Semi-Bounded Restrictions of Dirac Type Operators and the Unique Continuation Property

Christian Bär and Alexander Strohmaier

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Abstract

Let M be a connected Riemannian manifold and let D be a Dirac type operator acting on smooth compactly supported sections in a Hermitian vector bundle over M. Suppose D has a self-adjoint extension A in the Hilbert space of square-integrable sections. We show that any L^2-section φ contained in a closed A-invariant subspace onto which the restriction of A is semi-bounded has the unique continuation property: if φ vanishes on a non-empty open subset of M, then it vanishes on all of M.

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1 Introduction

There is a large class of unique continuation theorems for solutions to second order differential equations. For solutions to Dirac type equations on Riemannian manifolds there exists a number of methods to show the weak unique continuation property for solutions, i. e. a solution vanishing on a non-void open set vanishes identically ([4]).

The relativistic quantum theory of a single electron is described by the Hilbert space of 4-component spinors $\mathcal{H} := L^2(\mathbb{R}^3, \mathbb{C}^4)$ and the Dirac operator H_0 for mass $m > 0$, which is essentially self-adjoint on the dense subspace $C_c^\infty(\mathbb{R}^3, \mathbb{C}^4)$. If P_+ and P_- are the orthogonal projections onto the spectral subspaces for the positive and negative part of the spectrum of the Dirac operator, the physical state space of the theory is the space of rays in the Hilbert space $\mathcal{H}_+ := P_+\mathcal{H}$. The time evolution is given by the unitary group generated by the Dirac operator restricted to $\mathcal{H}_+$. For such a physical state $\psi \in \mathcal{H}_+$, $\|\psi\| = 1$, and any region $\mathcal{O} \subset \mathbb{R}^3$ the integral $\int_{\mathcal{O}} \langle \psi(x), \psi(x) \rangle d^3x$ is the probability to find the electron in $\mathcal{O}$. It is known for the case of the free Dirac operator that functions in $\mathcal{H}_+$ have the weak unique continuation property, see [13, Corollary 1.7]. Physically this means, that one cannot localize a single electron in a bounded region. In the fully quantized theory of the electron field this property has important consequences. The reason for this is that the quantization procedure requires the splitting of $\mathcal{H}$ into a direct sum $\mathcal{H}_+ \oplus \mathcal{H}_-$, see [13, Section 10.1] for details. The weak continuation property implies however that $P_+ C_c^\infty(\mathcal{O}, \mathbb{C}^4)$ is already dense in $\mathcal{H}_+$ for any non-void open set $\mathcal{O} \subset \mathbb{R}^3$. Hence the local information is destroyed by the splitting. As a consequence the vacuum vector turns out to be cyclic for the local field algebras. This Reeh-Schlieder property of the vacuum is a general property of quantum field theories in Minkowski space-time and can be proved in the Wightman framework even for interacting fields ([9]).

Replacing $\mathbb{R}^3$ by an arbitrary Riemannian manifold (M, g) it is natural to ask whether the weak unique continuation property still holds not only for solutions to Dirac type equations but also, as in the case of $\mathbb{R}^3$, for elements of certain spectral subspaces. In this note we will prove that such a property holds for a large class of generalized Dirac operators on an arbitrary connected Riemannian manifold. The physical interpretation is that single electrons cannot be localized even in curved space and the corresponding Dirac quantum field has the Reeh-Schlieder property. Our class of operators is large enough to allow coupling to arbitrary smooth Hermitian potentials. Note that in this situation only the space is curved which in a relativistic framework would correspond to an ultrastatic space-time, i. e. the space-time of the form $\mathbb{R} \times M$ with metric tensor $dt^2 - g$. There already exist results on the Reeh-Schlieder property for the free scalar field over such space-times ([14]) and recently also for quite general free quantum fields over space-times admitting a time-like Killing vector field ([12]). See also [10] for the connection of unique continuation theorems with the Reeh-Schlieder property within non-relativistic quantum mechanics.

The weak unique continuation property for elements of spectral subspaces provides a particularly straightforward way to understand the localization problem for single particles and the Reeh-Schlieder property for the corresponding quantum fields over ultrastatic space-times. Independently such theorems play an important role in Riemannian geometry since splittings of the type $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ also occur in an essential manner in boundary value problems for Dirac equations ([2]). This note can be considered a continuation of [3] since we are now able to remove the assumption in [3] that the underlying manifold be compact.

