A semi-classical K.A.M. theorem

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Abstract

We consider a semi-classical completely integrable system defined by a $\hbar$-pseudodifferential operator $\hat{H}$ on the torus $\mathbb{T}^d$. In order to study perturbed operators of the form $\hat{H} + \hbar^\kappa \hat{K}$, where $\hat{K}$ is an arbitrary pseudodifferential operator and $\kappa > 0$, we prove the conjugacy to a suitable normal form. This is then used to construct a large number of quasimodes.

1 Introduction

1.1 Semi-classical perturbations of completely integrable systems

One can probably state that the starting point of the study of completely integrable (CI in short) systems dates back to Marquis de Laplace's "La mécanique céleste" [21] in which he noticed the important role the first integrals of the motion could play in studying mechanical systems. It was Liouville [22] who then showed that the knowledge of sufficiently many first integrals in involution allows one to integrate completely the system of differential equations of classical mechanics. This result was actually local and one had to wait for the famous "Angle-action coordinates theorem" proved independently by Arnol'd [2] and the French mathematician Henri Mineur [25, 26], to get a more intrinsic and global understanding of Liouville's result. They proved the existence of a local fibration in invariant lagrangian tori on which the dynamics is linear : periodic or quasi-periodic. On the one hand, these CI systems present many elegant and natural geometrical structures (affine connection on each torus, integer affine structure on the space of tori, several natural bundles,...) and their dynamics is very

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simple. On the other hand, they are not generic, as was already suggested by Poincaré [29] who guessed that generic systems present some chaotic behaviour.

It is thus natural to investigate systems close to the CI ones. This strategy was actually considered by Poincaré [29] as one of the main problems of mechanics. It proved to be fruitful according to the celebrated K.A.M. theorem [20, 1, 27] which insures that a "large part of the completely integrable character" is preserved when a small perturbation εK is added to a completely integrable Hamiltonian H : namely the tori on which the completely integrable dynamics satisfies a certain diophantine relation are simply deformed into invariant lagrangian tori of $H + \varepsilon K$ without being destroyed by the perturbation. They are called *K.A.M. tori*.

In a different context, the theory of pseudodifferential operators (PDOs in short) and Fourier integral operators, initially developped by Hörmander [19], Duistermaat [15] and Maslov [24], and especially the semi-classical analysis with small parameter $\hbar$, developped later by many authors (see e.g. [18, 31, 14]) provide tools with which one can study quantum operators with the help of information coming from their associated classical system. It was in this context that the semi-classical CI systems were studied, first by Colin de Verdière [11] for the homogeneous theory (without parameter $\hbar$), and then by Charbonnel [9] for the $\hbar$-theory. These works provide a transposition to the semi-classical context of the momentum map as well as of the Arnold-Mineur-Liouville theorem. There is also a wide literature about semi-classical CI systems with singularities (see e.g. [12, 13, 36, 37]), but this is beyond the scope of the present paper.

In the semi-classical context, perturbations of any CI $\hbar$-PDO rely on two parameters : a parameter ε which controls the "intensity" of the perturbation and the parameter $\hbar$. In this paper, we will be interested in the regime $\varepsilon \sim \hbar^\kappa$ and thus consider perturbed operators of the form $\hat{H} + \hbar^\kappa \hat{K}$, where $\hat{H}$ is a CI PDO, $\kappa > 0$ and $\hat{K}$ is an arbitrary PDO.

1.2 PDO on the torus

Thanks to [11, 9], any PDO which admits a semi-classical momentum map can be conjugated to an operator on $T^*\mathbb{T}^d = \left\{(x, \xi) \mid x \in \mathbb{T}^d, \xi \in \mathbb{R}^d\right\}$ with a symbol depending only on ξ, microlocally in a neighbourhood of any connected component of any compact regular fiber of the momentum map. We will thus work directly in the angle-action coordinates and use a pseudodifferential calculus well-adapted to the classical situation. Because of the very particular structure of the torus $\mathbb{T}^d$, one is able to construct a pseudodifferential calculus involving globally defined (total) symbols. These operators are sometimes called "*periodic PDO*s" in the literature. Such PDOs have been studied by several authors [10, 16, 35] but always without a small parameter. In this paper, we use a $\hbar$-version of these theories.The basic properties of these operators will be given in Section 2.

We thus start with a completely integrable PDO $\hat{H}$ with symbol $H(\xi)$ being a CI Hamiltonian in the classical sense. Its spectrum is known to be $\{H(\hbar k) \mid k \in \mathbb{Z}^d\}$. We then consider a perturbed operator of the form $\hat{H} + \hbar^\kappa \hat{K}$. It depends on $\hbar$ and we want to investigate the associated family of spectra $\sigma_\hbar$ depending on $\hbar$. Actually, the use of the pseudodifferential calculus allows to investigate $\hbar^\infty$-quasimodes rather than genuine eigenvectors. Such a quasimode is a family of functions $\varphi_\hbar \in L^2(\mathbb{T}^d)$ together with a family of numbers $E_\hbar$ depending on $\hbar$, such that $\|\varphi_\hbar\|_{L^2} = 1$ and

$$\left\| \left(\hat{H} + \hbar^\kappa \hat{K} - E_\hbar \right) \varphi_\hbar \right\|_{L^2} = O(\hbar^\infty).$$

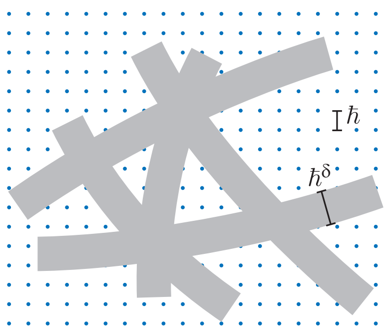
When the operator under consideration is self-adjoint, then $E_\hbar$ is $\hbar^\infty$-close to the spectrum $\sigma_\hbar$[1]. The main result of this paper is the construction of a large number of quasi-eigenvalues $E_\hbar$ of $\hat{H} + \hbar^\kappa \hat{K}$.

Theorem 1. *Let $\hat{H}$ be a CI PDO with symbol $H(\xi)$ and $\hbar^\kappa \hat{K}$ a perturbation. For all fixed $\delta \in]0,1[$, there exists a "quasi-resonant" zone $\mathcal{Z} \subset \mathbb{R}^d_\xi$ depending on $\hbar$ and of relative volume*

$$vol(\mathcal{Z}) \sim \hbar^{\delta-\varepsilon},$$

where ε can be taken arbitrarily small, and such that for all $k_\hbar \in \mathbb{Z}^d$ with $\hbar k_\hbar \in \mathbb{R}^d \setminus \mathcal{Z}$, there exists a $\hbar^\infty$-quasimode $(\varphi_\hbar, E_\hbar)$ of $\hat{H} + \hbar^\kappa \hat{K}$ with

- *$E_\hbar = H(\hbar k_\hbar) + \hbar^\kappa \langle\langle K \rangle\rangle (\hbar k_\hbar) + O(\hbar^{\kappa+\alpha})$, where $\alpha = \min(1-\delta, \kappa - 3\delta)$*
- *$\varphi_\hbar(x) = e^{ik_\hbar x} + O(\hbar^{\kappa-1-\delta})$.*



The assertion about the "relative volume" means that for each compact set $\mathcal{O} \subset \mathbb{R}^d$ the volume of $\mathcal{Z} \cap \mathcal{O}$ is of order $vol(\mathcal{Z}) \sim \hbar^{\delta-\varepsilon}$. The function $\langle\langle K \rangle\rangle$ denotes the average over the torus of the function $K(x,\xi)$. It is constant with respect to x and evaluated at the point $\xi = \hbar k_\hbar$. The picture

[1]Nevertheless, this does not imply that $\varphi_\hbar$ is close to an eigenvector, as first remarked by Arnol'd in [3]. There could even not be any eigenvalues. Only extra informations on the operator can insure that the spectrum is discrete, and consequently that $E_\hbar$ is $\hbar^\infty$-close to a genuine eigenvalue.

above represents the case $d = 2$, the quasi-resonant zone $\mathcal{Z}$ is in grey and the dots represent the lattice $\hbar\mathbb{Z}^d$.

In the semi-classical limit, the *non-resonant set* $\mathbb{R}^d \setminus \mathcal{Z}$ tends to the set of diophantine tori which are preserved by the perturbation, due to K.A.M. theory. This should be compared to the fact that all the eigenvalues $H(\hbar k_\hbar)$ of $\hat{H}$ with $\hbar k \in \mathbb{R}^d \setminus \mathcal{Z}$ are only slightly modified by the perturbation $\hbar^\kappa \hat{K}$ in the sense that there exists a $\hbar^\infty$-quasi-eigenvalue $\hbar^\kappa$-close to $H(\hbar k_\hbar)$. The first correction is the average of the symbol of the $\hbar^\kappa \hat{K}$. This result should be regarded as a semi-classical K.A.M. theorem.

1.3 A result by Feldman-Knörrer-Trubowitz

Before to start to investigate our problem, we should mention the article [17] written in the 90's by Feldman, Knörrer and Trubowitz (FKT in short). The authors studied the high energy asymptotics for the periodic Schrödinger operator $-\Delta + V$ on the torus. Even though this problem is not in the $\hbar$-pseudodifferential context, one can use the "usual" correspondence *Semi-classical limit* $\leftrightarrow$ *High frequency limit* in order to compare the results. Starting from the eigenvalue problem $-\Delta\varphi + V\varphi = \lambda\varphi$, setting $\lambda = \frac{E_\hbar}{\hbar^2}$ and multiplying everything by $\hbar^2$, the problem becomes

$$-\hbar^2\Delta\varphi + \hbar^2 V\varphi = E_\hbar\varphi$$

and the high energy limit $\lambda \to +\infty$ corresponds to investigating eigenvalues $E_\hbar$ of order 1 in the semi-classical limit $\hbar \to 0$. Under this correspondence, FKT's result appears as a special case of ours with the symbol of the completely integrable operator $-\hbar^2\Delta$ being $H(\xi) = \xi^2$ and the perturbation being a multiplication operator of order $\hbar^2$.

We now recall the relevant result of FKT. Let us consider the operator $-\Delta + V$ defined on the torus $\mathbb{R}^d/\Gamma$ where Γ is a generic lattice of $\mathbb{R}^d$, V is a periodic potential and $d \leq 3$. The corresponding unperturbed operator is simply $-\Delta$ and its eigenvectors are e^{ikx} with corresponding eigenvalues k^2, for each $k \in \Gamma^*$ and where the dual lattice Γ^* is the Fourier lattice.

Theorem 2 ([17]). *There exists an "exceptional subset" $S \subset \Gamma^*$ of density zero such that for all $k \in \Gamma^* \setminus S$ the following holds :*
- *There are 2 eigenvalues $\lambda_{\pm k}$ in the interval*

$$\left[k^2 + \langle\langle V \rangle\rangle - \frac{1}{|k|^{2-\varepsilon}}, k^2 + \langle\langle V \rangle\rangle + \frac{1}{|k|^{2-\varepsilon}}\right],$$

where $\langle\langle V \rangle\rangle$ denotes the average of V over the torus and $\varepsilon > 0$ can be taken arbitrarily small.
- *The corresponding eigenvectors $\Psi_{\pm k}$ verify*

$$\left\|\Psi_{\pm k} - \hat{\Pi}\Psi_{\pm k}\right\| = O\left(\frac{1}{|k|^{1-\varepsilon}}\right)$$

where $\hat{\Pi}$ *is the projector on the span of* $e^{\pm ikx}$.

The authors call *stable* the unperturbed eigenvalues k^2 for $k \notin S$ since in the large k limit, they are only slightly modified when the perturbation is added. The first correction is of order $O(1)$ and equals to the average of the "perturbation" V, and the next correction is of order $O\left(\frac{1}{|k|^{2-\varepsilon}}\right)$.

Our result (Theorem 1) thus extends this result to general CI PDOs, with general pseudodifferential perturbations of any order $\hbar^{\kappa}$ and with no restriction on the dimension d. Moreover, the *exceptional subset* S arising in FKT's result is defined in quite a tricky way. But it corresponds in our setting to the intersection of the lattice $\hbar\mathbb{Z}^d$ with the quasi-resonant zone $\mathcal{Z}$ with is quite intuitive and geometric as we will see in the sequel.

