SURGERY AND THE SPECTRUM OF THE DIRAC OPERATOR

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ABSTRACT. We show that for generic Riemannian metrics on a simply-connected closed spin manifold of dimension ≥ 5 the dimension of the space of harmonic spinors is no larger than it must be by the index theorem. The same result holds for periodic fundamental groups of odd order.

The proof is based on a surgery theorem for the Dirac spectrum which says that if one performs surgery of codimension ≥ 3 on a closed Riemannian spin manifold, then the Dirac spectrum changes arbitrarily little provided the metric on the manifold after surgery is chosen properly.

0. INTRODUCTION

Classical Hodge-deRham theory establishes a tight link between the analysis of the Laplace operator acting on differential forms of a compact Riemannian manifold and its topology. Specifically, the dimension of the space of harmonic k-forms is a topological invariant, the k^{th} Betti number.

The question arises whether a similar relation holds for other elliptic geometric differential operators such as the Dirac operator on a compact Riemannian spin manifold. It is not hard to see that the dimension h_g of the space of harmonic spinors is a conformal invariant, it does not change when one replaces the Riemannian metric g by a conformally equivalent one [10, Prop. 1.3]. Moreover, the Atiyah-Singer index theorem implies a topological lower bound on h_g.

Berger metrics on spheres of dimension $4k + 3$ provide examples showing that in general h_g depends on the metric and is not topological, see [10, Prop. 3.2] and [3, Thm. 3.1]. Also for surfaces of genus at least 3 the number h_g varies with the choice of metric [10, Thm. 2.6]. All known examples indicate that the following two conjectures should be true. On the one hand, we should have

Conjecture A. *Harmonic spinors are not topologically obstructed, i.e. on any compact spin manifold of dimension at least three there is a metric g such that $h_g > 0$.*

Perhaps h_g is even unbounded. Conjecture A has been shown to be true in dimensions $n \equiv 0, 1, 3 \mod 8$ by Hitchin [10, Thm. 4.5] using a topological approach, while the first author [3, Thm. A] proves it for $n \equiv 3, 7 \mod 8$ using an analytic approach.

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Conjecture A has been shown to hold for spheres of dimension $n \equiv 0 \mod 4$ with unbounded h_g by Seeger [23].

Conjecture A is not true in dimension 2. For example, on S^2 all metrics are conformally equivalent, hence h_g cannot change with the metric. In fact, $h_g = 0$ and one even has a simple geometric lower bound for all Dirac eigenvalues λ on the 2-sphere equipped with any metric $\lambda^2 \geq 4\pi/\text{area}(M)$, see [2, Thm. 2].

In this paper we do not deal with the rather exceptional metrics giving rise to large h_g but instead we study generic metrics. Versions of the second conjecture can be found in [3, 6, 13].

Conjecture B. *On any compact connected spin manifold for a generic metric h_g is no larger than it is forced to be by the index theorem.*

This means that up to a "small" set of exceptional metrics an analogue of classical Hodge-deRham theory also holds for the Dirac operator. In particular, in dimensions divisible by 4 Conjecture B claims that for a generic metric either the kernel or the cokernel of the chiral Dirac operator vanishes. Using a variational approach Maier [17] proved Conjecture B for dimension $n \leq 4$. In the present article we deal with the case of dimension $n \geq 5$. We show

Theorem 3.10. *Conjecture B is true for all simply connected spin manifolds of dimension at least five.*

Our approach, very different from Maier's, follows the strategy that has successfully been used to characterize manifolds admitting metrics of positive scalar curvature. We show that the class of manifolds for which Conjecture B holds is closed under surgery of codimension ≥ 3. Then we find a set of manifolds for which the conjecture holds and which generates the spin bordism ring Ω_*^{spin}. This set consists of manifolds of positive scalar curvature and a few more special manifolds. Results from bordism theory then imply the theorem.

Using more refined spin bordism rings one can also deal with certain classes of non-simply connected manifolds. As an example, we show

Theorem 3.12. *Suppose M is a compact connected spin manifold with $\dim M \geq 5$ and fundamental group π a periodic group of odd order. Then Conjecture B holds for M.*

By exhibiting concrete examples Kotschick [13] shows that the analogue of Conjecture B does not hold in the class of Kähler manifolds.

The most involved part of our technique is to show the invariance under surgery mentioned above. To get this we show that if one performs surgery of codimension ≥ 3 to a closed Riemannian spin manifold, then the lower part of the Dirac spectrum changes arbitrarily little provided the metric on the manifold after surgery is chosen properly. More precisely, we show

Theorem 1.2. *Let (M, g) be a closed Riemannian manifold equipped with a spin structure. Let $N \subset M$ be an embedded sphere of codimension $k \geq 3$ and with trivialized tubular neighborhood. Let $\widetilde{M}$ be obtained from M by surgery along N*

together with the resulting spin structure. Let $\varepsilon > 0$ *and* $\Lambda > 0$, $\pm\Lambda \notin \mathrm{spec}\,(D_g)$. *Then there exists a Riemannian metric* $\widetilde{g}$ *on* $\widetilde{M}$ *such that* D_g *and* $D_{\widetilde{g}}$ *are* (Λ, ε)-*spectral close.*

Here (Λ, ε)-spectral close means that the Dirac eigenvalues of M and $\widetilde{M}$ in the interval $(-\Lambda, \Lambda)$ differ at most by ε.

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1. Surgery and the Dirac operator

By a *Riemannian spin manifold* we mean a Riemannian manifold (M, g) together with the choice of a spin structure. Hence we can form the *spinor bundle* ΣM which is a Hermitian vector bundle over M of rank $2^{[n/2]}$ where n is the dimension of M. For even n the spinor bundle splits as

$$\Sigma M = \Sigma^+ M \oplus \Sigma^- M \tag{1}$$

into two subbundles of equal rank. They are called *half-spinor bundles* of positive and negative *chirality*. Sections of ΣM are *spinors*. The classical *Dirac operator*, sometimes also called *Atiyah-Singer operator*, is a formally self-adjoint elliptic differential operator of first order acting on the space of spinors. We denote the Dirac operator by D and if we want to emphasize the dependence on the Riemannian metric we write D_g. In even dimensions it interchanges chiralities, that is with respect to splitting (1) it is of the form

$$D_g = \begin{pmatrix} 0 & D_g^- \\ D_g^+ & 0 \end{pmatrix}.$$

For any differentiable function f and spinor φ on M we have the formula

$$D(f\varphi) = fD\varphi + \nabla f \cdot \varphi$$

where ∇f is the gradient of f and $X \cdot \varphi$ denotes *Clifford multiplication* of a tangent vector X with a spinor φ. Another important formula concerns the square of the Dirac operator,

$$D^2 = \nabla^*\nabla + \frac{1}{4}\,\mathrm{scal}$$

where ∇ is the Hermitian connection on ΣM induced by the Levi-Civita connection and scal denotes scalar curvature. This is usually called the *Lichnerowicz formula* [16] but was already known to Schrödinger in the 1930's [22].

We will always consider closed manifolds M. Standard elliptic theory then tells us that the spectrum of D is real and discrete. In particular, D_g^+ is a Fredholm operator and its index is given by the celebrated Atiyah-Singer index formula

$$\mathrm{ind}(D_g^+) = \widehat{A}(M)$$

where $\widehat{A}(M)$ is the $\widehat{A}$-*genus* of M constructed out of the Pontrjagin classes of M. There is a generalization of this index theorem to which we will return in Section 3. See [15] for a thorough introduction to spin geometry.

Next let us recall the concept of *surgery*. Let N be an embedded sphere of codimension k in M with a trivialized neighborhood, meaning there is a diffeomorphism Φ mapping $S^{n-k} \times D^k$ diffeomorphically onto its image in M such that $S^{n-k} \times \{0\}$ is mapped onto N. Surgery now consists of removing $\Phi(S^{n-k} \times D^k)$ and gluing in $D^{n-k+1} \times S^{k-1}$ along the common boundary $S^{n-k} \times S^{k-1}$. We will only perform surgeries in codimension $k \geq 3$. If $n-k \neq 1$, then $S^{n-k} \times S^{k-1}$ is simply connected (for $n-k=0$ the two components are simply connected) and carries a unique spin structure. This spin structure extends to the unique spin structure on $D^{n-k+1} \times S^{k-1}$. Thus any spin structure on M induces one on the boundary of $\Phi(S^{n-k} \times D^k)$ which then extends uniquely to one on $\widetilde{M}$, the manifold after surgery. If $n-k=1$, then $S^{n-k} \times S^{k-1}$ has two different spin structures but only one of them extends to $D^{n-k+1} \times S^{k-1}$. In this case it will be assumed that the trivialization Φ of the neighborhood of N is chosen such that it induces on the boundary the spin structure which extends. Then again we obtain a spin structure on $\widetilde{M}$.

