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G-exceptional vector bundles on $\mathbb{P}^2$ and representations of quivers

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ABSTRACT

It is known that the category of homogeneous bundles on $\mathbb{P}^2$ is equivalent to the category of representations of a quiver with relation. In this paper we make use of this equivalence to describe a family of G-exceptional bundles on $\mathbb{P}^2$ and to prove that they are stable. We also study the G-exceptionality of Fibonacci bundles on $\mathbb{P}^2$.

Contents

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1. Introduction

The problem of classifying holomorphic vector bundles on algebraic varieties has been a central point of interest of many mathematicians during at least the last four decades.

It is well known that the set of isomorphic classes of vector bundles on an algebraic variety X cannot be parametrized by an algebraic variety. To get around this problem one is forced to consider families of (semi) stable vector bundles. It was in the way to search for stable vector bundles on $\mathbb{P}^2$ that Drézet and Le Potier in [5] introduced the notion of exceptional vector bundle. Indeed, exceptional vector bundles were defined by Drézet and Le Potier as a class of vector bundles on $\mathbb{P}^2$ without deformations. These bundles appear as a sort of exceptional cases in the study of the stable vector bundles on $\mathbb{P}^2$. Later, the school of Rudakov generalized the concept of exceptional bundles to $\mathbb{P}^n$ and other varieties. Nowadays, there is an axiomatic presentation of exceptional vector bundles on algebraic varieties in the setting of derived categories of coherent sheaves (see for instance [3,12]).

Exceptional vector bundles are known to be stable on $\mathbb{P}^2$ [5], and on $\mathbb{P}^3$ [24]. See also [4,14,22] for other families of exceptional vector bundles which are known to be stable. Nevertheless, the stability of exceptional vector bundles on $\mathbb{P}^n$ and more in general on an algebraic variety X is still an open and difficult problem.

Fibonacci bundles on $\mathbb{P}^n$ have been recently introduced in [4] as a generalization of the Steiner exceptional bundles, namely of the exceptional bundles which admit a linear resolution. Fibonacci bundles are homogeneous and generated by mutations. In general, these bundles are not exceptional, since in particular they may have deformations (they are not rigid). Nevertheless, there exist interesting families of non-rigid bundles which do not have deformations in the category of homogeneous bundles (e.g. the so-called syzygy bundles).

This remark leads us to study a property analogous to the exceptionality in the category of homogeneous vector bundles. We will call such a notion G-exceptionality (see Definition 4.2). One of the main results of this paper is that the Fibonacci bundles on $\mathbb{P}^2$ are G-exceptional.

A further natural object of investigation is the stability of G-exceptional vector bundles. In order to tackle the problem of stability in the setting of homogeneous bundles, we can take advantage of the techniques provided by the theory of representations of quivers with relations. Indeed a celebrated result due to Bondal and Kapranov [2] and Hille [8], recently investigated also by Ottaviani and Rubei [16], states that results of classification of vector bundles and results of classification of representations of quivers are closely related. In fact, there is an equivalence between the category of homogeneous bundles on $\mathbb{P}^2$ and the category of representation of a certain quiver $\mathcal{Q}_{\mathbb{P}^2}$ with relations and this allows to translate the stability of a homogeneous vector bundle on $\mathbb{P}^2$ in terms of the stability of some representations of the quiver $\mathcal{Q}_{\mathbb{P}^2}$.

This equivalence is the key ingredient to prove our second main result. In particular we focus on a special case of Fibonacci bundles, which we call *almost square bundles*. We describe explicitly the representation of the quiver associated to an almost square bundle and by studying all the possible subrepresentations we are able to prove the stability of the bundles we are dealing with. In this way we follow the approach of [15] and [21], where the authors investigate certain families of bundles whose associated representations admit a simple description. In our case the main difficulty is that the representations associated to our bundles are quite complicated and so we need several technical steps in order to get our result.

According to the results so far obtained, we are led to investigate the same kind of problems in more generality, for example for all the Fibonacci bundles on $\mathbb{P}^2$ or for some special families of bundles on $\mathbb{P}^n$ for $n \geqslant 2$. Some of these problems will be discussed in a forthcoming paper.

Next we outline the structure of the paper. In Section 2 we recall some preliminary definitions and results concerning homogeneous bundles and the theory of representation of quivers with relations. In particular we state the relation between homogeneous vector bundles on $\mathbb{P}^2$ and representations of a certain quiver with relations $(\mathcal{Q}_{\mathbb{P}^2}, \mathcal{R}_{\mathbb{P}^2})$. In Section 3 we introduce the principal objects that we will study in subsequent sections: the Fibonacci bundles (Definition 3.4) and the almost square bundles on $\mathbb{P}^2$ (Definition 3.6). We also introduce a family of representations R_d of the quiver $(\mathcal{Q}_{\mathbb{P}^2}, \mathcal{R}_{\mathbb{P}^2})$, which will be proved (in Theorem 5.1 and Proposition 5.8) to be the representations associated to almost square bundles. In Section 4, we deal with the G-exceptionality of these bundles, proving that any almost square bundle on $\mathbb{P}^2$ is simple and that any Fibonacci bundle is G-exceptional. In the proof we use cohomological methods, inspired by [4]. Section 5 is devoted entirely to prove that any almost square bundle on $\mathbb{P}^2$ is stable. We first develop some technical lemmas that allows us to control the slope of the subrepresentation T of R_d. Then, in Theorem 5.7, we show that any subrepresentation T of R_d has slope less that the slope of R_d. This allows us to prove in Theorem 5.10 that any almost square bundle is stable.

Notation 1.1. Throughout this paper, we will work over the complex numbers. If there is no confusion, we will denote by $H^i(E)$ the ith cohomology group of a vector bundle E on a smooth projective variety X and by $h^i(E)$ its dimension. Analogously, for any two vector bundles E and F, we will denote by $\hom(E,F)$ (resp. $\mathrm{ext}^i(E,F)$) the dimension of $\mathrm{Hom}(E,F)$ (resp. $\mathrm{Ext}^i(E,F)$) as complex vector spaces and we will denote by $\chi(E,F) := \sum_i (-1)^i \mathrm{ext}^i(E,F)$.

We will write $\mathbb{P}^2 = \mathbb{P}(V^*)$ for some 3-dimensional complex vector space V and thus we will have $H^0(\mathcal{O}_{\mathbb{P}^2}(1)) = V^*$ and, for any integer $d > 0$, $H^0(\mathcal{O}_{\mathbb{P}^2}(d)) = S^d V^*$. We will denote $\mathcal{O} := \mathcal{O}_{\mathbb{P}^2}$ when there is no confusion.

2. Homogeneous vector bundles and representations of quivers

The goal of this section is to collect the results concerning homogeneous vector bundles and representations of quivers that we will use through this paper.

Homogeneous vector bundles: We recall here some well known facts on homogeneous vector bundles on rational homogeneous varieties. See [7] for more details on representation theory. In this paper we are mostly interested in the case of complex projective spaces $\mathbb{P}^n$, and in particular in the case $n = 2$, anyway the results we present here hold in much more generality.

It is well known that the complex projective space $\mathbb{P}^n$ can be realized as a rational homogeneous variety G/P, where $G = SL(n+1)$ and P is a parabolic subgroup. In the sequel when we will work on $\mathbb{P}^n$, we will assume $G = SL(n+1)$.

A rank r vector bundle E on $\mathbb{P}^n$ is called *G-homogeneous* (or simply homogeneous) if for any $g \in G$, $g^*E \cong E$. It is well known that any homogeneous bundle on $\mathbb{P}^n$ is associated to a representation ρ of the parabolic subgroup P. The *irreducible* homogeneous bundles E_λ are defined to be the homogeneous bundles associated to the irreducible representations of P with highest weight λ.

The irreducible homogeneous bundles on the projective plane $\mathbb{P}^2$ are classified and they are of the form $S^l Q(t)$ for some $l \in \mathbb{N}$ and $t \in \mathbb{Z}$, where $Q := T_{\mathbb{P}^2}(-1)$ is the tangent bundle on $\mathbb{P}^2$ twisted by $\mathcal{O}_{\mathbb{P}^2}(-1)$.

Remark 2.1. Any homogenous vector bundle E on $\mathbb{P}^n$ admits a filtration

$$0 \subset E_1 \subset \cdots \subset E_{k-1} \subset E_k = E$$

where each E_i/E_{i-1} is irreducible. The graded vector bundle $\mathrm{gr}(E) := \bigoplus_i E_i/E_{i-1}$ does not depend on the filtration.

Given a sheaf E on $\mathbb{P}^n$ of rank $\mathrm{rk}(E) \geqslant 1$, we define the *slope* of E as

$$\mu(E) := \frac{c_1(E)}{\mathrm{rk}(E)},$$

where we denote by $c_1(E)$ the integer such that $\mathcal{O}_{\mathbb{P}^n}(c_1(E))$ is the first Chern class of E. A vector bundle E on $\mathbb{P}^n$ is called *semistable* (in the sense of Mumford–Takemoto) if and only if for all nonzero subsheaves $F \subset E$ with $\mathrm{rk}(F) < \mathrm{rk}(E)$ we have

$$\mu(F) \leqslant \mu(E)$$

and if strict inequality holds, then E is said to be *stable*.

We say that a homogeneous vector bundle E on $\mathbb{P}^n$ is *multistable* if it is the tensor product of a stable homogenous bundle and an irreducible G-representation. It follows immediately by the definition that if a vector bundle is multistable and simple, then it is also stable.