1.4 Normal forms and special classes of symbols

The main tool leading to Theorem 1 is a suitable semi-classical normal form for the perturbed operator $\hat{H} + \hbar^{\kappa}\hat{K}$, i.e. a conjugation of $\hat{H} + \hbar^{\kappa}\hat{K}$ by an "almost unitary"[2] operator $\hat{U} : \hat{U}\left(\widehat{init}\right)\hat{U}^* = \widehat{NF} + O(\hbar^{\infty})$, where $\widehat{NF}$ will have good properties. The construction of this normal form is an iterative process whose first step amounts to look for some self-adjoint PDO $\hat{P}$ such that the conjugation by $\hat{U} = e^{i\hbar^{\kappa-1}\hat{P}}$

$$e^{i\hbar^{\kappa-1}\hat{P}}\left(\hat{H} + \hbar^{\kappa}\hat{K}\right)\left(e^{i\hbar^{\kappa-1}\hat{P}}\right)^* = \hat{H} + \hbar^{\kappa}\hat{A} + O\left(\hbar^{1+\kappa}\right)$$

yields a PDO $\hat{A}$ an operator having the "simplest" form as possible, as we discuss just below. One can then write down the corresponding equation for the symbols and check that the cancelation of lower order terms is equivalent to solving

$$X_H(P) = K - A,$$

where X_H is the Hamiltonian vector field associated with the CI Hamiltonian $H(\xi)$ and is thus tangent to each torus ξ. On each torus, the solutions of this *homological equation* depend strongly on the dynamics on the torus (periodic or quasiperiodic). For example, when the dynamics is periodic, then one can choose A to be the average of K along the trajectories and solve the equation with a P depending smoothly on x. On the other hand, when dynamics is quasi-periodic and satisfies a diophantine condition, then one can show that it is possible to solve the equation, with A being the average of K over the whole torus, and to get a P depending smoothly on x.

Unfortunately, one cannot solve this equation in that way, torus by torus, since CI Hamiltonians are generically *non-degenerate.* This property

[2]"almost unitary" means that it satisfies $\hat{U}^*\hat{U} = \mathbb{I} + O(\hbar^{\infty})$ and $\hat{U}\hat{U}^* = \mathbb{I} + O(\hbar^{\infty})$.

implies in particular that the vector field X_H "turns" when one moves in the space of tori. Close to each periodic torus lie some quasi-periodic tori, and vice versa. It is thus impossible to solve the homological equation torus by torus since this would provide a function $P(x, \xi)$ not even continuous with respect to ξ and thus of course not acceptable as a symbol.

However, it is possible in some sense to "interpolate" between regions close to periodic motions and regions close to quasi-periodic motions, and more generally between regions close to any kind of resonance (non-resonant, partially resonant, periodic...). This can be achieved by covering the momentum space $\mathcal{B} = \{\xi\}$ with *quasi-resonant* regions, i.e. neighbourhoods of resonant tori, as people usualy do in Nekhoroshev-like theorems [30, 6]. In order for this construction to be of some interest, the involved neighbourhoods must have a thickness depending on $\hbar$ as $\hbar^{\delta}$, with a positive δ, but on the other hand this obliges us to consider symbols $P(x, \xi)$ whose dependence on ξ becomes "bad" in the semi-classical limit $\hbar \to 0$. In fact, we will see that it is possible to solve the homological equation in the class Ψ^{δ} of symbols satisfying

$$\left|\partial_x^{\alpha} \partial_{\xi}^{\beta} \left(P_{\hbar}(x, \xi)\right)\right| \leq C_{\alpha,\beta} \hbar^{m-\delta|\beta|}.$$

These symbols are actually similar to those used by Sjöstrand in his study of the semi-excited states [34], and one can show that they indeed form an acceptable class of symbols provided $\delta < 1$, i.e. one has a composition law (Moyal's product [28]) and a continuity theorem *à la* Calderón-Vaillancourt [8].

1.5 Plan of the paper

In the next section, we give without proofs the basic results concerning the pseudodifferential calculus on the torus with symbols whose derivatives with respect to ξ can bring "bad" factors $\hbar^{-\delta}$ (i.e. belonging to the class Ψ^{δ}). Namely, we give the compositon law, the L^2-continuity theorem, the properties of adjoints and the functional calculus for such PDOs, and we refer to [33] for detailed proofs.

The Section 3.1 is devoted to the construction of the announced covering of the momentum space $\mathcal{B} = \{\xi\}$ by quasi-resonant zones with thickness going to 0 with $\hbar$. This construction is then used in Section 3.2 in order to define the notion of *quasi-resonant averaging* of functions. Roughly speaking, this permits to interpolate between the regions with different kinds of resonance, and thus provides a function which has a different average property in each of these regions.

Equipped with these tools we are then able to study the perturbed operator $\hat{H} + \hbar^{\kappa} \hat{K}$ using the semi-classical normal form of Theorem 33, in which the symbol in the normal form is the quasi-resonant average of some symbol related to K. Before, we give Lemma 32 which insures that one can solve the homological equation, arising at each step of the normal form iteration, in the class of symbols Ψ^{δ} previously defined.

Finally, as an application of this normal form, we show in Section 4 how to build a large number of quasimodes for the perturbed operator, as stated in Theorem 1.

2 Pseudodifferential operators on the affine torus

2.1 Classes of symbols for periodic $\hbar$-PDOs

We consider the cotangent bundle $T^{*}\mathcal{T}$ of the d-dimensional affine torus $\mathcal{T}=(\mathbb{R}/2\pi\mathbb{Z})^{d}$ and we will denote by (x,ξ) the canonical variables. In the following, we denote by $\Lambda^{*}=\mathbb{Z}^{d}$ the lattice of the Fourier variables, which is the 2π-dual lattice of the lattice of horizontal 1-periodic constant vector fields $\Lambda=2\pi\mathbb{Z}^{d}$, i.e. vector fields of the form $X=X^{j}\frac{\partial}{\partial x^{j}}$ with $X^{j}\in 2\pi\mathbb{Z}$ for all (x,ξ). We will often denote by k the Fourier variables and by $\tilde{f}\left(k,\xi\right)$ the Fourier series with respect to x of a function $f\left(x,\xi\right)$.

Definition 3. Let m and $\delta\geq 0$ be two real constants and $\mathcal{S}\subset T^{*}\mathcal{T}$ any subset. The **class of symbols $\boldsymbol{\Psi_{\delta}^{m}}$** $(\mathcal{S})$ is the set of $\hbar$-families of functions $P_{\hbar}\left(x,\xi\right)\in C^{\infty}\left(T^{*}\mathcal{T},\mathbb{C}\right)$, for $\hbar\in\left]0,1\right]$, such that for all multi-indices $\alpha,\beta\in\mathbb{Z}^{d}$, there exists a constant $C_{\alpha,\beta}>0$ such that for each point $(x,\xi)\in\mathcal{S}$ and each $\hbar\in\left]0,1\right]$, we have the following upper bound

$$\left|\partial_{x}^{\alpha}\partial_{\xi}^{\beta}\left(P_{\hbar}\left(x,\xi\right)\right)\right|\leq C_{\alpha,\beta}\hbar^{m-\delta|\beta|}.$$

When $\mathcal{S}=T^{*}\mathcal{T}$, we simply denote $\Psi_{\delta}^{m}=\Psi_{\delta}^{m}\left(T^{*}\mathcal{T}\right)$. As well, the class of usual symbols (i.e. for $\delta=0$) is simply denoted by $\Psi^{m}=\Psi_{0}^{m}\left(T^{*}\mathcal{T}\right)$. Moreover, it follows from the definition that $\Psi_{\delta}^{m}=\hbar^{m}\Psi_{\delta}^{0}$. On the other hand, the reader should keep in mind that when $\delta\neq 0$ those symbols may not have any well-defined principal symbol $\lim_{\hbar\to 0}P_{\hbar}$.

Since this is a *left quantization*, one can recover the symbol of a PDO by acting on the functions e^{ikx}.

Proposition 4. *If $\hat{P}\in\hat{\Psi}_{\delta}^{m}$ is a PDO of symbol $P_{\hbar}\left(x,\xi\right)$. then for all $k\in\Lambda^{*}$*

$$\hat{P}\left(e^{ikx}\right)=P_{\hbar}\left(x,\hbar k\right)e^{ikx}.$$

A PDO $\hat{P}$ is called a **negligible operator** if $\left\Vert \hat{P}\right\Vert_{\mathcal{L}(L^{2})}=O\left(\hbar^{\infty}\right)$. We denote this by$\hat{P}=O\left(\hbar^{\infty}\right)$. We say that two operators $\hat{A},\hat{B}\in\hat{\Psi}_{\delta}^{m}$ are **equivalent** if they satisfy$\hat{A}-\hat{B}=O\left(\hbar^{\infty}\right)$ and we denote this by $\hat{A}\cong\hat{B}$.

It is also convenient to have a criterion for a function to be in the class Ψ_{δ}^{m} expressed in terms of its Fourier series with respect to the x variable.

Proposition 5. *A function $P_\hbar(x,\xi)$ is a symbol in the class Ψ_δ^m if and only if its Fourier series $\widetilde{P}_\hbar(k,\xi)$ satisfies the following estimate. For each multi-index $\beta \in \mathbb{Z}^d$ and each positive integer s, there exists a constant $C(s,\beta) > 0$ such that for all $k \in \Lambda^*$, all $\xi \in \mathcal{B}$ and all $\hbar \in]0,1]$,*

$$\left|\partial_\xi^\beta \tilde{P}_\hbar(k,\xi)\right| \le C(s,\beta)\frac{\hbar^{m-\delta|\beta|}}{\left(1+|k|^2\right)^{\frac{s}{2}}}.$$

2.2 Composition, L^2 continuity and functional calculus

2.2.1 Asymptotic expansions

First of all, due to the presence of $\hbar^{-\delta}$ factors arising in the derivatives of symbols belonging to the class $\hat{\Psi}_\delta^m$, one is forced to consider more general asymptotic expansions than the usual ones (which read $\hbar^0 P_0(x,\xi) + \hbar^1 P_1(x,\xi) + ...$ and which are sometimes called *classical symbols*). Nevertheless, we will mainly deal with symbols of the following form.

Definition 6. Let $0 \le \delta < 1$. Let $P_j \in \Psi_\delta^{j(1-\delta)}$ be a sequence of symbols for $j = \mathbb{N}$. We say that a symbol $P_\hbar \in \Psi_\delta^0$ admits the **asymptotic expansion**

$$P_\hbar(x,\xi) \sim \sum_{j=0}^{\infty} P_j(x,\xi,\hbar)$$

if for each integer J, one has

$$P_\hbar(x,\xi) - \sum_{j=0}^{J-1} P_j(x,\xi,\hbar) \in \Psi_\delta^{J(1-\delta)}.$$

Such a symbol $P_\hbar$ is called **$\boldsymbol{\delta}$-classical**.

We point out that in general these asymptotic expansions are not unique since each term P_j necessarily depends on $\hbar$. On the other hand, one knows also that they are not convergent in general. Nevertheless, one can apply the Borel resummation process for such δ-classical symbols, as stated in the following proposition (see e.g. [14] or [33].)

Proposition 7. *Let $0 \le \delta < 1$. If $P_j \in \Psi_\delta^{j(1-\delta)}$, with $j = \mathbb{N}$, is a sequence of symbols then there exists a symbol $P_\hbar \in \Psi_\delta^0$ admitting the asymptotic expansion $P_\hbar \sim \sum P_j$.*

2.2.2 Composition and commutators

There is a composition law for the previously defined class of symbols Ψ_δ^m provided $\delta < 1$.