The main theorem will roughly say that the spectrum of the Dirac operator in an arbitrarily fixed range will change only very little if one performs surgery in codimension ≥ 3 and chooses the metric on the resulting manifold $\widetilde{M}$ carefully. To formulate this in a precise way we make the following definition.

Definition 1.1. Let $\varepsilon > 0$ and $\Lambda > 0$. We call two operators with discrete spectrum (Λ, ε)-*spectral close* if

- $\pm\Lambda$ are not eigenvalues of either operator.
- Both operators have the same total number m of eigenvalues in the interval $(-\Lambda, \Lambda)$.
- If the eigenvalues in $(-\Lambda, \Lambda)$ are denoted by $\lambda_1 \leq \cdots \leq \lambda_m$ and $\mu_1 \leq \cdots \leq \mu_m$ respectively (each eigenvalue being repeated according to its multiplicity), then $|\lambda_j - \mu_j| < \varepsilon$ for $j = 1, \ldots, m$.

Theorem 1.2. *Let (M, g) be a closed Riemannian manifold equipped with a spin structure. Let $N \subset M$ be an embedded sphere of codimension $k \geq 3$ and with trivialized tubular neighborhood. Let $\widetilde{M}$ be obtained from M by surgery along N together with the resulting spin structure. Let $\varepsilon > 0$ and $\Lambda > 0$, $\pm\Lambda \notin \mathrm{spec}\,(D_g)$. Then there exists a Riemannian metric $\widetilde{g}$ on $\widetilde{M}$ such that D_g and $D_{\widetilde{g}}$ are (Λ, ε)-spectral close.*

Remark 1.3. If U is an open neighborhood of N in M, then $\widetilde{M}$ is of the form

$$\widetilde{M} = (M \setminus U) \cup \widetilde{U}$$

where $\widetilde{U}$ is an open subset of $\widetilde{M}$. The proof will show that for any such U the metric $\widetilde{g}$ can be chosen in such a way that it coincides with g on $M \setminus U$.

By choosing $\Lambda_i > 0$ such that $\Lambda_i \to \infty$ and $\varepsilon_i > 0$ such that $\varepsilon_i \to 0$ for $i \to \infty$ we obtain the following corollary.

Corollary 1.4. *Let (M, g) be a closed Riemannian manifold equipped with a spin structure. Let $N \subset M$ be an embedded sphere of codimension $k \geq 3$ and with trivialized tubular neighborhood. Let $\widetilde{M}$ be obtained from M by surgery along N together with the resulting spin structure. Then there is a sequence of Riemannian metrics $\widetilde{g}_i$ on $\widetilde{M}$ such that the Dirac eigenvalues of $(\widetilde{M}, \widetilde{g}_i)$ converge to exactly the Dirac eigenvalues of (M, g).*

Again we note that the metrics $\widetilde{g}_i$ can be chosen such that they converge to g away from the sphere N in the following sense: To any compact subset $K \subset M$ which does not intersect N there is an isometric copy of K contained in $(\widetilde{M}, \widetilde{g}_i)$ for i sufficiently large.

The next section will be devoted to the proof of Theorem 1.2. Since the proof is somewhat involved the reader who is primarily interested in applications of the theorem may continue directly with Section 3 at a first reading.

2. Proof of the surgery theorem

The proof of Theorem 1.2 consists of two parts. In the first part, which is quite simple, we show that to any Dirac eigenvalue of the original manifold M in the range $(-\Lambda, \Lambda)$ there is a nearby eigenvalue of the manifold $\widetilde{M}$ after surgery. This is done by multiplying a corresponding eigenspinor with a suitable cut–off function and plugging the resulting spinor into the Rayleigh quotient for the Dirac operator on $\widetilde{M}$. Here the choice of metric on $\widetilde{M}$ is not so crucial. It suffices that the metric coincides with the original metric outside a sufficiently small neighborhood of the sphere along which the surgery is performed.

The second, more complicated, part consists of showing that $\widetilde{M}$ has no further Dirac eigenvalues in the interval $(-\Lambda, \Lambda)$. This requires a finer analysis of the behavior of eigenspinors. First we make the scalar curvature on M very large on a neighborhood U of the surgery sphere by a C^1-small deformation of the metric (Section 2.1). This property persists under surgery of codimension ≥ 3. Here we use the construction of Gromov and Lawson which shows that the class of manifolds admitting metrics of positive scalar curvature is closed under surgery of codimension ≥ 3. Hence scalar curvature is very large in the corresponding open subset $\widetilde{U}$ of $\widetilde{M}$. This implies that most of the L^2-norm of an eigenspinor is supported in the complement $\widetilde{M} \setminus \widetilde{U} = M \setminus U$ (Section 2.2). Then we do a finer analysis of the distribution of the L^2-norm on certain annular regions (Section 2.3). This finally allows us to multiply an eigenspinor of $\widetilde{M}$ with a cut-off function whose gradient is supported in such an annular region where the eigenspinor carries only very little L^2-norm. Therefore the bad error term in the Rayleigh quotient coming from the gradient of the cut-off function is under good control. It follows that to each eigenvalue of $\widetilde{M}$ there corresponds one of M and we are done.

It should be noted that the proof is not at all symmetric in M and $\widetilde{M}$ because the metric on $\widetilde{M}$ is not given beforehand and therefore various geometric quantities are not under a-priori control. To make the second part work we must choose the metric on $\widetilde{M}$ very carefully.

2.1. **Increasing scalar curvature along submanifolds.** In this subsection we show that the scalar curvature can be made large near a submanifold by a C^1-small deformation of the metric (which does not change the Dirac eigenvalues much). Then Lemma 2.2 of the next subsection will show that only little L^2-norm of eigenspinors is contained in the problematical region near the sphere along which we will perform surgery.

Proposition 2.1. *Let (M, g) be an n-dimensional Riemannian manifold, let $N \subset M$ be a compact submanifold of positive codimension. Let U be a compact neighborhood of N in M. Then there exists a sequence of smooth Riemannian metrics g_j on M such that*

- *g_j is conformally equivalent to g,*
- *$g_j = g$ on $M \setminus U$,*
- *$g_j \to g$ in the C^1-topology as $j \to \infty$,*
- *$\min\limits_N \mathrm{scal}_{g_j} \to \infty$ as $j \to \infty$,*
- *there exists a constant $S_0 \in \mathbb{R}$ such that on U and for all j*
$$\mathrm{scal}_{g_j} \geq S_0 \,.$$

Proof. The scalar curvature of a conformally equivalent metric $\widehat{g} = e^{2u} g$ is given by
$$\mathrm{scal}_{\widehat{g}} = e^{-2u} \left(\mathrm{scal}_g + 2(n-1)\,\Delta u - (n-2)\,(n-1)|du|^2 \right) ,$$
see [4, Thm. 1.159]. Hence it is sufficient to find a sequence of smooth functions u_j on M such that

- $u_j = 0$ on $M \setminus U$,
- $u_j \to 0$ uniformly,
- $du_j \to 0$ uniformly,
- Δu_j uniformly bounded from below and
- $\min\limits_N \Delta u_j \to \infty$ as $j \to \infty$.

To construct such functions we first assume that the normal bundle of N in M contains a trivial line bundle, that is we assume there is a smooth unit normal field ν on N. Choose a relatively compact neighborhood U_1 of N in U sufficiently small so that the Riemannian exponential map maps a neighborhood $\widehat{U}$ of the zero section in the normal bundle diffeomorphically onto U_1,
$$\exp|_{\widehat{U}} : \widehat{U} \xrightarrow{\approx} U_1 \subset\subset U \subset M \,.$$
We define a smooth function ρ by $\rho(x) = g\left((\exp|_{\widehat{U}})^{-1}(x), \nu\right)$ on U_1. On $M \setminus U$ we define ρ to be 1, and we extend ρ in some way to $U \setminus U_1$ so that we obtain a smooth function defined on all of M with the following properties.

- $\rho \equiv 1$ on $M \setminus U$,

- $\rho \equiv 0$ on N,
- $\operatorname{grad} \rho = \nu$ on N, in particular $|d\rho| \equiv 1$ on N,
- $|d\rho| \leq C_1$ on M,
- $|\Delta\rho| \leq C_2$ on M.

