A generalized Verdier-type Riemann-Roch theorem for Chern-Schwartz-MacPherson classes

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Abstract

In this paper we give a general formula for the defect appearing in the Verdier- type Riemann-Roch formula for Chern-Schwartz-MacPherson classes in the case of a regular embedding (and for suitable local complete intersection morphisms). Our proof of this formula uses the "constructible function version" of Verdier's specialization functor $Sp_{X\setminus Y}$ (for constructible (complexes of) sheaves), together with a specialization property of Chern-Schwartz-MacPherson classes and the corresponding Riemann-Roch theorem for smooth morphisms. As a very special case we get a general formula for the "Milnor-class" of a local complete intersection in a smooth manifold (generalizing many recent results of different authors). Our formula becomes very simple in the case of a regular embedding of codimension one, and gives a far reaching generalization of Parusiński-Pragacz's formula for the "Milnor-class" of a hypersurface in a (compact) smooth manifold.

Introduction

The Chern-Schwartz-MacPherson class c_* is the unique transformation from constructible functions to homology, commuting with proper push forward and satisfying for smooth X the normalization

$$c_*(1_X) = c^*(TX) \cap [X] \,, \tag{1}$$

with $c^*(TX)\cap$ the action of the total Chern-class of the tangent-bundle TX on the fundamental-class $[X]$ of X. Note that the uniqueness follows from

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resolution of singularities and the existence of such a transformation was conjectured by Deligne and Grothendieck (compare [Gr, p.361-368]). It was first constructed by MacPherson in the complex algebraic context in his classical paper [M], using his celebrated "Euler-obstruction" and "graph-construction". In his origional work MacPherson used as a suitable homology theory the Borel-Moore homology $H_*^{BM}(\cdot, \mathbb{Z})$ of the corresponding analytic space. But his proof also applies to the case, where one uses the algebraic Chow-homology theory A_* of Fulton [Ful] (compare [Ful, Ex. 19.1.7]). Moreover both approaches are linked by the cycle map $cl : A_* \to H_*^{BM}(\cdot, \mathbb{Z})$ [Ful, Chapter 19].

The analyticity of the graph construction was finally solved by Kwieciński in his thesis [Kw] so that MacPherson's proof also applies to the complex analytic context. Moreover, for complex spaces which can be embedded into complex manifolds, the MacPherson Chern-class $c_*(1_X)$ of X is isomorphic via Alexander duality (by work of Brasselet-Schwartz [CEP, Exp. 6]) to the classes introduced already before by Schwartz [Schw]. Finally a new approach to these Chern-classes (in the embeddable context) comes from the "micro-local viewpoint" through the theory of Lagrangian cycles [Fu2, Gi, Sab, Sch2]. This theory was originally described in terms of characteristic cycles of holonomic D-modules or constructible (complexes of) sheaves, but only depends on the associated constructible functions (For a direct approach to Lagrangian cycles in terms of constructible functions compare with [Fu2, Sch3]). Moreover, the Lagrangian approach has been extended in the algebraic context to (embeddable) schemes of finite type over a ground field of characteristic zero [Ken].

The Verdier-type Riemann-Roch formula one is looking for is the commutativity of the following diagram:

$$\begin{array}{ccc} F(Y) & \xrightarrow{c_*} & H_*(Y) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle c^*(T_f)\cap f^*} \\ F(X) & \xrightarrow[c_*]{} & H_*(X) \end{array} \tag{2}$$

for a (local complete intersection) morphism $f : X \to Y$, which has a global factorization into a regular embedding $i : X \to Z$ followed by a smooth morphism $p : Z \to Y$. Here we use the following notations:

- $f^* : F(Y) \to F(X);\ \alpha \mapsto \alpha \circ f$ is the usual pullback of constructible functions.

- $H_*(\cdot)$ the relevant homology theory $A_*(\cdot)$ (or $H_*^{BM}(\cdot, \mathbb{Z})$) depending on the algebraic (or complex analytic) context.

- $c^*(T_f)\cap := \left(c^*(i^*T_p) \cup (c^*(N_X Z))^{-1}\right)\cap$, with T_p the relative tangent bundle of the smooth morphism p and $N_X Z$ the normal bundle of the regular embedding $X \to Z$.