Definition 8. Let $A_\hbar, B_\hbar \in \Psi^0_\delta$. We define their **(left) Moyal product** $A_\hbar \# B_\hbar$ by

$$A_\hbar \# B_\hbar (x,\xi) = \frac{1}{(2\pi)^d} \int_{\mathcal{T}} dy \sum_{k\in\Lambda^*} e^{ik(x-y)} A_\hbar (x, \xi+\hbar k)\, B_\hbar (y,\xi)\,.$$

Lemma 9. *Let $0 \le \delta < 1$. Let $\hat{A}$ and $\hat{B}$ be two PDOs in the class $\hat{\Psi}^0_\delta$ with symbols $A_\hbar$ and $B_\hbar$. Then, the product $\hat{C} = \hat{A}\hat{B}$ is a PDO in the same class and its symbol $C_\hbar$ is equal to the Moyal product $C_\hbar = A_\hbar \# B_\hbar$. Moreover, the symbol $C_\hbar$ admits the following δ-classical asymptotic expansion*

$$A_\hbar \# B_\hbar \sim \sum_{j=0}^{\infty} C_j (\hbar)\,,$$

where the $C_j \in \Psi^{j(1-\delta)}_\delta$ are given by

$$C_j (x,\xi,\hbar) = \left(\frac{\hbar}{i}\right)^j \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial^\alpha_\xi A_\hbar (x,\xi)\, \partial^\alpha_x B_\hbar (x,\xi)\,.$$

From the previous lemma, one can easily get the symbol $A_\hbar \# B_\hbar - B_\hbar \# A_\hbar$ of the commutator $\left[\hat{A}, \hat{B}\right]$ of two PDOs. In case one of the two operators is in the class Ψ^0_0 (i.e. with $\delta = 0$) and does not depend on x, one has a slightly better expansion that will be useful in the construction of the normal form in the next section.

Proposition 10. *Let $0 \le \delta < 1$. Let $A_\hbar (\xi) \in \Psi^0_0$ be a symbol independant of x and $B_\hbar (x,\xi) \in \Psi^0_\delta$ some symbol. Then the commutator $C_\hbar = A_\hbar \# B_\hbar - B_\hbar \# A_\hbar$ is in the class Ψ^1_δ and admits an asymptotic expansion of the following form*

$$C_\hbar (x,\xi) \sim \frac{\hbar}{i} \{A, B\} + \sum_{j=2}^{\infty} C_j (x,\xi,\hbar)\,,$$

where the $C_j \in \Psi^j_\delta$ are given by

$$C_j (x,\xi,\hbar) = \left(\frac{\hbar}{i}\right)^j \sum_{|\gamma|=j} \frac{1}{\gamma!} \partial^\gamma_\xi A_\hbar (\xi)\, \partial^\gamma_x B_\hbar (x,\xi)$$

and where the asymptotic equivalence $\sim$ means that for each $J \in \mathbb{N}$, one has

$$C_\hbar (x,\xi) - \sum_{j=0}^{J-1} C_j (x,\xi,\hbar) \in \Psi^J_\delta.$$

2.2.3 L^2 continuity and adjoints

One can easily check that PDOs in the class $\hat{\Psi}_\delta^m$ are continuous from $C^\infty(\mathcal{T})$ to $C^\infty(\mathcal{T})$. Moreover, the fact that the symbols together with all its derivatives are uniformly bounded for $(x,\xi) \in T^*\mathcal{T}$ implies that Calderón-Vaillancourt's theorem still holds, even with the special dependance in ξ of these symbols.

Lemma 11. *Each PDO $\hat{P} \in \hat{\Psi}_\delta^0$ is continuous from $L^2(\mathcal{T})$ to $L^2(\mathcal{T})$. Moreover, its norm is bounded by*

$$\left\| \hat{P} \right\|_{\mathcal{L}(L^2(\mathcal{T}))} \leq C \sup_{|\gamma| \leq \frac{d+1}{2}} \sup_{x,\xi} \left| \partial_x^\gamma P_\hbar(x,\xi) \right|.$$

Let's turn now to the description of the symbol of an adjoint of a PDO.

Lemma 12. *Let $0 \leq \delta < 1$. For each $\hat{P} \in \hat{\Psi}_\delta^0$, the adjoint $\hat{P}^*$ is a PDO in the same class $\hat{\Psi}_\delta^0$ and its symbol, denoted by $P_\hbar^*$, is given by*

$$P_\hbar^*(x,\xi) = \sum_{k \in \Lambda^*} \frac{1}{(2\pi)^d} \int_{\mathcal{T}} dy e^{ik(x-y)} \bar{P}_\hbar(y, \xi + \hbar k)$$

and admits the following δ-classical asymptotic expansion $\sum_{j=0}^\infty P_j^*(x,\xi,\hbar)$ where the $P_j^* \in \Psi_\delta^{j(1-\delta)}$ are given by

$$P_j^*(x,\xi,\hbar) = \left(\frac{\hbar}{i}\right)^j \sum_{|\gamma|=j} \frac{1}{\gamma!} \partial_x^\gamma \partial_\xi^\gamma \bar{P}_\hbar(x,\xi).$$

For convenience, we say that $P_\hbar^*$ is the **adjoint of the symbol** $P_\hbar$. Moreover, a straightforward calculation shows that the Fourier series of the adjoint is given by the following expression.

Proposition 13. *Let $P_\hbar \in \Psi_\delta^0$ be a symbol and $P_\hbar^* \in \Psi_\delta^0$ its adjoint. Then their Fourier series are related as follows*

$$\widetilde{P_\hbar^*}(k,\xi) = \overline{\tilde{P}_\hbar}(-k, \xi + \hbar k).$$

2.2.4 Functional calculus

The basic tool for developping a functional calculus for PDOs in the class $\hat{\Psi}_\delta^m$ is the resolvent formula

$$f\left(\hat{P}\right) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(z)\, dz}{\left(z - \hat{P}\right)},$$

where $\mathcal{C} \subset \mathbb{C}$ is a cycle surrounding the spectrum of $\hat{P}$ which is bounded as stated in lemma 11. The problem is to show that $f\left(\hat{P}\right)$ is still in the class

$\hat{\Psi}_{\delta}^{m}$, what amounts to show that the resolvant $\left(z-\hat{P}\right)^{-1}$ is a PDO in the class $\hat{\Psi}_{\delta}^{m}$. The corresponding question in the framework of standard PDOs on $\mathbb{R}^{d}$ was solved by Beals [5]. Concerning PDOs on the torus, but *without small parameter*, the problem was also solved in a similar way [16, 10, 35]. But the answer to this question in the periodic $\hbar$-PDO context is not clear (see [33], p. 118 for a discussion of this issue).

Nevertheless, we will content ourselves with an *approximate* calculus and use the fact that there exists a PDO $\hat{Q}\in\hat{\Psi}_{\delta}^{m}(\mathcal{T})$ such that $\hat{Q}\cong f\left(\hat{P}\right)$, i.e. $\hat{Q}=f\left(\hat{P}\right)+\hat{R}$ with $\hat{R}=O\left(\hbar^{\infty}\right)$ a negligible operator. To prove this, one has to a use parametrix of $\left(z-\hat{P}\right)$ rather than the resolvant itself in the resolvent formula for $f\left(\hat{P}\right)$. However, this resolvent formula can be applied only for a bounded operator since it makes use of a cycle $\mathcal{C}$ surrounding the spectrum of $\hat{P}$. In the pseudodifferential context, the usual assumption is that $\hat{P}$ should be bounded uniformly with respect to $\hbar$, i.e. $\hat{P}$ should be in the class $\hat{\Psi}_{\delta}^{m}$ for $m\geq0$, and one takes in that case a fixed cycle $\mathcal{C}$. Actually, it is possible to use the resolvent formula for $m<0$ too, provided the cycle $\mathcal{C}_{\hbar}$ remains at distance $\hbar^{m}$ from the spectrum. Namely, by definition of $\hat{\Psi}_{\delta}^{m}$ we have $\left|P_{\hbar}\left(x,\xi\right)\right|\leq C_{0,0}\hbar^{m}$ and, provided $\mathcal{C}_{\hbar}$ is a cycle of radius $2C_{0,0}\hbar^{m}$, one can build a parametrix of $\left(z-\hat{P}\right)$ in the class $\hat{\Psi}_{\delta}^{-m}$ uniformly with respect to z varying in a compact neighbourhood of $\mathcal{C}_{\hbar}$. Despite the special feature of the cycle used here, the construction of the parametrix is done in a standard way (see e.g. [23] or [14]). This parametrix together with the resolvent formula allow us to define an approximate exponential.

Proposition 14. *Let $\hat{P}\in\hat{\Psi}_{\delta}^{m}$ be a self-adjoint PDO. There exists a PDO in $\hat{\Psi}_{\delta}^{0}$ denoted by $e^{i\hat{P}}$ and called the **exponential** of $i\hat{P}$ with the following properties :*

- $\left(e^{i\hat{P}}\right)^{*}\cong e^{-i\hat{P}}$
- $\left(e^{i\hat{P}}\right)^{*}e^{i\hat{P}}\cong e^{i\hat{P}}\left(e^{i\hat{P}}\right)^{*}\cong\mathbb{I}$
- $\frac{1}{i}\frac{d}{d\varepsilon}e^{i\varepsilon\hat{P}}\cong\hat{P}e^{i\varepsilon\hat{P}}\cong e^{i\varepsilon\hat{P}}\hat{P}$

The basic step in the construction of normal forms is the conjugation of a given PDO by a unitary (up to $O\left(h^{\infty}\right)$) PDO.

Proposition 15. *Let $m>-1$ and $0\leq\delta<\min\left(1,1+m\right)$. Let $\hat{A}\in\hat{\Psi}_{\delta}^{m}$ and $\hat{B}\in\hat{\Psi}_{\delta}^{0}$ be two PDOs and let us consider the conjugation $\hat{C}=e^{i\hat{A}}\hat{B}\left(e^{i\hat{A}}\right)^{*}$. Then $\hat{C}$ admits the following asymptotic expansion $\hat{C}\sim\sum_{n=0}^{\infty}\hat{C}_{n}$, where the*

$\hat{C}_n \in \Psi_\delta^{(m+1-\delta)n}$ *are given by*

$$\hat{C}_n = \frac{i^n}{n!} [\underbrace{\hat{A}, ..., \left[\hat{A}}_{n}, \hat{B}\right] ...] .$$

The asymptotic expansion means that, for each integer $N \geq 0$, the remainder of the series truncated at order N verifies $\hat{C} - \sum_{n=0}^{N-1} C_n \cong \hat{R}_N$ where $\hat{R}_N \in \hat{\Psi}_\delta^{(m+1-\delta)N}$.

When the operator $\hat{B}$ is in $\hat{\Psi}_0^0$ (i.e. with $\delta = 0$) and its symbol does not depend on the x variable, then one can get a slightly better estimate (the gain is a factor $\hbar^\delta$), that will be usefull subsequently.

Proposition 16. *Let $m > -1$ and δ satisfy $0 \leq \delta < \min(1, 1+m)$. Let $\hat{B} \in \Psi_0^0$ be a PDO with symbol $B_\hbar(\xi)$ not depending on x and $\hat{A} \in \Psi_\delta^m$ any PDO. Then the conjugation $\hat{C} = e^{i\hat{A}}\hat{B}\left(e^{i\hat{A}}\right)^*$ admits the following asymptotic expansion $\hat{C} \sim \sum_{n=0}^{\infty} \hat{C}_n$, where the $\hat{C}_n \in \Psi_\delta^{(m+1-\delta)n+\delta}$ are given by*

$$\hat{C}_n = \frac{i^n}{n!} [\underbrace{\hat{A}, ..., \left[\hat{A}}_{n}, \hat{B}\right] ...] .$$

The asymptotic expansion means that for each integer $N \geq 0$, the remainder of the series truncated at order N verifies

$$\hat{C} - \sum_{n=0}^{N-1} C_n \cong \hat{R}_N,$$

where $\hat{R}_N \in \hat{\Psi}_\delta^{(m+1-\delta)N+\delta}$.

3 Quasi-resonant normal form

3.1 Geometry of resonances

3.1.1 Nondegenerate Hamiltonians and resonances

A classical completely integrable Hamiltonian $H(\xi)$ generates a linear dynamics on each torus $\mathcal{T}_\xi$ that can be periodic, ergodic or also partially (in a sub-torus) ergodic. For each ξ, the resonant lattice of dH at the point ξ is defined by

$$\mathcal{R}_\xi = \{k \in \Lambda^*; dH_\xi(k) = 0\},$$

where Λ^* is the Fourier lattice. We thus have the following cases :

• $\dim \mathcal{R}_\xi = 0$: We say that ξ (or $\mathcal{T}_\xi$) is **non-resonant**. The dynamics induced by H is ergodic.