2 Notation and Result

Let (M, g) be a Riemannian manifold, let $E \to M$ be a Hermitian vector bundle over M. Denote the Hermitian metric on E by $\langle \cdot, \cdot \rangle$. The space of smooth sections in E will be denoted $C^\infty(M, E)$ and similar notation will be used for compactly supported smooth sections and for square-integrable sections. Let

$$D : C^\infty(M, E) \to C^\infty(M, E)$$

be a formally self-adjoint differential operator of first order. We call D of *Dirac type* if its principal symbol σ_D satisfies the Clifford relations, i. e.

$$\sigma_D(\xi) \circ \sigma_D(\eta) + \sigma_D(\eta) \circ \sigma_D(\xi) = -2g(\xi, \eta) \cdot \mathrm{Id}_{E_p}$$

for all $\xi, \eta \in T_p^*M$, $p \in M$. Then D is an elliptic differential operator of first order.

Example 1 Let $M = S^1$ be the circle and let E be the trivial complex line bundle over M. Then $D = i\frac{d}{dt}$ is of Dirac type. Fourier expansion of complex valued functions is nothing but the eigenvector expansion for D.

Example 2 More generally, any generalized Dirac operator in the sense of Gromov and Lawson ([7]) is of Dirac type.

Example 3 Let $M = \mathbb{R}^3$ and let D be the Dirac operator in an external field, $D = H = H_0 + V$ in the terminology of [13, Section 4]. Here the potential V to be added to the free Dirac operator H_0 may be any smooth function V with values in Hermitian matrices. Examples include electromagnetic vector potentials, and electric and magnetic anomalous moments.

Example 4 Since the underlying manifold M need not necessarily be complete we can also deal with singular potentials. Let us demonstrate this for the *Coulomb potential*. Put $M = \mathbb{R}^3 - \{0\}$, $V(x) := \frac{\gamma}{|x|}\,\mathrm{Id}$, and $D = H = H_0 + V$ as in the previous example. It is known ([13, Section 4]) that if the coupling constant γ is not too large, $|\gamma| < const$, then D is essentially self-adjoint on $C_c^\infty(M, \mathbb{C}^4)$. But even for larger coupling constant for which essential self-adjointness is known to break down there still exist self-adjoint extensions and our theorem can be applied.

Example 5 Let (M, g), E, and D be as above with D of Dirac type. Put $X := M \times \mathbb{R}$ with Riemannian metric $ds^2 = g + dt^2$ where t is the standard coordinate on $\mathbb{R}$. Hence X is the *cylinder* over M. The pull-back of E to X is again denoted by E. Define the Hermitian vector bundle $\mathcal{E} := E \oplus E$ over X. Then

$$\mathcal{D} := \begin{pmatrix} D & \frac{\partial}{\partial t} \\ -\frac{\partial}{\partial t} & -D \end{pmatrix}$$

acting on sections in $\mathcal{E}$ is of Dirac type over X.

Now let D be of Dirac type over the connected Riemannian manifold M. Let φ be in the kernel of D. By elliptic regularity theory φ is necessarily smooth. It is well-known that D has the *unique continuation property* meaning that if φ vanishes on a non-empty open subset $\mathcal{O} \subset M$, then φ vanishes everywhere, see e. g. [4, 5]. Of course this is also true for eigensections of D to other eigenvalues $\lambda \neq 0$ since one can simply replace D by $D - \lambda\,\mathrm{Id}$. This unique continuation property of Dirac type operators can also be shown by passing to the square of D and applying the classical Aronszajn theorem ([1]) for Laplace type operators.

We show here that the assumption $D\varphi = 0$ can be relaxed considerably. Roughly speaking, it is enough that φ lies in a D-invariant subspace of the Hilbert space of square-integrable sections $L^2(M, E)$ on which D is semibounded.