Next, we pick continuous functions $\widehat{\varphi}_j : \mathbb{R} \to \mathbb{R}$, smooth on $\mathbb{R} \setminus \{0\}$ such that

- $\widehat{\varphi}_j\,(0) = \frac{1}{j}$,
- $0 \leq \widehat{\varphi}_j \leq \frac{1}{j}$ on $\mathbb{R}$,
- $\widehat{\varphi}_j \equiv 0$ on $\mathbb{R} \setminus (-1, 1)$,
- $\widehat{\varphi}'_j = \frac{2}{j}$ on $[\,-\frac{1}{4}, 0)$, $\widehat{\varphi}'_j = -\frac{2}{j}$ on $(0, \frac{1}{4}\,]$,
- $|\widehat{\varphi}'_j| \leq \frac{2}{j}$ on $\mathbb{R} \setminus \{0\}$,
- $|\widehat{\varphi}''_j| \leq \frac{C_3}{j}$ on $\mathbb{R} \setminus \{0\}$.

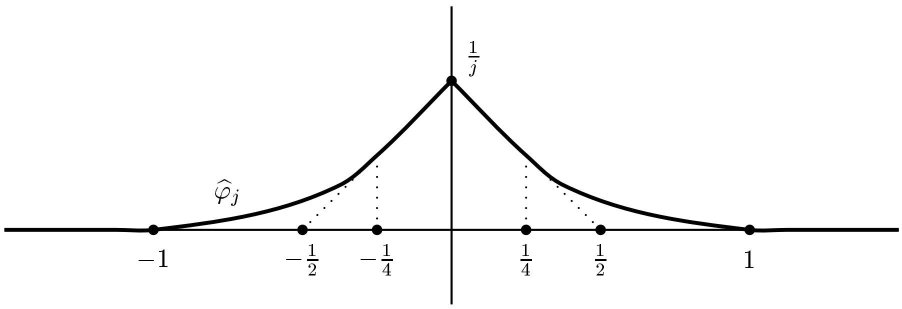

Fig. 1

Note that $\widehat{\varphi}_j$ can simply be chosen as $\frac{1}{j}\,\widehat{\varphi}_1$. We smooth out $\widehat{\varphi}_j$ in a neighborhood of 0 with size of order $\frac{1}{j}$ and obtain smooth functions $\varphi_j : \mathbb{R} \to \mathbb{R}$ with :

- $0 \leq \varphi_j \leq \frac{1}{j}$ on $\mathbb{R}$,
- $|\varphi'_j| \leq \frac{2}{j}$ on $\mathbb{R}$,
- $\varphi''_j \leq \frac{C_3}{j}$ on $\mathbb{R}$,
- $\varphi_j \equiv 0$ on $\mathbb{R} \setminus (-1, 1)$,
- $\varphi''_j\,(0) \leq -j$.

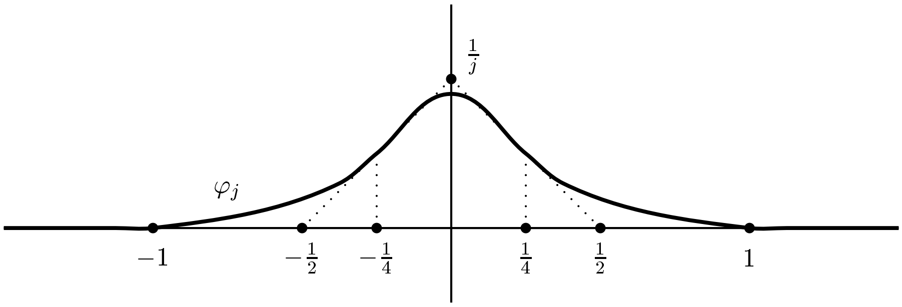

Fig. 2

One checks easily that $u_j := \varphi_j \circ \rho$ has all the required properties (using the formula $\Delta u_j = (\varphi_j' \circ \rho)\, \Delta\rho - (\varphi_j'' \circ \rho)\, |d\rho|^2$).

It remains to handle the case when the normal bundle of N in M has no trivial one-dimensional sub-bundle. Locally it always has. Hence we can cover N by open sets $V_k, k = 1, \ldots, \ell$, and find smooth functions $\rho_k : M \to \mathbb{R}$ having all the properties of the previous ρ on V_k instead of N. Let $\chi_1, \ldots, \chi_{\ell+1}$ be a partition of unity on M subordinate to the open covering $\exp(\pi^{-1}(V_1) \cap \widetilde{U}), \ldots ,\ \exp(\pi^{-1}(V_\ell) \cap \widetilde{U})$, $M \setminus N$. Then $\chi_1|_N, \ldots, \chi_\ell|_N$ form a partition of unity on N subordinate to the covering $V_1, \ldots, V_\ell$. Again, it is easy to check that

$$u_j := \sum_{k=1}^{\ell+1} \chi_k \cdot (\varphi_j \circ \rho_k)$$

has the required properties. □

2.2. **Distribution of $\mathbf{L^2}$-norm of eigenspinors and scalar curvature.** Next we show that eigenspinors try to avoid large scalar curvature, that is, most of their L^2-norm is supported in the region of the manifold where the scalar curvature is small.

Lemma 2.2. *Let M be a closed Riemannian spin manifold. Let $\Lambda, S_0, S_1 \in \mathbb{R},\ S_0 < S_1$. Suppose the scalar curvature of M satisfies $\mathrm{scal} \geq S_0$. Put*

$$M_+ := \{x \in M \mid \mathrm{scal}\,(x) \geq S_1\}.$$

Then for any spinor satisfying

$$\|D\psi\|^2_{L^2(M)} \leq \Lambda^2\, \|\psi\|^2_{L^2(M)}$$

the following inequality holds:

$$\int_{M_+} |\psi|^2 \mathrm{dvol} \leq \frac{4\Lambda^2 - S_0}{S_1 - S_0} \int_M |\psi|^2\, \mathrm{dvol}\ .$$

Proof. Using the Lichnerowicz formula $D^2 = \nabla^*\nabla + \frac{1}{4}\mathrm{scal}$ we obtain

$$\begin{aligned} 0 &\le \int_M |\nabla\psi|^2 \,\mathrm{dvol} = \int_M \langle \nabla^*\nabla\psi, \psi\rangle \,\mathrm{dvol} \\ &= \int_M \langle D^2\psi, \psi\rangle \,\mathrm{dvol} - \frac{1}{4}\int_M \mathrm{scal}\,|\psi|^2 \,\mathrm{dvol} \\ &\le \Lambda^2 \int_M |\psi|^2 \,\mathrm{dvol} - \frac{S_0}{4}\int_{M\setminus M_+} |\psi|^2 \,\mathrm{dvol} - \frac{S_1}{4}\int_{M_+} |\psi|^2 \,\mathrm{dvol}\,. \end{aligned}$$

Hence

$$\left(\frac{S_1}{4} - \Lambda^2\right)\int_{M_+} |\psi|^2 \,\mathrm{dvol} \le \left(\Lambda^2 - \frac{S_0}{4}\right)\int_{M\setminus M_+} |\psi|^2 \,\mathrm{dvol}$$

and therefore

$$\left(\frac{S_1}{4} - \frac{S_0}{4}\right)\int_{M_+} |\psi|^2 \,\mathrm{dvol} \le \left(\Lambda^2 - \frac{S_0}{4}\right)\int_{M} |\psi|^2 \,\mathrm{dvol}$$

which is the claim of the lemma. □

Remark 2.3. The lemma can be slightly improved to

$$\int_{M_+} |\psi|^2 \,\mathrm{dvol} \le \frac{4\,\frac{n-1}{n}\,\Lambda^2 - S_0}{S_1 - S_0}\int_M |\psi|^2 \,\mathrm{dvol} \tag{2}$$

with essentially the same proof. One combines the Lichnerowicz formula with the identity

$$|\nabla\psi|^2 = |P\psi|^2 + \frac{1}{n}\,|D\psi|^2$$

where P is the *twistor operator*. Estimating $\int_M |P\,\psi|^2\,\mathrm{dvol}$ (instead of $\int_M |\nabla\,\psi|^2\,\mathrm{dvol}$) by 0 one looses a little less. Note that the estimate is nontrivial only for

$$\Lambda^2 \le \frac{n}{n-1}\,\frac{S_1}{4}$$

because otherwise the coefficient on the right hand side of (2) is greater than 1. The estimate also shows that if $\Lambda^2 \le \frac{n}{n-1}\,\frac{S_0}{4}$, then $\psi = 0$. This way we recover Friedrich's eigenvalue estimate [8]:

$$\lambda^2 \ge \frac{n}{n-1}\,\frac{S_0}{4}\,.$$

In our application however S_0 will typically be negative.

