ANALYTIC EXTENSION OF VECTOR VALUED FUNCTIONS

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Dedicated to the memory of Professor Béla Szőkefalvi-Nagy

Abstract. Let D be a plane domain and X a complex Banach space. It is well known that the analyticity of some $F : D \to X$ with respect to the norm of X follows from the analyticity of the compositions of F with all functionals belonging to a closed linear subspace $\mathcal{F}$ of the dual of X, which determines the norm of X. Thus, roughly speaking, scalar analyticity implies vector valued analyticity.

We investigate here, whether in reasonable situations scalar analytic extendibility implies vector valued analytic extendibility. More precisely, we describe situations in which an X-valued map on a Jordan arc Γ in the boundary of D, whose compositions with all functionals belonging to an $\mathcal{F}$ as above have analytic extensions on D satisfying weak continuity properties at Γ, have X-valued analytic extensions on D, continuous at Γ.

Introduction

The following situation frequently occurs in problems of Functional Analysis:

Given a simply connected plane domain D, a Jordan arc Γ on the boundary of D, no part of which is slice in D, and a map F from Γ in a complex Banach space X, such that all compositions of F with the functionals belonging to a linear subspace $\mathcal{F}$ of the dual of X, which determines the norm of X, have continuous extension to $D \cup \Gamma$, analytic in the interior, we look for the existence of an extension of F to a $\sigma(X, \mathcal{F})$ - continuous map $\tilde{F} : D \cup \Gamma \to X$, which is analytic in the interior.

In certain cases the above problem can be easily settled. For example, let H be a complex Hilbert space, $B(H)$ the Banach space of all bounded linear operators on H, and $F : (-a, a) \to B(H)$ a map such that, for every ξ, $\eta \in H$, the functions $(a, b) \ni t \mapsto (F(t)\xi \,|\, \eta) \in \mathbb{C}$ have continuous extensions $F_{\xi\,,\,\eta}$ on a rectangle $(-a, a) + i\,[0, b)$, which are analytic in the interior. Then there exists a weak operator continuous extension $\tilde{F} : (-a, a) + i\,[0, b) \to B(H)$ of F, which is analytic in the interior. Indeed, replacing a, b by a', b' with $a < a' < b' < b$, we can assume withlout loss of generality that the functions $F_{\xi\,,\,\eta}$ are bounded.

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Then the closed graph theorem entails the existence of a constant $c > 0$ such that $\|F_{\xi\,,\eta}\| \leq c\,\|\xi\|\,\|\eta\|$ and, for every $z \in (-a,a) + i\,[0,b)$, $\tilde{F}(z)$ is defined by the sesqui-linear form $(\xi\,,\,\eta) \mapsto F_{\xi\,,\eta}(z)$.

Not every case is so easy. For example, if $A \subset B(H)$ is a C^*-algebra, ξ a cyclic vector for A, and $\mathbb{R} \ni t \mapsto \alpha_t$ a strongly continuous one-parameter group of $*$-automorphisms of A, such that the "two-point functions" $\mathbb{R} \ni t \mapsto (x\,\alpha_t(y)\,\xi\,|\,\xi) \in \mathbb{C}$, $x, y \in A$ have bounded, continuous extensions on the strip $\{z \in \mathbb{C};\ 0 \leq \ \mathrm{Im}\ < 1\}$, which are analytic in the interior, then the derivation of the corresponding analytic extendibility properties of the vector valued maps $\mathbb{R} \ni t \mapsto \alpha_t(x)\xi \in H$, of interest for the investigation of the relativistic KMS-condition, is much more involved (see [D-Z 2]).

Our approach of the general problem has the following strategy. We approximate F with analytically extendable mollifications relative to the arc length measure of Γ, for which we need the local rectifiability of Γ and a Pettis integrability property of F, and then show that the values in any $z \in D$ of the analytic extensions of these mollifications are norm-convergent. In the case of Krein completeness of $\sigma(X,\mathcal{F})$ (the $\sigma(X,\mathcal{F})$ - closed convex hull of any $\sigma(X,\mathcal{F})$ - compact set is $\sigma(X,\mathcal{F})$ - compact), the Pettis integrability of $\sigma(X,\mathcal{F})$ - continuous maps is automatic, so in this situation the case of arbitrary Γ can be reduced to the rectifiable case by using conformal representations (which, in general, does not preserve rectifiability). The main results of the paper are the analytic extension criteria Theorems 5.1 and 5.4, supplemented by the continuity criterion at the boundary Theorem 4.3.

The authors wish to thank H.Helson for the reference [He], which enabled them to replace a weaker first version of Theorem 5.4 with the present one. They are grateful also to W.Arendt for calling their attention on the reference [Are-Ni].

1. Preliminaries on weak topologies

We call *dual pair of normed vector spaces* a pair $(X,\mathcal{F})$ of complex normed vector spaces together with a bilinear functional

$$X \times \mathcal{F} \ni (x,\varphi) \mapsto < x,\varphi > \in \mathbb{C}$$

satisfying

$$\|x\| = \sup\{|< x,\varphi >|\,;\ \varphi \in \mathcal{F},\ \|\varphi\| \leq 1\}, \quad x \in X,$$
$$\|\varphi\| = \sup\{|< x,\varphi >|\,;\ x \in X,\ \|x\| \leq 1\}, \quad \varphi \in \mathcal{F}.$$

We denote, as usual, by $\sigma(X,\mathcal{F})$ the weak topology defined on X by the linear functionals $X \ni x \mapsto < x,\varphi > \in \mathbb{C}$, $\varphi \in \mathcal{F}$. We recall that a map $F : \Omega \to X$, defined on a locally compact Hausdorff topological space Ω, is called Pettis integrable with respect to a Radon measure μ on Ω and the topology $\sigma(X,\mathcal{F})$ if all functions $\Omega \ni \omega \mapsto < F(\omega)\,,\varphi > \in \mathbb{C}$, $\varphi \in \mathcal{F}$ belong to $\mathcal{L}^1(\mu)$ and there exists a (necessarily unique) $x_{F,\mu} \in X$ such that

$$\int\limits_{\Omega} < F(\omega)\,,\varphi > \,d\mu(\omega) = < x_{F,\mu}\,,\varphi > \text{ for all } \varphi \in \mathcal{F}\,.$$

In this case $x_{F,\mu}$ is denoted by

$$\sigma(X,\mathcal{F}) - \int\limits_{\Omega} F(\omega)\,d\mu(\omega)$$

and it is called the Pettis integral of F with respect to μ and $\sigma(X,\mathcal{F})$.

Relevant examples of dual pairs of normed spaces will be given in 1.5.

We say that a dual pair of normed vector spaces $(X, \mathcal{F})$ has the

- *Banach-Steinhaus Property* if
$$S \subset X\,,\ \sup\{|<x,\varphi>|\,;\ x \in S\} < +\infty \text{ for all } \varphi \in \mathcal{F}$$
implies
$$\sup\{\|x\|\,;\ x \in S\} < +\infty\,;$$
- *Krein Property* if the $\sigma(X, \mathcal{F})$ - closed convex hull of every $\sigma(X, \mathcal{F})$ - compact subset of X is $\sigma(X, \mathcal{F})$ -compact.

Let us recall the basic criterion for the Banach-Steinhaus property and formulate an immediate consequence of it:

1.1. Proposition. *Let $(X, \mathcal{F})$ be a dual pair of normed spaces.*

(i) If the norm of $\mathcal{F}$ is complete, then $(X, \mathcal{F})$ has the Banach-Steinhaus Property.

(ii) If $(X, \mathcal{F})$ has the Banach-Steinhaus Property, $\Omega \subset \mathbb{R}^n$ is open and $F : \Omega \to X$ is such that $\Omega \ni \mathbf{t} \mapsto < F(\mathbf{t})\,,\, \varphi >$ has locally bounded partial derivatives for all $\varphi \in \mathcal{F}$, then F is locally Lipschitz (in particular, norm-continuous).

Proof. (i) is the classical uniform boundedness principle (see [Hi-P], Theorem 2.8.6).

For (ii) let $\mathbf{a} \in \Omega$ and $\delta > 0$ be such that Ω contains the (compact) closure of $U_\delta(\mathbf{a}) = \{\mathbf{t} \in \Omega\,;\ \|\mathbf{t} - \mathbf{a}\| < \delta\}$. By the assumption on F, the function $U_\delta(\mathbf{a}) \ni \mathbf{t} \mapsto F_\varphi(\mathbf{t}) = < F(\mathbf{t})\,,\, \varphi >$ has bounded partial derivatives for all $\varphi \in \mathcal{F}$.

Now, denoting by $\mathbf{e_1}, ..., \mathbf{e_n}$ the canonical basis of $\mathbb{R}^n$ and using the mean value theorem, we get for all $\mathbf{t} \neq \mathbf{t}'$ in $U_\delta(\mathbf{a})$ and all $\varphi \in \mathcal{F}$:

$$\begin{aligned}
&\left| \operatorname{Re} \ < \frac{1}{\|\mathbf{t}-\mathbf{t}'\|} \left(F(\mathbf{t}) - F(\mathbf{t}')\right), \varphi > \right| = \\
&= \frac{1}{\|\mathbf{t}-\mathbf{t}'\|} \left| \operatorname{Re} \ (F_\varphi(\mathbf{t}) - F_\varphi(\mathbf{t}')) \right| = \\
&= \frac{1}{\|\mathbf{t}-\mathbf{t}'\|} \left| \operatorname{Re} \sum_{k=1}^{n} \left(F_\varphi\left(\mathbf{t} + \sum_{j=1}^{k-1} (t'_j - t_j)\mathbf{e_j}\right) - F_\varphi\left(\mathbf{t} + \sum_{j=1}^{k} (t'_j - t_j)\mathbf{e_j}\right)\right)\right| = \\
&= \frac{1}{\|\mathbf{t}-\mathbf{t}'\|} \left| \sum_{k=1}^{n} (t_k - t'_k) \operatorname{Re} \frac{\partial F_\varphi}{\partial t_k} \left(\sum_{j=1}^{k-1} t'_j \mathbf{e_j} + \tau_k \mathbf{e_k} + \sum_{j=k+1}^{n} t_j \mathbf{e_j} \right) \right| \leq \\
&\leq \left(\sum_{k=1}^{n} \| \frac{\partial F_\varphi}{\partial t_k} \|_\infty \right)^{\frac{1}{2}},
\end{aligned}$$

where τ_k is an appropriate real between t_k and t'_k, and $\|\frac{\partial F_\varphi}{\partial t_k}\|_\infty$ stands for the uniform norm of $\frac{\partial F_\varphi}{\partial t_k}$ on $U_\delta(\mathbf{a})$. A similar inequality holds for the imaginary part, so

$$\left| < \frac{1}{\|\mathbf{t}-\mathbf{t}'\|} \left(F(\mathbf{t}) - F(\mathbf{t}')\right), \varphi > \right| \leq 2 \left(\sum_{k=1}^{n} \| \frac{\partial F_\varphi}{\partial t_k} \|_\infty \right)^{\frac{1}{2}}.$$

By the assumed Banach-Steinhaus Property it follows that

$$\sup\left\{\frac{1}{\|\mathbf{t}-\mathbf{t}'\|}\|F(\mathbf{t})-F(\mathbf{t}'\|\,;\,\mathbf{t},\mathbf{t}'\in U_\delta(\mathbf{a})\,,\mathbf{t}\neq\mathbf{t}'\right\}<+\infty\,.$$

□

We notice that far reaching extensions of the classical uniform boundedness principle were given by H.Helson in [He].

Let $(X,\mathcal{F})$ be a dual pair of normed spaces having the Banach-Steinhaus Property and let us assume that the norm of X is complete. If $\Omega\subset\mathbb{C}$ is open and $U:\Omega\to X$ is such that $\Omega\ni z\mapsto <U(z)\,,\varphi>$ is harmonic for all $\varphi\in\mathcal{F}$, then Proposition 1.1 (ii) yields the norm-continuity of U. Thus, if $a\in\Omega$ and $\delta>0$ are such that the closure of $U_\delta(a)=\{z\in\mathbb{C};\ |z-a|<\delta\}$ is contained in Ω, then for every $z\in U_\delta(a)$ the Poisson integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\delta^2-|z-a|^2}{|\delta e^{i\,\theta}-(z-a)|^2}\,U(a+\delta e^{i\,\theta})\,d\theta$$

exists in the Riemann sense with respect to the norm of X and, taking into account that by the harmonicity assumption on U

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\delta^2-|z-a|^2}{|\delta e^{i\,\theta}-(z-a)|^2}<U(a+\delta e^{i\,\theta})\,,\varphi>\,d\theta=<U(z)\,,\varphi>\ \text{ for all }\varphi\in\mathcal{F}\,,$$

it is equal to $U(z)$. Consequently, U is harmonic with respect to the norm of X: it is locally expressible as a norm-convergent power series in the real part and the imaginary part of z and its Laplacian vanishes identically.

