\dot{x} = A(t)x$ to:

$$\dot{y} = B(t)y, \quad B(t) = G(t)^{-1}[A(t)G(t) - \dot{G}(t)]$$

i) $B(t)$ is bounded.
ii) $B(t)$ is a block-upper triangular matrix

Then $X(t) = G(t)R(t)$. Note: $G(t)$ is orthogonal and $R(t)$ is block-upper triangular. $X(t)$ symplectic $\Rightarrow R_{22}(t) = [R_{11}(t)^*]^{-1}$. All matrices are C^1, and the transformation $x = G(t)y$ takes $\dot{x} = A(t)x$ to:

$$\dot{y} = B(t)y, \quad B(t) = G(t)^{-1}[A(t)G(t) - \dot{G}(t)]$$

i) $B(t)$ is bounded.
ii) $B(t)$ is a block-upper triangular matrix
Can be proved: $G(B + B^*)G^* = A + A^*$. Hence $|B + B^*| = |A + A^*|$ and then $|B|^2 = 4n|A|^2 \Rightarrow |B| = 2\sqrt{n}|A|$.

Then $X(t) = G(t)R(t)$. Note: $G(t)$ is orthogonal and $R(t)$ is block-upper triangular. $X(t)$ symplectic $\Rightarrow R_{22}(t) = [R_{11}(t)^*]^{-1}$. All matrices are C^1, and the transformation $x = G(t)y$ takes $\dot{x} = A(t)x$ to:

$$\dot{y} = B(t)y, \quad B(t) = G(t)^{-1}[A(t)G(t) - \dot{G}(t)]$$

i) $B(t)$ is bounded.
ii) $B(t)$ is a block-upper triangular matrix
Can be proved: $G(B + B^*)G^* = A + A^*$. Hence $|B + B^*| = |A + A^*|$ and then $|B|^2 = 4n|A|^2 \Rightarrow |B| = 2\sqrt{n}|A|$.
It follows that $\dot{x} = A(t)x$ and $\dot{y} = B(t)y$ are kinematically similar.

Then $X(t) = G(t)R(t)$. Note: $G(t)$ is orthogonal and $R(t)$ is block-upper triangular. $X(t)$ symplectic $\Rightarrow R_{22}(t) = [R_{11}(t)^*]^{-1}$. All matrices are C^1, and the transformation $x = G(t)y$ takes $\dot{x} = A(t)x$ to:

$$\dot{y} = B(t)y, \quad B(t) = G(t)^{-1}[A(t)G(t) - \dot{G}(t)]$$

i) $B(t)$ is bounded.
ii) $B(t)$ is a block-upper triangular matrix
Can be proved: $G(B + B^*)G^* = A + A^*$. Hence $|B + B^*| = |A + A^*|$ and then $|B|^2 = 4n|A|^2 \Rightarrow |B| = 2\sqrt{n}|A|$.
It follows that $\dot{x} = A(t)x$ and $\dot{y} = B(t)y$ are kinematically similar.
Hence both $\dot{x} = A(t)x$ and $\dot{y} = B(t)y$ have an ED or not.

Then $X(t) = G(t)R(t)$. Note: $G(t)$ is orthogonal and $R(t)$ is block-upper triangular. $X(t)$ symplectic $\Rightarrow R_{22}(t) = [R_{11}(t)^*]^{-1}$. All matrices are C^1, and the transformation $x = G(t)y$ takes $\dot{x} = A(t)x$ to:

$$\dot{y} = B(t)y, \quad B(t) = G(t)^{-1}[A(t)G(t) - \dot{G}(t)]$$

i) $B(t)$ is bounded.
ii) $B(t)$ is a block-upper triangular matrix
Can be proved: $G(B + B^*)G^* = A + A^*$. Hence $|B + B^*| = |A + A^*|$ and then $|B|^2 = 4n|A|^2 \Rightarrow |B| = 2\sqrt{n}|A|$. It follows that $\dot{x} = A(t)x$ and $\dot{y} = B(t)y$ are kinematically similar. Hence both $\dot{x} = A(t)x$ and $\dot{y} = B(t)y$ have an ED or not. Since a fundamental matrix of $\dot{x} = A(t)x$ is $X(t)$, a fundamental matrix of $\dot{y} = B(t)y$ is $Y(t) = G^{-1}(t)X(t) = R(t)$. So $B(t) = \dot{R}(t)R(t)^{-1}$ is block upper triangular. Write $\dot{y} = B(t)y$ as