A basic result is the following criterion for the stability of homogeneous vector bundles on $\mathbb{P}^n$ (see [19] and [6]):

Theorem 2.2. *A homogeneous bundle E on $\mathbb{P}^n$ is semistable (resp. multistable) if and only if $\mu(F) \leqslant \mu(E)$ (resp. $\mu(F) < \mu(E)$) for any homogeneous subbundle F of E associated to a subrepresentation of the P-representation associated to E.*

Given a vector bundle E on $\mathbb{P}^n$, we recall that it is called *simple* if it satisfies $\mathrm{Hom}(E, E) \cong \mathbb{C}$, and *exceptional* if it is simple and satisfies $\mathrm{Ext}^i(E, E) = 0$ for any $i > 0$. A vector bundle satisfying $\mathrm{Ext}^1(E, E) = 0$ is called *rigid*. It is known that a rigid bundle is also homogeneous.

Given two homogenous bundles E and F on $\mathbb{P}^n$, we denote by $\mathrm{Ext}^i(E, F)^G$ the G-invariant part of the G-module $\mathrm{Ext}^i(E, F)$, that is the G-submodule where G acts trivially. We also denote $\chi(E, F)^G := \sum_i (-1)^i \mathrm{ext}^i(E, F)^G$, where $\mathrm{ext}^i(E, F)^G$ stands for the dimension of $\mathrm{Ext}^i(E, F)^G$.

Definition 2.3. Let E be a homogeneous vector bundle on an homogeneous variety G/P. We say that E is *G-simple* if $\mathrm{Hom}(E, E)^G \cong \mathbb{C}$, *$G$-rigid* if $\mathrm{Ext}^1(E, E)^G = 0$ and *G-exceptional* if it is G-simple and $\mathrm{Ext}^i(E, E)^G = 0$ for any $i > 0$.

Clearly, if a vector bundle E is exceptional, then it is also G-exceptional. Of course, the converse is not true.

Remark 2.4. It is clear that by definition $\mathrm{ext}^i(E, E)^G$ equals to the number of copies of the trivial representation $\mathbb{C}$ contained in the G-module $\mathrm{Ext}^i(E, E) \cong H^i(E \otimes E^*)$.

Representations of quivers: Now we will recall the definitions and state the main results that we will use concerning quivers and representations of quivers associated to homogeneous bundles. We will focus in particular on the case of $\mathbb{P}^2$.

This theory has been introduced by Bondal and Kapranov in [2] and generalized by Hille in [8] and [9]. We will adopt the same notation as in [16] and [15].

Definition 2.5. A quiver is an oriented graph $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$, where $\mathcal{Q}_0$ is the set of vertices and $\mathcal{Q}_1$ is the set of arrows. We define two maps $t, h : \mathcal{Q}_1 \to \mathcal{Q}_0$ such that for any arrow $a \in \mathcal{Q}_1$, $t(a)$ is the tail of a and $h(a)$ is the head of a. A *path* in $\mathcal{Q}$ is a formal composition of arrows $\beta_m \cdots \beta_1$ such that the tail of an arrow β_k is the head of β_{k-1}. A *relation* in $\mathcal{Q}$ is a linear combination of paths of $\mathcal{Q}$ with common head and common tail.

A *representation* of a quiver $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$ is a set of vector spaces $\{X_v\}_{v \in \mathcal{Q}_0}$ and a set of linear maps $\{\phi_\beta\}_{\beta \in \mathcal{Q}_1}$ where $\phi_\beta : X_{h(\beta)} \to X_{t(\beta)}$. Given a set of relation $\mathcal{R}$ in $\mathcal{Q}$, a *representation of a quiver $\mathcal{Q}$ with relations $\mathcal{R}$* is a representation of $\mathcal{Q}$ such that

$$\sum_k \lambda_k \phi_{\beta_{i_1}} \cdots \phi_{\beta_{i_k}} = 0$$

for any relation $\sum_k \lambda_k \beta_{i_1} \cdots \beta_{i_k} \in \mathcal{R}$. A *morphism* between two representations of the quiver $\mathcal{Q}$, $(X_v, \phi_\beta)_{v \in \mathcal{Q}_0, \beta \in \mathcal{Q}_1}$ and $(Y_v, \psi_\beta)_{v \in \mathcal{Q}_0, \beta \in \mathcal{Q}_1}$ is a set of linear maps $\{f_v : X_v \to Y_v\}$ such that, for every $\beta \in \mathcal{Q}_1$ from v to w, we have

$$\psi_\beta \circ f_v = f_w \circ \phi_\beta.$$

A *subrepresentation* of a representation $(X_v, \phi_\beta)_{v \in \mathcal{Q}_0, \beta \in \mathcal{Q}_1}$ of a quiver $\mathcal{Q}$ is a representation $(Y_v, \psi_\beta)_{v \in \mathcal{Q}_0, \beta \in \mathcal{Q}_1}$ of $\mathcal{Q}$ such that for any $v \in \mathcal{Q}_0$, $Y_v \subset X_v$ is a subvector space and for any arrow $\beta \in \mathcal{Q}_1$ from v to w, $\psi_\beta = \phi_\beta|_{Y_v}$. A representation $Y = (Y_v, \psi_\beta)_{v \in \mathcal{Q}_0, \beta \in \mathcal{Q}_1}$ of a quiver $\mathcal{Q}$ is called *quotient representation* of a representation $X = (X_v, \phi_\beta)_{v \in \mathcal{Q}_0, \beta \in \mathcal{Q}_1}$ of the same quiver if there is a surjective morphism from X to Y.

For a later use, we need to introduce the following terminology and notation. Notice that our definition of support is not standard.

Definition 2.6. We say that a representation $X = (X_v, \phi_\beta)_{v \in \mathcal{Q}_0, \beta \in \mathcal{Q}_1}$ has *multiplicity* m at a point v of $\mathcal{Q}_0$ if $\dim X_v = m$ and we will denote it by m_v^X. We call *support* of a representation X of a quiver $\mathcal{Q}$, the subset of $\mathcal{Q}_0$ containing the vertices where X has positive multiplicity. More precisely $\mathrm{Supp}(X) := \{v \in \mathcal{Q}_0 \mid m_v^X \geqslant 1\}$. We call *support with multiplicities*, and we denote by $\mathrm{Suppm}(X)$ the data $\mathrm{Supp}(X)$ and $(m_v^X)_{v \in \mathrm{Supp}(X)}$. The vector $(m_v^X)_{v \in \mathrm{Supp}(X)}$ is usually called *dimension vector* of the representation.

We will use the following notation concerning the support with multiplicities of given representations of a quiver $\mathcal{Q}$.

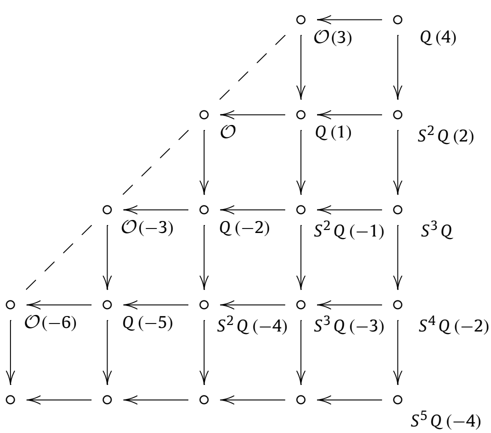
(a) Given two representations X and Y, such that $m_v^X \geqslant m_v^Y$ for any $v \in \mathcal{Q}_0$, we denote by $X \setminus Y$ the set of vertices of the support of X with multiplicities $(m_v^X - m_v^Y)_{v \in \mathrm{Supp}(X)}$.
(b) Given two representations X and Y, we will say that a set of vertices with multiplicities, that is a subset $S \subset \mathcal{Q}_0$ and a collection of nonnegative integers $(n_v)_{v \in S}$, is the *disjoint union* of $\mathrm{Suppm}(A)$ and $\mathrm{Suppm}(B)$, if we have $S = \mathrm{Supp}(A) \cup \mathrm{Supp}(B)$ and for each vertex $v \in S$, we have $n_v = m_v^X + m_v^Y$. If Z is a representation such that $\mathrm{Suppm}(Z) = S$, $(n_v)_{v \in S}$, we will also say that Z is the disjoint union of X and Y and we will write $Z = X \sqcup Y$.
(c) Given two representations X and Y, we denote by $X \cap Y$ the set of vertices with multiplicities given by the intersection $\mathrm{Supp}(X) \cap \mathrm{Supp}(Y)$ and by the multiplicities $\min\{m(A_v), m(B_v)\}$, for any $v \in \mathrm{Supp}(X) \cap \mathrm{Supp}(Y)$.

Definition 2.7. From now on we denote by $\mathcal{Q}_{\mathbb{P}^2}$ the quiver $(\mathcal{Q}_0, \mathcal{Q}_1)$ such that:

$$\mathcal{Q}_0 := \left\{ S^l Q(t) \;\middle|\; l \in \mathbb{N},\ t \in \mathbb{Z} \right\},$$

i.e. each vertex is identified with an irreducible homogeneous bundle on $\mathbb{P}^2$. The set of arrows $\mathcal{Q}_1$ is defined in the following way: there is an arrow β from the vertex $v \in \mathcal{Q}_0$ corresponding to $S^l Q(t)$ to the vertex $w \in \mathcal{Q}_0$ corresponding to $S^p Q(q)$ if and only if $\mathrm{Ext}^1(S^l Q(t), S^p Q(q))^G \neq 0$. This happens if and only if $(p,q) = (l-1, t-1)$ or $(p,q) = (l+1, t-2)$.