Similarly, if $(X,\mathcal{F})$, $\Omega\subset\mathbb{C}$ are as above and $F:\Omega\to X$ is such that $\Omega\ni z\mapsto <F(z)\,,\varphi>$ is analytic for all $\varphi\in\mathcal{F}$, then F is norm-continuous and, for every $a\in\Omega$, $\delta>0$ with the closure of $U_\delta(a)$ contained in Ω, we have

$$F(z)=\frac{1}{2\pi\,i}\int\limits_{\partial U_\delta(a)}\frac{1}{z-\zeta}\,F(\zeta)\,d\zeta\,,$$

where $\partial U_\delta(a)$ denotes the counterclockwise oriented boundary of $U_\delta(a)$ and the Cauchy integral exists in the Riemann sense with respect to the norm of X. Consequently, F is analytic with respect to the norm of X: it is locally expressible as a norm-convergent power series in z.

Thus we got the following consequence of the Banach-Steinhaus Property (see [Hi-P], Theorem 3.10.1):

1.2. Corollary. *Let $(X,\mathcal{F})$ be a dual pair of normed spaces having the Banach-Steinhaus Property and let us assume that the norm of X is complete. If $\Omega\subset\mathbb{C}$ is open and $U:\Omega\to X$ (respectively $F:\Omega\to X$) is such that $\Omega\ni z\mapsto <U(z)\,,\varphi>$ is harmonic (respectively $\Omega\ni z\mapsto <U(z)\,,\varphi>$ is analytic) for all $\varphi\in\mathcal{F}$, then U is harmonic (respectively F is analytic) with respect to the norm of X. Thus we can speak simply about harmonic and analytic X-valued maps.*

□

Clearly, the statement of Corollary 1.2 holds, with similar proof, for the harmonicity (respectively analiticity) of X-valued maps of more than two real variables (respectively more than one complex variable).

We notice also that a recent result of W.Arendt and N.Nikolski entails that, assuming the completeness of the norm of X , the conclusion of Corollary 1.2 implies the Banach-Steinhaus Property for $(X, \mathcal{F})$ (see [Are-Ni], Theorem 1.5).

The Krein Property of dual pairs of normed spaces is intimately related with the Pettis integrability of X-valued maps with respect to $\sigma(X, \mathcal{F})$:

1.3. Proposition. *Let $(X, \mathcal{F})$ be a dual pair of normed spaces. Then $(X, \mathcal{F})$ has the Krein Property if and only if every $\sigma(X, \mathcal{F})$ -continuous map from a compact Hausdorff topological space Ω in X is Pettis integrable with respect to any Radon measure on Ω and $\sigma(X, \mathcal{F})$. In particular, the Krein Property for $(X, \mathcal{F})$ implies the completeness of the norm of X .*

Proof. The necessity is well known (see e.g. [R], Theorem 3.27, [Sch], Exercise IV.39(a), [Arv], Proposition 1.2): if $(X, \mathcal{F})$ has the Krein Property, Ω is a compact Hausdorff topological space, $F : \Omega \to X$ is $\sigma(X, \mathcal{F})$ -continuous and μ is a Radon measure on Ω , then the $\sigma(X, \mathcal{F})$ - closed convex hull $\mathcal{K}$ of the $\sigma(X, \mathcal{F})$ - compact set $F(\Omega)$ is $\sigma(X, \mathcal{F})$ - compact and

$$\left| \int_\Omega < F(\omega) \, , \varphi > \, d\mu(\omega) \right| \leq \mu(\Omega) \, \sup\{ | < x, \varphi > | \, ; \, x \in \mathcal{K} \} \text{ for all } \varphi \in \mathcal{F} \, ,$$

so the linear functional $\mathcal{F} \ni \varphi \mapsto \int_\Omega < F(\omega) \, , \varphi > \, d\mu(\omega)$ is continuous with respect to the Mackey topology associated to $\sigma(X, \mathcal{F})$ and it follows that it is defined by some $x_{F,\mu} \in X$.

Now let us assume that every $\sigma(X, \mathcal{F})$ -continuous map from a compact Hausdorff topological space Ω in X is Pettis integrable with respect to any Radon measure on Ω and $\sigma(X, \mathcal{F})$. Let $\mathcal{K}$ be any $\sigma(X, \mathcal{F})$ - compact subset of X and let $\mathcal{P}(\mathcal{K})$ denote the weak*-compact convex set of all probability Radon measures on $\mathcal{K}$. By our assumption we can define the affine map

$$\mathcal{P}(\mathcal{K}) \ni \mu \mapsto \Phi(\mu) = \sigma(X, \mathcal{F}) \text{ - } \int_{\mathcal{K}} x \, d\mu(x) \in X$$

and it is easy to see that Φ is continuous with respect to the weak* topology on $\mathcal{P}(\mathcal{K})$ and $\sigma(X, \mathcal{F})$ on X . Consequently, $\Phi(\mathcal{P}(\mathcal{K}))$ is a $\sigma(X, \mathcal{F})$ - compact convex subset of X , containing

$$\{ \Phi(\delta_\varphi) \, ; \, \delta_\varphi \text{ the Dirac measure in } \varphi \in \mathcal{K} \} = \mathcal{K} \, .$$

Finally, let us assume that the above equivalent conditions are satisfied. For the completeness of the norm of X it is enough to prove that, for every sequence $(x_n)_{n \geq 1}$ in X with $\|x_n\| \leq 4^{-n}$ for all $n \geq 1$, the series $\sum_{n \geq 1} x_n$ is $\sigma(X, \mathcal{F})$ - convergent. Denoting by Ω the compact set $\{0, 1, \frac{1}{2}, \frac{1}{3}, ...\} \subset [0, 1]$, the function $F : \Omega \to X$ defined by

$$F(0) = 0 \text{ and } F(\tfrac{1}{n}) = 2^n \, x_n \text{ for } n \geq 1$$

is norm-continuous. On the other hand, the formulas

$$\mu(0) = 0 \text{ and } \mu(\{\tfrac{1}{n}\}) = 2^{-n} \text{ for } n \geq 1$$

define a Radon measure on Ω . Then F is Pettis integrable with respect to μ and $\sigma(X, \mathcal{F})$ and the partial sums of the series $\sum_{n \geq 1} x_n$ are clearly $\sigma(X, \mathcal{F})$ - convergent to the Pettis integral of F with respect to μ and $\sigma(X, \mathcal{F})$.

□

For Ω a locally compact Hausdorff topological space and μ a Radon measure on it, we call a map F from Ω into a topological space X μ-measurable in the Lusin sense if for every compact $K \subset \Omega$ and $\varepsilon > 0$ there exists a compact $L \subset K$ such that $\mu(K \setminus L) < \varepsilon$ and $F|L$ is continuous. Clearly, if $F : \Omega \to X$ is μ-measurable in the Lusin sense then the same holds for $G \circ F$ whenever G is a continuous map from X into a topological space. By the classical Lusin Theorem, a scalar valued function on Ω is μ-measurable if and only if it is μ-measurable in the Lusin sense. We notice that the same holds for maps from Ω into arbitrary metric spaces, according to a deep result of D.H.Fremlin (see [F] or [Ku-Pr]).

Proposition 1.3 implies the following Pettis integrability criterion (see [Arv], Proposition 1.2, [C-Z], Proposition 1.4):

1.4. Corollary. *Let $(X, \mathcal{F})$ be a dual pair of normed spaces having the Krein Property. Let further Ω be a locally compact Hausdorff topological space, μ a Radon measure on it, and $F : \Omega \to X$. If F is μ-measurable in the Lusin sense and $\Omega \ni \omega \mapsto \|F(\omega)\|$ is majorized by some function from $\mathcal{L}^1(\mu)$, then F is Pettis integrable.*

Proof. Let $f \in \mathcal{L}^1(\mu)$ be such that $\|F(\omega)\| \le f(\omega)$ for all $\omega \in \Omega$, choose compact sets $K_1 \subset K_2 \subset \ldots \subset \Omega$ with

$$\int_{\Omega \setminus K_n} f(\omega)\, d\mu(\omega) \le \frac{1}{n}\,,$$

and then compact sets $L_n \subset K_n$ such that $\mu(K_n \setminus L_n)$ is small enough to have

$$\int_{K_n \setminus L_n} f(\omega)\, d\mu(\omega) \le \frac{1}{n}$$

and $F|L_n$ is $\sigma(X, \mathcal{F})$ - continuous. By Proposition 1.3 there exist

$$x_n = \sigma(X, \mathcal{F}) - \int_{L_n} F(\omega)\, d\mu(\omega) \in X$$

and we have, for every $\varphi \in \mathcal{F}$,

$$\begin{aligned} | < x_j - x_k \,, \varphi > | &= | \int_{L_j} < F(\omega)\,,\, \varphi > \, d\mu(\omega) - \int_{L_k} < F(\omega)\,,\, \varphi > \, d\mu(\omega) | = \\ &= | \int_{(L_k \setminus L_j) \cup (L_j \setminus L_k)} < F(\omega)\,,\, \varphi > \, d\mu(\omega) | \le \\ &\le \|\varphi\| \int_{(\Omega \setminus L_j) \cup (\Omega \setminus L_k)} f(\omega)\, d\mu(\omega) \le \\ &\le \|\varphi\| \left(\frac{2}{j} + \frac{2}{k} \right) . \end{aligned}$$

Then $\|x_j - x_k\| \le \left(\frac{2}{j} + \frac{2}{k} \right)$, so $(x_n)_{n \ge 1}$ is a Cauchy sequence in X. Now it is easy to verify that the limit of this sequence is the Pettis integral of F with respect to μ and $\sigma(X, \mathcal{F})$.

□

1.5. Examples. Let X be a complex normed vector space. According to the Hahn-Banach theorem, the natural pairing between X and its dual X^* gives rise to the dual pairs (X, X^*), (X^*, X). By the uniform boundedness principle

(X, X^*) has always the Banach-Steinhaus Property,

(X^*, X) has the Banach-Steinhaus Property whenever X is complete.

By the Krein theorem, respectively the uniform boundedness principle and the Alaoglu theorem,

(X, X^*) and (X^*, X) have the Krein property whenever X is complete.

Now let be A a C^*-algebra and $M(A)$ its multiplier algebra, considered imbedded in the second dual A^{**} (see [Ped], 3.12 or [WO], Chapter 2). Then the natural pairing between A^* and A^{**} induces a pairing between A^* and $M(A)$, which makes $(A^*, M(A)$ and $(M(A), A^*)$ dual pairs of Banach spaces. We notice that $M(A)$ is the dual of A^* only in very particular cases and A^* is the dual of $M(A)$ only in the trivial case $M(A) = A$. By the completeness of the norms of A^* and $M(A)$, $(M(A), A^*)$ and $(A^*, M(A)$ have the Banach-Steinhaus Property. While the Krein Property of $(M(A), A^*)$ follows from Grothendieck's extension of Krein's theorem, the Krein Property of $(A^*, M(A)$ was proved only recently (see [D-Z 1], Theorem 1.1).

Further examples can be obtained from the above ones by passing "from vectors to operators". More precisely, let $(X, \mathcal{F})$ be any dual pair of normed spaces, and Y a complex normed vector space. Let us denote by $B(Y, X)$ the vector space of all bounded linear operators from Y into X, endowed with the operator norm. Putting

$< T, \sum_{j=1}^n y_j \otimes \varphi_j >= \sum_{j=1}^n < Ty_j, \varphi_j >$ for $T \in B(Y, X)$, $y_j \in Y$, $\varphi_j \in \mathcal{F}$,

it is easy to verify that

$\sum_{j=1}^n y_j \otimes \varphi_j \neq 0 \ \Rightarrow < T, \sum_{j=1}^n y_j \otimes \varphi_j >\neq 0$ for some $T \in B(Y, X)$.

Therefore the formula

$\| \sum_{j=1}^n y_j \otimes \varphi_j \| = \sup\{ | < T, \sum_{j=1}^n y_j \otimes \varphi_j > | \, ; T \in B(Y, X), \|T\| \leq 1\}$

defines a norm on $Y \otimes \mathcal{F}$, whith respect to which $(B(Y, X), Y \otimes \mathcal{F})$ becomes a dual pair of normed spaces.

We notice that in general the norm of $Y \otimes \mathcal{F}$ is not complete. In the case $\mathcal{F} = X^*$, $\sigma(B(Y, X), Y \otimes \mathcal{F})$ is the weak operator topology on $B(Y, X)$.

1.6. Proposition. *Let $(X, \mathcal{F})$ dual pair of normed spaces, and Y a complex normed vector space.*

(i) If $(X, \mathcal{F})$ has the Banach-Steinhaus Property and the norm of Y is complete, then $(B(Y, X), Y \otimes \mathcal{F})$ has the Banach-Steinhaus Property.