$$\begin{cases} \dot{y}_1 = B_{11}(t)y_1 + B_{12}(t)y_2 \\ \dot{y}_2 = B_{22}(t)y_2 \end{cases}$$

where $B_{11} = \dot{R}_{11}(t)R_{11}(t)^{-1}$ and

$$\begin{cases} \dot{y}_1 = B_{11}(t)y_1 + B_{12}(t)y_2 \\ \dot{y}_2 = B_{22}(t)y_2 \end{cases}$$

where $B_{11} = \dot{R}_{11}(t)R_{11}(t)^{-1}$ and $B_{22} = \ldots = -B_{11}^*$.

$$\begin{cases} \dot{y}_1 = B_{11}(t)y_1 + B_{12}(t)y_2 \\ \dot{y}_2 = B_{22}(t)y_2 \end{cases}$$

where $B_{11} = \dot{R}_{11}(t)R_{11}(t)^{-1}$ and $B_{22} = \ldots = -B_{11}^*$. Then $B_{22}(t)$ is lower diagonal with diagonal entries opposite to the diagonal entries of $B_{11}(t)$.

$$\begin{cases} \dot{y}_1 = B_{11}(t)y_1 + B_{12}(t)y_2 \\ \dot{y}_2 = B_{22}(t)y_2 \end{cases}$$

where $B_{11} = \dot{R}_{11}(t)R_{11}(t)^{-1}$ and $B_{22} = \ldots = -B_{11}^*$. Then $B_{22}(t)$ is lower diagonal with diagonal entries opposite to the diagonal entries of $B_{11}(t)$. Now, since $B_{12}(t)$ is bounded we are almost in position to apply the following result concerning ED.

$$\begin{cases} \dot{y}_1 = B_{11}(t)y_1 + B_{12}(t)y_2 \\ \dot{y}_2 = B_{22}(t)y_2 \end{cases}$$

where $B_{11} = \dot{R}_{11}(t)R_{11}(t)^{-1}$ and $B_{22} = \ldots = -B_{11}^*$. Then $B_{22}(t)$ is lower diagonal with diagonal entries opposite to the diagonal entries of $B_{11}(t)$. Now, since $B_{12}(t)$ is bounded we are almost in position to apply the following result concerning ED.

Theorem

Let $\dot{x} = \begin{pmatrix} A_1(t) & C(t) \\ 0 & A_2(t) \end{pmatrix} x$ be an upper triangular system, with $C(t)$ bounded. If the diagonal system $\begin{cases} \dot{x}_1 = A_1(t)x_1 \\ \dot{x}_1 = A_2(t)x_2 \end{cases}$ has an ED on $I = \mathbb{R}, \mathbb{R}_+, \mathbb{R}_-$ with projection of rank r then the triangular system has an ED on I with projection of rank r.

- Since $B_{22}(t) = -B_{11}^*(t)$ is lower diagonal we reverse order of the last n variables. This corresponds to replace y_2 with, say $J_n y_2$, $J_2^2 = \mathbb{I}$. The equations become $\dot{z}_1 = B_{11}(t)z_1 + B_{12}(t)J_n z_2, \quad \dot{z}_2 = J_n B_{22}(t)J_n z_2$, where $J_n B_{22}(t)J_n$ is upper triangular with the same diagonal elements as $B_{22}(t)$ which are the opposite of those of $B_{11}(t)$.

- Since $B_{22}(t) = -B_{11}^*(t)$ is lower diagonal we reverse order of the last n variables. This corresponds to replace y_2 with, say $J_n y_2$, $J_2^2 = \mathbb{I}$. The equations become $\dot{z}_1 = B_{11}(t)z_1 + B_{12}(t)J_n z_2, \quad \dot{z}_2 = J_n B_{22}(t) J_n z_2$, where $J_n B_{22}(t) J_n$ is upper triangular with the same diagonal elements as $B_{22}(t)$ which are the opposite of those of $B_{11}(t)$.
- the transformed system is SES on $\mathbb{R}_+$, hence so is the diagonal system $\dot{z}_1 = C_1(t)z_1, \quad \dot{z}_2 = -C_1(t)z_2$ where $C_1(t)$ is obtained from $B_{11}(t)$ changing to 0 all out-of-diagonal terms.