It is easily seen that the quiver $\mathcal{Q}_{\mathbb{P}^2}$ has three connected components $\mathcal{Q}_{\mathbb{P}^2}^{(1)}$, $\mathcal{Q}_{\mathbb{P}^2}^{(2)}$ and $\mathcal{Q}_{\mathbb{P}^2}^{(3)}$, given by the congruence class modulo $\frac{3}{2}$ of the slope of the homogeneous bundles corresponding to the vertices of the connected component. Every homogeneous bundle E on $\mathbb{P}^2$ splits as $E = \bigoplus_i E^{(i)}$ where the sum is over the connected components of $\mathcal{Q}_{\mathbb{P}^2}$ and $\mathrm{gr}(E^{(i)})$ contains only irreducible vector bundles corresponding to vertices of the connected component labeled by i. For convenience, we identify this component $\mathcal{Q}_{\mathbb{P}^2}^{(1)}$ with the following subset of $\mathbb{Z}^2$



Definition 2.8. We define $\mathcal{R}_{\mathbb{P}^2}$ as the set of relations on $\mathcal{Q}_{\mathbb{P}^2}$ given by the commutativity of the squares. More precisely, denoting by $\beta_{w,v}$ the arrow from v to w, the relations in $\mathcal{R}_{\mathbb{P}^2}$ are

$$\beta_{(x-1,y-1),(x-1,y)}\beta_{(x-1,y),(x,y)} - \beta_{(x-1,y-1),(x,y-1)}\beta_{(x,y-1),(x,y)}$$

for all $(x, y) \in \mathcal{Q}^{(i)}_{\mathbb{P}^2} \in \mathbb{Z}^2$ for some i, such that $(x-1, y) \in \mathcal{Q}_{\mathbb{P}^2}$ and

$$\beta_{(x-1,y-1),(x,y-1)}\beta_{(x,y-1),(x,y)}$$

for all $(x, y) \in \mathcal{Q}^{(i)}_{\mathbb{P}^2} \in \mathbb{Z}^2$ for some i, such that $(x-1, y) \notin \mathcal{Q}_{\mathbb{P}^2}$.

Any homogeneous bundle E on $\mathbb{P}^2$ defines an associated representation of the quiver $\mathcal{Q}_{\mathbb{P}^2}$ with relations $\mathcal{R}_{\mathbb{P}^2}$, in the following way:

Definition 2.9. Given a homogeneous vector bundle E on $\mathbb{P}^2$, according to Remark 2.1 we have the graded

$$\mathrm{gr}(E) = \bigoplus_{\lambda} E_\lambda \otimes V_\lambda$$

where $E_\lambda = S^l Q(t)$ for some $l \in \mathbb{N}$ and $t \in \mathbb{Z}$ and where V_λ is a k-dimensional complex vector space, being $k \geqslant 0$ the number of times that the irreducible homogenous bundle $S^l Q(t)$ occurs in the graded bundle $\mathrm{gr}(E)$. To the vertex of $\mathcal{Q}_{\mathbb{P}^2}$ corresponding to $E_\lambda = S^l Q(t)$ we associate the vector space $V_\lambda = \mathbb{C}^k$. To any arrow $\lambda \to \lambda'$ of the quiver $\mathcal{Q}_{\mathbb{P}^2}$ we associate a linear map $V_\lambda \to V_{\lambda'}$, defined by the G-invariant element of $\mathrm{Ext}^1(\mathrm{gr}(E), \mathrm{gr}(E))$ associated to the action of the nilpotent algebra on $\mathrm{gr}(E)$. See e.g. [16] for more details.

A key result is the following equivalence of categories due to Bondal–Kapranov and in a much more general setting due to Hille (see [2,8–10]).

Theorem 2.10. *The category of homogeneous bundles on $\mathbb{P}^2$ is equivalent to the category of finite dimensional representations of the quiver $\mathcal{Q}_{\mathbb{P}^2}$ with the relations $\mathcal{R}_{\mathbb{P}^2}$.*

According to Theorem 2.10, we will identify an homogeneous bundle E on $\mathbb{P}^2$ with its associated representation of the quiver $(\mathcal{Q}_{\mathbb{P}^2}, \mathcal{R}_{\mathbb{P}^2})$. In particular we will use the name support of a vector bundle E to refer the support with multiplicities of the representation associated to E.

Remark 2.11. Notice that the first Chern class of a homogeneous vector bundle E can be computed as the sum of the first Chern classes of the irreducible bundles corresponding to the vertices of the support of E multiplied by the multiplicities. Analogously, the rank of E is the sum of the ranks of the irreducible bundles corresponding to such vertices multiplied by the multiplicities.

The previous remark lead us to pose the following definition:

Definition 2.12. We define the slope (resp. first Chern class, rank) of a set of vertices with multiplicities as the slope (resp. first Chern class, rank) of the vector bundle whose support is that set of vertices with multiplicities.

The equivalence between the category of homogeneous bundles on $\mathbb{P}^2$ and the category of the representations of the quiver $(\mathcal{Q}_{\mathbb{P}^2}, \mathcal{R}_{\mathbb{P}^2})$ implies that any homogeneous subbundle F of a homogeneous bundle E on $\mathbb{P}^2$ is associated to a subrepresentation of the representation associated to E. Hence in view of Theorem 2.2 in order to prove the multistability of a homogeneous bundle E, it is enough to check that the slope of any subrepresentation of the representation associated to E is less than the slope of E.

It is immediate to deduce from the definition the following lemma,

Lemma 2.13. *Let E be a homogeneous vector bundle on $\mathbb{P}^2$ such that the set of vertices of the support of E is disjoint union of the sets of vertices of the supports of two representations X and Y. The following holds*:

(a) *If $\mu(X) = \mu(Y)$, then $\mu(E) = \mu(X) = \mu(Y)$.*
(b) *If $\mu(X) < \mu(Y)$, then $\mu(X) < \mu(E) < \mu(Y)$.*

To construct moduli spaces of representations of quivers according to Mumford's geometric invariant theory there is a suitable notion of semistability of quivers introduced in [13] by A. King (see also [11,20]). This notion of semistability turns out to be equivalent to the notion of Mumford–Takemoto semistability of the bundle and in this way one gets a moduli space of homogeneous semistable bundles E with fixed $\mathrm{gr}(E)$. More precisely according to [13]:

Definition 2.14. Let $\mathrm{mod} - k\mathcal{Q}$ be the abelian category of representations of a quiver $\mathcal{Q}$ and $\theta : K_0(\mathrm{mod} - k\mathcal{Q}) \to \mathbb{R}$ an additive function on the Grothendieck group. Any representation R of $\mathcal{Q}$ is called θ-semistable if $\theta(R) = 0$ and for every subrepresentation $R' \subseteq R$, $\theta(R') \geqslant 0$. R is called θ-stable if the only subrepresentations $R' \subseteq R$ with $\theta(R') = 0$ are R and 0.

To any homogeneous bundle E with dimension vector α and

$$\mathrm{gr}(E) = \bigoplus_{\lambda} E_\lambda \otimes V_\lambda,$$

there is associated a natural character $\mu(\alpha) = (\mu(\alpha)_\lambda)_\lambda$ given by

$$\mu(\alpha)_\lambda = c_1(E)rk(E_\lambda) - rk(E)c_1(E_\lambda).$$

This defines an additive function

$$\mu(\alpha) : K_0(\mathrm{mod} - k\mathcal{Q}) \to \mathbb{R}$$

such that for any F of dimension vector $(\beta_\lambda)_\lambda$,

$$\mu(\alpha)(F) = \sum_{\lambda} \beta_\lambda \mu(\alpha)_\lambda.$$

Keeping these notations, we have

Proposition 2.15. *Let E be a homogeneous vector bundle on $\mathbb{P}^2$ with dimension vector α corresponding to* $\mathrm{gr}(E)$. *Then*

(1) *E is semistable if and only if the representation of $\mathcal{Q}_{\mathbb{P}^2}$ associated to E is $\mu(\alpha)$-semistable.*
(2) *E is multistable if and only if the representation of $\mathcal{Q}_{\mathbb{P}^2}$ associated to E is $\mu(\alpha)$-stable.*

Proof. See Theorem 2.2 and [16, Theorems 7.1 and 7.2]. □

Remark 2.16. It is clear from the above result that the Mumford–Takemoto stability of a vector bundle E is a stronger property than the stability of the representation associated to E.

3. Fibonacci bundles and almost square bundles

In this section we introduce some families of homogeneous vector bundles and we describe the associated representation of the quiver. In particular we will recall the definition of *syzygy* bundles (Definition 3.1), of *Fibonacci* bundles (Definition 3.4) and we will introduce the *almost square* bundles (Definition 3.6). In next sections, we will study the G-exceptionality and the stability of such bundles.

Definition 3.1. For any integer $d > 0$, we denote by Syz_d the vector bundle on $\mathbb{P}^2$ defined as the cokernel of the evaluation map $\mathcal{O}(-d) \to \mathrm{Hom}(\mathcal{O}(-d), \mathcal{O})^* \otimes \mathcal{O}$, that is by the exact sequence

$$0 \to \mathcal{O}(-d) \to S^d V \otimes \mathcal{O} \to Syz_d \to 0. \tag{3.1}$$

The vector bundle Syz_d is called a *syzygy bundle*.

It is well known that syzygy bundles are stable homogeneous vector bundles: see for instance [1,17,18].

Lemma 3.2. *The graded vector bundle of Syz_d is given by*

$$\mathrm{gr}(Syz_d) = \bigoplus_{i=1}^{d} S^i Q\,(i-d) \tag{3.2}$$

and the representation of the quiver $(\mathcal{Q}_{\mathbb{P}^2}, \mathcal{R}_{\mathbb{P}^2})$ associated to Syz_d is given by

$$\begin{tikzcd} \underset{Q(-d+1)}{\circ} & \underset{S^2Q(-d+2)}{\circ} \arrow[l] & \circ \arrow[l] & \cdots\cdots & \underset{S^{d-2}Q(-2)}{\circ} & \underset{S^{d-1}Q(-1)}{\circ} \arrow[l] & \underset{S^dQ}{\circ} \arrow[l] \end{tikzcd}$$

with all the multiplicities equal to one and all the maps different from zero.