• $\dim \mathcal{R}_\xi > 0$: We say that ξ (or $\mathcal{T}_\xi$) is **resonant**. In this case, the dynamics is partially ergodic, i.e. ergodic in a sub-torus of dimension $d - \dim \mathcal{R}_\xi$. In particular, when $\dim \mathcal{R}_\xi = d - 1$, we say that ξ (or $\mathcal{T}_\xi$) is **periodic**.

From now on, the functions $dH_\xi(k)$ will be used to define the resonances and their neighbourhoods.

Definition 17. For each non-vanishing $k \in \Lambda^*$, we define the fonction $\Omega_k \in C^\infty(\mathcal{B})$ by $\Omega_k(\xi) = dH_\xi(k)$ and the associated **resonance surface** $\Sigma_k \subset \mathcal{B}$ by

$$\Sigma_k = \{\xi \in \mathcal{B}; \Omega_k(\xi) = 0\}.$$

The resonant set Σ_k will indeed be a hyper-surface as soon as we will impose H to be *non-degenerate*. Such a condition is a very common assumption in K.A.M. like or Nekhoroshev like theories which insures that the CI dynamics "varies enough" from one torus to another one. The nondegeneracy condition that we will use can is actually slightly weaker than Kolmogorov's one [20] or Arnol'd's one [4] and equivalent to Bryuno's one [7].

Definition 18. A completely integrable Hamiltonian $H(\xi)$ is said to be **non-degenerate** if for each non-vanishing $k \in \Lambda^*$ and each point $\xi \in \Sigma_k$, we have $d(\Omega_k)_\xi \neq 0$.

This implies that the set Σ_k is a codimension 1 submanifold of $\mathcal{B}$ and thus deserves its name "resonance surface". We will now give without proof two consequences of the nondegeneracy hypothesis. These are more or less well-known statements, but we can find a proof e.g. in [33, 32].

Proposition 19. *Let $H(\xi)$ be a nondegenerate completely integrable Hamiltonian. If $k_1, ..., k_n \in \Lambda^*$ are linearly independent, then in a neighbourhood of the intersection $\bigcap_i \Sigma_{X_i}$ the differentials $d\Omega_{k_j}$ are linearly independent. This implies that the submanifolds Σ_{k_j} are transverse.*

Proposition 20. *Let $H(\xi)$ be a non-degenerate completely integrable Hamiltonian. Then the set of tori on which the dynamics is periodic, is dense in $\mathcal{B}$.*

3.1.2 Quasi-resonant blocks

The first step in the construction of the announced quasi-resonant normal form, is to construct a covering of the momentum space $\mathcal{B}$ in regions attached to each particular kind of resonance. For each resonant torus $\mathcal{T}_\xi$, we consider a "small" neighbourhood and we remove from it a "sufficiently large" neighbourhood of higher order resonances, as Pöschel did in [30], in order to get the so-called *quasi-resonant blocks.* One the other hand, in our semi-classical context, one needs to let both notions "small" and "sufficiently large" depend on $\hbar$. We now elaborate on Pöschel's construction, yet incorporating $\hbar$ in the right place. For this, we will fix two exponents $\gamma > 0$ and $\delta > 0$ which control respectively the "amount" of resonances we consider and the "size" of the quasi-resonant zones.

Definition 21. A n-dimensional sub-lattice $\mathcal{R}$ of the Fourier lattice Λ^* is called a **resonance $\hbar^{-\gamma}$-lattice** (or simply a **$\hbar^{-\gamma}$-lattice**) if there exists a basis $(e_1, ..., e_n)$ of $\mathcal{R}$ such that $|e_j| \leq \hbar^{-\gamma}$ for all $j = 1..n$.

Similarly with Definition 17, we define the resonant manifold attached to each $\hbar^{-\gamma}$-lattice $\mathcal{R}$, making use of the function Ω_k previously defined.

Definition 22. For each resonance $\hbar^{-\gamma}$-lattice $\mathcal{R}$, the associated **resonance manifold** $\Sigma_{\mathcal{R}} \subset \mathcal{B}$ is defined by

$$\Sigma_{\mathcal{R}} = \{\xi \in \mathcal{B}; \forall k \in \mathcal{R} \Rightarrow \Omega_k(\xi) = 0\}.$$

For consistency of the notations, in the case of the trivial lattice $\mathcal{R} = 0$, we define Σ_0 to be the whole $\mathcal{B}$.

For a given $\hbar^{-\gamma}$-lattice $\mathcal{R}$, the resonant manifold $\Sigma_{\mathcal{R}}$ is thus the set of points ξ at which the resonant lattice of dH is exactely equal to $\mathcal{R}$. Moreover, we obviously have$\Sigma_{\mathcal{R}} = \bigcap_{k\in\mathcal{R}} \Sigma_k$ and the notation is still consistent when $\mathcal{R}$ is 1-dimensional, i.e. of the form $\mathcal{R} = \mathbb{Z}.k_0$, if we write $\Sigma_{\mathbb{Z}.k_0} = \Sigma_{k_0}$. Thanks to Proposition 19, the nondegeneracy hypothesis implies that for each $\hbar^{-\gamma}$-lattice $\mathcal{R}$ of dimension n, the manifold $\Sigma_{\mathcal{R}}$ is of codimension n in $\mathcal{B}$.

Definition 23. For each resonance $\hbar^{-\gamma}$-lattice $\mathcal{R}$ of dimension $n > 0$, the associated **resonance zone** $\mathcal{Z}_{\mathcal{R}} \subset \mathcal{B}$ is defined by

$$\mathcal{Z}_{\mathcal{R}} = \left\{\xi \in \mathcal{B}; \forall X \in \mathbb{R}.\mathcal{R} \Rightarrow \frac{|\Omega_X(\xi)|}{|X|} < \frac{2^n \hbar^{\delta-\gamma n}}{\mathrm{vol}(\mathcal{R})}\right\}.$$

We also define $\mathcal{Z}_0 = \mathcal{B}$.

The denominator $\mathrm{vol}(\mathcal{R})$ in the previous definition refers to the volume of a fundamental domain of the lattice $\mathcal{R}$.

Definition 24. We denote by $\mathcal{Z}_n^*$ the union of all resonance zones of order n. For $0 \leq n \leq d$, we have

$$\mathcal{Z}_n^* = \bigcup_{\dim \mathcal{R}=n} \mathcal{Z}_{\mathcal{R}}.$$

We call $\mathcal{Z}_n^*$ the **zone of n-resonances**.

Then we remove, from each resonance zone, a neighbourhood of all next order resonances and obtain the so-called resonance blocks.

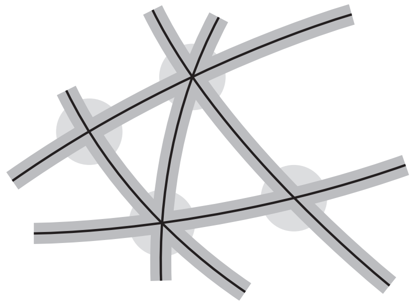
Definition 25. For each resonance $\hbar^{-\gamma}$-lattice $\mathcal{R}$ of dimension n, the associated **resonance block** $\mathcal{B}_{\mathcal{R}} \subset \mathcal{B}$ is defined by

$$\mathcal{B}_{\mathcal{R}} = \mathcal{Z}_{\mathcal{R}} \setminus \mathcal{Z}^*_{n+1},$$

where we defined $\mathcal{Z}^*_{d+1} = \emptyset$ for consistency of the notations. We also denote

$$\mathcal{B}^*_n = \bigcup_{\dim \mathcal{R}=n} \mathcal{B}_{\mathcal{R}}$$

the **block of n-resonances** and call $\mathcal{B}^*_0 = \mathcal{B}_0$ the **non-resonant block**.



The picture above represents the situation in dimension $d = 2$. The black lines are the resonance manifolds for 1-dimensional resonance lattices (i.e. the set of periodic tori). They intersect on resonance manifolds of 2-dimensional resonance lattices. The dark grey regions represent the associated zones of 1-resonances and the light grey regions are the zones of 2-resonances. This picture can also been understood as a 2-dimensional cross section of $\mathcal{B}$ in dimension $d = 3$.

The resonant zones are defined in such a way (with sizes increasing with the order) that all points ξ in a given block $\mathcal{B}_{\mathcal{R}}$ are "almost resonant" for all $k \in \mathcal{R}$ and not "almost resonant" for all $k \notin \mathcal{R}$. The precise statement of this assertion is given in the following lemma. We refer the reader to Pöschel's article ([30], p. 201) for the proof.

Lemma 26. *For each resonance $\hbar^{-\gamma}$-lattice $\mathcal{R}$ of dimension $0 \leq n < d$ and each $\xi \in \mathcal{B}_{\mathcal{R}}$, we have*

$$\forall k \notin \mathcal{R}, |k| \leq \hbar^{-\gamma} \Rightarrow \frac{|\Omega_k(\xi)|}{|k|} \geq \hbar^{\delta}.$$

This formula still holds for the non-resonant block $\mathcal{B}_0$.

On the other hand, the resonance blocks form a covering of the space $\mathcal{B}$ since they satisfy

$$\mathcal{B} = \mathcal{B}^*_0 \cup \mathcal{B}^*_1 ... \cup \mathcal{B}^*_d.$$

3.2 Quasi-resonant averaging

3.2.1 Resonant averaging

For any n-dimensional resonance lattice $\mathcal{R}$, one can consider the averaging of functions along the dual space of $\mathcal{R}$ in the following way.

Definition 27. For any function $f \in C^{\infty}(T^*\mathcal{T}, \mathbb{C})$ and any n-dimensional resonance lattice $\mathcal{R}$, we call the **average of f with respect to $\mathcal{R}$**, or the **$\mathcal{R}$-average of f**, the function $\mathcal{R}\text{-av}(f) \in C^{\infty}(T^*\mathcal{T}, \mathbb{C})$ defined by

$$\mathcal{R}\text{-av}(f)(x,\xi) = \int_0^1 dt_1 ... \int_0^1 dt_p\, f(x + t_1 X_1 + ... + t_{d-n} X_{d-n}, \xi),$$

where $(X_1, ..., X_{d-n})$ is any basis of $\mathcal{R}^{\circ} \subset \Lambda$. In particular, we will note $\langle\langle f \rangle\rangle = \{0\}\text{-av}(f)(\xi)$ the average of f over the whole torus.

One can easily check that this definition doesn't depend on the choice of the basis $(X_1, ..., X_{d-n})$. Moreover, it is easily shown that the Fourier series of such an averaged function has the simple form given below.

Proposition 28. *Let $\mathcal{R}$ be any resonance lattice. Let $f \in C^{\infty}(T^*\mathcal{T}, \mathbb{C})$ be any function and $\langle f \rangle = \mathcal{R}\text{-av}(f)$ its $\mathcal{R}$-average. If we denote by $\widetilde{f}$ the Fourier series of f then $\widetilde{\langle f \rangle}$, the Fourier series of $\langle f \rangle$, is given by the following expression.*

$$\widetilde{\langle f \rangle}(k,\xi) = \begin{cases} \widetilde{f}(k,\xi) \text{ for } k \in \mathcal{R} \\ 0 \text{ for } k \notin \mathcal{R}. \end{cases}$$

In particular, the Fourier series of $\langle\langle f \rangle\rangle$ verifies $\widetilde{\langle\langle f \rangle\rangle}(0,\xi) = \widetilde{f}(0,\xi)$ and vanishes for $k \neq 0$.

3.2.2 Quasi-resonant averaging

Let us consider the previously defined covering of $\mathcal{B}$ with resonant blocks $\mathcal{B}_{\mathcal{R}}$. For any symbol $K_\hbar \in \Psi_\delta^m$, we will construct a symbol in Ψ_δ^m which is an $\mathcal{R}$-averaged function in each blocks $\mathcal{B}_{\mathcal{R}}$ and which is moreover exactly the $\mathcal{R}$-average of $K_\hbar$ on $\Sigma_{\mathcal{R}} \cap \mathcal{B}_{\mathcal{R}}$. For the construction, we need a trunctature function that will be keeped fixed and that will localize in a neighbourhood of size $\hbar^\delta$ of the resonant blocks. Precisely, let us choose a function $\chi \in C_0^{\infty}(\mathbb{R})$ with value in $[0,1]$, symmetric, vanishing for $|t| \geq 1$ and such that $\chi - 1$ is flat at $t = 0$.