For a subset $S \subset \mathbb{R}$ denote the characteristic function by χ_S,

$$\chi_S(\lambda) = \begin{cases} 1, & \lambda \in S \\ 0, & \lambda \in \mathbb{R} - S. \end{cases}$$

Our main result is

Theorem *Let M be a connected Riemannian manifold, let E be a Hermitian vector bundle over M, and let D be of Dirac type acting on $C_c^\infty(M, E)$. Suppose D has a self-adjoint extension in $L^2(M, E)$ which we denote by A. Let $\mathcal{O} \subset M$ be a non-empty open subset of M. Let $\varphi \in L^2(M, E)$ such that there exists $\lambda_0 \in \mathbb{R}$ with $\chi_{[\lambda_0,\infty)}(A)\varphi = 0$ or $\chi_{(-\infty,\lambda_0]}(A)\varphi = 0$.*

Then the following conclusion holds:

$$\varphi|_{\mathcal{O}} = 0 \Rightarrow \varphi = 0 \textit{ on all of } M.$$

Example 6 If φ is a square-integrable eigensection for D, then clearly, $\chi_{[\lambda_0,\infty)}(A)\phi = 0$ for any λ_0 larger than the eigenvalue and we recover the classical unique continuation property of D mentioned above.

Example 7 In case $M = S^1$ and $D = i\frac{d}{dt}$ we may apply the theorem to all $\varphi \in L^2(S^1, \mathbb{C})$ whose Fourier coefficients c_n vanish for all n smaller than some n_0, i. e. all φ of the form

$$\varphi(t) = \sum_{n \geq n_0} c_n e^{int}.$$

The theorem says that any such φ must be identically zero if it vanishes on a non-empty open subset of S^1.

Example 8 More generally, if M is compact and connected , then the spectrum of A is discrete. We may split $L^2(M, E) = \mathcal{H}_{\leq 0} \oplus \mathcal{H}_{>0}$ where $\mathcal{H}_{\leq 0}$ denotes the sum of all eigenspaces for the non-positive eigenvalues and similarly $\mathcal{H}_{>0}$ corresponds to the positive eigenvalues. This splitting is important to define the Atiyah-Patodi-Singer boundary conditions for boundary value problems for Dirac operators ([2]). By the theorem no non-trivial $\varphi \in \mathcal{H}_{\leq 0}$ or $\varphi \in \mathcal{H}_{>0}$ can vanish on a non-empty open subset of M.

Remark Note that in the theorem no assumption of essential self-adjointness of D on $C_c^\infty(M, E)$ is made. Dirac type operators on a *complete* manifold are always essentially self-adjoint on smooth compactly supported sections. The well-known proofs of this fact for generalized Dirac operators in sense of Gromov and Lawson ([7]) easily carry over to our more general class of operators, see also [6] for a different approach. However, we only need existence of self-adjoint extensions not their uniqueness. Hence in our case the manifold M need not necessarily be complete, compare Example 4. Also see [8] for examples with cone singularities to which our theorem can be applied.

It should also be emphasized that φ need not lie in the domain of definition of D or of A. Only square-integrability and $\chi_{[\lambda_0,\infty)}(A)\varphi = 0$ or $\chi_{(-\infty,\lambda_0]}(A)\varphi = 0$ are assumed.

Also note that our theorem shows that the spectrum of any Dirac type operator is indeed unbounded from above and from below. Otherwise, every section in $L^2(M, E)$ would have the unique continuation property which is absurd.

3 The Proof

In this section we give the proof of the theorem. We keep the notation of the theorem and we assume $\varphi|_{\mathcal{O}} = 0$. We wish to show $\varphi = 0$.

Without loss of generality we may assume that $\lambda_0 = 1$ and that $\chi_{(-\infty,1]}(A)\varphi = 0$. In other words, φ lies in the image of the projection $\chi_{(1,\infty)}(A)$. Denote the upper half plane by

$$\mathfrak{H} := \{\zeta \in \mathbb{C} \mid \Im(\zeta) > 0\}.$$

Step 1: The family of functions f_z,

$$f_z(\lambda) = \begin{cases} \lambda^{-1} e^{iz\lambda}, & \lambda \geq 1 \\ 0, & \lambda < 1 \end{cases},$$

is uniformly bounded by 1 for all $z \in \bar{\mathfrak{H}}$ and for each fixed λ it is continuous in z. Therefore the family of bounded operators $f_z(A)$ is continuous in $z \in \bar{\mathfrak{H}}$ in the strong operator topology. Hence $z \mapsto f_z(A)\varphi$ is a continuous $L^2(M, E)$-valued function on $\bar{\mathfrak{H}}$.