Proof. By [15, Remark 23] it is easy to check that the graded bundle of $S^d V \otimes \mathcal{O}$ is

$$\mathrm{gr}(S^d V \otimes \mathcal{O}) = \bigoplus_{i=0}^{d} S^i Q\,(i-d),$$

and thus we get (3.2), since by definition Syz_d is the quotient of $S^d V \otimes \mathcal{O}$ by $\mathcal{O}(-d)$. The maps in the representation are all different from zero, because otherwise the associated bundle would be decomposable, and this is impossible because Syz_d is stable. $\square$

Remark 3.3. Any syzygy bundle on $\mathbb{P}^2$ is G-exceptional. Indeed, Syz_d is simple and hence G-simple. Moreover one can see that it is G-rigid by looking at the representation associated and observing that, since all the multiplicities in the representation are one, all the possible choices of the nonzero maps give isomorphic representations.

The syzygy bundles are special cases of the so-called Fibonacci bundles. Following [4], we call Fibonacci bundles a family of homogeneous bundles defined by means of mutations, which can be characterized from the fact that they admit a resolution whose coefficients are related to the numbers of Fibonacci. Let us recall the definition of the Fibonacci bundles.

Definition 3.4. The Fibonacci bundles (associated to the pair $(\mathcal{O}_{\mathbb{P}^2}(-d), \mathcal{O}_{\mathbb{P}^2})$) are the vector bundles C_k defined recursively as follows: $C_0 = \mathcal{O}(-d)$, $C_1 = \mathcal{O}$ and

$$0 \to C_{k-1} \xrightarrow{i_k} C_k \otimes \mathrm{Hom}(C_{k-1}, C_k) \to C_{k+1} \to 0, \quad \text{for } k \geqslant 1,$$

where i_k is the natural evaluation map. Notice that $C_2 = Syz_d$. It is possible to see that $\mathrm{Hom}(C_{k-1}, C_k) \cong S^d V^*$ if k is odd, $\mathrm{Hom}(C_{k-1}, C_k) \cong S^d V$ if k is even.

We refer the reader to [4] for the details of the construction and the definition in a more general context (see also [23]).

Remark 3.5. We recall the following characterization which explain the relation between these bundles and the Fibonacci numbers. The Fibonacci bundle C_k on $\mathbb{P}^2$ has the following resolution

$$0 \to \mathcal{O}(-d)^{a_{k-1}} \to \mathcal{O}^{a_k} \to C_k \to 0$$

where the sequence $\{a_k\}$ is defined as follows

$$a_0 = 0, \qquad a_1 = 1, \qquad a_{k+1} = \binom{d+2}{2} a_k - a_{k-1}.$$

In [4], the first author proved that these bundles are exceptional if and only if $d = 1, 2$, while for $d \geqslant 3$ a general deformation of C_k is simple, but C_k is not rigid.

Now we are going to concentrate our attention on the Fibonacci bundles on $\mathbb{P}^2$ of type C_3, that we will also call *almost square bundles*. Also in this case, as in case of syzygy bundles, we are able to describe their corresponding representation of the quiver $(\mathcal{Q}_{\mathbb{P}^2}, \mathcal{R}_{\mathbb{P}^2})$.

Definition 3.6. Let $d \geqslant 1$ be an integer. According to Definition 3.4, the Fibonacci bundle C_3 is the cokernel of the natural map:

$$\mathcal{O} \to \mathrm{Hom}(\mathcal{O}, Syz_d)^* \otimes Syz_d \cong S^d V^* \otimes Syz_d.$$

We call *almost square bundle* the dual of such bundles, that is the bundle $E_d \cong C_3^*$ given by the exact sequence

$$0 \to E_d \to S^d V \otimes Syz_d^* \to \mathcal{O} \to 0. \tag{3.3}$$

The choice of the name is motivated by the shape of the associated representation, see Definition 3.8 below.

Lemma 3.7. *The graded vector bundle associated to $S^d V \otimes Syz_d^*$ is*

$$\mathrm{gr}\big(S^d V \otimes Syz_d^*\big) = \bigoplus_{j=1}^{d} \bigoplus_{i=0}^{d} \left(\bigoplus_{k=0}^{\min(i,j)} S^{i+j-2k}(k+i-2j) \right).$$

Proof. By [15, Remark 23] the graded bundle of $S^d V \otimes \mathcal{O}$ is

$$\mathrm{gr}\big(S^d V \otimes \mathcal{O}\big) = \bigoplus_{i=0}^{d} S^i Q\,(i-d).$$

On the other hand, $\mathrm{gr}(Syz_d^*) = \mathrm{gr}(Syz_d)^*$ and thus by Lemma 3.2

$$\mathrm{gr}(Syz_d)^* = \bigoplus_{j=1}^{d} \big(S^j Q\,(j-d)\big)^* = \bigoplus_{j=1}^{d} S^j Q\,(d-2j)$$

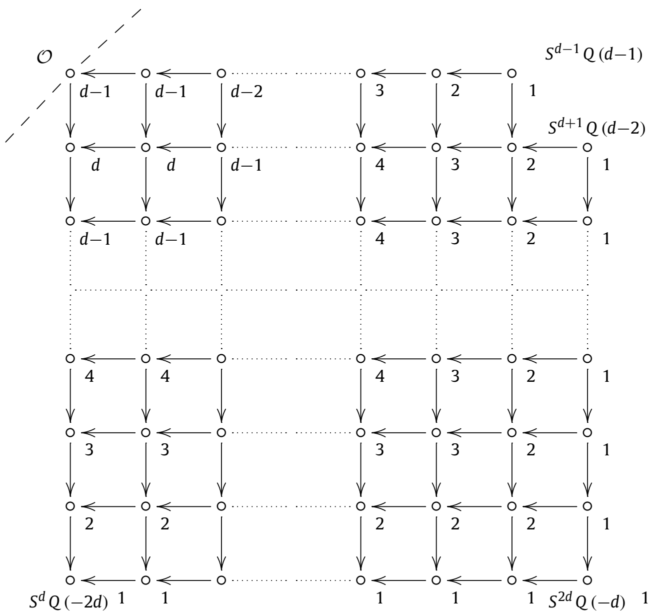
where the last equality follows from the fact that since Q is a rank two vector bundle with $c_1(Q) = 1$ then $Q^* \cong Q\,(-1)$. Thus,

$$\begin{aligned}
\mathrm{gr}\big(S^d V \otimes Syz_d^*\big) &= \mathrm{gr}\big(S^d V \otimes \mathcal{O}\big) \otimes \mathrm{gr}\big(Syz_d^*\big) \\
&= \left(\bigoplus_{i=0}^{d} S^i Q\,(i-d)\right) \otimes \left(\bigoplus_{j=1}^{d} S^j Q\,(d-2j)\right) \\
&= \bigoplus_{j=1}^{d} \bigoplus_{i=0}^{d} \left(\bigoplus_{k=0}^{\min(i,j)} S^{i+j-2k}(k+i-2j)\right)
\end{aligned}$$

where the last equality follows by Pieri's formula. □

We define now a representation R_d of the quiver $\mathcal{Q}_{\mathbb{P}^2}$. In Theorem 5.1 and Proposition 5.8 below we will prove that this representation R_d is exactly the unique representation associated to an almost square bundle E_d on $\mathbb{P}^2$.

Definition 3.8. Let $R_d = (U^d_{i,j}, \varphi^d_{i,j}, \psi^d_{i,j})$ be a representation of the quiver $\mathcal{Q}_{\mathbb{P}^2}$ defined as follows. The support of R_d is contained in a square with the vertices corresponding to $\mathcal{O}$, $S^{d+1}Q\,(d+1)$, $S^d(-2d)$, $S^{2d}Q\,(-d)$. For any fixed d we label the vertices (i,j) denoting by $(1,1)$ the vertex $S^{2d}Q\,(-d)$, by $(1,d+1)$ the vertex $S^dQ\,(-2d)$, by $(d,1)$ the vertex $S^{d+1}Q\,(d-2)$, by $(d+1,2)$ the vertex $S^{d-1}Q\,(d-1)$ and by $(d+1,d+1)$ the vertex $\mathcal{O}$.



We denote by $U^d_{i,j}$ the vector space corresponding to the vertex (i,j), for $1 \leqslant i, j \leqslant d+1$. The dimensions of such vector spaces are as follows:

$$a^d_{i,j} := \dim U^d_{i,j} = \begin{cases} i & \text{for } 1 \leqslant i \leqslant j \leqslant d, \\ j & \text{for } 1 \leqslant j < i \leqslant d, \end{cases}$$
$$a^d_{d+1,j} := \dim U^d_{d+1,j} = \begin{cases} d-1 & \text{for } j = d+1, \\ j-1 & \text{for } 1 \leqslant j \leqslant d. \end{cases} \tag{3.4}$$

In the picture, we have written the dimension $a_{i,j}$ of the vector space corresponding to the vertex (i,j). We denote by $\varphi^d_{i,j}$ the horizontal map from $U^d_{i,j}$ to $U^d_{i,j+1}$ and by $\psi^d_{i,j}$ the vertical map from $U^d_{i,j}$ to $U^d_{i-1,j}$. Moreover these maps satisfy the following conditions:

any map has maximal rank, (3.5)

any possible composition of maps has maximal rank, (3.6)

the direct sum of the maps $\varphi^d_{d,1}$ and $\psi^d_{d+1,2}$ has rank 2. (3.7)

4. *G*-exceptional vector bundles

The main goal of this section if to prove the G-exceptionality of the Fibonacci bundles on $\mathbb{P}^2$. We will prove that a Fibonacci bundle on $\mathbb{P}^2$ is G-exceptional, in spite it is not exceptional. Moreover, we will prove that any almost square bundle on $\mathbb{P}^2$ is also simple, and not only G-simple.

Remark 4.1. Since the anticanonical line bundle on $\mathbb{P}^2$ is ample, it is easy to see, by Serre duality, that any G-simple bundle E satisfies also $\operatorname{Ext}^2(E,E)^G = 0$.