2.3. **$\mathbf{L^2}$-norm on annular regions.** In the following we will look at the "distance spheres" of a compact submanifold N of a Riemannian manifold M given by

$$S_N(r) := \{x \in M \,|\, \mathrm{dist}\,(x, N) = r\}$$

for $r > 0$. For $0 \le R_1 < R_2$ we define the annular region

$$A_N(R_1, R_2) := \{x \in M \mid R_1 \le \mathrm{dist}(x, N) \le R_2\} = \bigcup_{R_1 \le r \le R_2} S_N(r)\,.$$

Let ν denote the unit normal vector field of $S_N(r)$ pointing away from N, let $W = -\nabla\nu$ denote the Weingarten map (or shape operator) and let $H = \frac{1}{n-1}\,\mathrm{tr}(W)$ be the mean curvature of $S_N(r)$ with respect to ν. The following lemma tells us that under a suitable condition only little L^2-norm of a spinor is contained in the annulus $A_N(r,2r)$ compared to the bigger annulus $A_N(r,(2r)^{1/11})$ when r is small.

Lemma 2.4. *Let M be an n-dimensional Riemannian spin manifold and let $N \subset M$ be a compact submanifold of codimension $k \geq 3$. Then there exists $0 < R < 1$ depending on M and N such that for any $0 < r \leq \frac{1}{2}R^{11}$ and any smooth spinor φ defined on $A_N(r,(2r)^{1/11})$ the following estimate holds*

$$\frac{\|\varphi\|^2_{L^2(A_N(r,2r))}}{\|\varphi\|^2_{L^2(A_N(r,(2r)^{1/11}))}} \quad \leq \quad 10\; r^{5/2}$$

provided

$$\mathrm{Re}\int_{S_N(\rho)} \langle \nabla_\nu \varphi, \varphi\rangle \, dA \geq 0$$

holds for all $\rho \in [r,(2r)^{1/11}]$.

Proof. From

$$\frac{d}{d\rho}\, dA \;=\; -(n-1)\, H\, dA$$

and

$$\frac{d}{d\rho}\, |\varphi|^2 \;=\; 2\,\mathrm{Re}\,\langle \nabla_\nu\varphi,\varphi\rangle$$

we conclude

$$\begin{aligned} \frac{d}{d\rho}\int_{S_N(\rho)} |\varphi|^2\, dA \quad &= \quad \mathrm{Re}\int_{S_N(\rho)} \{2\langle\nabla_\nu\varphi,\varphi\rangle - (n-1)\, H\, |\varphi|^2\}\, dA \\ &\geq \quad -(n-1)\int_{S_N(\rho)} H\, |\varphi|^2\, dA. \end{aligned} \tag{3}$$

For $p \in N$ and ν a unit normal vector to N at p let P_ν denote the orthogonal projection

$$P_\nu \,:\, \nu^\perp \longrightarrow \nu^\perp \cap T_pN^\perp \;.$$

In particular, we have the orthogonal decomposition

$$T_pM \;=\; T_pN \oplus \mathbb{R}\,\nu \oplus \mathrm{im}(P_\nu) \;.$$

Let $P_{\nu,\rho}$ denote the parallel translate of P_ν along $c(\rho) = \exp(\rho\,\nu)$. Then W is of the form

$$W\,(c(\rho)) \;=\; -\frac{1}{\rho}\, P_{\nu,\rho} + C\,(\rho) \;,$$

where $C\,(\rho)$ extends smoothly to $\rho = 0$, compare [7]. Since the rank of P_ν is equal to $k-1$ we conclude that

$$(n-1)H = -\frac{k-1}{\rho} + \mathrm{O}_{M,N}\;(1)\;.$$

For $0 < \rho \leq R$, where R is sufficiently small, we have

$$|\mathrm{O}_{M,N}\;(1)| \leq \frac{1}{4\rho}$$

and therefore

$$(n-1)H \leq -\frac{k-5/4}{\rho}.$$

Plugging this into (3) we get

$$\frac{d}{d\rho}\int_{S_N(\rho)}|\varphi|^2\,dA \geq \frac{k-5/4}{\rho}\int_{S_N(\rho)}|\varphi|^2\,dA \geq \frac{7/4}{\rho}\int_{S_N(\rho)}|\varphi|^2\,dA.$$

since $k \geq 3$. Therefore

$$\frac{d}{d\rho}\ln\int_{S_N(\rho)}|\varphi|^2\,dA \geq \frac{7/4}{\rho}$$

and hence for $r_1 \leq r_2$

$$\begin{aligned}
\ln\left(\frac{\int_{S_N(r_2)}|\varphi|^2\,dA}{\int_{S_N(r_1)}|\varphi|^2\,dA}\right) &= \ln\left(\int_{S_N(r_2)}|\varphi|^2\,dA\right) - \ln\left(\int_{S_N(r_1)}|\varphi|^2\,dA\right)\\
&\geq \frac{7}{4}\int_{r_1}^{r_2}\frac{d\rho}{\rho}\\
&= \frac{7}{4}\ln\left(\frac{r_2}{r_1}\right).
\end{aligned}$$

Exponentiation yields

$$\frac{\int_{S_N(r_2)}|\varphi|^2\,dA}{\int_{S_N(r_1)}|\varphi|^2\,dA} \geq \left(\frac{r_2}{r_1}\right)^{7/4}. \tag{4}$$

From (3) and $H < 0$ it follows that the function $\rho \mapsto \int_{S_N(\rho)}|\varphi|^2\,dA$ is monotonically increasing, so we have on the one hand

$$\|\varphi\|^2_{L^2(A_N(r,2r))} = \int_r^{2r}\int_{S_N(\rho)}|\varphi|^2\,dA\,d\rho \leq r\int_{S_N(2r)}|\varphi|^2\,dA. \tag{5}$$

On the other hand by (4) with $r_1 = 2r$ and $r_2 = \rho$,

$$\begin{aligned}
\|\varphi\|^2_{L^2(A_N(2r,(2r)^{1/11}))} &= \int_{2r}^{(2r)^{1/11}}\int_{S_N(\rho)}|\varphi|^2\,dA\,d\rho\\
&\geq \int_{S_N(2r)}|\varphi|^2\,dA\int_{2r}^{(2r)^{1/11}}\left(\frac{\rho}{2r}\right)^{7/4}\,d\rho\\
&= \int_{S_N(2r)}|\varphi|^2\,dA\cdot\frac{8r}{11}\left[\left(\frac{1}{2r}\right)^{5/2}-1\right]. \qquad (6)
\end{aligned}$$

Combining (5) and (6) we obtain

$$\begin{aligned}
\frac{\|\varphi\|^2_{L^2(A_N(r,(2r)^{1/11}))}}{\|\varphi\|^2_{L^2(A_N(r,2r))}} &= 1+\frac{\|\varphi\|^2_{L^2(A_N(2r,(2r)^{1/11}))}}{\|\varphi\|^2_{L^2(A_N(r,2r))}}\\
&\geq 1+\frac{8}{11}\left[\left(\frac{1}{2r}\right)^{5/2}-1\right]\\
&> \frac{1}{10}\left(\frac{1}{r}\right)^{5/2},
\end{aligned}$$

and the lemma follows. ∎

2.4. **Proof of Theorem 1.2.** By decreasing ε if necessary we may assume that $(\Lambda-\varepsilon,\Lambda+\varepsilon)$ and $(-\Lambda-\varepsilon,-\Lambda+\varepsilon)$ contain no eigenvalues of D_g and that any two distinct eigenvalues in $(-\Lambda,\Lambda)$ are at distance at least 2ε. Write $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$ for the eigenvalues of D_g in $(-\Lambda,\Lambda)$, each eigenvalue being repeated according to its multiplicity. Denote the distance tube of radius r about N by $U_N(r)$, that is

$$U_N(r) = \{x \in M \mid \mathrm{dist}\,(x,N) < r\} .$$

We will first show that there exists $R > 0$ such that the Dirac operator $D_{\widetilde{g}}$ on any closed Riemannian spin manifold $(\widetilde{M},\widetilde{g})$ containing an isometric copy of $M \setminus U_N(r)$ (together with its spin structure) has eigenvalues $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_m$ with $|\lambda_i - \mu_i| < \varepsilon$ provided $r < R$. It may however have further eigenvalues in the interval $(-\Lambda,\Lambda)$. This will be excluded later under additional assumptions.

To prove the existence of the eigenvalues μ_i let $\chi_r : M \to \mathbb{R}$ be a smooth cut–off function with the following properties

- $0 \leq \chi_r \leq 1$ on M,
- $\chi_r \equiv 0$ on $U_N(r)$,
- $\chi_r \equiv 1$ on $M \setminus U_N(2r)$,
- $|\nabla\chi_r| \leq \frac{2}{r}$ on M.