(ii) If $(X, \mathcal{F})$ has the Krein Property and $(B(Y, X), Y \otimes \mathcal{F})$ has the Banach-Steinhaus Property, then $(B(Y, X), Y \otimes \mathcal{F})$ has the Krein Property.

Proof. (i) follows immediately by using the Banach-Steinhaus Property of $(X, \mathcal{F})$ and the classical uniform boundedness principle in $B(Y, X)$.

Taking into account Proposition 1.3, for (ii) it is enough to prove that, under the assumptions from (ii), every $\sigma(B(Y, X), Y \otimes \mathcal{F})$-continuous map F from a compact Hausdorff topological space Ω into $B(Y, X)$ is Pettis integrable with respect to any Radon measure μ on Ω and $\sigma(B(Y, X), Y \otimes \mathcal{F})$. But then F is norm-bounded and the Pettis integrals

$$\sigma(X, \mathcal{F}) - \int_{\Omega} F(\omega) y \, d\mu(\omega) \in X, \ y \in Y$$

exist. Denoting by $\|F\|_\infty$ the uniform norm of F, we have for every $\varphi \in \mathcal{F}$

$$| \int_{\Omega} < F(\omega)y, \varphi > \, d\mu(\omega) | \leq \int_{\Omega} | < F(\omega)y, \varphi > | \, d\mu(\omega) \leq \mu(\Omega) \, \|F\|_\infty \, \|y\| \, \|\varphi\| \ ,$$

so

$$Y \ni y \mapsto \sigma(X, \mathcal{F}) - \int_{\Omega} F(\omega) y \, d\mu(\omega) \in X$$

is a linear operator T of norm $\leq \mu(\Omega)\,\|F\|_\infty$. Clearly, T is the Pettis integral of F with respect to μ and $\sigma(B(Y,X)\,,\,Y\otimes\mathcal{F})$.

□

2. Scalar analytic extendibility versus vector valued analytic extendibility

In all this section $(X,\mathcal{F})$ will denote a dual pair of normed spaces. In the case of a complete norm on X, harmonicity and analyticity of X-valued maps will be understood with respect to the norm of X.

For any $S\subset\mathbb{C}$, ∂S will stand for the boundary of S, and $d(z\,,\,\partial S)$ for the distance of $z\in\mathbb{C}$ to ∂S. For $a\in\mathbb{C}$ and $r>0$, $U_r(a)$ will denote the disk $\{z\in\mathbb{C},\ |z-a|<r\}$. By complex integrals on $\partial U_r(0)$ we shall always use the counterclockwise orientation of $\partial U_r(0)$.

For any set S, normed vector space Y and $F:S\to Y$, $\|F\|_\infty$ will stand for the uniform norm $\sup\{\|F(s)\|\,,\ s\in S\}\in[0,+\infty]$. The Banach space of all bounded, continuous complex functions on $S\subset\mathbb{C}$, endowed with the uniform norm, will be denoted by $\mathcal{C}_b(S)$. If $S\subset\mathbb{C}$ and $f:S\to\mathbb{C}$, we call f *regular* whenever it is continuous and its restriction to the interior of S is analytic. Clearly, the set of all bounded, regular complex functions on $S\subset\mathbb{C}$, endowed with the uniform norm, is a complex Banach space $\mathcal{R}_b(S)$. We shall also denote:

$$\mathbf{S}_\beta=\{z\in\mathbb{C};\ |\operatorname{Im} z\,|<|\beta|\,,\ \beta\cdot(\operatorname{Im} z\,)>0\,\}\,,\quad \beta\in\mathbb{R}.$$

The aim of this section is to find reasonable conditions, under which a map $F:\mathbb{R}\to X$ such that $\mathbb{R}\ni t\mapsto\,<F(t)\,,\,\varphi>\,\in\mathbb{C}$ has a (necessarily unique) extension to some $F_\varphi\in\mathcal{R}_b(\mathbf{S}_1\cup\mathbb{R})$ for all $\varphi\in\mathcal{F}$, can be extended to a bounded, $\sigma(X,\mathcal{F})$ -continuous map $\mathbf{S}_1\cup\mathbb{R}\to X$, which is analytic in the interior. Using conformal representations, we shall see in Section 5 that the obtained result still holds if we replace $\mathbf{S}_1$ with more general simply connected domains, and $\mathbb{R}$ with a suitable Jordan arc on the boundary of the domain. We have chosen $\mathbf{S}_1$ and $\mathbb{R}$ for model because of the easy way to integrate on $\mathbb{R}$.

Let us first prove an elementary inequality for bounded, regular functions on $\mathbf{S}_1\cup\mathbb{R}$:

2.1. Lemma. *For every $f\in\mathcal{R}_b(\mathbf{S}_1\cup\mathbb{R})$ we have*

$$\left|\sqrt{\frac{n}{\pi}}\int_{-\infty}^{+\infty}e^{-n(t-z)^2}f(t)\,dt-f(z)\right|\leq\frac{4\,\|f\|_\infty}{d(z\,,\,\partial\mathbf{S}_1)\cdot\sqrt{\pi\,n}}\,,\quad z\in\mathbf{S}_1\,,\ n>0\,.$$

Proof. For arbitrary $0<\delta<r<d(z\,,\,\partial\mathbf{S}_1)$ we have

$$\left|\sqrt{\frac{n}{\pi}}\int_{-\infty}^{+\infty}e^{-n(t-z)^2}f(t)\,dt-f(z)\right|=\left|\sqrt{\frac{n}{\pi}}\int_{-\infty}^{+\infty}e^{-n\,t^2}(f(t+z)-f(z))\,dt\right|\leq$$
$$\leq\sqrt{\frac{n}{\pi}}\int\limits_{|t|\geq\delta}e^{-n\,t^2}2\,\|f\|_\infty\,dt+\sqrt{\frac{n}{\pi}}\left|\int_{-\delta}^{+\delta}e^{-n\,t^2}(f(t+z)-f(z))\,dt\right|\,,$$

But

$$\sqrt{n}\int\limits_{|t|\geq\delta}e^{-n\,t^2}\,dt=\int\limits_{|s|\geq\delta\sqrt{n}}e^{-s^2}\,ds\leq\frac{2}{\delta\sqrt{n}}\int\limits_{\delta\sqrt{n}}^{+\infty}e^{-s^2}\,s\,ds=\frac{e^{-n\,\delta^2}}{\delta\sqrt{n}}$$

implies

$$\sqrt{\frac{n}{\pi}} \int\limits_{|t|\geq\delta} e^{-n\,t^2} 2\,\|f\|_\infty\,dt \leq \frac{\|f\|_\infty}{\delta\cdot\sqrt{\pi\,n}}\,2\,e^{-n\,\delta^2}$$

and, using the Cauchy integral formula, we get also

$$\sqrt{\frac{n}{\pi}}\left|\int_{-\delta}^{+\delta} e^{-n\,t^2}\left(f(t+z)-f(t)\right)dt\right| =$$

$$= \sqrt{\frac{n}{\pi}}\left|\int_{-\delta}^{+\delta} e^{-n\,t^2}\left(\frac{1}{2\,\pi\,i}\int\limits_{\partial U_r(z)} f(\zeta)\left(\frac{1}{\zeta-z-t}-\frac{1}{\zeta-z}\right)d\zeta\right)dt\right| =$$

$$= \sqrt{\frac{n}{\pi}}\frac{1}{2\,\pi}\left|\int\limits_{\partial U_r(z)} f(\zeta)\left(\int_{-\delta}^{+\delta}\frac{e^{-n\,t^2}\cdot t}{(\zeta-z-t)\cdot(\zeta-z)}\,dt\right)d\zeta\right| \leq$$

$$\leq \sqrt{\frac{n}{\pi}}\frac{1}{2\,\pi}2r\pi\,\|f\|_\infty\,\frac{1}{(r-\delta)\delta}\int_{-\delta}^{+\delta} e^{-n\,t^2}\cdot t\,dt =$$

$$= \frac{\|f\|_\infty}{\delta\cdot\sqrt{\pi\,n}}\,\frac{r}{r-\delta}\,(1-e^{-n\,\delta^2})\,.$$

Consequently,

$$\left|\sqrt{\frac{n}{\pi}}\int_{-\infty}^{+\infty} e^{-n(t-z)^2} f(t)\,dt - f(z)\right| \leq \frac{\|f\|_\infty}{\delta\cdot\sqrt{\pi\,n}}\left(2\,e^{-n\,\delta^2}+\frac{r}{r-\delta}\,(1-e^{-n\,\delta^2})\right) =$$

$$= \frac{\|f\|_\infty}{\delta\cdot\sqrt{\pi\,n}}\left(\frac{r}{r-\delta}+\frac{r-2\,\delta}{r-\delta}\,e^{-n\,\delta^2}\right).$$

Now, putting $\delta=\frac{r}{2}$ and taking into account that r can be chosen arbitrarily close to $d(z\,,\,\partial\mathbf{S}_1)$, the desired inequality follows.

□

Now we prove that, roughly speaking, vector valued analytic extendibility means scalar analytic extendibility together with a boundedness property. We shall se later that this boundedness property we get often for free.

2.2. Proposition. *Let us assume that the norm of X is complete and let $F:\mathbb{R}\to X$ be norm-bounded, Pettis integrable with respect to all bounded, absolutely continuous Radon measures on $\mathbb{R}$ and $\sigma(X,\mathcal{F})$, and such that*

> $\mathbb{R}\ni t\mapsto\, < F(t)\,,\,\varphi > \,\in\mathbb{C}$ *has an extension to some* $F_\varphi\in\mathcal{R}_b(\mathbf{S}_1\cup\mathbb{R})$ *for all* $\varphi\in\mathcal{F}$.

Then

> F *has a norm-bounded,* $\sigma(X,\mathcal{F})$ *-continuous extension* $\tilde{F}:\mathbf{S}_1\cup\mathbb{R}\to X$ *, which is analytic in the interior,*

if and only if

> *the linear map* $\mathcal{F}\ni\varphi\mapsto F_\varphi\in\mathcal{R}_b(\mathbf{S}_1\cup\mathbb{R})$ *is bounded,*

in which case the norm of the above linear map is equal to $\|\tilde{F}\|_\infty$.

Proof. The necessity is clear: the norm of $\mathcal{F} \ni \varphi \mapsto F_\varphi \in \mathcal{R}_b(\mathbf{S}_1 \cup \mathbb{R})$ will be equal to $\|\tilde{F}\|_\infty$, hence finite.

Now let us assume that the norm c of $\mathcal{F} \ni \varphi \mapsto F_\varphi \in \mathcal{R}_b(\mathbf{S}_1 \cup \mathbb{R})$ is finite and define the analytic maps $F_n : \mathbb{C} \to X$, $n \geq 1$ by

$$F_n(z) = \sqrt{\frac{n}{\pi}}\ \sigma(X, \mathcal{F}) \ \ - \int_{-\infty}^{+\infty} e^{-n(t-z)^2} F(t)\, dt .$$

Let $z \in \mathbf{S}_1$ and $n \, , \, k \geq 1$ be arbitrary. By Lemma 2.1 we have for every $\varphi \in \mathcal{F}$

$$\begin{aligned} \| < F_n(z) - F_k(z) \, , \, \varphi > \| &\leq \| < F_n(z) \, , \, \varphi > - F_\varphi(z) \| + \| < F_k(z) \, , \, \varphi > - F_\varphi(z) \| \leq \\ &\leq \frac{4\, c}{d(z \, , \, \partial \mathbf{S}_1) \cdot \sqrt{\pi}} \left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{k}} \right) \cdot \|\varphi\| , \end{aligned}$$

so

$$\|F_n(z) - F_k(z)\| \leq \frac{4\, c}{d(z \, , \, \partial \mathbf{S}_1) \cdot \sqrt{\pi}} \left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{k}} \right) .$$

Therefore the sequence $(F_n)_{n \geq 1}$ is uniformly Cauchy on every compact subset of $\mathbf{S}_1$, so it converges uniformly on the compacts in $\mathbf{S}_1$ to some analytic map $F_\infty : \mathbf{S}_1 \to X$. Since

$$\begin{aligned} < F_n(z) \, , \, \varphi > &= \sqrt{\frac{n}{\pi}} \int_{-\infty}^{+\infty} e^{-n(t-z)^2} < F(t) \, , \, \varphi > \, dt = \\ &= \sqrt{\frac{n}{\pi}} \int_{-\infty}^{+\infty} e^{-n\, t^2} F_\varphi(t + z)\, dt \longrightarrow F_\varphi(z) , \quad \varphi \in \mathcal{F} , \end{aligned}$$

we have

$$< F_\infty (z) \, , \, \varphi > = F_\varphi(z) \ \text{ for all } \varphi \in \mathcal{F} .$$

In particular, $\|F_\infty\|_\infty \leq c$.