- Finally $f^* : H_*(Y) \to H_*(X)$ is the Gysin transformation for such a morphism, i.e. the composition of the smooth pullback p^* and the Gysin transformation i^* for the regular embbeding i (which is defined with the help of the deformation to the normal cone $N_X Z$). For the construction of i^* in the algebraic (or analytic) context compare with [Ful, Ch.5,Ch.6] (or [GA, Exp.IX]).

Note that $c^*(T_f)\cap$ and $f^* : H_*(Y) \to H_*(X)$ are independent of the chosen factorization $f = p \circ i$ of f. This commutativity is motivated by a corresponding result of Verdier [GA, p.214, thm.7.1] for the "Todd-homology transformation" τ_* of Baum-Fulton-MacPherson (compare with [Ful, thm.18.2(3)] and especially also with [FM, p.11,p.122] for the origin of the name for such a formula).

A similar Verdier-Riemann-Roch theorem for Chern-Schwartz-MacPherson classes c_* is true for a smooth morphism p ([Y, thm.2.2] and compare also with [FM, p.111]), i.e. the following diagram is commutative for a smooth morphism $p : X \to Y$:

$$\begin{array}{ccc} F(Y) & \xrightarrow{c_*} & H_*(Y) \\ {\scriptstyle p^*}\downarrow & & \downarrow{\scriptstyle c^*(T_p)\cap p^*} \\ F(X) & \xrightarrow[c_*]{} & H_*(X) \end{array} , \tag{3}$$

with T_p the relative tangent bundle of p. Note that Yokura works in the complex algebraic context, but his simple proof applies also to the analytic (or algebraic context over a ground field of characteristic zero), since it uses resolution of singularities, a smooth base-change property (for constructible functions and for the corresponding homology theory), together with the defining properties (normalization, proper push forward) of the Chern-classes c_* (and for a Lagrangian approach compare with [Sch2]).

But in general the diagram (2) is not commutative! The factorization

$$\begin{array}{ccc} F(Y) & \xrightarrow{c_*} & H_*(Y) \\ {\scriptstyle p^*}\downarrow & & \downarrow{\scriptstyle c^*(T_p)\cap p^*} \\ F(Z) & \xrightarrow{c_*} & H_*(Z) \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle c^*(N_X Z)^{-1}\cap i^*} \\ F(X) & \xrightarrow[c_*]{} & H_*(X) \end{array} \tag{4}$$

implies that the problems come from the case of a regular embedding $i : X \hookrightarrow Y$. For the special case of a smooth ambient space Y

$$(c^*(N_X Y))^{-1} \cap i^*(c_*(1_Y)) = (c^*(N_X Y))^{-1} \cap c^*(i^* TY) \cap [X]$$

is the Fulton-Johnson Chern class $c^{FJ}(X)$ of X [Ful, Ex.4.2.6], which is in general different from the Chern-class $c_*(1_X) = c_*(i^* 1_Y)$ of X (compare [Y, Ex.3.1]). The difference class

$$M_0(X) := c^{FJ}(X) - c_*(1_X) \tag{5}$$

is called the "Milnor class" of X (up to suitable sign-conventions) and has been recently studied in some cases by many authors using quite different methods. Compare for example with [Al, BLSS, BLSS2, PP, Su, OY, Y3] and especially with the nice survey articles [Br, Y2] of Brasselet and Yokura. We will explain the relations to our results in the end of this work. Note that these authors use a different sign-convention for their Milnor-class $M(X)$, and for this reason we use the notation $M_0(X)$ (following [Y3]). Moreover, in most cases they consider (only) the case of a "global" complete intersection X in Y, i.e. X is defined by a cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ \downarrow & & \downarrow{\scriptstyle s} \\ Y & \xrightarrow[0]{} & E\,, \end{array}$$

with s a section of a vector-bundle $E \to Y$ which is generically transverse to the zero-section $0 : Y \to E$. So in this case one gets $N_X Y = E|X$.