Theorem 4.2. *Any almost square bundle E_d is simple and G-exceptional.*

Proof. In the cases $d = 1, 2$, by [4] we know that E_d is exceptional. So, we can assume that $d \geqslant 3$. We want to prove that $\operatorname{Hom}(E_d, E_d) \cong \mathbb{C}$. Applying the functor $\operatorname{Hom}(-, E_d)$ to the sequence (3.3), we get

$$\operatorname{Hom}\bigl(S^d V \otimes Syz^*_d, E_d\bigr) \to \operatorname{Hom}(E_d, E_d) \to \operatorname{Ext}^1(\mathcal{O}, E_d).$$

We show first that the group $\operatorname{Hom}(Syz^*_d, E_d)$ vanishes. Indeed applying the functor $\operatorname{Hom}(Syz^*_d, -)$ to the sequence (3.3) we get

$$0 \to \operatorname{Hom}(Syz^*_d, E_d) \to S^d V^* \otimes \operatorname{Hom}(Syz^*_d, Syz^*_d) \xrightarrow{f} \operatorname{Hom}(Syz^*_d, \mathcal{O}).$$

Since the bundle Syz_d is simple and the map f in this sequence is the canonical isomorphism $S^d V^* \cong H^0(Syz_d)$, we get $\operatorname{Hom}(Syz^*_d, E_d) = 0$.

On the other hand, we prove now that $\operatorname{Ext}^1(\mathcal{O}, E_d) \cong \mathbb{C}$ and the simplicity of E_d will follow. Taking the cohomology of the sequence (3.3) we get

$$S^d V \otimes H^0(Syz^*_d) \to H^0(\mathcal{O}) \to H^1(E_d) \to S^d V \otimes H^1(Syz^*_d)$$

By the sequence in Definition 3.4 it is easy to check that

$$H^0(Syz^*_d) = H^1(Syz^*_d) = 0$$

and so we conclude that

$$\mathrm{Ext}^1(\mathcal{O}, E_d) \cong H^1(E_d) \cong H^0(\mathcal{O}) \cong \mathbb{C}.$$

This proves that $\mathrm{Hom}(E_d, E_d) \cong \mathbb{C}$. By Remark 4.1, it also follows that $\mathrm{Ext}^2(E_d, E_d) = 0$.

We want to prove now that $\mathrm{Ext}^1(E_d, E_d)^G = 0$. Since we have $\mathrm{Hom}(E_d, E_d)^G \cong \mathbb{C}$ and $\mathrm{Ext}^2(E_d, E_d)^G = 0$ it is enough to prove that $\chi(E_d, E_d)^G = 1$. By applying again the functor $\mathrm{Hom}(-, E_d)$ to the sequence (3.3), we have

$$\chi(E_d, E_d)^G = \chi\big(S^d V \otimes Syz_d^*, E_d\big)^G - \chi(E_d)^G.$$

We have showed that $H^1(E_d) \cong \mathbb{C}$. In the same way it is easy to prove that

$$H^0(E_d) = H^2(E_d) = 0,$$

and so it follows that

$$\chi(E_d) = \chi(E_d)^G = -1.$$

We want to prove now that $\chi(S^d V \otimes Syz_d^*, E_d)^G = 0$. Applying now the functor $\mathrm{Hom}(S^d V \otimes Syz_d^*, -)$ to the sequence (3.3), we get

$$\chi\big(S^d V \otimes Syz_d^*, E_d\big)^G = \chi\big(S^d V \otimes Syz_d^*, S^d V \otimes Syz_d^*\big)^G - \chi\big(S^d V \otimes Syz_d^*, \mathcal{O}\big)^G.$$

Since we know that Syz_d is a G-exceptional bundle, we have

$$\chi\big(S^d V \otimes Syz_d^*, S^d V \otimes Syz_d^*\big)^G = 1.$$

Hence, it only remains to prove that $\chi(S^d V \otimes Syz_d^*, \mathcal{O})^G \cong \chi(S^d V^* \otimes Syz_d)^G = 1$. Tensoring by $S^d V^*$ the sequence defining Syz_d we get

$$0 \to \mathcal{O}(-d) \otimes S^d V^* \to \mathcal{O} \otimes S^d V \otimes S^d V^* \to S^d V^* \otimes Syz_d \to 0.$$

Clearly

$$H^i\big(\mathcal{O} \otimes S^d V \otimes S^d V^*\big) = 0 \quad \text{for } i = 1, 2,$$
$$H^0\big(\mathcal{O} \otimes S^d V \otimes S^d V^*\big) \cong S^d V \otimes S^d V^*,$$
$$H^j\big(\mathcal{O}(-d) \otimes S^d V^*\big) = 0 \quad \text{for } i = 0, 1,$$

and since $d \geqslant 3$, by Serre's duality

$$H^2\big(\mathcal{O}(-d) \otimes S^d V^*\big) \cong H^0\big(\mathcal{O}(d-3) \otimes S^d V\big)^* \cong S^{d-3} V \otimes S^d V^*.$$

Hence, since by the Littlewood–Richardson rule, for any $d \geqslant 3$, the $SL(V)$-module $S^{d-3} V \otimes S^d V^*$ does not contain $\mathbb{C}$ and $S^d V \otimes S^d V^*$ contains one copy of $\mathbb{C}$, we obtain

$$H^2\big(\mathcal{O}(-d) \otimes S^d V^*\big)^G = 0 \quad \text{and} \quad \dim H^0\big(\mathcal{O} \otimes S^d V \otimes S^d V^*\big)^G = 1.$$

Then, we conclude that $\chi(\mathcal{O}(-d)\otimes S^dV^*)^G=0$ and $\chi(\mathcal{O}\otimes S^dV\otimes S^dV^*)^G=1$ and this implies

$$\chi\big(S^dV\otimes Syz_d^*,\mathcal{O}\big)^G=1$$

which concludes our proof. □

Remark 4.3. The same kind of computations as in the proof of the previous theorem allows us to show that

$$\operatorname{Ext}^1(E_d,E_d)\cong S^dV\otimes Ad\big(S^dV\big)\otimes S^{d-3}V$$

and for this reason E_d is not rigid (hence not exceptional) as soon as $d\geqslant 3$. Nevertheless $\operatorname{Ext}^1(E_d,E_d)$ as an $SL(V)$-module does not contain any summand isomorphic to $\mathbb{C}$ and so we have $\operatorname{Ext}^1(E_d,E_d)^G=0$.

The following technical lemma will be useful to prove the G-exceptionality of any Fibonacci bundle on $\mathbb{P}^2$.

Lemma 4.4. *For any $k\geqslant 1$, let C_k be a Fibonacci bundle on $\mathbb{P}^2$. Then the following holds*:

(i) $\chi(C_k,C_k)^G=1$,
(ii) $\chi(C_k\otimes S^dV,C_{k-1})^G=0$ *for k odd*, $\chi(C_k\otimes S^dV^*,C_{k-1})^G=0$ *for k even*,
(iii) $\chi(C_{k-1},C_k\otimes S^dV)^G=1$ *for k odd*, $\chi(C_{k-1},C_k\otimes S^dV^*)^G=1$ *for k even.*

Proof. We will prove it by induction on k. Recall that $C_0=\mathcal{O}(-d)$, $C_1=\mathcal{O}$, $C_2=Syz_d$ and $C_3=E_d^*$. It is easy to check directly that the relations (i)–(iii) hold for $k=1,2$.

Now assume that the relations hold for C_h and C_{h-1} with $h\leqslant k$. Assume k odd, then the Fibonacci bundle C_{k+1} is defined by the exact sequence:

$$0\to C_{k-1}\to C_k\otimes S^dV\to C_{k+1}\to 0. \tag{4.1}$$

Applying the functor $\operatorname{Hom}(C_k\otimes S^dV,-)$ to this sequence we get

$$\chi\big(C_k\otimes S^dV,C_{k+1}\big)^G=\chi\big(C_k\otimes S^dV,C_k\otimes S^dV\big)^G-\chi\big(C_k\otimes S^dV,C_{k-1}\big)^G$$

and by induction hypotheses (i) and (ii) we get

$$\chi\big(C_k\otimes S^dV,C_{k+1}\big)^G=1-0=1,$$

that is condition (iii) in case $k+1$ (even). Applying now the functor $\operatorname{Hom}(-,C_k\otimes S^dV)$ to the sequence (4.1) we get

$$\chi\big(C_{k+1},C_k\otimes S^dV\big)^G=\chi\big(C_k\otimes S^dV,C_k\otimes S^dV\big)^G-\chi\big(C_{k-1},C_k\otimes S^dV\big)^G.$$

Since by hypothesis of induction

$$\chi\big(C_k\otimes S^dV,C_k\otimes S^dV\big)^G=1\quad\text{and}\quad\chi\big(C_{k-1},C_k\otimes S^dV\big)^G=1,$$

we get

$$\chi\big(C_{k+1},C_k\otimes S^dV\big)^G=0$$

which proves condition (ii) in case $k+1$ (even). Let us apply now $\mathrm{Hom}(C_{k-1}, -)$ to the same sequence (4.1) and we get

$$\chi(C_{k-1}, C_{k+1})^G = \chi\left(C_{k-1}, C_k \otimes S^d V\right)^G - \chi(C_{k-1}, C_{k-1})^G = 1 - 1 = 0$$

where we have used once again hypothesis of induction.

Finally applying the functor $\mathrm{Hom}(-, C_{k+1})$ we get

$$\chi(C_{k+1}, C_{k+1})^G = \chi\left(C_k \otimes S^d V, C_{k+1}\right)^G - \chi(C_{k-1}, C_{k+1})^G = 1 - 0$$

and this proves equality (i) in case $k+1$. The case k even is analogous. $\square$

The following result proves that any Fibonacci bundle C_k on $\mathbb{P}^2$ is G-exceptional.