- Since $B_{22}(t) = -B_{11}^*(t)$ is lower diagonal we reverse order of the last n variables. This corresponds to replace y_2 with, say $J_n y_2$, $J_2^2 = \mathbb{I}$. The equations become $\dot{z}_1 = B_{11}(t)z_1 + B_{12}(t)J_n z_2, \quad \dot{z}_2 = J_n B_{22}(t) J_n z_2$, where $J_n B_{22}(t) J_n$ is upper triangular with the same diagonal elements as $B_{22}(t)$ which are the opposite of those of $B_{11}(t)$.
- the transformed system is SES on $\mathbb{R}_+$, hence so is the diagonal system $\dot{z}_1 = C_1(t)z_1, \quad \dot{z}_2 = -C_1(t)z_2$ where $C_1(t)$ is obtained from $B_{11}(t)$ changing to 0 all out-of-diagonal terms.
- This diagonal system reads $\dot{z}_j = c_j z_j$, $1 \leq j \leq 2n$ (little change of notation, hopefully you don't mind) with $c_{j+n} = -c_j$, $1 \leq j \leq n$.

It follows that $\{1, \dots, 2n\} = \mathcal{I} \dot{\cup} \mathcal{I}^c$, where

$$i \in \mathcal{I}, j \in \mathcal{I}^c \;\Leftrightarrow\; \int_s^t c_i(u) - c_j(u) du \leq K - \alpha(t - s)$$

for any $s \leq t$.

It follows that $\{1,\ldots,2n\} = \mathcal{I}\dot{\cup}\mathcal{I}^c$, where

$$i \in \mathcal{I}, j \in \mathcal{I}^c \iff \int_s^t c_i(u) - c_j(u)du \leq K - \alpha(t-s)$$

for any $s \leq t$. One proves that i and $i+n \mod 2n$ cannot belong both to $\mathcal{I}$ or $\mathcal{I}^c$.

It follows that $\{1, \ldots, 2n\} = \mathcal{I} \dot{\cup} \mathcal{I}^c$, where

$$i \in \mathcal{I}, j \in \mathcal{I}^c \ \Leftrightarrow \ \int_s^t c_i(u) - c_j(u) du \leq K - \alpha(t-s)$$

for any $s \leq t$. One proves that i and $i + n \mod 2n$ cannot belong both to $\mathcal{I}$ or $\mathcal{I}^c$. Let $i \in \mathcal{I}_1$. Then $i + n \in \mathcal{I}_2$ and so

$$2 \int_s^t c_i(u) du = \int_s^t c_i(u) - c_{i+n}(u) \leq K - \alpha(t-s)$$

the diagonal system (and hence also the original) has an ED on $\mathbb{R}_+$ with projection of rank n. Similar arguments works on $\mathbb{R}_-$.

It follows that $\{1,\ldots,2n\} = \mathcal{I}\dot{\cup}\mathcal{I}^c$, where

$$i \in \mathcal{I}, j \in \mathcal{I}^c \;\Leftrightarrow\; \int_s^t c_i(u) - c_j(u)du \leq K - \alpha(t-s)$$

for any $s \leq t$. One proves that i and $i+n \mod 2n$ cannot belong both to $\mathcal{I}$ or $\mathcal{I}^c$. Let $i \in \mathcal{I}_1$. Then $i+n \in \mathcal{I}_2$ and so

$$2\int_s^t c_i(u)du = \int_s^t c_i(u) - c_{i+n}(u) \leq K - \alpha(t-s)$$

the diagonal system (and hence also the original) has an ED on $\mathbb{R}_+$ with projection of rank n. Similar arguments works on $\mathbb{R}_-$. When $I = \mathbb{R}$, the system has an ED on both half-lines. A careful study of the intersection of the stable space for $\mathbb{R}_+$ and the unstable space for $\mathbb{R}_-$ show that they intersect in the 0 vector and hence the *ED* is on $\mathbb{R}$.

The end

Thanks for the attention

References

COPPEL, W.A.: *Dichotomies in stability theory*, Lecture Notes in Math. 629, Springer Verlag, Berlin, 1978

PALMER, K.J.: *Exponential dichotomy, exponential separation and diagonalizability of linear systems of ordinary differential equations*, J. Differential Equations, 43, 1982, 184-203

PÖTZCHE, C.: *Geometric theory of discrete non-autonomous dynamical systems*, Lecture Notes in Math. 2002, Springer Verlag, Berlin, 2010

B.F., PALMER, K. J. .: *Criteria for exponential dichotomy for triangular systems.*, J. Math. Analisys Appl. 2015, 525-543