In this paper we explain in a very general context the non-commutativity of (2) in the case of a regular embedding $i : X \hookrightarrow Y$ using the "constructible function version" of Verdier's specialization to the normal cone (i.e. of Verdier's specialization transformation $Sp_{X\backslash Y}$ [V] for constructible (complexes of) sheaves):

$$Sp_{X\backslash Y} : F(Y) \to F_{mon}(C_X Y) \ , \tag{6}$$

with $F_{mon}(C_X Y)$ the group of "monodromic" constructible functions on the normal cone (i.e. constructible functions on $C_X Y$, which are invariant under the natural $\mathbb{C}^*$-action on this cone over X). Note that this definition works for any closed embedding $i : X \hookrightarrow Y$ of a subspace (with $C_X Y = N_X Y$ the normal bundle in the case of a regular embedding). We explain in the next chapter the definition and basic properties of $Sp_{X\backslash Y}$ (following [V]). Let us only mention that $Sp_{X\backslash Y}(\alpha)|X = \alpha|X$ for any $\alpha \in F(Y)$, where we consider X as a subspace of the normal cone $C_X Y$ as the "zero- or vertex-"section $k : X \hookrightarrow C_X Y$. Let $\pi : C_X Y \to X$ be the projection.

Then we introduce the "generalized vanishing cycles" Φ_i of i as the transformation

$$\Phi_i := Sp_{X\backslash Y} - \pi^* i^* : F(Y) \to F_{mon}(C_X Y) \ . \tag{7}$$

Note that $\Phi_i(\alpha)$ is a monodromic constructible function, which is zero on X (for any $\alpha \in F(Y)$). The notation "generalized vanishing cycles" is motivated by the following observation (compare [V, (Sp6) Normalisation,p.353] and the next chapter): Assume $i : X \hookrightarrow Y$ is a regular embedding of codimension one (i.e. a hypersurface on Y locally given by one equation $\{g = 0\}$). By the above properties of Φ_i (and the fact that $\pi : C_X Y = N_X Y \to X$ is a line-bundle), there exists a welldefined transformation $\mu : F(Y) \to F(X)$, with

$$\Phi_i = (\pi^* - k_*) \circ \mu \ . \tag{8}$$

Here $k_* : F(X) \to F(N_X Y)$ is just extension by zero. If in addition X is (locally) given by the equation $\{g = 0\}$ on Y, then μ is just the "constructible function version" of Deligne's vanishing cycle functor ϕ_g for constructible (complexes of) sheaves. Especially, this vanishing (and also the corresponding nearby) cycle functor is independent of the chosen local equation of X. This was one of the motivations of Verdier for introducing his specialization functor $Sp_{X\backslash Y}$ (compare [V, p.332]).

Now we are ready to formulate the main result of this paper. For simplicity we work for the rest of this introduction in

- the complex algebraic context with algebraic constructible functions $F(\cdot)$ on the associated complex analytic spaces of seperated schemes of finite type over $spec(\mathbb{C})$ and $H_*(\cdot)$ one of the two possible homology theories (i.e. the Chow-homology $A_*(\cdot)$ or the Borel-Moore homology $H_*^{BM}(\cdot, \mathbb{Z})$ of the associated complex analytic space),
- or in the complex analytic context with **compact** analytic spaces.

Then our main result is the following generalized Verdier-type Riemann-Roch theorem for Chern-Schwartz-MacPherson classes (asked for in [Y, Y2, BY]):

Theorem 0.1. *Consider a regular embedding $i : X \hookrightarrow Y$ of complex "algebraic" (or compact analytic) spaces. Then one has for the "defect" in the corresponding Verdier-type Riemann-Roch-formula the following explicit description:*

$$c^*(N_XY)^{-1} \cap i^*(c_*(\cdot)) - c_*(i^*(\cdot)) = c^*(N_XY)^{-1} \cap k^*\Big(c_*(\Phi_i(\cdot))\Big) \ . \tag{9}$$

Here $k : X \hookrightarrow N_XY$ is the "zero-section" of the normal-bundle $\pi : N_XY \to X$ and k^* on homology is the inverse of the isomorphism $\pi^* : H_*(X) \to H_*(N_XY)$.

The above "defect" formula follows by the definition of Φ_i directly from the following two results:

$$c_*(\cdot) = c^*(N_XY)^{-1} \cap k^*\Big(c_*(\pi^*(\cdot))\Big) \ , \tag{10}$$

which is a simple application of the corresponding Verdier Riemann-Roch theorem (3) for the smooth projection $\pi : N_XY \to X$:

$$c_*(\pi^*(\cdot)) = c^*(T_\pi) \cap \pi^*(c_*(\cdot)) \ ,$$

with $k^*(c^*(T_\pi) \cap \pi^*) = c^*(k^*T_\pi)\cap$ (since k^*T_π is isomorphic to N_XY, and $k^*\pi^* = id$). This part works without any compactness assumptions.