Since all norms on a finite dimensional vector space are equivalent there exists a constant $C_1 > 0$ such that

$$\|\varphi\|_{L^\infty(M)} \leq C_1\, \|\varphi\|_{L^2(M)}$$

for all eigenspinors φ of D_g with eigenvalue in $(-\Lambda,\Lambda)$. Moreover, there is a constant $C_2 > 0$ such that

$$\mathrm{vol}\,(U_N(2r)) \leq C_2\, r^k .$$

Let φ_i be an eigenspinor on M for the eigenvalue λ_i. Since $\chi_r\,\varphi_i$ has its support in $M \setminus U_N(r)$ it can also be regarded as a spinor on $\widetilde{M}$. We plug it into the Rayleigh quotient of $D_{\widetilde{g}} - \lambda_i$:

$$\begin{aligned}
\frac{\|(D_{\widetilde{g}} - \lambda_i)\,(\chi_r\,\varphi_i)\|^2_{L^2(\widetilde{M})}}{\|\chi_r\,\varphi_i\|^2_{L^2(\widetilde{M})}} &= \frac{\|\nabla\chi_r \cdot \varphi_i\|^2_{L^2(U_N(2r))}}{\|\chi_r\,\varphi_i\|^2_{L^2(M)}} \\
&\leq \frac{\frac{4}{r^2}\,\|\varphi_i\|^2_{L^2(U_N(2r))}}{\|\varphi_i\|^2_{L^2(M)} - \|\varphi_i\|^2_{L^2(U_N(2r))}} .
\end{aligned}$$

From

$$\begin{aligned}
\|\varphi_i\|^2_{L^2(U_N(2r))} &\leq \|\varphi_i\|^2_{L^\infty(U_N(2r))}\,\mathrm{vol}\,(U_N(2r)) \\
&\leq C_1^2\,\|\varphi_i\|^2_{L^2(M)}\,C_2\,r^k
\end{aligned}$$

we conclude

$$\begin{aligned}\frac{\|(D_{\widetilde{g}}-\lambda_i)\,(\chi_r\,\varphi_i)\|^2_{L^2(\widetilde{M})}}{\|\chi_r\,\varphi_i\|^2_{L^2(\widetilde{M})}} &\le \frac{\frac{4}{r^2}\,C_1^2\,C_2\,r^k}{1-C_1^2\,C_2\,r^k}\\ &= \frac{4\,C_1^2\,C_2\,r^{k-2}}{1-C_1^2\,C_2\,r^k}\\ &< \varepsilon^2\end{aligned}$$

for r sufficiently small because $k \ge 3$. Therefore the operator $D_{\widetilde{g}} - \lambda_i$ has an eigenvalue in the interval $(-\varepsilon, \varepsilon)$, hence $D_{\widetilde{g}}$ has an eigenvalue μ_i in $(\lambda_i - \varepsilon, \lambda_i + \varepsilon)$. If λ_i has multiplicity ℓ this yields ℓ eigenvalues of $D_{\widetilde{g}}$ in $(\lambda_i - \varepsilon, \lambda_i + \varepsilon)$.

It remains to be shown that in the situation of the theorem $\widetilde{g}$ can be chosen so that the operator $D_{\widetilde{g}}$ has no more than m eigenvalues in $(-\Lambda, \Lambda)$. According to Proposition 2.1 there is a metric on M arbitrarily close to g in the C^1-topology such that with respect to this metric

- scal $\ge S_0$ on all of M,
- scal $\ge 2S_1$ on a neighborhood U_0 of N with S_1 arbitrarily large.

Since the eigenvalues of D_g depend continuously on the Riemannian metric with respect to the C^1-topology (see for example [3, Prop. 7.1]) we may assume without loss of generality that the scalar curvature of g itself has these properties with S_1 so large that

$$\frac{S_1 - 4\Lambda^2}{S_1 - S_0} \ge \left(\frac{\Lambda + \varepsilon/2}{\Lambda + \varepsilon}\right)^2 .$$

Now choose $r > 0$ so small that

- $r < \frac{\varepsilon^4}{2^8\cdot 100\cdot (m+1)^4}$,
- $U_N((2r)^{1/11}) \subset U_0$,
- $(2r)^{1/11}$ is no larger than the R in Lemma 2.4.

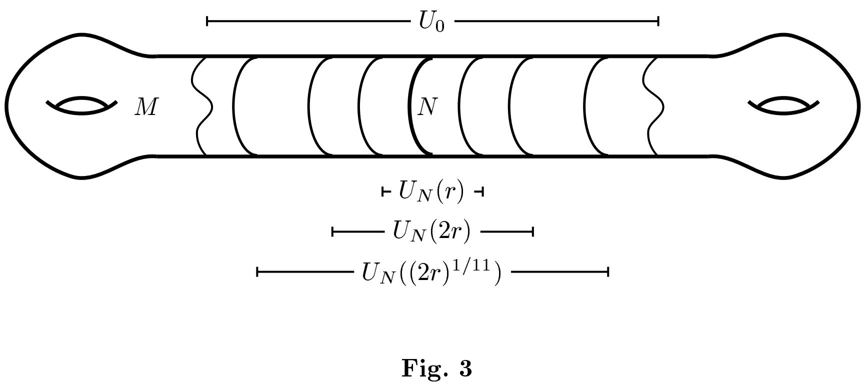

Fig. 3

We perform the surgery along N in the neighborhood $U = U_N(r)$. Hence $\widetilde{M}$ is of the form $\widetilde{M} = (M \setminus U_N(r)) \cup \widetilde{U}$. Surgery in codimension ≥ 3 does not decrease scalar curvature too much if the metric $\widetilde{g}$ on $\widetilde{M}$ is chosen properly, see [9, Proof of Theorem A] and [21, Proof of Theorem 3.1]. We may assume

- $\mathrm{scal}_{\widetilde{g}} \geq S_0$ on all of $\widetilde{M}$,
- $\mathrm{scal}_{\widetilde{g}} \geq S_1$ on $\widetilde{U}$.

We will show that with these choices of r and $\widetilde{g}$ the Dirac operator of $\widetilde{M}$ can have no more than m eigenvalues in $(-\Lambda, \Lambda)$. Assume the contrary, that is, assume there is an $(m+1)$-dimensional space $\widetilde{\mathcal{H}}$ of spinors on $\widetilde{M}$ spanned by eigenspinors with eigenvalues in $(-\Lambda, \Lambda)$. The function χ_r can also be considered as a cut–off function on $\widetilde{M}$ since its support is contained in $M \setminus U_N(r)$ which is contained in $\widetilde{M}$. For every spinor φ on $\widetilde{M}$ the spinor $\chi_r \varphi$ can be considered a spinor also on M. The space

$$\mathcal{H} := \{\chi_r \varphi \mid \varphi \in \widetilde{\mathcal{H}}\}$$

has the same dimension $m+1$ as $\widetilde{\mathcal{H}}$ by the unique-continuation property of the Dirac operator (see for example [5]) and we consider it as a space of spinors on M. We will show that

$$\frac{\|D_g \psi\|^2_{L^2(M)}}{\|\psi\|^2_{L^2(M)}} < (\Lambda + \varepsilon)^2$$

for all nonzero $\psi \in \mathcal{H}$. Then D_g must have at least $m+1$ eigenvalues in $(-\Lambda - \varepsilon, \Lambda + \varepsilon)$ which is a contradiction.

Let $\psi := \chi_r \varphi \in \mathcal{H}$. We write $\varphi = \varphi_1 + \cdots + \varphi_{m+1}$ with $D_{\widetilde{g}}$-eigenspinors φ_j (some possibly zero) for the eigenvalues $\mu_j \in (-\Lambda, \Lambda)$. By the assumption on the scalar curvature of $\widetilde{g}$ (recall that scal $\geq S_1$ on $\widetilde{U}$ and scal $\geq 2\,S_1$ on $A_N(r, 2r)$) and by Lemma 2.2 applied to $\widetilde{M}$ we have

$$\|\varphi\|^2_{L^2(\widetilde{U} \cup A_N(r,2r))} \leq \frac{4\Lambda^2 - S_0}{S_1 - S_0} \|\varphi\|^2_{L^2(\widetilde{M})}.$$

Thus

$$\begin{aligned}
\|\psi\|^2_{L^2(M)} &= \|\chi_r \varphi\|^2_{L^2(M)} \geq \|\varphi\|^2_{L^2(M \setminus U_N(2r))} \\
&\geq \left[1 - \frac{4\Lambda^2 - S_0}{S_1 - S_0}\right] \cdot \|\varphi\|^2_{L^2(\widetilde{M})} \\
&= \frac{S_1 - 4\Lambda^2}{S_1 - S_0} \|\varphi\|^2_{L^2(\widetilde{M})} \\
&\geq \left(\frac{\Lambda + \frac{\varepsilon}{2}}{\Lambda + \varepsilon}\right)^2 \|\varphi\|^2_{L^2(\widetilde{M})}.
\end{aligned}$$