We conclude that the map

$$\mathbf{S}_1 \cup \mathbb{R} \ni z \longmapsto \tilde{F}(z) = \begin{cases} F(z) & \text{if } z \in \mathbb{R} , \\ F_\infty(z) & \text{if } z \in \mathbf{S}_1 \end{cases}$$

is $\sigma(X, \mathcal{F})$ -continuous and norm-bounded, and $\tilde{F} | \mathbf{S}_1 = F_\infty$ is analytic.

□

Proposition 2.2 makes reasonable to say that $(X, \mathcal{F})$ has the

- *Dominated Analytic Extension Property* if, for any norm-bounded $F : \mathbb{R} \to X$ which is Pettis integrable with respect to all bounded, absolutely continuous Radon measures on $\mathbb{R}$ and $\sigma(X, \mathcal{F})$, and such that $\mathbb{R} \ni t \mapsto < F(t) \, , \, \varphi > \in \mathbb{C}$ has an extension to some $F_\varphi \in \mathcal{R}_b(\mathbf{S}_1 \cup \mathbb{R})$ for all $\varphi \in \mathcal{F}$, the linear map $\mathcal{F} \ni \varphi \mapsto F_\varphi \in \mathcal{R}_b(\mathbf{S}_1 \cup \mathbb{R})$ is bounded.

Using the classical closed graph theorem, it is easily seen that $(X, \mathcal{F})$ has the Dominated Analytic Extension Property whenever the norm of $\mathcal{F}$ is complete.

Proposition 2.2 implies the following supplement to Proposition 1.6 :

2.3. Corollary. *Let us assume that $(X, \mathcal{F})$ has the Dominated Analytic Extension Property and the norm of X is complete. Then, for every complex Banach space Y, $(B(Y,X), Y \otimes \mathcal{F})$ has the Dominated Analytic Extension Property.*

Proof. Let $F : \mathbb{R} \to B(Y,X)$ be norm-bounded, Pettis integrable with respect to all bounded, absolutely continuous Radon measures on $\mathbb{R}$ and $\sigma(B(Y,X), Y \otimes \mathcal{F})$, and such that $\mathbb{R} \ni t \mapsto < F(t)y , \varphi >$ has an extension to some $F_{y,\varphi} \in \mathcal{R}_b(\mathbf{S}_1 \cup \mathbb{R})$ for all $y \in Y$ and $\varphi \in \mathcal{F}$.

The complex vector space $\mathcal{R}_b(\mathbf{S}_1 \cup \mathbb{R};\ X)$ of all norm-bounded, $\sigma(X, \mathcal{F})$-continuous maps $\mathbf{S}_1 \cup \mathbb{R} \to X$ with analytic restriction to $\mathbf{S}_1$, endowed with the uniform norm, is a Banach space and Proposition 2.2 implies, for every $y \in Y$, the existence of an extension $F_y \in \mathcal{R}_b(\mathbf{S}_1 \cup \mathbb{R};\ X)$ of $\mathbb{R} \ni t \mapsto F(t)y \in X$.

It turns out that the graph of the linear operator $Y \ni y \mapsto F_y \in \mathcal{R}_b(\mathbf{S}_1 \cup \mathbb{R};\ X)$ is closed:

$$\|y_n - y\| \to 0 \text{ and } \|F_{y_n} - f\| \to 0$$

imply for every $\varphi \in \mathcal{F}$ and $t \in \mathbb{R}$

$$\begin{aligned} & |< (f - F_y)(t) , \varphi > | \leq \\ \leq & |< (f - F_{y_n}(t)) , \varphi > | + |< (F_{y_n} - F_y)(t) , \varphi > | \leq \\ \leq & \|f - F_{y_n}\|_\infty \cdot \|\varphi\| + \|F(t)\| \cdot \|y_n - y\| \cdot \|\varphi\| \longrightarrow 0 . \end{aligned}$$

Thus the closed graph theorem yields the existence of some constant $c > 0$ with

$$\|F_y\|_\infty \leq c \cdot \|y\| , \quad y \in Y .$$

It follows that the norms of the linear operators

$$\tilde{F}(z) : Y \ni y \mapsto F_y(z) \in X , \quad z \in \mathbf{S}_1 \cup \mathbb{R}$$

are $\leq c$, so we got a map $\tilde{F} : \mathbf{S}_1 \cup \mathbb{R} \to B(Y,X)$ with $\|\tilde{F}\|_\infty \leq c$. For every $y \in Y$ and $\varphi \in \mathcal{F}$,

$$F_{y,\varphi}(t) = < F(t)y , \varphi > = < F_y(t) , \varphi > , \quad t \in \mathbb{R}$$

implies

$$F_{y,\varphi}(z) = < F_y(z) , \varphi > = < \tilde{F}(z)y , \varphi > , \quad z \in \mathbf{S}_1 \cup \mathbb{R}.$$

Therefore we have for a generic element $\sum_{j=1}^n y_j \otimes \varphi_j \in Y \otimes \mathcal{F}$ and every $z \in \mathbf{S}_1 \cup \mathbb{R}$:

$$\begin{aligned} |\sum_{j=1}^n F_{y_j,\varphi_j}(z)| = |\sum_{j=1}^n < \tilde{F}(z)y_j , \varphi_j > | = \\ = |< \tilde{F}(z) , \sum_{j=1}^n y_j \otimes \varphi_j > | \leq \\ \leq c \cdot \|\sum_{j=1}^n y_j \otimes \varphi_j\| . \end{aligned}$$

□

In particular, Corollary 2.3 supplies us with examples of dual pairs $(X, \mathcal{F})$ of normed spaces having the Dominated Analytic Extension Property, but with a non-complete norm on $\mathcal{F}$.

3. Preliminaries on conformal maps

Assuming that a conformal representation of $U_1(0)$ onto a simply connected domain D can be extended continuously to $\overline{U_1(0)}$, we investigate here, in which way this extension parametrizes arcs in the boundary of D.

A *Jordan arc* J is the range of an injective, continuous map $f : [0,1] \to \mathbb{C}$ and then $f(0)$, $f(1)$ are called the endpoints of J, while all other points of J are called inner points of J. Then we say that f parametrizes J and we denote the set of all inner points of J by $\overset{\circ}{J}$. An *open Jordan arc* is the range of an injective, continuous map $(0,1) \to \mathbb{C}$. It should not necessarily have endpoints.

An *endcut* of a domain $D \subset \mathbb{C}$ is a Jordan arc, one endpoint of which belongs to the boundary of D, and all other points to D. We say that $f : D \to \mathbb{C}$ has the *asymptotic value* $w \in \mathbb{C}$ at $z \in \partial D$ if there exists an endcut C of D with one endpoint equal to z, such that

$$\lim_{\overset{\circ}{C} \ni \zeta \to z} f(\zeta) = w \,.$$

A *crosscut* of a domain $D \subset \mathbb{C}$ is a Jordan arc, whose both endpoints belong to the boundary of D and whose inner points are in D.

A *Jordan curve* is the range of an injective, continuous map $\partial U_1(0) \to \mathbb{C}$. If J is a Jordan curve then, according to the Jordan curve theorem (see e.g. [D], Theorem XVII.5.4 or [N], Theorem V.10.2), $\mathbb{C} \backslash J$ has exactly two connected components, one of which is bounded and simply connected, and the common boundary of the two components is J. The bounded component is called the *Jordan domain* defined by J.

By the well known Riemann mapping theorem, for every simply connected domain $\mathbb{C} \neq D \subset \mathbb{C}$ there exists a conformal map Φ of $U_1(0)$ onto D. Moreover, according to [Pom], Theorem 2.1, if the boundary $\hat{\partial} D$ of D in the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is locally connected, then Φ has a continuous extension to a map from $\overline{U_1(0)}$ in $\hat{\mathbb{C}}$, which we shall still denote by Φ. Clearly, Φ will map $U_1(0)$ conformally onto D, and $\partial U_1(0)$ onto $\hat{\partial} D$.

Let us first prove a lemma concerning continuous extensions of conformal maps up to the boundary:

3.1. Lemma. *Let $D \subset \mathbb{C}$ be a simply connected domain such that $\hat{\partial} D$ is locally connected, and let Φ denote an arbitrary conformal map of $U_1(0)$ onto D. Let further $J \subset \partial D = \mathbb{C} \cap \hat{\partial} D$ be a Jordan arc, parametrized by $f : [0,1] \to J$, such that $\overset{\circ}{J} = f((0,1))$ is an open subset of ∂D, hence also of $\hat{\partial} D$. Then either*

(1) *there exist reals $\theta_1 < \theta_2 < \theta_1 + 2\pi$ such that Φ maps $U_1(0) \cup \{e^{i\theta}\,;\, \theta \in [\theta_1\,,\, \theta_2]\}$ homeomorphically onto $D \cup J$, and $\{e^{i\theta}\,;\, \theta \in [\theta_2\,,\, \theta_1 + 2\pi]\}$ onto $\hat{\partial} D \backslash \overset{\circ}{J}$,*

or

(2) *there are reals $\theta_1 < \theta_2 \leq \theta_1{}' < \theta_2{}' \leq \theta_1 + 2\pi$ such that $\Phi(\{e^{i\theta}\,,\, \theta \in (\theta_1\,,\, \theta_2)\}) = \Phi(\{e^{i\theta}\,,\, \theta \in (\theta_1{}'\,,\, \theta_2{}')\}) = \overset{\circ}{J}$, the function $[\theta_1\,,\, \theta_2] \ni \theta \mapsto f^{-1}(\Phi(e^{i\theta})) \in [0,1]$ is strictly increasing, while the function $[\theta_1{}'\,,\, \theta_2{}'] \ni \theta \mapsto f^{-1}(\Phi(e^{i\theta})) \in [0,1]$ is strictly decreasing, and Φ maps $\{e^{i\theta}\,;\, \theta \in [\theta_2\,,\, \theta_1{}'] \cup [\theta_2{}'\,,\, \theta_1 + 2\pi]\}$ onto $\hat{\partial} D \backslash \overset{\circ}{J}$.*

Moreover, (1) can only occur if $\overset{\circ}{J}$ *is disjoint to the interior of the closure of* D *, and (2) only if* $\overset{\circ}{J}$ *is contained in the interior of the closure of* D *.*

Proof. According to [Pom], Proposition 2.5, $\varPhi$ maps the connected components of $\partial U_1(0)\backslash\varPhi^{-1}(\{f(0)\})$ exactly onto the connected components of $\hat{\partial} D\backslash\{f(0)\}$. Therefore there are reals $\varphi_1 < \varphi_2 \le \varphi_1 + 2\pi$ such that

$\varPhi(\{e^{i\theta}\,;\,\varphi_1 < \theta < \varphi_2\})$ is the component of $\hat{\partial} D\backslash\{f(0)\}$ containing $f((0,1])$,
$\varPhi(\{e^{i\theta}\,;\,\varphi_2 \le \theta \le \varphi_1 + 2\pi\})$ is disjoint to $f((0,1])$.

It is similarly seen that there exist reals $\varphi_1 \le \psi_1 < \varphi_1 + 2\pi$ and $\psi_1 < \psi_2 \le \psi_1 + 2\pi$ such that

$\varPhi(\{e^{i\theta}\,;\,\psi_1 < \theta < \psi_2\})$ is the component of $\hat{\partial} D\backslash\{f(1)\}$ containing $f([0,1))$,
$\varPhi(\{e^{i\theta}\,;\,\psi_2 \le \theta \le \psi_1 + 2\pi\})$ is disjoint to $f([0,1))$.

Then $\varPhi(\{e^{i\theta}\,;\,\varphi_2 \le \theta \le \varphi_1 + 2\pi \text{ or } \psi_1 < \theta \le \psi_1 + 2\pi\})$ is disjoint to $f((0,1))$, so

$$\varPhi(\{e^{i\theta}\,;\,\varphi_1 < \theta < \varphi_2\} \cap \{e^{i\theta}\,;\,\psi_1 < \theta < \psi_2\}) \text{ contains } f((0,1)).$$

But a short inspection shows that

$$\{e^{i\theta}\,;\,\varphi_1 < \theta < \varphi_2\} \cap \{e^{i\theta}\,;\,\psi_1 < \theta < \psi_2\} = \{e^{i\theta}\,;\,\theta \in I_1 \cup I_2\},$$

where $I_1 = (\varphi_1\,,\ \min(\varphi_2\,,\,\psi_2 - 2\pi))$ and $I_2 = (\psi_1\,,\ \min(\varphi_2\,,\,\psi_2))$, hence

$$(*)\qquad \varPhi(\{e^{i\theta}\,;\,\theta \in I_1\}) \cup \varPhi(\{e^{i\theta}\,;\,\theta \in I_2\}) \supset f((0,1)),$$

$$(**)\qquad \varPhi(\{e^{i\theta}\,;\,\theta \in (\varphi_1\,,\,\varphi_1 + 2\pi]\,,\,\theta \notin I_1 \cup I_2\}) \text{ is disjoint to } f((0,1)).$$

We notice that if $I \subset \mathbb{R}$ is any (necessarily bounded) open interval such that $\varPhi(\{e^{i\theta}\,;\,\theta \in I\})$ intersects $\overset{\circ}{J}$, but does not contain the endpoints of J, then

$$\varPhi(\{e^{i\theta}\,;\,\theta \in I\}) \subset \overset{\circ}{J}.$$

Indeed, since $\overset{\circ}{J}$ is open in $\hat{\partial} D$, $\{\theta \in I\,,\,\varPhi(e^{i\theta}) \in \overset{\circ}{J}\}$ is an open subset of I. On the other hand, $\{\theta \in I\,,\,\varPhi(e^{i\theta}) \in \overset{\circ}{J}\} = \{\theta \in I\,,\,\varPhi(e^{i\theta}) \in J\}$ is also closed in I and not empty by assumption, so it coincides with I by the connectedness of this.