The other result is the formula

$$i^*(c_*(\cdot)) = k^*\Big(c_*(Sp_{X\backslash Y}(\cdot))\Big) \ , \tag{11}$$

which follows from the definition of the Gysin homomorphism $i^* = k^* \circ sp$ (based on the deformation to the normal cone) and the specialization property

$$sp \circ c_* = c_* \circ Sp_{X\backslash Y} \tag{12}$$

of the Chern-class transformation c_* (Compare with [Ful, p.352] and [GA, thm.6.5,p.211] for the corresponding specialization property of the "Todd-homology transformation" τ_* of Baum-Fulton-MacPherson).

This specialization property (12) of c_* follows from a corresponding result with respect to the "constructible function counterpart" of Deligne's nearby cycle functor (used in the definition of $Sp_{X\backslash Y}$). In the analytic context this goes back to Verdier [CEP, Exp.7, thm.5.1], and it is exactly at this point where we use in the analytic context our compactness assumption. Other proofs of this specialization property for c_* (in the embedded context) have been given by the theory of Lagrangian cycles in [Fu, Gi, Sab, Sch3]. Moreover, this Lagrangian approach was extended by Kennedy [Ken2, thm.2)] to the algebraic context (based on the results of Sabbah [Sab]).

So we see that the proof of theorem 0.1 is very simple, once we use the Verdier Riemann-Roch theorem (3) for smooth morphisms and the specialization property (12) of the Chern-class transformation c_*. The main point of our paper is therefore just to explain the definition of the specialization functor $Sp_{X\backslash Y}$ and the "generalized vanishing cycles" Φ_i for constructible functions, and to show how to use them for the applications.

Our "defect-formula" (9) becomes very simple in the following two cases:

Corollary 0.1. *Consider a regular embedding $i : X \hookrightarrow Y$ as before and suppose that $\alpha \in F(Y)$ has the property $\Phi_i(\alpha) = 0$ (i.e. $\pi^* i^*(\alpha) = Sp_{X\backslash Y}(\alpha)$). Then one has a corresponding Verdier-type Riemann-Roch-theorem for α:*

$$c_*(i^*(\alpha)) = c^*(N_X Y)^{-1} \cap i^*(c_*(\alpha)) \ .$$

The assumption on α in the above corollary is for example satisfied, if X and Y are smooth and X is transverse to a complex analytic (or algebraic) Whitney b-regular stratification adapted to α (i.e. α is (locally) constant on the strata of this stratification). But we will not give a proof of this fact (which uses Fourier-transformation of constructible functions (or sheaves)). Let us just remark that in this case we have

$$c^*(N_X Y)^{-1} = c^*(TX) \cup c^*(i^* TY)^{-1} \ ,$$

and a micro-local proof of the corresponding Verdier-type Riemann-Roch theorem was given in [Sch2] (based on the theory of (non-characteristic pullback of)Lagrangian cycles).

The other case is given by the

Corollary 0.2. *Consider a regular embedding $i : X \hookrightarrow Y$ of codimension one. Then one has for the "defect" in the corresponding Verdier-type Riemann-Roch-formula the following simple description:*

$$c^*(N_XY)^{-1} \cap i^*(c_*(\cdot)) - c_*(i^*(\cdot)) = c^*(N_XY)^{-1} \cap c_*(\mu(\cdot)) \ , \tag{13}$$

with $\mu : F(Y) \to F(X)$ defined as in (8) .

Note that this follows from (9) and the definition (8) of μ by the following easy calculations:

$$c_*(\Phi_i(\cdot)) = c_*\Big(\pi^*(\mu(\cdot))\Big) - c_*\Big(k_*(\mu(\cdot))\Big)$$

$$= c^*(T_\pi) \cap \pi^*\Big(c_*(\mu(\cdot))\Big) - k_*\Big(c_*(\mu(\cdot))\Big) \ .$$

For the last equality we use the Verdier-type Riemann-Roch theorem for the smooth projection π (for the first term), and the functoriality of c_* with respect to the proper map k (for the second term). Then the claim follows from

$$k^*\Big(c^*(T_\pi) \cap \pi^*\Big) - k^*k_* = c^*(k^*T_\pi) \cap k^*\pi^* - k^*k_*$$

$$= c^*(N_XY) \cap - c^1(N_XY) \cap = \Big(1 + c^1(N_XY) - c^1(N_XY)\Big) \cap = 1 \cap \ ,$$

where we used the self-intersection formula $k^*k_* = c^1(N_XY)\cap$ (compare [Ful, prop.2.6(c), cor.6.3] for the algebraic case).