Secondly,

$$\begin{aligned}
\|D_g\,\psi\|_{L^2(M)} &= \|D_{\widetilde{g}}\,(\chi_r\,\varphi)\|_{L^2(\widetilde{M})}\\
&= \|\nabla\chi_r\cdot\varphi+\chi_r\,D_{\widetilde{g}}\,\varphi\|_{L^2(\widetilde{M})}\\
&\le \|\nabla\chi_r\cdot\varphi\|_{L^2(\widetilde{M})}+\|D_{\widetilde{g}}\,\varphi\|_{L^2(\widetilde{M})}\\
&\le \frac{2}{r}\,\|\varphi\|_{L^2(A_N(r,2r))}+\Lambda\,\|\varphi\|_{L^2(\widetilde{M})}\,. \qquad (7)
\end{aligned}$$

Now we justify that we can apply Lemma 2.4 to the eigenspinors φ_j. Fix $\rho\in[r,(2r)^{1/11}]$. We have to show

$$\mathrm{Re}\int_{S_N(\rho)}\langle\nabla_\nu\varphi_j,\varphi_j\rangle\,dA\ge 0.$$

Set $\widehat{M}:=\widetilde{U}\cup A_N(r,\rho)$. Then $\widehat{M}\subset\widetilde{M}$ is a compact manifold with boundary $\partial\widehat{M}=S_N(\rho)$ and scal $\ge S_1$ on $\widehat{M}$.

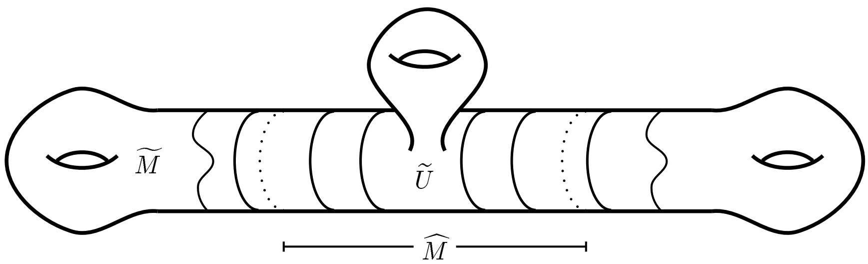

Fig. 4

The Lichnerowicz formula and a partial integration yield

$$\begin{aligned}
\mu_j^2\,\|\varphi_j\|^2_{L^2(\widehat{M})} &= \int_{\widehat{M}}\langle D^2\varphi_j,\varphi_j\rangle\,\mathrm{dvol}\\
&= \int_{\widehat{M}}\langle\nabla^*\nabla\varphi_j,\varphi_j\rangle\,\mathrm{dvol}+\frac{1}{4}\int_{\widehat{M}}\mathrm{scal}\,|\varphi_j|^2\,\mathrm{dvol}\\
&\ge \|\nabla\varphi_j\|^2_{L^2(\widehat{M})}-\int_{\partial\widehat{M}}\langle\nabla_\nu\varphi_j,\varphi_j\rangle\,dA+\frac{S_1}{4}\,\|\varphi_j\|^2_{L^2(\widehat{M})}.
\end{aligned}$$

Hence

$$\int_{S_N(\rho)}\langle\nabla_\nu\varphi_j,\varphi_j\rangle\,dA\ge\left(\frac{S_1}{4}-\Lambda^2\right)\,\|\varphi_j\|^2_{L^2(\widehat{M})}\ge 0.$$

Thus Lemma 2.4 can be applied and yields

$$\begin{aligned}
\|\varphi_j\|^2_{L^2(A_N(r,2r))} &\le 10\,r^{5/2}\,\|\varphi_j\|^2_{L^2(A_N(r,(2r)^{1/11}))}\\
&\le 10\,r^{5/2}\,\|\varphi_j\|^2_{L^2(\widetilde{M})}\\
&\le 10\,r^{5/2}\,\|\varphi\|^2_{L^2(\widetilde{M})}\,.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{2}{r}\,\|\varphi\|_{L^2(A_N(r,2r))} &\le \frac{2}{r}\,\big\{\|\varphi_1\|_{L^2(A_N(r,2r))} + \cdots + \|\varphi_{m+1}\|_{L^2(A_N(r,2r))}\big\} \\
&\le \frac{2}{r}\,(m+1)\,\sqrt{10\cdot r^{5/2}}\,\|\varphi\|_{L^2(\widetilde{M})} \\
&= 2\sqrt{10}\,(m+1)\,r^{1/4}\,\|\varphi\|_{L^2(\widetilde{M})} \\
&< \frac{\varepsilon}{2}\,\|\varphi\|_{L^2(\widetilde{M})}\,.
\end{aligned}$$

Plugging this into (7) gives

$$\|D_g\psi\|_{L^2(M)} < \left(\Lambda + \frac{\varepsilon}{2}\right)\,\|\varphi\|_{L^2(\widetilde{M})}$$

and therefore

$$\frac{\|D_g\psi\|^2_{L^2(M)}}{\|\psi\|^2_{L^2(M)}} < \frac{\left(\Lambda+\frac{\varepsilon}{2}\right)^2\,\|\varphi\|^2_{L^2(\widetilde{M})}}{\left(\frac{\Lambda+\frac{\varepsilon}{2}}{\Lambda+\varepsilon}\right)^2\,\|\varphi\|^2_{L^2(\widetilde{M})}} = (\Lambda+\varepsilon)^2$$

which completes the proof of Theorem 1.2.

3. Generic metrics and harmonic spinors

The *alpha-genus* is a ring homomorphism $\alpha_* : \Omega_*^{\mathrm{spin}} \to KO^{-*}(\mathrm{pt})$ from the spin bordism ring to the real K-theory ring. This means that the alpha-genus $\alpha(M)$ of a spin manifold M depends only on the spin bordism class of M and that it is additive with respect to disjoint union and multiplicative with respect to product of spin manifolds. Choosing generators we have

$$KO^{-n}(\mathrm{pt}) = \begin{cases} \mathbb{Z} & \text{if } n \equiv 0,4 \mod 8, \\ \mathbb{Z}/2 & \text{if } n \equiv 1,2 \mod 8, \\ 0 & \text{otherwise.} \end{cases}$$

In this identification $\alpha_{8k} = \widehat{A}_{8k}$ and $\alpha_{8k+4} = \frac{1}{2}\widehat{A}_{8k+4}$ where $\widehat{A}_* : \Omega_*^{\mathrm{spin}} \to \mathbb{Z}$ is the $\widehat{A}$-genus.

There is a generalization $\mathcal{D}$ of the Dirac operator called the *Cl_n-linear Atiyah-Singer operator* [15]. This operator acts on sections of a bundle associated to the spin structure with fiber the Clifford algebra Cl_n and commutes with the right action of Cl_n on this bundle. The operator $\mathcal{D}$ has a "Clifford index" $\mathrm{ind}_n(\mathcal{D}) \in KO^{-n}(\mathrm{pt})$. This bundle of Clifford algebras decomposes into a sum of copies of the spinor bundle and the Clifford index is related to the index of the Dirac operator acting on sections of the spinor bundle as follows.

$$\mathrm{ind}_n(\mathcal{D}) = \begin{cases} \mathrm{ind}(D^+) & \text{if } n \equiv 0 \mod 8, \\ \frac{1}{2}\,\mathrm{ind}(D^+) & \text{if } n \equiv 4 \mod 8, \\ \dim\ker D \mod 2 & \text{if } n \equiv 1 \mod 8, \\ \dim\ker D^+ \mod 2 & \text{if } n \equiv 2 \mod 8. \end{cases}$$

The index theorem for $\mathcal{D}$ states that $\mathrm{ind}_n(\mathcal{D})$ coincides with $\alpha_n(M)$ and this gives a lower bound on the dimension of the kernel of the Dirac operator D on a manifold M,

$$\dim\ker D \geq \begin{cases} |\widehat{A}(M)| & \text{if } n \equiv 0,4 \mod 8, \\ |\alpha(M)| & \text{if } n \equiv 1 \mod 8, \\ 2|\alpha(M)| & \text{if } n \equiv 2 \mod 8. \end{cases}$$

The left hand side of this inequality depends on the Riemannian metric while the right hand side is a differential topological invariant. In the following definition we give a name to those Riemannian metrics for which equality holds.

Definition 3.1. A Riemannian metric g on a compact spin manifold M is called D-*minimal* if the kernel of the Dirac operator D_g is no larger than it is forced to be by the index theorem.