Moreover, assuming additionally that $\theta \mapsto \varPhi(e^{i\theta})$ carries the endpoints of $I = (\alpha\,,\,\beta)$ in the endpoints of J (we notice that the equality $\varPhi(\{e^{i\theta}\,;\,\theta \in I\}) = \overset{\circ}{J}$ would imply this), the function $[\alpha\,,\,\beta] \ni \theta \mapsto g(\theta) = f^{-1}(\varPhi(e^{i\theta})) \in [0,1]$ follows to be strictly increasing if $g(\alpha) = 0$, $g(\beta) = 1$, and strictly decreasing if $g(\alpha) = 1$, $g(\beta) = 0$. Let us consider only the first case, the treatment of the second case being completely similar.

If g would not be increasing, then it would exist $\theta' < \theta''$ in $(\alpha\,,\,\beta)$ such that $g(\theta') > g(\theta'')$. By the intermediate value theorem it would then follow that every value $t \in (g(\theta'')\,,\,g(\theta'))$ is taken at least once in each one of the intervals $(\alpha\,,\,\theta')$, $(\theta'\,,\,\theta'')$, $(\theta''\,,\,\beta)$, that is three times in $(\alpha\,,\,\beta)$. But having uncountably many such values t, this would contradict [Pom], Corollary 2.19. Consequently g must be increasing. Now if it would not be strictly increasing, then it would be constant on some non-empty subinterval of I and by the Privalov uniqueness theorem (see e.g. [K], III:D.2 or [Pom], Corollary 6.14) it would follow that $\varPhi$ is constant, what is clearly not true.

Now, going back to $(*)$, it implies that either only one of the sets $\varPhi(\{e^{i\theta}\,;\,\theta \in I_1\})$ and $\varPhi(\{e^{i\theta}\,;\,\theta \in I_2\})$ intersects $f((0,1))$, or both of them.

Assuming that only $\varPhi(\{e^{i\theta}\,;\,\theta \in I_1\})$ intersects $f((0,1))$, $(*)$ implies that it contains $f((0,1))$ and then the above remarks yield first that $\varPhi(\{e^{i\theta}\,;\,\theta \in I_1\}) =$

$f((0,1))$, and then that $\theta \mapsto \Phi(e^{i\theta})$ maps the closure of I_1 homeomorphically onto J. Taking into account also $(**)$, it is easily seen that (1) holds with $\theta_1 = \varphi_1$ and $\theta_2 = \min(\varphi_2\,,\,\psi_2 - 2\pi)$. Similarly, if only $\Phi(\{e^{i\theta}\,;\,\theta \in I_2\})$ intersects $f((0,1))$, then (1) holds with $\theta_1 = \psi_1$ and $\theta_2 = \min(\varphi_2\,,\,\psi_2)$.

Let us now assume that both sets $\Phi(\{e^{i\theta}\,;\,\theta \in I_1\})$ and $\Phi(\{e^{i\theta}\,;\,\theta \in I_2\})$ intersect $f((0,1))$. By the first one of the above remarks and by $(*)$, their union must then be equal to $f((0,1))$. But

$$I_1 \neq \emptyset \quad\Rightarrow\quad \varphi_1 < \psi_2 - 2\pi \quad\Rightarrow\quad \varphi_2 \leq \varphi_1 + 2\pi < \psi_2\,,$$
$$I_1 \neq \emptyset \quad\Rightarrow\quad \psi_1 < \varphi_2 \quad\Rightarrow\quad \psi_2 - 2\pi \leq \psi_1 < \varphi_2\,,$$

so $I_1 = (\varphi_1\,,\,\psi_2 - 2\pi)$ and $I_2 = (\psi_1\,,\,\varphi_2)$. Thus $\theta \mapsto \Phi(e^{i\theta})$ acts on the endpoints of I_1 and I_2 as follows:

$$\Phi(e^{i\varphi_1}) = f(0)\,,\ \Phi(e^{i(\psi_2 - 2\pi)}) = f(1)\,,$$
$$\Phi(e^{i\psi_1}) = f(1)\,, \Phi(e^{i\varphi_1}) = f(0)\,.$$

Using the second one of the above remarks and $(**)$, we conclude that (2) holds with $\theta_1 = \varphi_1\,,\ \theta_2 = \psi_2 - 2\pi\,,\ \theta_1{}' = \psi_1\,,\ \theta_2{}' = \varphi_2$.

Next we show that (1) implies that $\mathring{J}$ is disjoint to the interior of $\overline{D}$. Indeed, assuming the contrary, it would exist some $\theta_\circ \in (\theta_1\,,\,\theta_2)$ and $0 < r <$ distance $(\Phi(e^{i\theta_\circ})\,,\,\partial D \backslash \mathring{J})$ with $U_r(\Phi(e^{i\theta_\circ})) \subset \overline{D}$ and it would follow

$$U_r(\Phi(e^{i\theta_\circ})) \subset \overline{D} \backslash (\partial D \backslash \mathring{J}) = D \cup \mathring{J}\,.$$

But the restriction Ψ of Φ to $U_1(0) \cup \{e^{i\theta}\,;\,\theta_1 < \theta < \theta_2\}$ is a homeomorphism onto $D \cup \mathring{J}$ and Ψ^{-1} would then then map the open subset $U_r(\Phi(e^{i\theta_\circ}))$ of $\mathbb{C}$ homeomorphically onto some subset of $\overline{U_1(0)}$ containing $e^{i\theta_\circ}$, which can not be open. This would contradict Brouwer's theorem on the invariance of open sets (see e.g. [D], Theorem XVII.3.1 or [N], Theorem V.13.2).

To complete the proof, it remains only to show that (2) implies that $\mathring{J}$ is contained in the interior of $\overline{D}$. If C (respectively C') is the chord joining $e^{i\theta_1}$ and $e^{i\theta_2}$ (respectively $e^{i\theta_1{}'}$ and $e^{i\theta_2{}'}$) then $\Phi(C) \cup J$ (respectively $\Phi(C') \cup J$) is a Jordan curve and let D_1 (respectively D_2) denote the corresponding Jordan domain. Since Φ is injective on the region U_1 between C and the arc $\{e^{i\theta}\,,\,\theta \in [\theta_1\,,\,\theta_2]\,\}$ and it maps ∂U_1 homeomorphically onto $\Phi(C) \cup J$, it follows that $\Phi(U_1) = D_1$ (see e.g. [Pom], Theorem 1.9). Similarly, Φ maps the region U_2 between C' and the arc $\{e^{i\theta}\,,\,\theta \in [\theta_1{}'\,,\,\theta_2{}']\,\}$ onto D_2. In particular, $D_1 \cup D_2 \subset D$.

Now let G denote the Jordan domain defined by the Jordan curve $\Phi(C) \cup \Phi(C')$ and choose some $z_\circ$ in the intersection of the unbounded components of the sets $\mathbb{C} \backslash (\Phi(C) \cup J)$, $\mathbb{C} \backslash (\Phi(C') \cup J)$, $\mathbb{C} \backslash (\Phi(C) \cup \Phi(C'))$. We prove that then

$$(***) \qquad\qquad G \subset D_1 \cup D_2 \cup \mathring{J}\,.$$

For let us assume that there exists some $z \in G \backslash (D_1 \cup D_2 \cup \mathring{J})$. Then z belongs to the unbounded component $\mathbb{C} \backslash (\Phi(C) \cup J \cup D_1)$ of $\mathbb{C} \backslash (\Phi(C) \cup J)$, so z and $z_\circ$ are not separated by $\Phi(C) \cup J$. Similarly, they are not separated also by $\Phi(C') \cup J$. Since $(\Phi(C) \cup J) \cup (\Phi(C') \cup J) = J$ is connected, the Janiszewski theorem (see e.g. [D], Theorem XVII.5.2 or [N], Theorem V.9.2) implies that z and $z_\circ$ are not separated by $\Phi(C) \cup J \cup \Phi(C')$. But this is false, because they were chosen in different components of $\mathbb{C} \backslash (\Phi(C) \cup \Phi(C'))$.

Finally we prove that actually $D_1 \cup D_2 \cup \overset{\circ}{J} = G$, what implies first $\overline{G} = \overline{D_1 \cup D_2} \subset \overline{D}$, and then $\overset{\circ}{J} \subset G \subset$ the interior of $\overline{D}$. For we notice that, the intersection of the connected sets $D_1 \cup \overset{\circ}{J}$ and $D_2 \cup \overset{\circ}{J}$ being not empty, their union $D_1 \cup D_2 \cup \overset{\circ}{J}$ is connected. On the other hand, since G is not contained in $\overset{\circ}{J}$, there exists $z \in G \backslash \overset{\circ}{J}$ and by (***) it follows that $z \in (D_1 \cup D_2 \cup \overset{\circ}{J}) \cap G$. Therefore the union $D_1 \cup D_2 \cup \overset{\circ}{J} \cup G \subset \mathbb{C} \backslash (\Phi(C) \cup \Phi(C'))$ is connected, hence it is equal to the connected component G and it follows that $D_1 \cup D_2 \cup \overset{\circ}{J} \subset G$.

□

The following result will allow to reduce the investigation of the boundary behaviour of maps defined on rather general simply connected domains in $\mathbb{C}$, as well as of extension possibilities from the boundary of such domains, to the case when the domain is a Jordan domain:

3.2. Proposition. *Let $D \subset \mathbb{C}$ be a simply connected domain such that $\hat{\partial} D$ is locally connected. Let further $\Gamma \subset \partial D = \mathbb{C} \cap \hat{\partial} D$ be an open Jordan arc, which is open in ∂D and not contained in the interior of the closure of D.*

(i) *For any conformal map Φ of $U_1(0)$ onto D there exist reals $\theta_1 < \theta_2 \leq \theta_1 + 2\pi$ such that Φ maps $U_1(0) \cup \{e^{i\theta}, \theta \in (\theta_1, \theta_2)\}$ homeomorphically onto $D \cup \Gamma$, and $\{e^{i\theta}; \theta \in [\theta_2, \theta_1 + 2\pi]\}$ onto $\hat{\partial} D \backslash \Gamma$. Moreover, if Γ is locally rectifiable, then the homeomorphism $\Phi | \{e^{i\theta}, \theta \in (\theta_1, \theta_2)\}$ onto Γ is absolutely bicontinuous relative to the arc length measures.*

(ii) *Assuming that C is a crosscut of D with endpoints $a(C)$, $b(C)$ belonging to Γ, let J_C denote the Jordan arc on Γ between $a(C)$ and $b(C)$, and D_C the Jordan domain defined by the Jordan curve $C \cup J_C$. Then $D_C \subset D$ and $D_C \cup \overset{\circ}{J}_C$ is open in $\overline{D}$, so every point of $\overset{\circ}{J}_C$ has a neighborhood N in $\mathbb{C}$ with $N \cap D \subset D_C$.*

(iii) *For every compact $K \subset D \cup \Gamma$ there exists a rectifiable crosscut C of D with endpoints on $\Gamma \backslash K$, such that $K \backslash \Gamma$ is contained in the Jordan domain defined by the union of C with the Jordan subarc of Γ having the same endpoints as C, and K is contained in the closure of the above Jordan domain.*

(iv) *There exists a sequence $C_1, C_2, \dots$ of rectifiable crosscuts of D with endpoints on Γ, such that, denoting by J_n the Jordan subarc of Γ having the same endpoints as C_n, and by D_n the Jordan domain defined by $C_n \cup J_n$,*
$$J_n \subset \overset{\circ}{J}_{n+1} \text{ and } D_n \cup \overset{\circ}{C}_n \subset D_{n+1} \text{ for } n \geq 1, \quad \bigcup_{n \geq 1} J_n = \Gamma, \quad \bigcup_{n \geq 1} D_n = D.$$

Proof. Let us first notice that, according to the last remarks in Lemma 3.1, Γ is actually disjoint to the interior of $\overline{D}$.