Definition 29. For any $\delta > 0$ and any symbol $K_\hbar \in \Psi_\delta^m$, we define $A_\hbar$ the **$\hbar^{\delta}$-average** of $K_\hbar$ by

$$\tilde{A}_\hbar(k,\xi) = \chi\left(\frac{\Omega_k(\xi)}{|k|\,\hbar^\delta}\right) \tilde{K}_\hbar(k,\xi)$$

for all $\xi \in \mathcal{B}$ and all non-vanishing $k \in \Lambda^*$, and $\tilde{A}_\hbar(0,\xi) = \tilde{K}_\hbar(0,\xi)$ for all $\xi \in \mathcal{B}$.

The following lemma tells us that the $\hbar^\delta$-average has the property that for each resonance $\hbar^{-\gamma}$-lattice $\mathcal{R}$, on the resonant manifold $\Sigma_{\mathcal{R}} \cap \mathcal{B}_{\mathcal{R}}$ the $\hbar^\delta$-average of $K_\hbar \in \Psi_\delta^m$ is equal (up to $O(\hbar^\infty)$) to the $\mathcal{R}$-average of $K_\hbar$, and in the resonant block $\mathcal{B}_{\mathcal{R}}$ it is a $\mathcal{R}$-averaged function.

Lemma 30. *The $\hbar^\delta$-average $A_\hbar$ of any symbol $K_\hbar \in \Psi_\delta^m$ is in the class Ψ_δ^m and has the following properties. For each resonance $\hbar^{-\gamma}$-lattice $\mathcal{R}$, we have :*

$$A_\hbar - \mathcal{R}\text{-}av\,(K_\hbar) \in \Psi_\delta^\infty\left(\mathcal{T} \times (\Sigma_{\mathcal{R}} \cap \mathcal{B}_{\mathcal{R}})\right) \quad \textit{and} \quad A_\hbar - \mathcal{R}\text{-}av\,(A_\hbar) \in \Psi_\delta^\infty\left(\mathcal{T} \times \mathcal{B}_{\mathcal{R}}\right).$$

Proof. Let us first show that $A_\hbar$ is indeed in the class Ψ_δ^m. First of all, for each multi-index $\beta \in \mathbb{N}^d$, the derivative of the Fourier series of $A_\hbar$ is given by

$$\partial_\xi^\beta \tilde{A}_\hbar(k,\xi) = \sum_{\beta' \le \beta} C_{\beta'}^{\beta} \left(\partial_\xi^{\beta-\beta'} \widetilde{K}_\hbar(k,\xi)\right) \partial_\xi^{\beta'} \left(\chi\left(\frac{\Omega_k(\xi)}{|k|\,\hbar^\delta}\right)\right)$$

for non-vanishing $k \in \Lambda^*$ and simply $\partial_\xi^\beta \tilde{A}_\hbar(0,\xi) = \partial_\xi^\beta \tilde{K}_\hbar(0,\xi)$ for $k = 0$. On the other hand, according to Proposition 5, the fact that $K_\hbar$ is in Ψ_δ^m implies for all s the estimates

$$\left|\partial_\xi^{\beta-\beta'} \tilde{K}_\hbar(k,\xi)\right| \le C\left(s, \beta-\beta'\right) \frac{\hbar^{m-\delta\left|\beta-\beta'\right|}}{\left(1+|k|^2\right)^{\frac{s}{2}}},$$

where $C\left(s,\beta-\beta'\right)$ is a positive constant. However, one gets easily convinced that the derivatives of the function χ are of the form

$$\left|\partial_\xi^{\beta'} \left(\chi\left(\frac{\Omega_k(\xi)}{|k|\,\hbar^\delta}\right)\right)\right| \le \sum_{n=1}^{\left|\beta'\right|} c(n)\, \hbar^{-\delta n} \chi^{(n)}\left(\frac{\Omega_k(\xi)}{|k|\,\hbar^\delta}\right),$$

where the constants $c(n)$ depend only on H and its derivatives. We then notice that $\hbar^{-\delta n} \le \hbar^{-\delta\left|\beta'\right|}$ and that all the derivatives $\chi^{(n)}$ are bounded (thanks to the fact that $\chi^{(n)} \in C_0^\infty(\mathbb{R})$), what implies the estimate

$$\left|\partial_\xi^{\beta'} \left(\chi\left(\frac{\Omega_k(\xi)}{|k|\,\hbar^\delta}\right)\right)\right| \le \hbar^{-\delta\left|\beta'\right|} C\left(\beta'\right)$$

for all k, all $\hbar$ and all ξ. This shows that

$$\left|\partial_\xi^\beta \tilde{A}_\hbar(k,\xi)\right| \le C'(s,\beta) \frac{\hbar^{m-\delta\left|\beta-\beta'\right|} \hbar^{-\delta\left|\beta'\right|}}{\left(1+|k|^2\right)^{\frac{s}{2}}} = C'(s,\beta) \frac{\hbar^{m-\delta|\beta|}}{\left(1+|k|^2\right)^{\frac{s}{2}}},$$

where $C'(s,\beta)$ is a positive constant. Using again Proposition 5, we deduce that $A_\hbar$ is a symbol in the class Ψ_δ^m.

Let us now prove that $A_\hbar$ is an $\mathcal{R}$-averaged function, up to $O(\hbar^\infty)$, in each resonant block $\mathcal{B}_\mathcal{R}$. For this, let's define the remainder $R_\hbar = A_\hbar - \mathcal{R}\text{-av}(A_\hbar)$, which is in the class Ψ_δ^m since $A_\hbar$ and $\mathcal{R}\text{-av}(A_\hbar)$ are. The Fourier series of $R_\hbar$ is thus given by $\widetilde{R}_\hbar(k,\xi) = \chi\left(\frac{\Omega_k(\xi)}{|k|\hbar^\delta}\right)\tilde{K}_\hbar(k,\xi)$ for $k \in \mathcal{R}$ and $\widetilde{R}_\hbar(k,\xi) = 0$ for $k \in \mathcal{R}$. For each $\beta \in \mathbb{N}^d$ we now estimate $\partial_\xi^\beta \tilde{R}_\hbar(k,\xi)$ at each point $\xi \in \mathcal{B}_\mathcal{R}$. For all $k \in \mathcal{R}$ with $|k| \leq \hbar^{-\gamma}$, one simply has $\partial_\xi^\beta \tilde{R}_\hbar(k,\xi) = 0$ everywhere. For all $k \notin \mathcal{R}$ with $|k| \leq \hbar^{-\gamma}$, one has $\left|\frac{\Omega_k(\xi)}{|k|\hbar^\delta}\right| \geq 1$ at each point $\xi \in \mathcal{B}_\mathcal{R}$ thanks to Lemma 26. The truncature function $\chi(t)$ vanishing for $t \geq 1$, it follows that $\partial_\xi^\beta \tilde{R}_\hbar(k,\xi) = 0$ for all $k \notin \mathcal{R}$ with $|k| \leq \hbar^{-\gamma}$ and all $\xi \in \mathcal{B}_\mathcal{R}$. Finally, for all k with $|k| > \hbar^{-\gamma}$, we simply use the fact that $R_\hbar \in \Psi_\delta^m(\mathcal{T})$, what implies (Proposition 5) that

$$\left|\partial_\xi^\beta \tilde{R}_\hbar(k,\xi)\right| \leq C(s,\beta)\frac{\hbar^{m-\delta|\beta|}}{\left(1+|k|^2\right)^{\frac{s}{2}}}, \tag{1}$$

where $C(s,\beta)$ is a positive constant.

Reconstructing the Fourier series of $R_\hbar$, one then gets for each $\alpha \in \mathbb{N}^d$

$$\partial_x^\alpha \partial_\xi^\beta R_\hbar(x,\xi) = \sum_{|k|>\hbar^{-\gamma}} e^{ikx} i^{|\alpha|} k^\alpha \partial_\xi^\beta \tilde{R}_\hbar(k,\xi),$$

and thus for each $\xi \in \mathcal{B}_\mathcal{R}$ one has

$$\begin{aligned}\left|\partial_x^\alpha \partial_\xi^\beta R_\hbar(x,\xi)\right| &\leq C(s,\beta)\hbar^{m-\delta|\beta|}\sum_{|k|>\hbar^{-\gamma}}\frac{|k|^{|\alpha|}}{\left(1+|k|^2\right)^{\frac{s}{2}}} \\ &\leq C(\alpha,\beta,N)\hbar^{m-\delta|\beta|}\hbar^{\gamma N},\end{aligned}$$

where we have defined N by $s = |\alpha| + d + N$. This estimate holds for each s and thus for each N. This implies that $R_\hbar \in \Psi_\delta^\infty(\mathcal{T}\times\mathcal{B}_\mathcal{R})$.

Let's now turn to the second property, namely on $\Sigma_\mathcal{R} \cap \mathcal{B}_\mathcal{R}$, $A_\hbar$ is equal up to $O(\hbar^\infty)$ to $\mathcal{R}\text{-av}(K_\hbar)$. To prove this, we define another remainder $S_\hbar = A_\hbar - \mathcal{R}\text{-av}(K_\hbar)$ whose Fourier series is given by

$$\widetilde{S}_\hbar(k,\xi) = \begin{cases} \left(\chi\left(\frac{\Omega_k(\xi)}{|k|\hbar^\delta}\right) - 1\right)\tilde{K}_\hbar(k,\xi) & \text{for } k \in \mathcal{R} \\ \chi\left(\frac{\Omega_k(\xi)}{|k|\hbar^\delta}\right)\tilde{K}_\hbar(k,\xi) & \text{for } k \notin \mathcal{R}\end{cases}$$

As before, we will estimate $\partial_\xi^\beta \tilde{S}_\hbar(k,\xi)$ for each $\beta \in \mathbb{N}^d$ at each point $\xi \in \mathcal{B}_\mathcal{R}$. For all $k \in \mathcal{R}$ with $|k| \leq \hbar^{-\gamma}$, one has $\Omega_k(\xi) = 0$ at each point $\xi \in \Sigma_\mathcal{R} \cap \mathcal{B}_\mathcal{R}$ by definition of Σ_k. On the other hand, the function $\chi(t) - 1$ is flat at $t = 0$. Thus, for all $k \in \mathcal{R}$ one has $\partial_\xi^\beta \tilde{S}_\hbar(k,\xi) = 0$ at each $\xi \in \Sigma_\mathcal{R} \cap \mathcal{B}_\mathcal{R}$. For the cases $k \notin \mathcal{R}$ with $|k| \leq \hbar^{-\gamma}$ and $k \notin \mathcal{R}$ with $|k| > \hbar^{-\gamma}$, we argue as before and obtain that $S_\hbar \in \Psi_\delta^\infty(\mathcal{T}\times(\Sigma_\mathcal{R} \cap \mathcal{B}_\mathcal{R}))$. $\square$

The $\hbar^\delta$-average $A_\hbar$ of a symbol $K_\hbar$ has nevertheless a drawback : it is not self-adjoint even when $K_\hbar$ is. To solve this, we have to show that the self-adjoint part $\frac{1}{2}(A_\hbar + A_\hbar^*)$ has the same average properties as $A_\hbar$ has.