In dimensions $3,5,6,7 \mod 8$ the index of D always vanishes and a metric g is D-minimal if $\ker D_g = 0$.

Denote by h_g the dimension of the kernel of the Dirac operator D_g and if the dimension is even denote by h_g^+, h_g^- the dimensions of the kernels of D_g^+, D_g^-. On an n-dimensional manifold M with a D-minimal metric g the numbers h_g, h_g^+, h_g^- are determined by α according to the following table.

$n \mod 8$	$\alpha(M)$	
$0,4$	≥ 0	$h_g^+ = \widehat{A}, \quad h_g^- = 0$
	< 0	$h_g^+ = 0, \quad h_g^- = -\widehat{A}$
1	0	$h_g = 0$
	1	$h_g = 1$
2	0	$h_g^+ = h_g^- = 0$
	1	$h_g^+ = h_g^- = 1$
$3,5,6,7$	0	$h_g = 0$

With this terminology Conjecture B from the introduction can be phrased as

Conjecture B. *On any compact connected spin manifold a generic metric is D-minimal.*

The first result in this direction is by Anghel [1] who studies twisted Dirac operators on a manifold with a fixed Riemannian metric and shows that in dimensions ≤ 4 a generic choice of connection on the twisting bundle gives a minimal kernel. The idea of the proof is to show that a non-minimal connection can be deformed to give nearby minimal connections. Such deformations are easily found only under the strong restriction on dimension. Using the formulas by Bourguignon and Gauduchon [6] for the variation of the spinor bundle and the Dirac operator with the Riemannian metric Maier [17] adapts the method of Anghel to prove Conjecture B in the same dimensions.

Denote by $\mathcal{R}(M)$ the space of smooth Riemannian metrics on a compact spin manifold M and by $\mathcal{R}_{\min}(M)$ the subset of D-minimal metrics. We have not yet made clear what we mean by the term "generic". Standard results from perturbation theory give the following proposition (see [12, Section VII.1.3], [1], [17]).

Proposition 3.2. *$\mathcal{R}_{\min}(M)$ is open in the C^1-topology on $\mathcal{R}(M)$ and if it is non-empty it is dense in all C^k-topologies, $k \geq 1$.*

We take "generic" to mean "in a set which is open in the C^1-topology on $\mathcal{R}(M)$ and dense in all C^k-topologies, $k \geq 1$". Denote by $\mathcal{M}_{\min}$ the class of compact spin manifolds possessing at least one D-minimal metric. It follows from Proposition 3.2 that Conjecture B is true precisely for manifolds in $\mathcal{M}_{\min}$.

The rest of this paper is devoted to a study of the class $\mathcal{M}_{\min}$.

3.1. **Properties of the class $\mathcal{M}_{\min}$.** Note that the conclusion of Conjecture B may not hold if the manifold is not connected. Let M be a manifold with $\alpha(M) \neq 0$ and let $-M$ be the same manifold with reversed orientation. Then the disjoint union $M + (-M)$ has $\alpha(M + (-M)) = 0$ but the kernel of the Dirac operator cannot be trivial because it then would be trivial also on M. The following proposition tells us that we can take disjoint unions (and later connected sums) in the class $\mathcal{M}_{\min}$ whenever there is no cancellation of the alpha-genera.

Proposition 3.3. *Let $M_1, M_2 \in \mathcal{M}_{\min}$ with $\dim M_1 = \dim M_2 = n$. Suppose*

(1) $n \equiv 0, 4 \mod 8$ *and* $\alpha(M_1)\alpha(M_2) \geq 0$, *or*
(2) $n \equiv 1, 2 \mod 8$ *and* $\alpha(M_1)\alpha(M_2) = 0$, *or*
(3) $n \equiv 3, 5, 6, 7 \mod 8$.

Then the disjoint union $M_1 + M_2 \in \mathcal{M}_{\min}$.

Proof. We only treat the first case $n = 4k$ and $\alpha(M_1)\alpha(M_2) \geq 0$. The other cases are similar.

Let g_1 and g_2 be D-minimal metrics on M_1 and M_2 and let g be the metric on $M_1 + M_2$ given by g_1 and g_2. Suppose $\alpha(M_1 + M_2) = \alpha(M_1) + \alpha(M_2) = 0$, then $\alpha(M_1) = \alpha(M_2) = 0$ since we assume $\alpha(M_1)\alpha(M_2) \geq 0$. This means that $h_{g_1} = h_{g_2} = 0$ so $h_g = 0$ and g is D-minimal. Next suppose $\alpha(M_1 + M_2) = \alpha(M_1) + \alpha(M_2) > 0$. Then we cannot have $\alpha(M_1) \leq 0$ and $\alpha(M_2) \leq 0$. By assumption $\alpha(M_1)$ and $\alpha(M_2)$ do not have different signs and we must have $\alpha(M_1) \geq 0$ and $\alpha(M_2) \geq 0$. Since g_1 and g_2 are D-minimal we have $h_g^- = h_{g_1}^- + h_{g_2}^- = 0 + 0 = 0$ so g is D-minmimal. The case $\alpha(M_1 + M_2) < 0$ is analogous to $\alpha(M_1 + M_2) > 0$. $\square$

A metric g with positive scalar curvature is D-minimal since the Lichnerowicz formula forces $h_g = 0$. This gives a rich source of manifolds in the class $\mathcal{M}_{\min}$.

Proposition 3.4. *Let M be a compact spin manifold which has a metric of positive scalar curvature. Then $M \in \mathcal{M}_{\min}$.*

A manifold with a positive scalar curvature metric has vanishing alpha-genus [10]. We will need a set of manifolds in $\mathcal{M}_{\min}$ whose alpha-genera generate $KO^{-*}(\mathrm{pt})$.

Proposition 3.5. *For each* $n \equiv 0, 1, 2, 4 \mod 8$, $n \geq 1$, *there is a* $V_n \in \mathcal{M}_{\min}$ *with* $\dim(V_n) = n$ *and* $\alpha(V_n) = 1$.

Proof. We prove this by induction starting with $n = 1, 2, 4, 8$.

Let V_1 be the circle with the spin structure which is not spin bordant to zero, let $V_2 = V_1 \times V_1$, and let V_4 be the K3 surface. Let V_8 be a compact Riemannian 8-manifold with holonomy Spin(7) [11]. For V_1 we have $\alpha(V_1) = 1$ and $h = 1$ so $V_1 \in \mathcal{M}_{\min}$. By conformal invariance any metric g on V_2 has $h_g^+ = h_g^- = 1$ and since $\alpha(V_2) = 1$ we have $V_2 \in \mathcal{M}_{\min}$. The manifold V_4 has $\widehat{A}(V_4) = 2\alpha(V_4) = 2$ so for any metric g we have $h_g^+ \geq 2$. Let g_0 be a Ricci flat metric on V_4. By the Lichnerowicz formula all harmonic spinors are parallel for this metric, so $h_{g_0}^+ \leq \operatorname{rank}(\Sigma^+ V_4) = 2$. This means that $h_{g_0}^+ = 2$ so g_0 is D-minimal and $V_4 \in \mathcal{M}_{\min}$. The manifold V_8 with a Spin(7)-holonomy metric g_8 has a one-dimensional space of parallel spinors [25] and since it is Ricci-flat there are no other harmonic spinors. This means that $\alpha(V_8) = \widehat{A}(V_8) = 1 = h_{g_8}^+$ so g_8 is D-minimal and $V_8 \in \mathcal{M}_{\min}$.

Next suppose that for some $n = 8k + 1, 8k + 2, 4k + 4$, $k \geq 0$ we have a manifold V_n with the desired properties. Set $V_{n+8} := V_8 \times V_n$ with the product metric $g_{n+8} = g_8 + g_n$, where g_n is a D-minimal metric on V_n. Then $\alpha(V_{n+8}) = \alpha(V_8)\alpha(V_n) = 1$. The spinor bundle on V_{n+8} is isomorphic to the outer tensor product of the spinor bundles on V_8 and V_n, $\Sigma V_{n+8} = \Sigma V_8 \widehat{\otimes} \Sigma V_n$, and the Dirac operators are related by $D_{g_{n+8}}^2 = D_{g_8}^2 \otimes 1 + 1 \otimes D_{g_n}^2$. From this we see that the harmonic spinors on the product are given precisely by products of harmonic spinors on the factors. In particular we have $h_{g_{n+8}} = h_{g_8} \cdot h_{g_n} = h_{g_n}$. Since g_n is D-minimal g_{n+8} is a D-minimal metric on V_{n+8} and $V_{n+8} \in \mathcal{M}_{\min}$. □

The most important step in our study of the class $\mathcal{M}_{\min}$ is the following application of the surgery theorem for the Dirac spectrum.