Theorem 4.5. *For any $k \geqslant 1$, let C_k be a Fibonacci bundle on $\mathbb{P}^2$. Then the following holds*:

(i) $\hom(C_k, C_k)^G = 1$, $\mathrm{ext}^i(C_k, C_k)^G = 0$ *for* $i = 1, 2$,
(ii) $\mathrm{ext}^2(C_k \otimes W_k, C_{k-1})^G = 0$,
(iii) $\hom(C_{k-1}, C_k \otimes W_k)^G = 1$, $\mathrm{ext}^i(C_{k-1}, C_k \otimes W_k)^G = 0$, *for* $i = 1, 2$,

where $W_k \cong S^d V$ if k is odd and $W_k \cong S^d V^*$ if k is even.

Proof. The proof is by induction on k. If $k = 1, 2$ it is easy to check directly the statements.

Now assume that the relations hold for C_h and C_{h-1} with $h \leqslant k$. Assume k odd, let C_{k+1} be the Fibonacci bundle defined by the exact sequence:

$$0 \to C_{k-1} \to C_k \otimes S^d V \to C_{k+1} \to 0.$$

Applying the functor $\mathrm{Hom}(C_k \otimes S^d V, -)$ to this sequence and using induction hypotheses (i) and (ii) we get $\mathrm{ext}^1(C_k \otimes S^d V, C_{k+1})^G = 0$ and $\mathrm{ext}^2(C_k \otimes S^d V, C_{k+1})^G = 0$. Since by Lemma 4.4 we know that $\chi(C_k \otimes S^d V, C_{k+1})^G = 1$, it follows that $\hom(C_k \otimes S^d V, C_{k+1})^G = 1$. Thus we obtain the statement (iii) in case $k+1$.

Applying now the functor $\mathrm{Hom}(-, C_k \otimes S^d V)$ to the same sequence we get

$$\mathrm{Ext}^1\left(C_{k-1}, C_k \otimes S^d V\right)^G \to \mathrm{Ext}^2\left(C_{k+1}, C_k \otimes S^d V\right)^G \to \mathrm{Ext}^2\left(C_k \otimes S^d V, C_k \otimes S^d V\right)^G$$

and the statement (ii) in case $k+1$ immediately follows by the assumptions (i) and (iii). Applying $\mathrm{Hom}(C_{k-1}, -)$ to the sequence, we get $\hom(C_{k-1}, C_{k+1})^G = 0$ and $\mathrm{ext}^i(C_{k-1}, C_{k+1})^G = 0$ for $i = 1, 2$.

Finally applying the functor $\mathrm{Hom}(-, C_{k+1})$ and using condition (iii) we obtain equality (i) in case $k+1$. The case k even is analogous. $\square$

From Theorem 4.5 we immediately get that

Corollary 4.6. *For any $k \geqslant 1$, the Fibonacci bundle C_k on $\mathbb{P}^2$ is G-exceptional.*

5. Stability of the almost square bundles

The main goal of this section is to prove that any almost square bundle E_d on $\mathbb{P}^2$ is stable. As a key ingredient, we will use the fact that we are able to describe exactly the representation of the quiver $(\mathcal{Q}_{\mathbb{P}^2}, \mathcal{R}_{\mathbb{P}^2})$ associated to the homogeneous bundle E_d. Indeed we have:

Theorem 5.1. *The representation of the quiver $(\mathcal{Q}_{\mathbb{P}^2}, \mathcal{R}_{\mathbb{P}^2})$ associated to any almost square bundle E_d on $\mathbb{P}^2$ is of type R_d.*

Proof. Let R' be the representation associated to E_d given by the correspondence stated in Definition 2.9. By Lemma 3.7, the graded vector bundle associated to $S^dV \otimes Syz_d^*$ is

$$\mathrm{gr}\big(S^dV \otimes Syz_d^*\big) = \bigoplus_{j=1}^{d}\bigoplus_{i=0}^{d}\left(\bigoplus_{k=0}^{\min(i,j)} S^{i+j-2k}(k+i-2j)\right).$$

Thus, it is easily seen that the support with multiplicities of R' is equal to the support with multiplicities of a representation of type R_d. Let us adopt for the arrows and the vector spaces of R' the same notation as in Definition 3.8.

Now we will show that the maps of the representation R' must verify all the properties (3.5), (3.6), (3.7). In order to do this we will show that if the maps of R' do not satisfy one of these conditions, then there exists a nontrivial subrepresentation of R', which is also a quotient representation of R'. This will imply that such representation is a direct sum of R' and so, the vector bundle E_d splits, and this contradicts the simplicity of E_d.

Assume first that R' does not satisfy property (3.7). In that case, we can consider a subrepresentation which has multiplicity 1 at any vertex of the support of R' and all the maps different from zero. Indeed, it is enough to take at the vertex $(d,2)$ a 1-dimensional subspace containing the image of $\varphi^d_{d,1} \oplus \psi^d_{d+1,2}$, and then restrict all the vector spaces at the following vertices to the corresponding images. By the commutativity of the diagram, we get everywhere 1-dimensional spaces. It is easy to see that such subrepresentation is also a quotient representation, and we are done.

Assume now that a map $\chi: V \to W$ of R' does not have maximal rank, thus contradicting property (3.5). Assume $\dim V \leqslant \dim W$. Then if the map χ is not injective, we can take a subrepresentation supported at $0 \neq \ker(\chi) \subset V$ and we consequently restrict all the vector spaces of the support of R' to the corresponding images and preimages with respect to all the maps. By the commutativity of the diagram we get a nontrivial subrepresentation, which is also a quotient representation, that is a direct summand of R' and we get a contradiction as above. Assume now that $\dim V \geqslant \dim W$. If the map χ is not surjective, we can take a subrepresentation supported at V and at $0 \neq \mathrm{Im}(\chi) \subset W$. Restricting all the other spaces to the corresponding images and preimages, we conclude as above.

From property (3.5) it follows immediately that the property (3.6) holds for any composition of maps, except possibly for the compositions χ_j of the following form:

$$\chi_j := \psi^d_{j,j} \circ \cdots \circ \psi^d_{d,j} \circ \psi^d_{d+1,j} : U^d_{d+1,j} \to U^d_{j-1,j} \quad \text{for some } 2 \leqslant j \leqslant d, \text{ or} \tag{5.1}$$

$$\chi_{d+1} := \psi^d_{d+1,d+1} \circ \psi^d_{d,d+1} : U^d_{d+1,d+1} \to U^d_{d-1,d+1}. \tag{5.2}$$

Assume then that χ_j is not injective, for some $2 \leqslant j \leqslant d+1$. Then we can consider a subrepresentation of R' supported at $0 \neq \ker(\chi) \subset U^d_{d+1,j}$. By consequently restricting all the vector spaces of the support of R' to the images and to the preimages with respect to all the maps, we will obtain a subrepresentation, which in particular, by the commutativity of the diagram, has multiplicity 0 at the vertex $(j-1,j)$ for $2 \leqslant j \leqslant d$ and at the vertex $(d,d+1)$ for $j = d+1$. Moreover such a subrepresentation is also a quotient representation, and this concludes the proof. □

The next basic lemma characterize the subrepresentations of a representation R_d.

Lemma 5.2. *Let $\{b_{i,j}\}$ be a collection of integers for $1 \leqslant i, j \leqslant d+1$ such that: $b_{i,j} \leqslant a^d_{i,j} = \dim U^d_{i,j}$. Then there exists a subrepresentation of R_d whose support has multiplicities $\{b_{i,j}\}$ if and only if the following conditions hold*:

$$b_{d,2} \geqslant b_{d,1} + b_{d+1,2} \tag{5.3}$$

and for any (i,j) *we have*

$$b_{i,j} \leqslant b_{i,j+1} \quad \textit{and} \quad b_{i,j} \leqslant b_{i-1,j} + 1. \tag{5.4}$$

Moreover the equality $b_{i,j} = b_{i-1,j} + 1$ *is possible only in the following cases*:

(i) $i < j$, $(i,j) \neq (d, d+1)$,
(ii) $i = j \leqslant d-1$, *if* $b_{d+1,j} \leqslant b_{i-1,j}$,
(iii) $(i,j) = (d,d)$, *if* $b_{d+1,k} \leqslant b_{d-1,d}$, *for* $k = d, d+1$,
(iv) $(i,j) = (d, d+1)$, *if* $b_{d+1,d+1} \leqslant b_{d-1,d+1}$.

Proof. We first prove that the conditions listed in the statement are necessary. Assume that $\{V_{i,j}\}$ is the support of a subrepresentation of R_d and set $b_{i,j} = \dim V_{i,j}$.

By definition the representation R_d satisfies condition (3.5). In particular the horizontal maps $\varphi^d_{i,j} : U^d_{i,j} \to U^d_{i,j+1}$ are injective. The vertical maps $\psi^d_{i,j} : U^d_{i,j} \to U^d_{i-1,j}$ are injective if $i > j$ or if $(i,j) = (d+1, d+1)$, while if $i \leqslant j$ and $i \neq d+1$ we have $\dim\ker(\psi^d_{i,j}) = 1$. It follows immediately that the conditions (5.4) hold. Moreover (5.3) follows from the property (3.7).

Assume now that $b_{i,j} = b_{i-1,j} + 1$. Clearly, since the space $V_{i,j}$ contains $\ker(\psi^d_{i,j}) \neq 0$, we have $i \leqslant j$ and $i \neq d+1$. From the property (3.6) it follows that the maps χ_j defined by (5.1) and (5.2) are injective. Hence it immediately follows that if $i = j \leqslant d-1$, then we have $\dim V_{d+1,j} \leqslant \dim V_{i-1,j}$, namely we have (ii). Analogously we prove case (iv). In order to check (iii), we also note that the maps $\varphi^d_{d+1,d}$ and $\varphi^d_{d,d}$ are surjective and, since the diagram is commutative, then we have $\dim V_{d+1,k} \leqslant \dim V_{d-1,d}$, for $k = d, d+1$.