Lemma 31. *Let the parameters γ and δ satisfy $\gamma + \delta < 1$. Let $K_\hbar \in \Psi_\delta^m$ be any self-adjoint symbol and $A_\hbar \in \Psi_\delta^m$ its $\hbar^\delta$-average as in Definition 29. Then the adjoint $A_\hbar^* \in \Psi_\delta^m$ has the same properties as $A_\hbar$ (as given in Lemma 30), as well as its self-adjoint part $\frac{1}{2}(A_\hbar + A_\hbar^*)$.*

Proof. According to Proposition 13, the Fourier series of $A_\hbar^*$ is given by

$$\begin{aligned}\tilde{A}_\hbar(k,\xi) &= \overline{\tilde{A}_\hbar}(-k,\xi+\hbar k) \\ &= \overline{\chi}\left(\frac{\Omega_{-k}(\xi+\hbar k)}{|-k|\,\hbar^\delta}\right)\overline{\tilde{K}_\hbar}(-k,\xi+\hbar k),\end{aligned}$$

for all nonvanishing k and simply $\tilde{A}_\hbar^*(0,\xi) = \overline{\tilde{K}_\hbar}(0,\xi)$. Using then the facts that $K_\hbar$ is self-adjoint, that $\Omega_{-k} = -\Omega_k$ and that the function χ is real and symetric, we obtain

$$\tilde{A}_\hbar^*(k,\xi) = \chi\left(\frac{\Omega_k(\xi+\hbar k)}{|k|\,\hbar^\delta}\right)\tilde{K}_\hbar(k,\xi),$$

for all non-vanishing k and $\tilde{A}_\hbar^*(0,\xi) = \tilde{K}_\hbar(0,\xi)$. We now prove that $A_\hbar^* - \mathcal{R}\text{-av}(A_\hbar^*) \in \Psi_\delta^\infty(\mathcal{T}\times\mathcal{B}_\mathcal{R})$. We introduce the remainder $R_\hbar = A_\hbar^* - \mathcal{R}\text{-av}(A_\hbar^*)$, which is proved to be in the class Ψ_δ^m in the same way as we proved it for $A_\hbar$ itself. The Fourier series of $R_\hbar$ is given by $\widetilde{R}_\hbar(k,\xi) = \chi\left(\frac{\Omega_k(\xi+\hbar k)}{|k|\hbar^\delta}\right)\tilde{K}_\hbar(k,\xi)$ for all $k \notin \mathcal{R}$ and 0 otherwise.

For each $\beta \in \mathbb{N}^d$ we have to estimate $\partial_\xi^\beta \tilde{R}_\hbar(k,\xi)$ at each point $\xi \in \mathcal{B}_\mathcal{R}$. For all $k \in \mathcal{R}$ with $|k| \le \hbar^{-\gamma}$, one simply has $\partial_\xi^\beta \tilde{R}_\hbar(k,\xi) = 0$ everywhere. For all $k \notin \mathcal{R}$ with $|k| \le \hbar^{-\gamma}$, one has $\left|\frac{\Omega_k(\xi)}{|k|\hbar^\delta}\right| \ge 1$ at each point $\xi \in \mathcal{B}_\mathcal{R}$ thanks to Lemma 26. Nevertheless, we have to evaluate Ω_k at $\xi + \hbar k$ and not at ξ. However, the bound $|k| \le \hbar^{-\gamma}$ insures that $\frac{\Omega_k(\xi+\hbar k)}{|k|\hbar^\delta} = \frac{\Omega_k(\xi)}{|k|\hbar^\delta} + O\left(\hbar^{1-\delta-\gamma}\right)$. The relation between δ and γ then implies that $\hbar^{1-\delta-\gamma}$ is small when $\hbar \to 0$, and using the fact that the truncature function $\chi(t)$ is flat when $|t| \ge 1$, we obtain $\chi\left(\frac{\Omega_k(\xi+\hbar k)}{|k|\hbar^\delta}\right) = O\left(\hbar^{(1-\delta-\gamma)\infty}\right) = O(\hbar^\infty)$. A similar argument for the derivatives $\chi^{(n)}$ yields $\partial_\xi^\beta \tilde{R}_\hbar(k,\xi) = O\left(\hbar^{-|\beta|\delta}\hbar^{(1-\delta-\gamma)\infty}\right) = O(\hbar^\infty)$ for all $k \notin \mathcal{R}$ with $|k| \le \hbar^{-\gamma}$ and all $\xi \in \mathcal{B}_\mathcal{R}$. Finally, for all k with $|k| > \hbar^{-\gamma}$, we use as before the fact that $R_\hbar \in \Psi_\delta^m(\mathcal{T})$, what implies (Proposition 5) that

$$\left|\partial_\xi^\beta \tilde{R}_\hbar(k,\xi)\right| \le C(s,\beta)\,\frac{\hbar^{m-\delta|\beta|}}{\left(1+|k|^2\right)^{\frac{s}{2}}},$$

where $C(s,\beta)$ is a positive constant.

Reconstructing the Fourier series of $R_\hbar$, one gets for each $\alpha \in \mathbb{N}^d$

$$\partial_x^\alpha \partial_\xi^\beta R_\hbar(x,\xi) = \sum_{|k|\leq \hbar^{-\gamma}} e^{ikx} i^{|\alpha|} k^\alpha \partial_\xi^\beta \tilde{R}_\hbar(k,\xi) + \sum_{|k|>\hbar^{-\gamma}} e^{ikx} i^{|\alpha|} k^\alpha \partial_\xi^\beta \tilde{R}_\hbar(k,\xi),$$

and thus for each $\xi \in \mathcal{B}_\mathcal{R}$ one has

$$\left|\partial_x^\alpha \partial_\xi^\beta R_\hbar(x,\xi)\right| \leq O(\hbar^\infty) \sum_{|k|\geq \hbar^{-\gamma}} |k|^{|\alpha|} + C(s,\beta)\, \hbar^{m-\delta|\beta|} \sum_{|k|\geq \hbar^{-\gamma}} \frac{|k|^{|\alpha|}}{\left(1+|k|^2\right)^{\frac{s}{2}}}.$$

This holds for all s and thus one has $\left|\partial_x^\alpha \partial_\xi^\beta R_\hbar(x,\xi)\right| = O(\hbar^\infty)$. This proves that $R_\hbar \in \Psi_\delta^\infty(\mathcal{T} \times \mathcal{B}_\mathcal{R})$.

We now let the reader check that, following the same arguments, one can show that $A_\hbar^* - \mathcal{R}\text{-av}(K_\hbar^*) \in \Psi_\delta^\infty(\mathcal{T} \times \Sigma_\mathcal{R})$. This shows that $A_\hbar^*$ has the same average properties as $A_\hbar$, as well as $\frac{1}{2}(A_\hbar + A_\hbar^*)$. □

From now on, the function $A_\hbar$ in Definition 29 is called the **$\hbar^{\delta}$-average** of $K_\hbar$ and the one in Lemma 31 is called the **self-adjoint $\hbar^{\delta}$-average** of $K_\hbar$.

3.3 Semi-classical normal form

Let's now turn to the study of the *homological equation* arising at each step of the construction of the normal form given in Theorem 33.

Lemma 32 (Homological equation). *Let $H(\xi) \in \Psi^0$ be a non-degenerate CI Hamiltonian. Let $K_\hbar \in \Psi_\delta^m$ be any symbol and $A_\hbar \in \Psi_\delta^m$ its $\hbar^\delta$-average. Then there exists a symbol $P_\hbar \in \Psi_\delta^{m-\delta}$ solution of the equation*

$$\{P_\hbar, H\} + K_\hbar - A_\hbar = 0.$$

Proof. We first write the Fourier series of the homological equation, i.e.

$$i\Omega_k(\xi)\, \tilde{P}_\hbar(k,\xi) + \tilde{K}_\hbar(k,\xi) - \tilde{A}_\hbar(k,\xi) = 0. \tag{2}$$

For $k = 0$, the equation is fullfilled since $\Omega_0(\xi) = 0$ and et $\tilde{A}_\hbar(0,\xi) = \tilde{K}_\hbar(0,\xi)$. We can choose for example $\tilde{P}_\hbar(k,\xi) = 0$. For all $k \neq 0$, the Fourier series of the $\hbar^\delta$-average is given by $\tilde{A}_\hbar(k,\xi) = \chi\left(\frac{\Omega_k(\xi)}{|k|\hbar^\delta}\right) \tilde{K}_\hbar(k,\xi)$. We then notice that the function $\phi(t) = \frac{1-\chi(t)}{t}$ is smooth. This implies that the function $P_\hbar(x,\xi)$ defined by

$$\tilde{P}_\hbar(k,\xi) = \frac{i\tilde{K}_\hbar(k,\xi)}{|k|\,\hbar^\delta} \phi\left(\frac{\Omega_k(\xi)}{|k|\,\hbar^\delta}\right)$$

is well-defined and satisfies Equation 2. Moreover, proceeding as in Proposition 30 for proving that the $\hbar^\delta$-average is a symbol in the class Ψ_δ^m, and using the smoothness of ϕ, one shows that for all $\alpha, \beta \in \mathbb{N}^d$,

$$\left|\partial_x^\alpha \partial_\xi^\beta P_\hbar(x,\xi)\right| \leq C(\alpha,\beta)\,\hbar^{m-\delta-\delta|\beta|}.$$

This proves that $P_\hbar \in \Psi_\delta^{m-\delta}$. $\square$

Theorem 33 (Semi-classical normal form). *Let us consider a PDO $\hat{H} \in \hat{\Psi}^0$ with non-degenerate CI symbol $H(\xi) \in \Psi^0$ and any self-adjoint perturbation $\hbar^\kappa \hat{K}_0 \in \hat{\Psi}_\delta^\kappa$, with $\kappa > 0$. Let us choose small parameters $\gamma \geq 0$ and $\delta \geq 0$ such that $\delta < 1-\gamma$ and $\delta < \frac{\kappa}{3}$, and let us consider the covering of $\mathcal{B}$ with resonance blocks as described in Section 3.1.2. Then there exist*
• a PDO $\hat{U} \in \hat{\Psi}_\delta^0$ satisfying $\hat{U} = \mathbb{I} + O\left(\hbar^{\kappa-1-\delta}\right)$, $\hat{U}^*\hat{U} = \mathbb{I} + O(\hbar^\infty)$ and $\hat{U}\hat{U}^* = \mathbb{I} + O(\hbar^\infty)$
• and a self-adjoint PDO $\hat{K} \in \hat{\Psi}_\delta^0$ satisfying $\hat{K} = \hat{K}_0 + O(\hbar^\alpha)$, whith $\alpha = \min(1-\delta, \kappa-3\delta)$,
such that

$$\hat{U}\left(\hat{H} + \hbar^\kappa \hat{K}_0\right)\hat{U}^* = \hat{H} + \hbar^\kappa \hat{A} + O(\hbar^\infty), \tag{3}$$

where the symbol of $\hat{A} \in \hat{\Psi}_\delta^0$ is the self-adjoint $\hbar^\delta$-average of the symbol $K_\hbar$.

Proof. Let us define $\alpha = \min(1-\delta, \kappa-3\delta)$ which is the relevant exponent that will appear in the expansions we will deal with. It is positive because of the restrictions on δ and κ. We first prove that there exist self-adjoint PDOs $\hat{P}_0 \in \hat{\Psi}_\delta^{-\delta}(\mathcal{T})$ and $\hat{K}_1 \in \hat{\Psi}_\delta^\alpha(\mathcal{T})$ such that

$$e^{i\hbar^{\kappa-1}\hat{P}_0}\left(\hat{H} + \hbar^\kappa \hat{K}_0\right)\left(e^{i\hbar^{\kappa-1}\hat{P}_0}\right)^* = \hat{H} + \hbar^\kappa \hat{A}_0 + \hbar^\kappa \hat{K}_1, \tag{4}$$

where $A_0(\hbar) \in \Psi_\delta^0$ is the self-adjoint $\hbar^\delta$-average of $K_0(\hbar)$.
• Indeed, Proposition 16 tells us that

$$\begin{aligned} e^{i\hbar^{\kappa-1}\hat{P}_0}\hat{H}\left(e^{i\hbar^{\kappa-1}\hat{P}_0}\right)^* &= \hat{H} + i\hbar^{\kappa-1}\left[\hat{P}_0, \hat{H}\right] + O\left(\hbar^{2(\kappa-1-\delta+1-\delta)+\delta}\right), \\ &= \hat{H} + i\hbar^{\kappa-1}\left[\hat{P}_0, \hat{H}\right] + O\left(\hbar^{2\kappa-3\delta}\right). \end{aligned}$$

On the other hand, one can apply Proposition 10 which insures that the symbol of $\left[\hat{P}_0, \hat{H}\right]$ is equal to $\frac{\hbar}{i}\{P_0, H\} + O\left(\hbar^{2-\delta}\right)$. This yields

$$\begin{aligned} e^{i\hbar^{\kappa-1}\hat{P}_0}\hat{H}\left(e^{i\hbar^{\kappa-1}\hat{P}_0}\right)^* &= \hat{H} + i\hbar^{\kappa-1}\left(\frac{\hbar}{i}\widehat{\{P_0, H\}}\right) + O\left(\hbar^{1+\kappa-\delta}\right) + O\left(\hbar^{2\kappa-3\delta}\right) \\ &= \hat{H} + \hbar^\kappa \widehat{\{P_0, H\}} + O\left(\hbar^{\kappa+\alpha}\right), \end{aligned}$$

where we have used the previously defined parameter α.
• Similarly, Proposition 15 tells us that

$$\begin{aligned} e^{i\hbar^{\kappa-1}\hat{P}_0}\hbar^\kappa \hat{K}_0\left(e^{i\hbar^{\kappa-1}\hat{P}_0}\right)^* &= \hbar^\kappa \hat{K}_0 + O\left(\hbar^\kappa \hbar^{\kappa-1-\delta+1-\delta}\right). \\ &= \hbar^\kappa \hat{K}_0 + O\left(\hbar^{\kappa+\alpha}\right), \end{aligned}$$

where we have used that $\kappa-2\delta>\kappa-3\delta\geq\alpha$ and thus $\hbar^{\kappa-2\delta}\ll\hbar^{\alpha}$.