Proposition 3.6. $\mathcal{M}_{\min}$ *is closed under surgery in codimension* ≥ 3.

Proof. Let $M \in \mathcal{M}_{\min}$ and suppose $\widetilde{M}$ is obtained from M by surgery in codimension ≥ 3. Let g be a D-minimal metric on M and choose $\Lambda > 0$ so that the interval $[-\Lambda, \Lambda]$ does not contain any non-zero eigenvalues of D_g. By Theorem 1.2 there is a metric $\widetilde{g}$ on $\widetilde{M}$ with the property that $D_{\widetilde{g}}$ has precisely h_g many eigenvalues in the interval $[-\Lambda, \Lambda]$ (counting multiplicity). Hence $h_{\widetilde{g}} \leq h_g$. Since surgery does not change the spin bordism class of a manifold we have $\alpha(\widetilde{M}) = \alpha(M)$ and therefore $h_{\widetilde{g}} \geq h_g$ because g is D-minimal. We conclude that $h_{\widetilde{g}} = h_g$ and therefore the metric $\widetilde{g}$ is also D-minimal. □

3.2. **Simply connected manifolds.** We want to use Proposition 3.6 to construct new manifolds in the class $\mathcal{M}_{\min}$. Without restrictions on the codimensions we can start with a spin manifold N and do a sequence of surgeries ending up with a spin manifold M if and only if N and M are spin bordant. The problem in our situation is that Proposition 3.6 allows only surgeries of codimension at least three. The same difficulty one encounters in the study of manifolds admitting metrics of positive scalar curvature.

Gromov and Lawson [9] show that these surgeries are sufficient to handle simply-connected manifolds (where in simply connected it is included that the manifold is connected).

Theorem 3.7. [9] *Suppose that M is a compact simply connected spin manifold of dimension ≥ 5 and suppose that M is spin bordant to a spin manifold N. Then M can be obtained from N by a sequence of surgeries of codimension ≥ 3.*

Using this theorem and the Surgery Theorem 1.2 we get the following result relating the spectrum of the Dirac operator for two ends of a spin bordism:

Theorem 3.8. *Let M be a compact simply connected spin manifold of dimension ≥ 5 and let M be spin bordant to a spin manifold N. Let h be a Riemannian metric on N and let $\Lambda, \varepsilon > 0$ be such that $\pm\Lambda \notin \mathrm{spec}(D_h)$. Then there is a Riemannian metric g on M such that D_h and D_g are (Λ, ε)-spectral close.*

We will now show that the manifolds from Propositions 3.4 and 3.5 are sufficiently plenty to allow us to find a manifold in $\mathcal{M}_{\min}$ in each spin bordism class.

Proposition 3.9. *Any compact spin manifold is spin bordant to a manifold in $\mathcal{M}_{\min}$.*

Proof. Let M be a compact spin manifold of dimension n and let $p = \alpha(M)$. If $p \geq 0$ let pV_n be the disjoint union of p copies of the manifold V_n from Proposition 3.5 and if $p < 0$ let pV_n be the disjoint union of $-p$ copies of $-V_n$. We have $\alpha(M - pV_n) = \alpha(M) - p\alpha(V_n) = 0$ so by Theorem B of Stolz [24] $M - pV_n$ is spin bordant to a manifold E which allows a positive scalar curvature metric. This means that M is spin bordant to $E + pV_n$ which by Propositions 3.3 and 3.4 is in $\mathcal{M}_{\min}$. ☐

We are now ready to prove our main result on Conjecture B.

Theorem 3.10. *Conjecture B is true for all simply connected spin manifolds of dimension at least five.*

Proof. Suppose M is a simply connected spin manifold with $\dim M \geq 5$. From the previous proposition we know that M is spin bordant to a manifold in $\mathcal{M}_{\min}$. Theorem 3.7 tells us that this bordism can be decomposed into a sequence of surgeries of codimension at least three. Using Proposition 3.6 we conclude that $M \in \mathcal{M}_{\min}$ and that the conjecture holds for M. ☐

3.3. **Non-simply connected manifolds.** In the study of manifolds allowing positive scalar curvature metrics a machinery has been built up to analyse when two manifolds are related by surgeries of codimension ≥ 3, see for example [19, 21]. We are now going to give one example of how results from this area can be used to prove Conjecture B for certain classes of fundamental groups.

For non-simply connected manifolds one uses a refinement of the ordinary spin bordism groups Ω_*^{spin}. Let π be a group and let $B\pi$ be a classifying space for π. The bordism groups $\Omega_*^{\mathrm{spin}}(B\pi)$ consists of bordism classes of pairs (M, f) where M is a compact spin manifold and $f : M \to B\pi$ is a map. In particular, if M has fundamental group π we get such a pair (M, f) since the universal cover of M is a π-bundle over M and corresponds to a map $f : M \to B\pi$.

In [18] Rosenberg proves the following generalization of the bordism theorem 3.7:

Theorem 3.11. *Let M be a compact spin manifold of dimension ≥ 5 with fundamental group π. Let f be the classifying map of the universal cover of M. Suppose that (M, f) is equivalent to (N, f') in $\Omega_*^{\mathrm{spin}}(B\pi)$ (where f' is any map $N \to B\pi$). Then M can be obtained from N by a sequence of surgeries in codimension ≥ 3.*

For manifolds M and N satisfying the conditions of this theorem we get a bordism result for the Dirac spectrum just as Theorem 3.8.

A finite group π is called *periodic* if it has periodic cohomology, that is if there is $d > 0$ so that $H^i(\pi) = H^{i+d}(\pi)$ for all $i > 0$. This includes cyclic groups of all orders.

Theorem 3.12. *Suppose M is a compact connected spin manifold with $\dim M \geq 5$ and fundamental group π a periodic group of odd order. Then Conjecture B holds for M.*

Proof. Let $f : M \to B\pi$ be the classifying map of the universal cover of M. The class $(M, f) \in \Omega_*^{\mathrm{spin}}(B\pi)$ can be written as

$$(M, f) = (M, f) - (M, \mathrm{const}) + (M, \mathrm{const}).$$

Composition with the constant map $B\pi \to \mathrm{pt}$ gives a map $\Omega_n^{\mathrm{spin}}(B\pi) \to \Omega_n^{\mathrm{spin}}(\mathrm{pt})$ and the first two terms $(M, f) - (M, \mathrm{const})$ constitute an element in the kernel $\widetilde{\Omega}_n^{\mathrm{spin}}(B\pi)$ of this map. From Theorem 1.8 and fact (1.2) in [14] we know that this class can be represented by a manifold M' with positive scalar curvature and fundamental group π. Since M' has a metric of positive scalar curvature $M' \in \mathcal{M}_{\min}$. By doing surgery on embedded circles we can make the third term (M, const) simply connected without changing its class in $\Omega_*^{\mathrm{spin}}(B\pi)$. The resulting manifold N is in $\mathcal{M}_{\min}$ by Theorem 3.10.

We thus have that (M, f) is spin bordant to $(M', f') + (N, \mathrm{const})$ over $B\pi$. Since $\alpha(M') = 0$ Proposition 3.3 tells us that the disjoint union $M' + N \in \mathcal{M}_{\min}$. Theorem 3.11 and Proposition 3.6 finish the proof. □

Remark 3.13. The crucial point of the proof is the fact that for odd order periodic groups π the kernel $\widetilde{\Omega}_n^{\mathrm{spin}}(B\pi)$ can be represented by manifolds allowing positive scalar curvature metrics. This fact — and the theorem — holds for any odd order finite group for which the Gromov-Lawson-Rosenberg conjecture is true, see Section 3 of [20].

Remark 3.14. The torus T^n with a flat metric (and the trivial spin structure) has a large space of harmonic spinors, even though $\alpha(T^n) = 0$. It is not clear when in general the product of two manifolds in $\mathcal{M}_{\min}$ is again in $\mathcal{M}_{\min}$. By taking products we can however prove the following.

On the torus T^n with any spin structure there is a metric for which the kernel of the Dirac operator is trivial.

Namely, from the solution of Conjecture B for manifolds of low dimensions by Maier [17] we know that (for any spin structure) there is a metric g_3 on T^3 without nontrivial harmonic spinors. Write $T^n = T^3 \times T^{n-3}$ and set $g_n = g_3 + g_{n-3}$, where g_{n-3} is any metric on T^{n-3}. Since D_{g_3} does not have zero as an eigenvalue it follows from the Pythagorean formula for the eigenvalues of the product metric g_n that D_{g_n} has no zero eigenvalue.

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