Now we need to check that the conditions above are also sufficient. Assume that $\{b_{i,j}\}$ is a collection of integers as above. Then it is easy to see that there exists a subrepresentation whose support has multiplicities $(b_{i,j})$. Indeed, starting from the vertex $(1, d+1)$ we can choose a subspace $V_{1,d+1}$ of $U^d_{1,d+1}$ of dimension $b_{1,d+1}$. Then we choose $V_{2,d+1}$ and $V_{1,d}$, such that their images are contained in $V_{1,d+1}$, and using the commutativity of the diagram we can go on and choose all the other subspaces $V_{i,j} \subseteq U^d_{i,j}$ of dimension $b_{i,j}$. The conditions on the integers $b_{i,j}$ allow us to choose these spaces $V_{i,j}$ for any i, j such that

$$\varphi_{i,j}(V_{i,j}) \subseteq V_{i,j+1} \quad \text{and} \quad \psi_{i,j}(V_{i,j}) \subseteq V_{i-1,j} + 1,$$

and

$$\ker\bigl(\psi^d_{i,j}\bigr) \subseteq V_{i,j}$$

whenever $b_{i,j} = b_{i-1,j} + 1$. This clearly implies that the collection $\{V_{i,j}\}$ can be the support of a subrepresentation of R_d. $\square$

Now we are going to state some definitions and prove some technical lemmas that we will needed later on.

Definition 5.3. Let S_d be the representation of $(\mathcal{Q}_{\mathbb{P}^2}, \mathcal{R}_{\mathbb{P}^2})$ such that the support of S_d is the support of R_d with all multiplicities equal to one and all the maps are nonzero constants (and thus equal to one). For any $2 \leqslant k \leqslant d$, let P^k_d be the representation of $(\mathcal{Q}_{\mathbb{P}^2}, \mathcal{R}_{\mathbb{P}^2})$ such that the support of P^k_d is the support of S_d with the following multiplicities. For any vertex (i,j):

$$m(i,j) = \begin{cases} 1, & i = 1, \\ 1, & j = 1, \\ 1, & i = d+1,\ 2 \leqslant j \leqslant k, \\ 2 & \text{elsewhere}. \end{cases}$$

Remark 5.4.

(a) It follows from [15, Lemma 40] that the vector bundle F_d associated to S_d is multistable, that is, that any subrepresentation of S_d has slope less than the slope of S_d.
(b) By [15, Remark 23 and the Four Terms Lemma], the vector bundle F_d can be seen as the kernel of the natural projection

$$S^{2d,d}V \otimes \mathcal{O} \xrightarrow{\pi} S^d Q(d)$$

where $S^{2d,d}V$ is an irreducible Schur representation (see for instance [7]).

With the above notations

Lemma 5.5. *For any integer $d > 0$, the following holds*:

(a) $\mathrm{rk}(R_d) = \binom{d+2}{2}^2 - \binom{d+2}{2} - 1$ *and* $c_1(R_d) = -\frac{d(d+1)(d+2)}{2}$.
(b) $\mathrm{rk}(S_d) = d(d+1)(d+2)$ *and* $c_1(S_d) = -\frac{3}{2}d(d+1)$.
(c) $\mathrm{rk}(P_d^k) = d((d+1)(d+2) + \frac{(d-1)(2d+1)}{2}) + \frac{(d-k)(d-k+1)}{2}$ *and*

$$c_1(P_d) = -\frac{3d(d+1)}{2} - \frac{d(d-1)(d+1)}{2} + \frac{(d-k)(d-k+1)(d-k-1)}{2}.$$

(d) $\mu(R_{d-1}) < \mu(R_d) < \mu(S_d)$.
(e) $\mu(P_d^k) < \mu(R_d)$ *for $d \geqslant 2$ and $2 \leqslant k \leqslant d$.*

Proof. Once we have proved (a)–(c), the items (d) and (e) follow after a straightforward computation, keeping in mind that, by definition, given a representation R we have $\mu(R) = \frac{c_1(R)}{\mathrm{rk}(R)}$. So, we will prove the first three items and we left the proof of the remaining to the reader.

(a) By Theorem 5.1 we have $\mathrm{rk}(R_d) = \mathrm{rk}(E_d)$ and $c_1(R_d) = c_1(E_d)$. Recall that the vector bundle E_d is given by the short exact sequence

$$0 \to E_d \to S^d V \otimes Syz_d^* \to \mathcal{O} \to 0 \tag{5.5}$$

where Syz_d is the syzygy bundle on $\mathbb{P}^2$ defined by the exact sequence

$$0 \to \mathcal{O}(-d) \to S^d V \otimes \mathcal{O} \to Syz_d \to 0. \tag{5.6}$$

From the exact sequence (5.6) we get that

$$\mathrm{rk}(Syz_d) = \binom{d+2}{2} - 1 \quad \text{and} \quad c_1(Syz_d) = d.$$

Using this equalities together with the exact sequence (5.5) we obtain:

$$\mathrm{rk}(R_d) = \binom{d+2}{2}\mathrm{rk}\big(Syz_d^*\big) - 1 = \binom{d+2}{2}^2 - \binom{d+2}{2} - 1$$

and

$$c_1(R_d) = \binom{d+2}{2}c_1\big(Syz_d^*\big) = -d\frac{(d+1)(d+2)}{2}.$$

(b) First of all notice that

$$R_d = R_{d-1} \sqcup S_d$$

from which we deduce that

$$\mathrm{rk}(S_d) = \mathrm{rk}(R_d) - \mathrm{rk}(R_{d-1}) \quad \text{and} \quad c_1(S_d) = c_1(R_d) - c_1(R_{d-1}).$$

Thus, using (a) we have

$$\mathrm{rk}(S_d) = \binom{d+2}{2}^2 - \binom{d+2}{2} - 1 - \binom{d+1}{2}^2 + \binom{d+1}{2} + 1 = d(d+1)(d+2)$$

and

$$c_1(S_d) = -d\frac{(d+1)(d+2)}{2} + (d-1)\frac{(d+1)(d)}{2} = -\frac{3}{2}d(d+1).$$

(c) Let R be the rectangle of base $d-1$, height $d-2$ and $Q(-2)$ as the highest vertex of the left-hand side and let R' be the rectangle of base $d-k-1$, height 0 and $\mathcal{O}$ as the highest vertex of the left-hand side (i.e. R' is the left-hand side of the first row of R_d of length $d-k-1$). By construction

$$P_d^k = S_d \sqcup R \sqcup R'$$

and therefore

$$rk\big(P_d^k\big) = rk(S_d) + rk(R) + rk(R'), \tag{5.7}$$

$$c_1\big(P_d^k\big) = c_1(S_d) + c_1(R) + c_1(R'). \tag{5.8}$$

By [15, Lemma 28],

$$rk(R) = d(d-1)\left(d + \frac{1}{2}\right) \quad \text{and} \quad c_1(R) = \frac{-d(d-1)(d+1)}{2},$$

$$rk(R') = \frac{(d-k)(d-k+1)}{2} \quad \text{and} \quad c_1(R) = \frac{(d-k)(d-k+1)(d-k-1)}{2}.$$

Now, we conclude by substituting these equalities together with (b) in (5.8) and in (5.7). □

Lemma 5.6. *For any integer d and any proper subrepresentation G of S_d, the following inequality holds*

$$\mu(G) < \mu(R_d).$$

Proof. Denote by p and q the vertex corresponding to $S^{d-1}Q(d-1)$ and $S^{d+1}Q(d-2)$ respectively and we will denote by the same letter the corresponding representation with one vertex of multiplicity one. By Lemma 5.5,

$$\mu(S_d) = -\frac{3}{2}\frac{d(d+1)}{d(d+1)(d+2)}.$$

Therefore, since $\mathrm{rk}(S^jQ(l)) = j+1$ and $c_1(S^jQ(l)) = \frac{(2l+j)(j+1)}{2}$ we get

$$\mu(S_d \setminus p) = \frac{-3d^2}{d(d^2+3d+1)},$$
$$\mu(S_d \setminus q) = \frac{-3(d^2+d-1)}{(d+2)(d^2+d-1)},$$
$$\mu(S_d \setminus (p \sqcup q)) = -\frac{3}{2}\frac{(3d^2+d-2)}{(d^3+3d^2-2)}$$

and from these equalities together with Lemma 5.5 (1) it is easy to see that

$$\mu(S_d \setminus p) < \mu(R_d), \qquad \mu(S_d \setminus q) < \mu(R_d) \quad \text{and} \quad \mu\big(S_d \setminus (p \sqcup q)\big) < \mu(R_d). \tag{5.9}$$

Now let G be a subrepresentation of S_d. If G contains p and q then $G = S_d$ and it is no a proper subrepresentation. If $G = S_d \setminus p$ or $G = S_d \setminus q$ or $G = S_d \setminus (p \sqcup q)$, then by (5.9) and Lemma 5.5(4) we get

$$\mu(G) < \mu(R_d) < \mu(S_d)$$

and we are done. If $G \subsetneq S_d \setminus (p \sqcup q)$, then the inequality

$$\mu(G) < \mu(R_d) < \mu(S_d)$$

follows from the fact that by [15, Theorem 36], $S_d \setminus (p \sqcup q)$ is stable and hence

$$\mu(G) < \mu\big(S_d \setminus (p \sqcup q)\big) < \mu(S_d). \qquad \square$$

Now we are ready to prove our main technical result.