We then write the symbol of Equation (4) and, dividing by $\hbar^{\kappa}$, one sees that we have to solve

$$\{P_0,H\}+K_0-A_0=O\left(\hbar^{\alpha}\right). \tag{5}$$

Actually, Lemma 32 insures that we can find a symbol $P_0^{'}\in\Psi_{\delta}^{-\delta}$ such that we have exactely $\left\{P_0^{'},H\right\}+K_0-A_0^{'}=0$, with $A_0^{'}\left(\hbar\right)\in\Psi_{\delta}^{0}$ the $\hbar^{\delta}$-average of $K_0\left(\hbar\right)$. Nevertheless, neither $A_0^{'}$ nor $P_0^{'}$ are self-adjoint. The adjoint of this equation is $\left\{P_0^{'},H\right\}^{*}+K_0-\left(A_0^{'}\right)^{*}=0$ since K_0 is self-adjoint. Moreover, using Lemma 9 and Proposition 10, one sees that $\left\{P_0^{'},H\right\}^{*}=\left\{\left(P_0^{'}\right)^{*},H\right\}+O\left(\hbar^{1-\delta}\right)$. This implies that $P_0=\frac{1}{2}\left(P_0^{'}+\left(P_0^{'}\right)^{*}\right)$ satisfies Equation (5) with $A_0=\frac{1}{2}\left(A_0^{'}+\left(A_0^{'}\right)^{*}\right)$ being the self-adjoint $\hbar^{\delta}$-average of $K_0\left(\hbar\right)$, thanks to Lemma 31 and to the fact that $\hbar^{1-\delta}\leq\hbar^{\alpha}$. The quantized of P_0 thus satisfies Equation (4) with a self-adjoint $\hat{K}_1\in\hat{\Psi}_{\delta}^{\alpha}$. If we define $\hat{U}_0=e^{i\hbar^{\kappa-1}\hat{P}_0}\in\hat{\Psi}_{\delta}^{0}$, we have $\hat{U}_0^{*}\hat{U}_0=\mathbb{I}+O\left(\hbar^{\infty}\right)$, $\hat{U}_0\hat{U}_0^{*}=\mathbb{I}+O\left(\hbar^{\infty}\right)$ and

$$\hat{U}_0\left(\hat{H}+\hbar^{\kappa}\hat{K}_0\right)\hat{U}_0^{*}=\hat{H}+\hbar^{\kappa}\hat{A}_0+\hbar^{\kappa}\hat{K}_1. \tag{6}$$

This equation is the initial step of the following induction process. Let us suppose that at the step $n\geq 0$, we have found self-adjoint PDOs $\hat{K}_1,...,\hat{K}_{n+1}$, with $\hat{K}_j\in\hat{\Psi}_{\delta}^{j\alpha}$ and PDOs $\hat{U}_0,...,\hat{U}_n\in\hat{\Psi}_{\delta}^{0}$ satisfying $\hat{U}_j^{*}\hat{U}_j=\mathbb{I}+O\left(\hbar^{\infty}\right)$ and $\hat{U}_j\hat{U}_j^{*}=\mathbb{I}+O\left(\hbar^{\infty}\right)$, such that

$$\hat{U}_n...\hat{U}_0\left(\hat{H}+\hbar^{1}\hat{K}\right)\hat{U}_0^{*}...\hat{U}_n^{*}=\hat{H}+\hbar^{1}\sum_{j=0}^{n}\hat{A}_j+\hbar^{1}\hat{K}_{n+1}, \tag{7}$$

where $A_j\left(\hbar\right)\in\Psi_{\delta}^{j\alpha}$ is the $\hbar^{\delta}$-average of $K_j\left(\hbar\right)$. Then we look for PDOs $\hat{K}_{n+2}\in\hat{\Psi}_{\delta}^{(n+2)\alpha}$ and $\hat{U}_{n+1}\in\hat{\Psi}_{\delta}^{0}$ with the same properties and satisfying

$$\hat{U}_{n+1}\hat{U}_n...\hat{U}_0\left(\hat{H}+\hbar^{\kappa}\hat{K}\right)\hat{U}_0^{*}...\hat{U}_n^{*}\hat{U}_{n+1}^{*}=\hat{H}+\hbar^{\kappa}\sum_{j=0}^{n+1}\hat{A}_j+\hbar^{\kappa}\hat{K}_{n+2}, \tag{8}$$

where $A_{n+1}\left(\hbar\right)\in\Psi_{\delta}^{(n+1)\alpha}$ is the $\hbar^{\delta}$-average of $K_{n+1}\left(\hbar\right)$. Looking for $\hat{U}_{n+1}$ of the form $\hat{U}_{n+1}=e^{i\hbar^{\kappa-1}\hat{P}_{n+1}}$, with $\hat{P}_{n+1}\in\hat{\Psi}_{\delta}^{-\delta+(n+1)\alpha}$ a self-adjoint PDO, and inserting Equation (7) into Equation (8), we get

$$e^{i\hbar^{\kappa-1}\hat{P}_{n+1}}\left[\hat{H}+\hbar^{\kappa}\sum_{j=0}^{n}\hat{A}_j+\hbar^{\kappa}\hat{K}_{n+1}\right]\left(e^{i\hbar^{\kappa-1}\hat{P}_{n+1}}\right)^{*}=\hat{H}+\hbar^{\kappa}\sum_{j=0}^{n+1}\hat{A}_j+\hbar^{\kappa}\hat{K}_{n+2}. \tag{9}$$

We now apply Propositions 15 and 16 for each term inside the bracket [ ].
• First, Proposition 16 tells us that

$$\begin{aligned}\hat{U}_{n+1}\left[\hat{H}\right]\hat{U}_{n+1}^{*} &= \hat{H}+i\hbar^{\kappa-1}\left[\hat{P}_{n+1},\hat{H}\right]+O\left(\hbar^{2(\kappa-1-\delta+(n+1)\alpha+1-\delta)+\delta}\right)\\ &= \hat{H}+i\hbar^{\kappa-1}\left[\hat{P}_{n+1},\hat{H}\right]+O\left(\hbar^{\kappa+(n+2)\alpha}\right),\end{aligned}$$

were we have used $\hbar^{\kappa-3\delta}\leq\hbar^{\alpha}$ and $\hbar^{(2n+3)\alpha}\ll\hbar^{(n+2)\alpha}$ provided $n\geq 0$.
• On the other hand, one can apply Proposition 10 which insures that the symbol of $\left[\hat{P}_{n+1},\hat{H}\right]$ equals to $\frac{\hbar}{i}\left\{P_{n+1},H\right\}+O\left(\hbar^{2-\delta+(n+1)\alpha}\right)$, i.e.

$$\frac{\hbar}{i}\left\{P_{n+1},H\right\}+O\left(\hbar^{1+(n+2)\alpha}\right),$$

where we have used $\hbar^{1-\delta}\leq\hbar^{\alpha}$. This implies that the symbol of $\hat{U}_{n+1}\left[\hat{H}\right]\hat{U}_{n+1}^{*}$ is

$$H+\hbar^{\kappa}\left\{P_{n+1},H\right\}+O\left(\hbar^{\kappa+(n+2)\alpha}\right).$$

• Then, Proposition 15 provides, for all $j=0..n$,

$$\begin{aligned}\hat{U}_{n+1}\left[\hbar^{\kappa}\hat{A}_{j}\right]\hat{U}_{n+1}^{*} &= \hbar^{\kappa}\hat{A}_{j}+O\left(\hbar^{\kappa}\hbar^{j\alpha}\hbar^{\kappa-1-\delta+(n+1)\alpha+1-\delta}\right)\\ &= \hbar^{\kappa}\hat{A}_{j}+O\left(\hbar^{\kappa+(n+2)\alpha}\right).\end{aligned}$$

where we have used $\hbar^{\kappa-2\delta}\ll\hbar^{\kappa-3\delta}\leq\hbar^{\alpha}$ and $\hbar^{j\alpha}\leq 1$.
• Finally, Proposition 15 yields

$$\begin{aligned}\hat{U}_{n+1}\left[\hbar^{\kappa}\hat{K}_{n+1}\right]\hat{U}_{n+1}^{*} &= \hbar^{\kappa}\hat{K}_{n+1}+O\left(\hbar^{\kappa}\hbar^{(n+1)\alpha}\hbar^{\kappa-1-\delta+(n+1)\alpha+1-\delta}\right)\\ &= \hbar^{\kappa}\hat{K}_{n+1}+O\left(\hbar^{\kappa+(2n+3)\alpha}\right)\\ &= \hbar^{\kappa}\hat{K}_{n+1}+O\left(\hbar^{\kappa+(n+2)\alpha}\right),\end{aligned}$$

where we have used $\hbar^{\kappa-2\delta}\ll\hbar^{\alpha}$ and $\hbar^{(2n+3)\alpha}\ll\hbar^{(n+2)\alpha}$ provided $n\geq 0$.

If we consider these different estimates, if we take the symbol of Equation (9) and if we divide by $\hbar^{\kappa}$, we see that we have to solve

$$\left\{P_{n+1},H\right\}+K_{n+1}-A_{n+1}=O\left(\hbar^{(n+2)\alpha}\right), \tag{10}$$

where $A_{n+1}(\hbar)$ is the $\hbar^{\delta}$-average of $K_{n+1}(\hbar)$. Then we use Lemma 32 which insures that we can find a symbol $P_{n+1}^{'}\in\Psi_{\delta}^{-\delta+(n+1)\alpha}$, such that we have exactely $\left\{P_{n+1}^{'},H\right\}+K_{n+1}-A_{n+1}^{'}=0$, where $A_{n+1}^{'}(\hbar)$ is the $\hbar^{\delta}$-average of $K_{n+1}(\hbar)$. Using the same technique as for the initial step $n=0$, we show that $P_{n+1}=\frac{1}{2}\left(P_{n+1}^{'}+\left(P_{n+1}^{'}\right)^{*}\right)$ satisfies Equation (10) with

$A_{n+1} = \frac{1}{2}\left(A'_{n+1} + \left(A'_{n+1}\right)^*\right)$ being the self-adjoint $\hbar^\delta$-average of $K_{n+1}(\hbar)$. The quantized of P_{n+1} thus satisfies Equation (9) with a self-adjoint $\hat{K}_{n+2} \in \hat{\Psi}_\delta^{(n+2)\alpha}$. This concludes the iterative process.