Now fix $z_0 \in \mathfrak{H}$. The family of functions g_z defined by

$$g_z(\lambda) := \begin{cases} \lambda^{-1} \frac{e^{iz\lambda} - e^{iz_0\lambda}}{z - z_0}, & z \neq z_0 \\ ie^{iz_0\lambda}, & z = z_0 \end{cases}$$

for $\lambda \geq 1$ and $g_z(\lambda) = 0$ for $\lambda < 1$ is also uniformly bounded and continuous in z. This shows

$$\lim_{z\to z_0} \frac{f_z(A)\varphi - f_{z_0}(A)\varphi}{z - z_0}$$

exists and hence $z \mapsto f_z(A)\varphi$ is holomorphic on $\mathfrak{H}$.

Step 2: Fix a non-empty relatively compact open subset $\tilde{\mathcal{O}} \subset\subset \mathcal{O}$. Let $u \in C_c^\infty(M,E)$ with support contained in $\tilde{\mathcal{O}}$. By finiteness of propagation speed, see Section 4, there exists $\varepsilon > 0$ such that the support of $e^{itA}u$ is contained in $\mathcal{O}$ for all $t \in [-\varepsilon, \varepsilon]$. Hence for all t with $|t| \leq \varepsilon$ we have

$$\begin{aligned} 0 &= (\varphi, e^{-itA}u) \\ &= (e^{itA}\varphi, u) \\ &= (Af_t(A)\varphi, u) \\ &= (f_t(A)\varphi, Au). \end{aligned}$$

Here we used that by assumption φ lies in the image of the projection $\chi_{(1,\infty)}(A)$ and hence $e^{izA}\varphi = Af_z(A)\varphi$. Since $z \mapsto (f_z(A)\varphi, Au)$ is continuous on $\bar{\mathfrak{H}}$ and holomorphic on $\mathfrak{H}$ and vanishes on $[-\varepsilon, \varepsilon]$ an application of the Schwarz reflection principle ([3, Lemma 2]) shows

$$(f_z(A)\varphi, Au) = 0$$

for all $z \in \bar{\mathfrak{H}}$. In particular for $z = it$ with $t > 0$ we obtain

$$(e^{-tA}\varphi, u) = (f_{it}(A)\varphi, Au) = 0$$

for all $u \in C_c^\infty(M,E)$ with support contained in $\tilde{\mathcal{O}}$. Hence $e^{-tA}\varphi$ vanishes on $\tilde{\mathcal{O}}$ for all $t > 0$.

Step 3: Now look at the half cylinder $X = M \times (0,\infty)$ with product metric $g + dt^2$, Hermitian vector bundle $\mathcal{E} := E \oplus E$ over X, and Dirac type operator

$$\mathcal{D} := \begin{pmatrix} D & \frac{\partial}{\partial t} \\ -\frac{\partial}{\partial t} & -D \end{pmatrix}$$

as in Example 5. We define a distributional section Φ_1 in E over X by

$$\Phi_1(v) := \int_0^\infty (e^{-tA}\varphi, v(\cdot,t))_{L^2(M,E)} dt$$

for all $v \in C_c^\infty(X,E)$. The differential equation

$$\frac{d}{dt} e^{-tA}\varphi = -Ae^{-tA}\varphi$$

shows that the distributional section

$$\Phi := \begin{pmatrix} \Phi_1 \\ \Phi_1 \end{pmatrix}$$

in $\mathcal{E}$ satisfies

$$\mathcal{D}\Phi = 0$$

in the distributional sense. In particular, Φ is smooth by elliptic regularity theory.

Since Φ vanishes on the open subset $\tilde{\mathcal{O}} \times (0, \infty)$ of X the standard unique continuation property of $\mathcal{D}$ implies $\Phi = 0$. Hence $\Phi_1 = 0$, i. e. $e^{-tA}\varphi = 0$ for all $t > 0$, and the limit $t \searrow 0$ yields $\varphi = 0$. $\square$

4 Finite Propagation Speed

In the second step of the proof we needed what is known as "finite propagation speed" for possibly incomplete manifolds. For the sake of completeness we give a justification for this by reducing it to the compact case. A direct approach can be found in [6].