Theorem 5.7. *Given any subrepresentation T of R_d, we have $\mu(T) < \mu(R_d)$.*

Proof. We will proceed by induction on $d \geqslant 1$. By definition, the representation R_1 is the following:

$$\underset{Q(-2)}{\circ} \longleftarrow \underset{S^2Q(-2)}{\circ}$$

Then the unique subrepresentation T of R_1 is given by the vertex corresponding to $Q(-2)$ and we immediately check that $\mu(T) = -\frac{3}{2} < \mu(R_1) = -\frac{3}{5}$.

Assume now that R_{d-1} satisfies the statement for $d \geqslant 2$ and we are going to prove that the same is true for R_d.

Now, let T be a subrepresentation of R_d and denote by $V_{i,j}$, for $1 \leqslant i, j \leqslant d+1$, the vector spaces where T is supported. We consider the following three cases A, B and C according to the shape of T.

Case A. There exists at least a pair (i, j), for $1 \leqslant i, j \leqslant d+1$ and $(i, j) \neq (d+1, 1)$, such that $V_{i,j} = 0$, and for any $2 \leqslant i \leqslant d+1$ and $1 \leqslant j \leqslant d+1$, if $V_{i,j} \neq 0$, then we have $V_{i-1,j} \neq 0$.

Let T_1 be a representation whose support with multiplicities is $T \cap S_d$ and with all nonzero maps.

Claim 1. *T_1 is a proper subrepresentation of S_d.*

Proof. Since all the multiplicities of vertices of S_d are one, it is enough to prove that if $V_{i,j} \neq 0$, then $V_{i-1,j} \neq 0$ and $V_{i,j+1} \neq 0$. But this is clear since T is a subrepresentation of R_d and we are under the hypotheses of case A. The fact that T_1 is proper is a direct consequence of the assumptions in this case. $\square$

Claim 2. *There exists a subrepresentation of R_{d-1} supported on $T_2 := T \setminus T_1$*

Proof. We denote by $b_{i,j}$ the multiplicities $m_{i,j}^{T_2}$ of T_2 at each vertex (i, j) Notice that

$$b_{i,j} \leqslant a_{i-1,j-1}^{d-1} = \dim U_{i-1,j-1}^{d-1} = \dim U_{i,j}^{d} - 1.$$

It is easy to check that all the other conditions of Lemma 5.2 are also satisfied. Hence, by Lemma 5.2 there exists a subrepresentation of R_{d-1} whose support with multiplicities is T_2. □

Since $T = T_1 \sqcup T_2$, if $\mu(T_1) \leqslant \mu(T_2)$, by Lemma 2.13 we get

$$\mu(T_1) \leqslant \mu(T) \leqslant \mu(T_2).$$

On the other hand, since by Claim 2, T_2 is a subrepresentation of R_{d-1} and by hypothesis of induction R_{d-1} is stable, we have

$$\mu(T_2) < \mu(R_{d-1}).$$

Thus,

$$\mu(T) < \mu(R_{d-1}) < \mu(R_d)$$

where the last inequality follows from Lemma 5.5(d).

Assume now $\mu(T_2) < \mu(T_1)$. Then, by Lemma 2.13, we have

$$\mu(T_2) < \mu(T) < \mu(T_1).$$

On the other hand, by Claim 1, T_1 is a proper subrepresentation of S_d. Thus, by Lemma 5.6

$$\mu(T) < \mu(T_1) < \mu(R_d)$$

and this finishes the case A.

Case B. There exists at least a pair (i, j), for $1 \leqslant i, j \leqslant d+1$ and $(i, j) \neq (d+1, 1)$, such that $V_{i,j} = 0$, and there exists at least a $V_{i,j} \neq 0$, such that $V_{i-1,j} = 0$.

We split this case in two further subcases:

Case B1. Assume $V_{1,j} = 0$ for all $1 \leqslant j \leqslant d+1$.

In that case, we prove the following claim.

Claim 3. *T is a subrepresentation of R_{d-1}.*

Proof. Indeed it is easy to check that $V_{i,1} = 0$ for any i and since $V_{1,2} = 0$ we also have $V_{d+1,2} = 0$. Moreover, if $\dim(V_{i,j}) = a_{i,j}^d$, then we would have $\dim(V_{1,j}) = 1$ which is a contradiction. So we have $\dim(V_{i,j}) < a_{i,j}^d$. Thus the support of T is contained in the support of R_{d-1}. Now it is easy to see that T is a subrepresentation of R_{d-1} by using Lemma 5.2. □

By hypothesis of induction R_{d-1} is stable, thus by Claim 3,

$$\mu(T) < \mu(R_{d-1}) < \mu(R_d)$$

where the last inequality follows form Lemma 5.5(d).

Case B2. Assume that there exists a $V_{1,j} \neq 0$.

Let i_0 be the maximal $i \geqslant 1$ such that $V_{i,1} \neq 0$ or let $i_0 = 1$ if for any i, $V_{i,1} = 0$. Let T_1 be the maximal staircase contained in $T \setminus (S_{d-i_0} \cap T)$. Clearly T_1 is a proper subrepresentation of S_d.

Claim 4. *There exists a subrepresentation of R_{d-1} whose support with multiplicities is $T_2 := T \setminus T_1$.*

Proof. Denote by $b_{i,j}$ the multiplicities of T_2. First of all notice that

$$b_{1,j} = 0 \quad \text{and} \quad b_{i,1} = 0$$

for any i, j. Indeed if $b_{1,j} = 1$, then it would implies that the staircase T_1 has multiplicity 0 at the vertex $(1, j)$. But it is easy to see that this would contradict the maximality of the staircase T_1. On the other hand, we also have

$$b_{d+1,2} = 0.$$

Indeed, if $b_{d+1,2} \neq 0$ then we would have in particular $\dim V_{d+1,2} = 1$ but this is impossible since we are under the assumptions of case B.

Assume now that $b_{i,j} = a^d_{i,j}$, which implies $\dim(V_{i,j}) = a^d_{i,j}$. Then we would have

$$\dim V_{hk} = a^d_{hk}$$

for any $1 \leqslant h \leqslant i$ and $j \leqslant k \leqslant d+1$. But this in particular implies that the vertex (i, j) is contained in the staircase T_1 and so $b_{i,j} = a^d_{i,j-1}$ which is a contradiction.

Thus, the support of T_2 is contained in the support of R_{d-1}. In addition, it can be easily checked that all the assumptions of Lemma 5.2 are satisfied. Hence there is a subrepresentation of R_{d-1} whose support with multiplicities is T_2 and Claim 4 is proved. $\square$

Since $T = T_1 \sqcup T_2$, with T_1 a proper subrepresentation of S_d and T_2 a subrepresentation of R_{d-1} we conclude with the same argument as in case A that

$$\mu(T) < \mu(R_d).$$

Case C. For any pair (i, j), for $1 \leqslant i, j \leqslant d+1$ and $(i, j) \neq (d+1, 1)$, we have $V_{i,j} \neq 0$.

First of all notice that $P^d_d \subset T$. Denote by $T_1 = T \cap P^1_d$. It is immediate to observe that there exists some k, $1 \leqslant k \leqslant d$ such that

$$T_1 = P^d_k.$$

Thus, by Lemma 5.5(e)

$$\mu(T_1) < \mu(R_d).$$

Claim 5. *There exists a subrepresentation of R_{d-1} whose support with multiplicities is $T_2 := T \setminus T_1$.*

Proof. It is clear that the support of T_2 is contained in the support of R_{d-1}. On the other hand, notice that since $T_1 = P^k_d$, then $\dim V_{d+1,j} = 1$, for any $2 \leqslant j \leqslant k+1$. Hence, it is easy to check that all the conditions in Lemma 5.2 are satisfied by the multiplicities of T_2. $\square$

Once again, since $T = T_1 \sqcup T_2$, with T_2 a subrepresentation of R_{d-1} and $\mu(T_1) < \mu(R_d)$, we conclude as in the above cases A and B. This concludes the proof of the theorem. $\square$

A first consequence of the previous theorem is that the properties of a representation of type R_d define a unique (up to isomorphism) representation. This implies, by Theorem 5.1, that any representation of type R_d is isomorphic to the representation associated to the almost square bundle E_d.

Proposition 5.8. *Any two representations R and R' of type R_d are isomorphic.*

Proof. Notice that, since the invariant Euler characteristic is a topological invariant and hence only depends on the support of the representation, we have

$$\chi(R, R')^G = \chi(R, R)^G = \chi(E_d, E_d)^G = 1$$

where the last equality follows from Theorem 4.2. Let us denote by E (resp. E') the homogeneous vector bundle associated to R (resp. R'). They have the same rank and Chern classes. Notice that E and E' are multistable bundles by Theorems 2.2 and 5.7. Hence, by Serre duality, we have

$$\mathrm{ext}^2(R, R')^G = \mathrm{ext}^2(E, E')^G = \mathrm{hom}(E', E(-3))^G = 0.$$

Then, it follows that $\mathrm{hom}(R, R')^G \geqslant 1$ and thus there exists a nontrivial morphism of representations $f : R \to R'$. This morphism must be an isomorphism since otherwise the subrepresentations $\ker(f)$ or $\mathrm{Im}(f)$ would contradict Theorem 5.7 for R or R'. □

Remark 5.9. The previous proposition also implies that the moduli space of homogeneous bundles containing an almost square bundle is a reduced point. For more details on the moduli problem of homogeneous bundles see [13], and [16, Section 7].

We are finally in a position to prove the main result of this section.

Theorem 5.10. *Any almost square bundle on $\mathbb{P}^2$ is stable.*

Proof. Since by Theorem 4.2 E_d is simple, it is enough to prove that it is multistable. By Proposition 5.8, we know that R_d is the representation of the quiver $\mathcal{Q}_{\mathbb{P}^2}$ associated to E_d, hence by Theorem 2.2 to prove that E_d is multistable it is enough to see that for any subrepresentation T of R_d, $\mu(T) < \mu(R_d)$. But this is true by Theorem 5.7 and this concludes the proof. □

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