Applying now Lemma 3.1 to an arbitrary conformal representation $\Phi : U_1(0) \to D$ and to the Jordan subarcs of Γ, it is easily seen that the first statement in (i) holds. In the remaining part of the proof Φ, θ_1, θ_2 will be as in (i).

If C is as in (ii), then there are uniquely defined $\theta_1 < \varphi_1 \neq \varphi_2 < \theta_2$ such that $\Phi(e^{i\varphi_1}) = a(C)$ and $\Phi(e^{i\varphi_2}) = b(C)$. Without loss of generality we can assume that $\varphi_1 < \varphi_2$. Then the restriction of Φ to $U_1(0) \cup e^{i[\varphi_1, \varphi_2]}$ is a homeomorphism Ψ onto $D \cup J_C$ and Φ maps $e^{i[\varphi_2, \varphi_1 + 2\pi]}$ onto $\hat{\partial} D \backslash \overset{\circ}{J}_C$.

Let U_1 denote the Jordan domain defined by the Jordan curve $C_1 = \Psi^{-1}(C) \cup e^{i\,[\varphi_1\,,\,\varphi_2]}$, and U_2 that one defined by $C_2 = \Psi^{-1}(C) \cup e^{i\,[\varphi_2\,,\,\varphi_1 + 2\,\pi]}$. Clearly, U_1 and U_2 are contained in $U_1(0)$. Actually holds

$$U_1(0) = U_1 \cup U_2 \cup \Psi^{-1}(\overset{\circ}{C})\,,\ \text{so}\ D = \varPhi(U_1) \cup \varPhi(U_2) \cup \overset{\circ}{C}\,.$$

Indeed, if z would belong to $U_1(0) \backslash (U_1 \cup U_2 \cup \Psi^{-1}(\overset{\circ}{C}))$, then it would not be separated from 2 by anyone of C_1 and C_2, so the Janiszewski theorem (see again [D], Theorem XVII.5.2 or [N], Theorem V.9.2) would imply that z and 2 are not separated by $C_1 \cup C_2 = \partial U_1(0) \cup \Psi^{-1}(C)$, what is clearly false. On the other hand, since $\varPhi$ is injective on U_1 and maps C_1 homeomorphically onto $C \cup J_C$, it maps U_1 onto D_C (see e.g. [Pom], Theorem 1.9), so

$$D = D_C \cup \varPhi(U_2) \cup \overset{\circ}{C}\,.$$

By the aboves we have

$$D \cup \hat{\partial} D = D_C \cup \varPhi(U_2) \cup \overset{\circ}{C} \cup \hat{\partial} D\,,$$
$$\varPhi(\overline{U_2}) = \varPhi\left(U_2 \cup \Psi^{-1}(\overset{\circ}{C}) \cup e^{i\,[\varphi_2\,,\,\varphi_1 + 2\,\pi]}\right) = \varPhi(U_2) \cup \overset{\circ}{C} \cup (\hat{\partial} D \backslash \overset{\circ}{J}_C)\,,$$

where D_C, $\varPhi(U_2)$, $\overset{\circ}{C}$, $\hat{\partial} D$ are mutually disjoint. Therefore

$$(D \cup \hat{\partial} D) \backslash \varPhi(\overline{U_2}) = D_C \cup \overset{\circ}{J}_C$$

and it follows that $D_C \cup \overset{\circ}{J}_C$ is open in $D \cup \hat{\partial} D \supset D \cup \partial D$.

For (iii) we notice that $\varPhi^{-1}(K)$ is a compact subset of $U_1(0) \cup e^{i\,(\theta_1\,,\,\theta_2)}$, so there exists $0 < \delta < \frac{1}{2}(\theta_2 - \theta_1)$ such that

$$\{r\, e^{i\,\theta}\,;\, r \geq 1 - \delta\,,\ \theta_2 - \delta \leq \theta \leq \theta_1 + 2\,\pi + \delta\,\} \cap \varPhi^{-1}(K) = \emptyset\,.$$

Indeed, if we define the function g on $[0,1] \times [\theta_1\,,\,\theta_1 + 2\,\pi]$ by putting

$$g(r\,,\,\theta) = \max\,(\,1 - r\,,\ \min\,(\,\theta_2 - \theta\,,\ \theta - \theta_1\,))\,,$$

then g takes a minimum value $\delta_\circ > 0$ on the compact set $\{(r\,,\,\theta) \in [0,1] \times [\theta_1\,,\,\theta_1 + 2\,\pi]\,;\, r\,e^{i\,\theta} \in \varPhi^{-1}(K)\,\}$ and we can choose for δ any number in $(0\,,\,\delta_\circ)$, which is $< \frac{1}{2}(\theta_2 - \theta_1)$.

According to [Pom], Corollary 4.18, there exist $\theta_1 < \alpha_1 < \theta_1 + \delta$ and $\theta_2 - \delta < \alpha_2 < \theta_2$ such that

$$\text{length}\,(\varPhi(\,[0,1] \cdot e^{i\,\alpha_1}\,)) \leq \text{distance}\,(\varPhi(0)\,,\partial D) \cdot \exp \tfrac{A}{\delta} < +\infty\,,$$
$$\text{length}\,(\varPhi(\,[0,1] \cdot e^{i\,\alpha_2}\,)) \leq \text{distance}\,(\varPhi(0)\,,\partial D) \cdot \exp \tfrac{A}{\delta} < +\infty\,,$$

where A is an absolute constant. Therefore the crosscut

$$C = \varPhi\left(([1 - \delta\,,\,1] \cdot e^{i\,\alpha_1}) \cup ((1 - \delta) \cdot e^{i\,[\alpha_1\,,\,\alpha_2]}) \cup ([1 - \delta\,,\,1] \cdot e^{i\,\alpha_2})\right)$$

of D is rectifiable. Moreover, since $\varPhi^{-1}(K \backslash \Gamma)$ is contained in the Jordan domain $U_{1-\delta}(0) \cup ([1 - \delta\,,\,1) \cdot e^{i\,(\alpha_1\,,\,\alpha_2)})$ and this is mapped by $\varPhi$ onto the Jordan domain defined by $C \cup \varPhi(e^{i\,[\alpha_1\,,\,\alpha_2]})$ (see e.g. [Pom], Theorem 1.9), $K \backslash \Gamma$ is contained in the Jordan domain defined by $C \cup \varPhi(e^{i\,[\alpha_1\,,\,\alpha_2]})$. Similarly, K is contained in the closure of the above Jordan domain.

For (iv) we have just to choose compact sets $L_1 \subset L_2 \subset \ldots \subset D \cup \Gamma$ such that $\bigcup\limits_{n \geq 1} L_n = D \cup \Gamma$, and to apply (iii) first with $K = L_1$ to get C_1, and then with $K = L_n \cup \overline{\text{the Jordan domain defined by } C_n \cup J_n}$ to get C_{n+1}.

Finally, in order to prove the second statement in (i), let us assume that Γ is locally rectifiable. Let J be any Jordan arc on Γ. Reproducing the proof of statement (iii) in the case $K = J$, we get $\theta_1 < \beta_1 < \beta_2 < \theta_2$ and a rectifiable crosscut B of $U_1(0)$ with endpoints $e^{i\,\beta_1}$, $e^{i\,\beta_2}$, such that $\varPhi(B)$ is rectifiable and $J \subset \varPhi(e^{i\,(\beta_1\,,\,\beta_2)})$. Since $\varPhi$ maps the Jordan curve $B \cup e^{i\,[\beta_1\,,\,\beta_2]}$ onto the Jordan curve $\Phi(B)\cup\Phi(e^{i\,[\beta_1\,,\beta_2]})$, it maps the Jordan domain defined by $B\cup e^{i\,[\beta_1\,,\beta_2]}$ onto the Jordan domain defined by $\Phi(B)\cup\Phi(e^{i\,[\beta_1\,,\beta_2]})$ (see e.g. [Pom], Theorem 1.9). But the two Jordan curves being rectifiable, the Riemann mapping theorem, the Carathéodory theorem (see e.g. [K], II:C.1 or [Pom], Theorem 2.6) and the F. and M. Riesz theorem (see e.g. [K], II:D.2 or [Pom], Theorem 6.8) imply that the homeomorphism $\Phi|B\cup e^{i\,[\beta_1\,,\beta_2]}$ onto $\Phi(B)\cup\Phi(e^{i\,[\beta_1\,,\beta_2]})$ is absolutely bicontinuous relative to the arc length measures. In particular, the homeomorphism $\Phi|e^{i\,[\beta_1\,,\beta_2]}$ onto $\Phi(e^{i\,[\beta_1\,,\beta_2]})\supset J$ is absolutely bicontinuous relative to the arc length measures. Since the Jordan arc $J\subset\Gamma$ was arbitrary, this yields the desired statement.

$\square$

4. Continuity properties of vector valued analytic maps at the boundary

The goal of this section is to give conditions under which harmonic or analytic maps taking values in a Banach space and having certain weak continuity properties at the boundary, allow norm-continuous extensions on the closure.

In all this section $(X,\mathcal{F})$ will denote a dual pair of normed spaces, and τ a locally convex vector space topology between $\sigma(X,\mathcal{F})$ and the norm-topology of X , such that the closed unit ball of X is τ-complete (the existence of such τ implies the completeness of the norm of X). Trivial example for such τ is the norm-topology of X in the case of a complete norm on X . But there are also other relevant examples:

- the strict topology on $M(A)$ for $(M(A)\,,\,A^*)$, where A is a C^*-algebra;
- the strong operator topology on $B(Z,Y)$ for $(B(Z,Y)\,,\,Z\otimes\mathcal{G})$, where $(Y,\mathcal{G})$ is a dual pair of normed spaces with the norm of Y complete.

"Almost everywhere" (abreviated: a.e.) in a rectifiable curve in $\mathbb{C}$ (e.g. in a segment or arc) will always be understood with respect to the arc length measure.

The following result concerning the Dirichlet problem is "folklore", but we sketch the proof for the convenience of the reader:

4.1. Lemma. *Every norm-bounded, τ-continuous $F:\partial U_1(0)\to X$ has a τ-continuous extension $\tilde F:\overline{U_1(0)}\to X$, which is harmonic in the interior.*

Proof. Standard arguments show that the Poisson integrals

$$P_F(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-|z|^2}{|e^{i\,\theta}-z|^2}\,F(e^{i\,\theta})\,d\theta\in X\,,\quad z\in U_1(0)$$

exist in the Riemann sense with respect to τ . Define $\tilde F:\overline{U_1(0)}\to X$ by

$$\tilde F(z)=\begin{cases}P_F(z)\,, & \text{for } z\in U_1(0)\\ F(z)\,, & \text{for } z\in\partial U_1(0)\,.\end{cases}$$

By the classical theory of the Dirichlet problem, the composition of $\tilde F$ with every bounded linear functional on X is continuous on $\overline{U_1(0)}$ and harmonic on $U_1(0)$, so, taking into account Corollary 1.2, it remains only to show that

$$\lim_{U_1(0)\ni z\to e^{i\,\theta}}p(P_F(z)-F(e^{i\,\theta}))=0 \text{ for every } \tau\text{-continuous seminorm } p \text{ on } X$$
$$\text{and for every } 0\le\theta<2\pi\,.$$

But this can be done similarly as in the case of scalar valued functions, using the properties of the Poisson kernel and the τ-continuity of F (see e.g. [A-B-R], proof of Theorem 1.14).

□

Next we prove a criterion for the τ-continuity at the boundary of harmonic maps from the unit disc in X :

4.2. Lemma. *Let $U : \overline{U_1(0)} \to X$ be a map such that*

- *$U|U_1(0)$ is harmonic and norm-bounded,*
- *$U|\partial U_1(0)$ is τ-continuous,*
- *for every $\varphi \in \mathcal{F}$ we have*
  $\lim\limits_{1>r\to 1} < U(r\, e^{i\,\theta}) \, , \varphi > = < U(e^{i\,\theta}) \, , \varphi >$ *for a.e.* $0 \leq \theta < 2\pi$.

Then U is τ-continuous.

Proof. By Lemma 4.1, the norm-bounded, τ- continuous map $F = U|\partial U_1(0)$ has a τ- continuous extension $\tilde{F} : \overline{U_1(0)} \to X$, which is harmonic in the interior. Then $V = U - \tilde{F} : \overline{U_1(0)} \to X$ is

- harmonic and norm-bounded on $U|U_1(0)$,
- vanishing on $U|\partial U_1(0)$,
- such that, for every $\varphi \in \mathcal{F}$, the function $U_1(0) \ni z \mapsto V_\varphi(z) =< V(z) \, , \varphi >$ has zero radial limit in a.e. point of $\partial U_1(0)$.