For any closed subset A of M let $U_r(A)$ be the closed r-neighborhood of A, i. e.

$$U_r(A) = \{x \in M \mid \operatorname{dist}(x, A) \leq r\}.$$

Proposition *Let M be a Riemannian manifold, let E be a Hermitian vector bundle over M, and let D be of Dirac type acting on $C_c^\infty(M, E)$. Suppose D has a self-adjoint extension in $L^2(M, E)$ which we denote A. Let $\varphi \in C_c^\infty(M, E)$.*

Then there exists $\varepsilon > 0$ such that for all $t \in [0, \varepsilon)$

$$\varphi_t := e^{itA}\varphi$$

is smooth in all variables and

$$\operatorname{supp}(\varphi_t) \subset U_t(\operatorname{supp}(\varphi)).$$

If M is complete, then the statement is true for $\varepsilon = \infty$.

Proof. The proposition is well-known if the underlying manifold is compact, see e. g. [11, Prop. 5.5] for a proof. By assumption $\operatorname{supp}(\varphi)$ is compact. Hence there exists $\varepsilon > 0$ such that $U_\varepsilon(\operatorname{supp}(\varphi))$ is still compact. We pick a compact manifold Y containing an isometric image of $U_\varepsilon(\operatorname{supp}(\varphi))$ together with a Hermitian bundle and Dirac type operator extending E and D over $U_\varepsilon(\operatorname{supp}(\varphi))$. (This can e. g. be obtained by choosing an open subset $X \subset M$ containing $U_\varepsilon(\operatorname{supp}(\varphi))$ with compact closure $\bar{X}$ and smooth boundary ∂X. Then let Y be the double of X and smooth out all data along ∂X.) Since φ has support contained in $U_\varepsilon(\operatorname{supp}(\varphi))$ we can consider it as a section over Y by extending it by 0. Let φ_t denote the solution of

$$\frac{\partial}{\partial t}\varphi_t = iD\varphi_t$$

over Y with $\varphi_0 = \varphi$. Then φ_t is smooth in all variables and has support contained in $U_\varepsilon(\mathrm{supp}(\varphi))$ for $t \in [0, \varepsilon)$. By this last property we can consider φ_t as a smooth section over M, again extending by zero. Considered as elements of $L^2(M, E)$ we have

$$\frac{d}{dt}\varphi_t = iD\varphi_t = iA\varphi_t$$

and hence $\varphi_t = e^{itA}\varphi$.

In case M is complete ε can be chosen arbitrarily large. $\square$

5 Concluding Remarks

Our main argument makes use of the unique continuation theorem for solutions to Dirac type equations on Riemannian manifolds (on the half cylinder X). In order to use this theorem we had to analytically continue the functions $t \mapsto (f_t(A)\varphi, Au)$ (see Step 2 in the proof) to "imaginary time". This was possible due to semi-boundedness of the spectrum of the Dirac operator on the spectral subspaces under consideration. In quantum field theory this method is commonly referred to as "Wick rotation".

It should be noted that our theorem can be extended to even more general subspaces. Clearly analyticity of the function $(f_z(A)\varphi, Au)$ in the whole upper half plane is not necessary, but it suffices to have boundedness and analyticity in a strip of the form $\{\zeta \in \mathbb{C} \mid 0 < \Im(\zeta) < \beta\}$ for some $\beta > 0$. Such subspaces arise naturally in thermal quantum field theory.

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Fachbereich Mathematik
Universität Hamburg
Bundesstr. 55
20146 Hamburg
Germany

E-Mail: `baer@math.uni-hamburg.de`
WWW: `http://www.math.uni-hamburg.de/home/baer/`

and

Institut für theoretische Physik
Universität Leipzig
Augustusplatz 10/11

04109 Leipzig
Germany

E-Mail: alexander.strohmaier@itp.uni-leipzig.de
WWW: http://www.physik.uni-leipzig.de/˜strohmai/