Then, if we define the sequence $\hat{V}_n$ by $\hat{V}_0 = \hat{U}_0$ and $\hat{V}_n = \hat{U}_n...\hat{U}_0 - \hat{U}_{n-1}...\hat{U}_0$, one can see that $\hat{V}_0 \in \hat{\Psi}^0_\delta$, $\hat{V}_n \in \hat{\Psi}_\delta^{\kappa-1-\delta+n\alpha}$ and $\hat{U}_n...\hat{U}_0 = \hat{V}_0 + ... + \hat{V}_n$. Using Borel's construction (Proposition 7), we can construct a PDO $\hat{U} \in \hat{\Psi}^0_\delta$ which admits the asymptotic expansion $\hat{U} \sim \sum_n \hat{V}_n$. By construction, it verifies $\hat{U}^*\hat{U} = \mathbb{I} + O(\hbar^\infty)$ and $\hat{U}\hat{U}^* = \mathbb{I} + O(\hbar^\infty)$. As well, there exists a self-adjoint PDO $\hat{K}$ in the class $\hat{\Psi}^0_\delta$ verifying $\hat{K} \sim \sum_n \hat{K}_n$. Moreover, if we define $A_\hbar$ as the $\hbar^\delta$-average of the symbol $K_\hbar$, we can see that $\hat{M} \sim \sum_n \hat{M}_n$. All that implies that Equation (3) is satisfied. □

4 Application : quasimodes

As an example of application of the normal form given in Theorem 33, we can easily construct quasimodes associated with the block $\mathcal{B}_0$.

Theorem 34 (Non-resonant quasimodes). *Let $\hat{H} \in \hat{\Psi}^0$ be a PDO with non-degenerate CI symbol $H(\xi) \in \Psi^0$ and $\hbar^\kappa \hat{K}_0 \in \hat{\Psi}^\kappa_\delta$ any self-adjoint perturbation with $\kappa > 0$. Let us fix small parameters $\gamma \geq 0$ and $\delta \geq 0$ such that $\delta < 1 - \gamma$ and $\delta < \frac{\kappa}{3}$, and let us consider the covering of $\mathcal{B}$ with resonance blocks.*

Then for each family $k_\hbar \in \Lambda^*$ such that $\hbar k_\hbar$ remains in the non-resonant block $\mathcal{B}_0$, there is a $\hbar^\infty$-quasimode $\varphi_\hbar$ of the perturbed operator

$$\left\|\left(\hat{H} + \hbar^\kappa \hat{K}_0 - E_\hbar\right)\varphi_\hbar\right\| = O(\hbar^\infty)$$

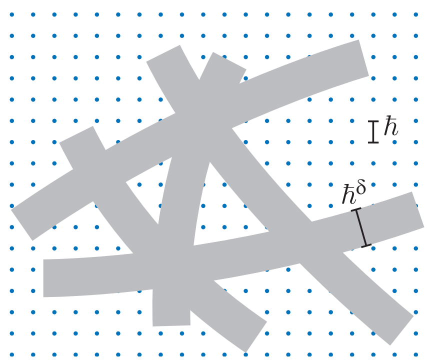
with the property $\varphi_\hbar = e^{ik_\hbar x} + O\left(\hbar^{\kappa-1-\delta}\right)$ and with quasi-eigenvalue

$$E_\hbar = H(\hbar k_\hbar) + \hbar^\kappa F_\hbar(\hbar k_\hbar),$$

where the x-independant symbol $F_\hbar \in \Psi^0_\delta$ is given by

$$F_\hbar(\hbar k_\hbar) = \langle\langle K_0 \rangle\rangle(\hbar k_\hbar) + O(\hbar^\alpha),$$

with $\langle\langle K_0\rangle\rangle(\xi)$ being the average of K_0 over the torus and α being defined by $\alpha = \min(1-\delta, \kappa - 3\delta)$.



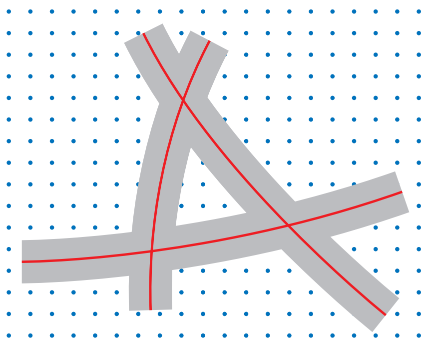
The picture above illustrates the case $d = 2$. The grey region is the zone $\mathcal{Z}_1^*$ of 1-resonances and the dots stand for the lattice $\hbar\mathbb{Z}^d$.

Proof. Indeed, according to Theorem 33, the perturbed operator is conjugate to its normal form $\widehat{NF} = \hat{H} + \hbar^\kappa \hat{A} + O(\hbar^\infty)$, where $A_\hbar$ is the self-adjoint $\hbar^\delta$-average of $K_\hbar$, which is a symbol satisfying $\hat{K} = \hat{K}_0 + O(\hbar^\alpha)$. Moreover, in the block $\mathcal{B}_0$, Lemma 31 tells us that $A_\hbar$ is simply $A_\hbar = \langle\langle K_\hbar \rangle\rangle + O(\hbar^\infty)$. On the other hand, as the averaged symbol $\langle\langle K_\hbar \rangle\rangle$ is independent on x, the eigenvalues of $\hat{H} + \hbar^\kappa \widehat{\langle\langle K_\hbar \rangle\rangle}$ are given by $E_k(\hbar) = H(\hbar k) + \hbar^\kappa \langle\langle K_\hbar \rangle\rangle (\hbar k)$, for each $k \in \Lambda^*$, and the associated eigenvectors are simply the exponential functions e^{ikx}. These functions are thus also $\hbar^\infty$-quasimodes of the $\widehat{NF}$ for each family $k_\hbar$ such that $\hbar k_\hbar \in \mathcal{B}_0$ for all $\hbar$, and the quasi-eigenvalues are $E_{k_\hbar}(\hbar)$. Then, applying the operators $\hat{U}$ which conjugate the perturbed operator to $\widehat{NF}$, we get $\hbar^\infty$-quasimodes $\varphi_\hbar = \hat{U}^* \left(e^{ik_\hbar x}\right)$ of the perturbed operator, with the same quasi-eigenvalues. Finally, we notice that according to the properties of $\hat{U}$, the quasimodes have the form $\varphi_\hbar = e^{ik_\hbar x} + O\left(\hbar^{\kappa-1-\delta}\right)$. Moreover, according to the expression of $K_\hbar$, the eigenvalues have the expression

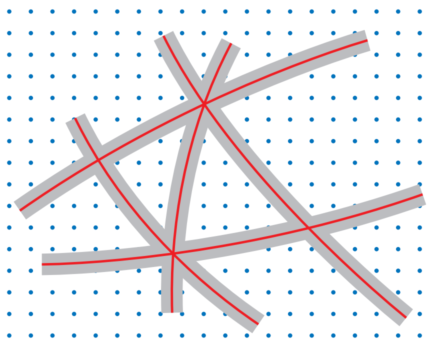
$$E_{k_\hbar}(\hbar) = H(\hbar k_\hbar) + \hbar^\kappa \langle\langle K_0 \rangle\rangle (\hbar k_\hbar) + O\left(\hbar^{\kappa+\alpha}\right).$$

$\square$

These quasi-eigenvalues are very easily constructed but the number we can construct depends on the size of the block $\mathcal{B}_0$ as $\hbar$ goes to zero. This size depends on the parameters γ and δ, which control respectively the amount of resonance zones we consider and their width (see below).



"small" δ and γ



"big" δ and γ

With an appropriate choice of γ and δ, one can insure that the relative volume of $\mathcal{B}_0$ tends to 1 as $\hbar$ goes to zero.

Proposition 35. *Let $\gamma > 0$ and $\delta > 0$ such that $\delta > 2d\gamma$. For each compact set $\mathcal{O} \subset \mathcal{B}$ and each $n = 1..d$, the volume of the block of n-resonances $\mathcal{B}_n^*$ is of order*

$$vol\left(\mathcal{B}_n^* \cap \mathcal{O}\right) = \left(\hbar^{n(\delta - 2d\gamma)}\right).$$

Proof. Indeed, the non-degeneracy condition for H implies that there exist two constants T and C such that for all $k \neq 0$ and all ξ satisfying $\frac{\Omega_k(\xi)}{|k|} \leq T$, one has $\left|d\frac{\Omega_k(\xi)}{|k|}\right| \geq C$. For $\hbar$ small enough ($\hbar^{\delta-\gamma n} \leq T$) and for each $k \neq 0$, the points ξ in the resonant zone $\mathcal{Z}_k \cap \mathcal{O}$ satisfy $\left|d\frac{\Omega_k(\xi)}{|k|}\right| \geq C$. The function $\frac{\Omega_k(\xi)}{|k|}$ is thus suitable to measure lengths and this implies that the volume of the resonant zone $Z_{\mathcal{R}} \cap \mathcal{O}$, for any n-dimensional lattice $\mathcal{R}$, is bounded by $vol\left(Z_{\mathcal{R}} \cap \mathcal{O}\right) = O\left(\left(\hbar^{\delta-\gamma n}\right)^n\right)$ uniformly with respect to $\mathcal{R}$. On the other hand, the volume of $\mathcal{B}_k \cap \mathcal{O}$ is bounded in the same way since $\mathcal{B}_k \subset \mathcal{Z}_k$.

The bloc $\mathcal{B}_1^* \cap \mathcal{O}$ of 1-resonances is the union of the $\mathcal{B}_k \cap \mathcal{O}$ for all non-vanishing k with $|k| \leq \hbar^{-\gamma}$. If we bound the sum over all primitive k with $|k| \leq \hbar^{-\gamma}$ by the sum over all k with $|k| \leq \hbar^{-\gamma}$, we get the following estimate

$$vol\left(\mathcal{B}_1^* \cap \mathcal{O}\right) \leq C\hbar^{\delta-\gamma} \sum_{|k| \leq \hbar^{-\gamma}} = O\left(\hbar^{\delta-\gamma-d\gamma}\right).$$

Similarly, for $n = 2..d$, the block $\mathcal{B}_n^* \cap \mathcal{O}$ of n-resonances is the union of the blocks $\mathcal{B}_{\mathcal{R}} \cap \mathcal{O}$ for all the resonance $\hbar^{-\gamma}$-lattices $\mathcal{R}$, i.e. the sub-lattices which admit a basis $(e_1, ..., e_n)$ composed of vector with norm $|e_j| \leq \hbar^{-\gamma}$. The volume of $\mathcal{B}_n^* \cap \mathcal{O}$ is then roughly bounded by

$$vol\left(\mathcal{B}_n^* \cap \mathcal{O}\right) \leq C^{'}\hbar^{n(\delta-n\gamma)} \sum_{\substack{|e_1| \leq \hbar^{-\gamma} \\ \vdots \\ |e_n| \leq \hbar^{-\gamma}}} = O\left(\hbar^{n(\delta-n\gamma)-\gamma dn}\right).$$

This estimate is really too rough when $n \geq \frac{d}{2}$ since, for example, the number of lattices of d-resonances is equal to 1 rather than to $\hbar^{-\gamma d^2}$! For $n \geq \frac{d}{2}$, we will rather count the orthogonal lattices, what gives a number of lattices of order $\hbar^{-\gamma d(d-n)}$ and thus a volume of order $vol\left(\mathcal{B}_n^* \cap \mathcal{O}\right) = O\left(\hbar^{n(\delta-n\gamma)-\gamma d(d-n)}\right)$ for $n \geq \frac{d}{2}$. On the other hand, the inequality involving γ and δ implies the following :

- When $0 < n < \frac{d}{2}$, we have

$$n\left(\delta - n\gamma\right) - \gamma dn > n\left(\delta - \frac{d}{2}\gamma - \gamma d\right) > n\left(\delta - 2\gamma d\right).$$

- When $\frac{d}{2} \leq n \leq d$, we have $n\left(\delta - n\gamma\right) - \gamma d\left(d-n\right) = n\left(\delta - n\gamma + d\gamma - \gamma\frac{d^2}{n}\right)$

and the bracket is estimated by

$$\delta - n\gamma + d\gamma - \gamma\frac{d^2}{n} \geq \delta - d\gamma + d\gamma - \gamma\frac{2d^2}{d} = \delta - 2\gamma d.$$

This thus shows that the volume $vol\left(\mathcal{B}_n^* \cap \mathcal{O}\right)$ is of order $O\left(\hbar^{n(\delta-2\gamma d)}\right) \ll 1$ for all $n = 1..d$. $\square$

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