But bounded harmonic functions on the open unit disc, which have zero radial limits a.e. on the unit circle, must vanish identically (see e.g. [A-B-R], Theorems 6.12 (b) and 6.31, or [K], the second theorem in I:C and the fourth theorem in I:D.2), so it follows that V_φ vanishes identically for every $\varphi \in \mathcal{F}$. Consequently V is vanishing also on $U_1(0)$ and we conclude that $U = \tilde{F}$ is τ-continuous.

□

Now we are ready to prove a general result concerning the τ-continuity at the boundary of analytic maps with values in X :

4.3. Theorem. *Let $D \subset \mathbb{C}$ be a simply connected domain such that $\hat{\partial} D$ is locally connected, $\Gamma \subset \partial D$ a locally rectifiable open Jordan arc, which is open in ∂D and not contained in the interior of the closure of D , and $F : D \cup \Gamma \to X$ a map with the properties:*

- *$F|D$ is analytic and norm-bounded,*
- *$F|\Gamma$ is τ-continuous,*
- *for every $\varphi \in \mathcal{F}$ there exists a set $N_\varphi \subset \Gamma$ of zero arc length measure such that $D \ni \zeta \mapsto < F(\zeta) \, , \varphi >$ has the asymptotic value $< F(z) \, , \varphi >$ at z for every $z \in \Gamma \backslash N_\varphi$.*

Then F is τ-continuous.

Proof. Taking into account statements (ii) and (iii) in Proposition 3.2, we can assume without loss of generality that $\Gamma = \overset{\circ}{J}$ for some rectifiable Jordan arc $J \subset \partial D$ and that there exists a rectifiable crosscut C of D with the same endpoints a , b as J , being enough to prove the τ-continuity of Φ on $D_C \cup \overset{\circ}{J}$, where $D_C \subset D$ denotes the Jordan domain defined by $C \cup J$.

Let $\Phi : U_1(0) \to D_C$ be a conformal representation. By the Carathéodory theorem (see e.g. [K], II:C.1 or [Pom], Theorem 2.6) Φ has an extension to a homeomorphism of $\overline{U_1(0)}$ onto $\overline{D_C}$, which will be still denoted by Φ. Moreover, by the F. and M. Riesz theorem (see e.g. [K], II:D.2 or [Pom], Theorem 6.8) a subset S of $\partial U_1(0)$ has arc length measure zero if and only if $\Phi(S)$ has arc length measure zero on ∂D_C. Therefore, if we define $U : \overline{U_1(0)} \to X$ by putting

$$U(z) = (\Phi(z) - a)(\Phi(z) - b)F(\Phi(z)) \text{ for } z \in \overline{U_1(0)} \backslash \{\Phi^{-1}(a)\,,\, \Phi^{-1}(b)\}\,,$$
$$U(\Phi^{-1}(a)) = U(\Phi^{-1}(b)) = 0\,,$$

then

- $U|U_1(0)$ is analytic and norm-bounded,
- $U|\partial U_1(0)$ is τ-continuous,
- $U|U_1(0) \cup (\partial U_1(0) \backslash \Phi^{-1}(J))$ is τ-continuous,
- for every $\varphi \in \mathcal{F}$ and for every $z \in \Phi^{-1}(\overset{\circ}{J})$, not belonging to the set $\Phi^{-1}(N_\varphi)$ of zero arc length measure, $U_1(0) \ni \zeta \mapsto < U(\zeta)\,,\, \varphi >$ has the asymptotic value $< \Phi(z)\,,\, \varphi >$ at z.

But, according to a theorem of Lindelöf (see e.g. [Col-L] Theorem 2.3 or [Con], Theorem 13.5.4 or, for a more general result, [Pom], Theorem 4.3), if a bounded analytic function on $U_1(0)$ has the asymptotic value w at $z \in \partial U_1(0)$ then it has the angular limit w at z. Consequently, the last two properties of U imply:

- for every $\varphi \in \mathcal{F}$ we have
  $\lim\limits_{1>r\to 1} < U(r\, e^{i\,\theta})\,,\, \varphi > = < U(e^{i\,\theta})\,,\, \varphi >$ for a.e. $0 \le \theta < 2\pi$.

Now we have only to apply Lemma 4.2 to conclude the τ-continuity of U, which clearly implies the τ-continuity of $F|D_C \cup \overset{\circ}{J}$.

□

We notice that the proof of Theorem 4.3 does not work if we assume only the harmonicity of $F|D$, because in contrast to analyticity, harmonicity is not preserved by multiplication with analytic functions, needed in the passage from F to U.

Taking into account statement (i) in Proposition 3.2, Theorem 4.3 implies immediately:

4.4. Corollary. *Let $D \subset \mathbb{C}$ be a simply connected domain such that $\hat{\partial} D$ is locally connected, $\Gamma \subset \partial D$ an open Jordan arc, which is open in ∂D and not contained in the interior of the closure of D, and $F : D \cup \Gamma \to X$ a map with the properties:*

- *$F|D$ is analytic and norm-bounded,*
- *$F|\Gamma$ is τ-continuous,*
- *for every $\varphi \in \mathcal{F}$ and every $z \in \Gamma$, $D \ni \zeta \mapsto < F(\zeta)\,,\, \varphi >$ has the asymptotic value $< F(z)\,,\, \varphi >$ at z.*

Then F is τ-continuous.

□

We recall that $f \in L^\infty(\partial U_1(0))$ is a.e. angular limit of a bounded analytic function $\tilde{f}$ on $U_1(0)$ if and only if $\int_0^{2\pi} f(e^{i\,\theta}) e^{i\,n\,\theta}\, d\theta = 0$ for all integers $n \ge 1$, in which case $\tilde{f}$ is the Poisson integral of f and $\|\tilde{f}\|_\infty$ is equal to the norm $\|f\|_\infty$ of f in $L^\infty(\partial U_1(0))$ (see e.g. in [Ho] the discussion at the end of Chapter 3 or [Con], Theorem 20.1.5). Therefore the set of all these f is a weak*closed linear subspace $\mathcal{H}^\infty(\partial U_1(0))$ of $L^\infty(\partial U_1(0))$. In particular, the closed unit ball of $\mathcal{H}^\infty(\partial U_1(0))$

is weak*compact. Furthermore, using the Poisson formula, it is easily seen that the map $\{f \in \mathcal{H}^\infty(\partial U_1(0))\,;\, \|f\|_\infty \leq 1\} \ni f \mapsto \tilde{f}$ is continuous with respect to the weak*topology on the closed unit ball of $\mathcal{H}^\infty(\partial U_1(0))$ and the topology of the uniform convergence on the compact subsets of $U_1(0)$.

Now let $\theta_1 < \theta_2 \leq \theta_1 + 2\pi$ be arbitrary. Since the restriction map $\mathcal{H}^\infty(\partial U_1(0)) \ni f \mapsto f|e^{i(\theta_1,\theta_2)} \in L^\infty(e^{i(\theta_1,\theta_2)})$ is weak*continuous and injective, it maps the closed unit ball of $\mathcal{H}^\infty(\partial U_1(0))$ bijectively onto the weak*compact set

$$\{f|e^{i(\theta_1,\theta_2)}\,;\, f \in \mathcal{H}^\infty(\partial U_1(0))\,,\, \|f\|_\infty \leq 1\} \subset L^\infty(e^{i(\theta_1,\theta_2)})$$

and this bijection is a homeomorphism with respect to the weak*topologies. It follows that the intersection of the above set with $\mathcal{C}_b(e^{i(\theta_1,\theta_2)})$

$$\left\{ f \in \mathcal{C}_b(e^{i(\theta_1,\theta_2)})\,;\, \begin{array}{l} f \text{ is a.e. angular boundary value of some} \\ \text{analytic } \tilde{f} : U_1(0) \to \mathbb{C} \text{ with } \|\tilde{f}\|_\infty \leq 1 \end{array} \right\},$$

which is by Theorem 4.3 (applied to $X = \mathbb{C}$) equal to

$$\mathcal{C}_{\theta_1,\theta_2} = \left\{ f \in \mathcal{C}_b(e^{i(\theta_1,\theta_2)})\,;\, \begin{array}{l} f \text{ has an extension } \tilde{f} \in \mathcal{R}_b(U_1(0) \cup e^{i(\theta_1,\theta_2)}) \\ \text{with } \|\tilde{f}\|_\infty \leq 1 \end{array} \right\},$$

is weak*closed in $\mathcal{C}_b(e^{i(\theta_1,\theta_2)})$ and the map $\mathcal{C}_{\theta_1,\theta_2} \ni f \mapsto \tilde{f}\,|U_1(0)$ is continuous with respect to the weak*topology on $\mathcal{C}_{\theta_1,\theta_2}$ and the topology of the uniform convergence on the compact subsets of $U_1(0)$.

Thus, using statement (i) in Proposition 3.2, we get:

4.5. Corollary. *Let $D \subset \mathbb{C}$ be a simply connected domain such that $\hat{\partial} D$ is locally connected, and $\Gamma \subset \partial D$ an open Jordan arc, which is open in ∂D and not contained in the interior of the closure of D. Then the set*

$$\mathcal{C}_\Gamma = \{f \in \mathcal{C}_b(\Gamma)\,;\, f \textit{ has an extension } \tilde{f} \in \mathcal{R}_b(D \cup \Gamma) \textit{ with } \|\tilde{f}\|_\infty \leq 1\}$$

is uniformly closed in $\mathcal{C}_b(\Gamma)$ and the map $\mathcal{C}_\Gamma \ni f \mapsto \tilde{f}\,|D$ is continuous with respect to the topology of the uniform convergence on $\mathcal{C}_\Gamma$ and the topology of the uniform convergence on the compact subsets of D. Moreover, if Γ is locally rectifiable and λ denotes the arc length measure on Γ, then the above statements hold with the topology of the uniform convergence on $\mathcal{C}_b(\Gamma)$ replaced by $\sigma(L^\infty(\lambda)\,,\, L^1(\lambda))$.

□

5. General analytic extension criteria for vector valued functions

This section contains the main results of the paper. In it $(X, \mathcal{F})$ will always denote a dual pair of normed spaces, $D \subset \mathbb{C}$ be a simply connected domain such that $\hat{\partial} D$ is locally connected, and $\Gamma \subset \partial D$ an open Jordan arc, which is open in ∂D and not contained in the interior of the closure of D.

5.1. Theorem (on analytic extension to a fixed domain). *Let us assume that $(X, \mathcal{F})$ has the Banach-Steinhaus Property and the Dominated Analytic Extension Property, the norm of X is complete, Γ is locally rectifiable, and $F : \Gamma \to X$ is locally Pettis integrable with respect to any Radon measure on Γ, which is absolutely continuous relative to the arc length measure, and $\sigma(X, \mathcal{F})$. If*

$$\Gamma \ni z \mapsto < F(z)\,,\, \varphi > \in \mathbb{C} \textit{ has a regular extension on } D \cup \Gamma \textit{ for all } \varphi \in \mathcal{F}\,,$$

then

> *F has a $\sigma(X,\mathcal{F})$ - continuous extension $\tilde{F}: D\cup\Gamma \to X$, which is analytic in the interior.*

Moreover:

- *If the regular extension of $\Gamma \ni z \mapsto < F(z)\,,\, \varphi > \in \mathbb{C}$ on $D\cup\Gamma$ is bounded for every $\varphi \in \mathcal{F}$, then $\tilde{F}$ is bounded.*
- *If F is continuous with respect to a locally convex vector space topology τ between $\sigma(X,\mathcal{F})$ and the norm-topology of X , which is complete on the closed unit ball of X , then $\tilde{F}$ is τ-continuous.*

Proof. Let C be any rectifiable crosscut of D with endpoints $a\,,\,b$ on Γ , J_C the Jordan arc on Γ between a and b , and D_C the Jordan domain defined by $C\cup J_C$. We recall that $D_C \subset D$ by statement (ii) of Proposition 3.2. By the Banach- Steinhaus Property of $(X,\mathcal{F})$ and the integrability assumption on F , the restriction $F|J_C$ is norm-bounded and Pettis integrable with respect to any Radon measure on J_C , which is absolutely continuous relative to the arc length measure, and $\sigma(X,\mathcal{F})$.

Let $\Phi_C : U_1(0) \to D_C$ be a conformal representation. By the Carathéodory theorem (see e.g. [K], II:C.1 or [Pom], Theorem 2.6) Φ_C has an extension to a homeomorphism of $\overline{U_1(0)}$ onto $\overline{D_C}$, which will be still denoted by Φ_C . By the F. and M. Riesz theorem (see e.g. [K], II:D.2 or [Pom], Theorem 6.8) $\Phi_C|\partial U_1(0)$ is an absolutely bicontinuous homeomorphism between $\partial U_1(0)$ and $C\cup J_C$ relative to the arc length measures. Let $\theta_1 < \theta_2 < \theta_1 + 2\,\pi$ be reals such that Φ_C maps $e^{i\,[\theta_1\,,\,\theta_2]}$ onto J_C .

Now let $\Psi_{\theta_1\,,\,\theta_2}$ be a conformal representation of $\mathbf{S}_1$ onto $U_1(0)$ which maps $\mathbb{R}$ onto $e^{i\,(\theta_1\,,\,\theta_2)}$. For example, we can take

$$\Psi_{\theta_1\,,\,\theta_2} : \zeta \longmapsto e^{i\,\theta_2}\,\frac{e^{\pi\,\zeta} + e^{i\,\frac{\theta_1-\theta_2}{2}}}{e^{\pi\,\zeta} + e^{-i\,\frac{\theta_1-\theta_2}{2}}}\,.$$

Clearly, $\Psi_{\theta_1\,,\,\theta_2}|\mathbb{R}$ is an absolutely bicontinuous homeomorphism between $\mathbb{R}$ and $e^{i\,(\theta_1\,,\,\theta_2)}$ relative to the arc length measures. Therefore $\Phi_C \circ \Psi_{\theta_1\,,\,\theta_2}$ carries the bounded Radon measures on $\mathbb{R}$, which are absolutely continuous relative to the Lebesgue measure, onto the bounded Radon measures on $\overset{\circ}{J}_C$, which are absolutely continuous relative to the arc length measure. It follows that $F\circ\Phi_C\circ\Psi_{\theta_1\,,\,\theta_2} : \mathbb{R} \to X$ is norm-bounded and Pettis integrable with respect to all bounded, absolutely continuous Radon measures on $\mathbb{R}$ and $\sigma(X,\mathcal{F})$.

By our assumptions on $(X,\mathcal{F})$ and F we can apply Proposition 2.2 to

$$F_C : \mathbb{R} \ni t \longmapsto F\circ\Phi_C\circ\Psi_{\theta_1\,,\,\theta_2}(t) \in X$$

and deduce the existence of a bounded, $\sigma(X,\mathcal{F})$ - continuous extension $\tilde{F}_C : \mathbf{S}_1 \cup \mathbb{R} \to X$ of F , which is analytic in the interior. The map

$$D_C \cup \overset{\circ}{J}_C \ni z \longmapsto \tilde{F}_C \circ \Psi^{-1}_{\theta_1\,,\,\theta_2} \circ \Phi_C^{-1}(z) \in X$$

will be a $\sigma(X,\mathcal{F})$ - continuous extension of

$$\overset{\circ}{J}_C \ni z \longmapsto F(z) \in X\,,$$

which is analytic in the interior.

Using statement (iv) in Proposition 3.2 and the above part of the proof, it follows the existence of a $\sigma(X,\mathcal{F})$ - continuous extension of F , which is analytic in the interior.

Let next assume that the regular extensions of the functions $\Gamma \ni z \mapsto < F(z), \varphi > \in \mathbb{C}$ on $D \cup \Gamma$ are bounded. We notice that then F must be norm-bounded. For $\Phi : U_1(0) \to D$ any conformal representation, let θ_1 , θ_2 be as in statement (i) of Proposition 3.2. Choosing some $\theta_1 < \alpha < \beta < \theta_2$, let $\Psi : \mathbf{S}_1 \to U_1(0)$ be a conformal representation mapping $\mathbb{R}$ onto $e^{i\,(\alpha\,,\beta)}$. Then $\Phi \circ \Psi | \mathbb{R}$ is an absolutely bicontinuous homeomorphism with respect to the arc length measures, hence it carries the bounded, absolutely continuous Radon measures on $\mathbb{R}$ onto the bounded Radon measures on its image, which are absolutely continuous relative to the arc length measure. It follows that the norm-bounded map $\mathbb{R} \ni t \longmapsto F \circ \Phi \circ \Psi(t) \in X$ is Pettis integrable with respect to all bounded, absolutely continuous Radon measures on $\mathbb{R}$. Now, using the Dominated Analytic Extension Property of $(X, \mathcal{F})$, it is easy to verify that $\tilde{F} \circ \Phi \circ \Psi$, hence also $\tilde{F}$ is norm-bounded.

Finally, the last statement follows easily using statement (i) in Proposition 3.2 and Corollary 4.4.

□

We notice that, assuming that $(X, \mathcal{F})$ has also the Krein property, the statement of the above theorem holds without requiring the local rectifiability of Γ and the local Pettis integrability property of F. Indeed, according to statement (i) in Proposition 3.2, we can assume in the proof without loss of generality that $D = U_1(0)$. But then Γ is rectifiable and the local Pettis integrability property of F follows from its $\sigma(X, \mathcal{F})$ - continuity, the Banach-Steinhaus Property of $(X, \mathcal{F})$ and the Krein Property of $(X, \mathcal{F})$.

Next we prepare the proof of a criterion on analytic extendibility to a not previously fixed domain. First we prove a result of interest on its own:

5.2. Proposition. *Let $F : \Gamma \to X$ be a norm-bounded, $\sigma(X, \mathcal{F})$ -continuous map. Then the sets*

$$\mathcal{F}_{F\,,\,c} = \left\{ \varphi \in \mathcal{F}\,;\, \begin{array}{l} \Gamma \ni \zeta \mapsto < F(\zeta)\,,\, \varphi > \ \textit{has an extension} \\ F_\varphi \in \mathcal{R}_b(D \cup \Gamma) \ \textit{with}\ \|F_\varphi\|_\infty \leq c \end{array} \right\}, \ c > 0$$

are norm closed in $\mathcal{F}$ and each map $\mathcal{F}_{F\,,\,c} \ni \varphi \mapsto F_\varphi\,|D$ is continuous with respect to the norm topology on $\mathcal{F}$ and the topology of the uniform convergence on the compact subsets of D. In particular,

$$\left\{ \varphi \in \mathcal{F}\,;\, \begin{array}{l} \Gamma \ni \zeta \mapsto < F(\zeta)\,,\, \varphi > \ \textit{has a} \\ \textit{bounded, regular extension on}\ D \cup \Gamma \end{array} \right\} = \bigcup_{n \geq 1} \mathcal{F}_{F\,,\,n}$$

is a linear subspace of $\mathcal{F}$, which is a Borel set with respect to the Borel structure defined by the norm of $\mathcal{F}$. Moreover, assuming additionally that Γ is locally rectifiable with arc length measure λ and F is Pettis integrable with respect to any bounded Radon measure on Γ, which is absolutely continuous relative to λ, and $\sigma(X, \mathcal{F})$, the sets $\mathcal{F}_{F\,,\,c}$ are even $\sigma(\mathcal{F}, X)$-closed in $\mathcal{F}$ and the maps $\mathcal{F}_{F\,,\,c} \ni \varphi \mapsto F_\varphi\,|D$ are continuous with respect to $\sigma(\mathcal{F}, X)$ on $\mathcal{F}$ and the topology of the uniform convergence on the compact subsets of D.

Proof. The first statement is an immediate consequence of Corollary 4.5.

Also the second statement follows from Corollary 4.5, once we prove the continuity of the linear map $T : \mathcal{F} \to \mathcal{C}_b(\Gamma)$, defined by $T(\varphi)(\zeta) =< F(\zeta)\,,\, \varphi >$, with respect to the topologies $\sigma(\mathcal{F},X)$ and $\sigma(L^\infty(\lambda)\,,\,L^1(\lambda))$, that is the $\sigma(\mathcal{F},X)$-continuity of the linear functionals

$$\mathcal{F}\ni\varphi\mapsto\int_\Gamma g(\zeta)\ <F(\zeta)\,,\,\varphi>\ d\lambda(\zeta)\ ,\ g\in L^1(\Gamma)\ .$$

But this follows plainly from the equality

$$\int_\Gamma g(\zeta)\ <F(\zeta)\,,\,\varphi>\ d\lambda(\zeta)=<\sigma(X,\mathcal{F})-\int_\Gamma g(\zeta)\,F(\zeta)\,d\lambda(\zeta)\,,\,\varphi>,$$

valid for every $g\in L^1(\Gamma)$ and every $\varphi\in\mathcal{F}$.

□

We recall that, according to a result due to H.Helson (see [He], (i)), for any sequence $Y_1\subset Y_2\subset\ldots$ of linear subspaces of a Banach space Y , each one of which is Borel set with respect to the Borel structure defined by the norm of Y and such that $\bigcup\limits_{n\geq1}Y_n=Y$, there exists $n_\circ\geq1$ such that $Y_{n_\circ}=Y$. The proof of the next statement uses this result and is inspired from the proof of a theorem of D.A.Herrero given in [He], (iii).

5.3. Lemma. *Let $J\subset\Gamma$ be a compact, connected set (that is a Jordan arc or a singleton) and $F:\Gamma\to X$ a norm-bounded, $\sigma(X,\mathcal{F})$ -continuous map. If the norm of $\mathcal{F}$ is complete and*

> *for every $\varphi\in\mathcal{F}$ there exists an open set $J\subset G_\varphi\subset\mathbb{C}$ such that $\Gamma\cap G_\varphi\ni\zeta\mapsto<F(\zeta)\,,\,\varphi>$ has a bounded, regular extension on $(D\cup\Gamma)\cap G_\varphi$*

then there exists an open set $J\subset G\subset\mathbb{C}$ such that

> *$\Gamma\cap G\ni\zeta\mapsto<F(\zeta)\,,\,\varphi>$ has an extension to some $F_\varphi\in\mathcal{R}_b((D\cup\Gamma)\cap G)$ for all $\varphi\in\mathcal{F}$.*

Proof. By statement (i) in Proposition 3.2 we can assume without loss of generality that $D=U_1(0)$, $\Gamma=e^{i\,(\theta_1\,,\,\theta_2)}$ and $J=e^{i\,[\varphi_1\,,\,\varphi_2]}$ for some $\theta_1<\varphi_1\leq\varphi_2<\theta_2\leq\theta_1+2\,\pi$. Choosing an integer $n_\circ\geq1$ with $\frac{1}{n_\circ}<\ \min\,(\varphi_1-\theta_1\,,\,\theta_2-\varphi_2)$, let us consider the open sets

$$G_n=\{\,r\,e^{i\,\theta}\,;\,r\in(1-\tfrac{1}{n}\,,\,1+\tfrac{1}{n})\,,\,\theta\in(\varphi_1-\tfrac{1}{n}\,,\,\varphi_2+\tfrac{1}{n})\,\}\,,\quad n\geq n_\circ\,.$$

Then every open subset of $\mathbb{C}$ containing J contains some G_n .

Now, according to Proposition 5.2, each one of the linear subspaces

$$\mathcal{F}_n=\left\{\varphi\in\mathcal{F}\,;\begin{array}{l}\Gamma\cap G_n\ni\zeta\mapsto<F(\zeta)\,,\,\varphi>\ \text{has a}\\ \text{bounded, regular extension on }(D\cup\Gamma)\cap G_n\end{array}\right\}\subset\mathcal{F}\,,\quad n\geq n_\circ$$

is a Borel set with respect to the Borel structure defined by the norm of $\mathcal{F}$. But, by the assumptions of the lemma, $\bigcup\limits_{n\geq n_\circ}\mathcal{F}_n=\mathcal{F}$, so the above quoted result of H.Helson implies that $\mathcal{F}=\mathcal{F}_n$ for some $n\geq n_\circ$. Put $G=G_n$ for this n .

□

Now, Lemma 5.3 and Theorem 5.1 imply immediately:

5.4. Theorem (on analytic extension to an undetermined domain). *Let us assume that the norms of X and $\mathcal{F}$ are complete, Γ is locally rectifiable, $J\subset\Gamma$ is a compact, connected set, and the map $F:\Gamma\to X$ is norm-bounded, $\sigma(X,\mathcal{F})$ -continuous and locally Pettis integrable with respect to any Radon measure on Γ , which is absolutely continuous relative to the arc length measure, and $\sigma(X,\mathcal{F})$. If*

> *for every $\varphi\in\mathcal{F}$ there exists an open set $J\subset G_\varphi\subset\mathbb{C}$ such that $\Gamma\cap G_\varphi\ni\zeta\mapsto<F(\zeta)\,,\,\varphi>$ has a bounded, regular extension on $(D\cup\Gamma)\cap G_\varphi$*

then there exists an open set $J \subset G \subset \mathbb{C}$ such that

> $\Gamma \cap G \ni z \mapsto F(z) \in X$ *has a bounded,* $\sigma(X, \mathcal{F})$ *- continuous extension on* $(D \cup \Gamma) \cap G$ *, which is analytic in the interior.*

□

Similarly to Theorem 5.1, also the statement of Theorem 5.4 holds without assuming the local rectifiability of Γ and the local Pettis integrability property of F whenever $(X, \mathcal{F})$ has additionally the Krein Property.

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