If we apply this corollary to 1_Y for a smooth Y, we get as a very special case the formula

$$M_0(X) = c^*(N_XY)^{-1} \cap c_*(\mu(1_Y)) \ . \tag{14}$$

As we will explain later on, the constructible function $\mu(1_Y)$ is exactly the corresponding function μ used in [PP] (up to the sign $(-1)^{dim(Y)-1}$, by which our definition of $M_0(X)$ differs from the convention used in [PP]). So in this special case we get back the main result of Parusiński-Pragacz [PP, thm.0.2] giving a nice formula of the Milnor class of a ("global") hypersurface inside a (compact) complex manifold (which was conjectured by Yokura). Even in this special case, we think that our more general approach is simpler than the proof given in [PP], which is based on deep results about suitable characteristic cycles. Parusiński-Pragacz used already in [PP] a specialization argument (i.e. Verdier's specialization result for c_*) for the proof of another formula, which for projective Y implies a slightly weaker version of the above formula for $M_0(X)$ (by some lengthy calculation, compare [Y, p.363-367]).

But up to now, this specialization property of c_* was not applied to the specialization map

$$Sp_{X\backslash Y} : F(Y) \to F_{mon}(C_X Y) ,$$

which is so basic for our approach (and even so natural, if one looks at the definition of the Gysin map in homology for regular embeddings). In the special case of the constructible function 1_Y, we get by theorem 0.1 the following generalization of the formula of Parusiński-Pragacz for a general regular embedding $i : X \hookrightarrow Y$ of complex "algebraic" (or compact analytic) spaces, with Y smooth:

$$M_0(X) = c^*(N_X Y)^{-1} \cap k^*\Big(c_*(\Phi_i(1_Y))\Big) , \qquad (15)$$

with $k : X \to N_X Y$ the zero-section. The difference to the codimension one case comes from the fact that in general the function $\Phi_i(1_Y) \in F_{mon}(N_X Y)$ is much more complicated (compared to the codimension one case)! Newertheless, in the end of this paper we will explain, that one can deduce many (known) properties of $M_0(X)$ from this formula (15).

Let us end this introduction with a general "defect-formula" for a local complete intersection morphism $f : X \to Y$, which follows directly from the factorization (4), the Verdier-formula (3) for smooth morphisms and theorem 0.1:

Corollary 0.3. *Consider a local complete intersection morphism $f = p \circ i : X \hookrightarrow Y$ with a global factorization into a regular embedding $i : X \to Z$ followed by a smooth morphism $p : Z \to Y$ of complex "algebraic" (or compact analytic) spaces. Then one has for the "defect" in the corresponding Verdier-type Riemann-Roch-formula the following explicit description:*

$$c^*(T_f) \cap f^*(c_*(\cdot)) - c_*(f^*(\cdot)) = c^*(N_X Z)^{-1} \cap k^*\Big(c_*(\ \Phi_i(p^*(\cdot))\ )\Big) , \quad (16)$$

with $c^*(T_f)\cap := \Big(c^*(i^*T_p) \cup (c^*(N_X Z))^{-1}\Big)\cap$. Here $k : X \hookrightarrow N_X Z$ is the "zero-section" of the normal-bundle $\pi : N_X Z \to X$ and k^* on homology is the inverse of the isomorphism $\pi^* : H_*(X) \to H_*(N_X Z)$.

1 The Verdier specialization functor for constructible functions

In this section we explain some basic constructions for constructible functions. Here we work with a complex analytic space X, and a function $\alpha : X \to \mathbb{Z}$ is called (analytically) constructible, if it satisfies one of the following two equivalent properties:

1. α is a locally finite sum $\alpha = \sum_j n_j \cdot 1_{Z_j}$, with $n_j \in \mathbb{Z}$ and 1_{Z_j} the indicator function of the closed analytic subset Z_j of X.
2. α is (locally) constant on the strata of a complex analytic Whitney b-regular stratification of X (which we then call a stratification adapted to α).

Note that on a compact analytic space a locally finite sum as above is indeed a finite sum.

This notation is closely related to the much more sophisticated language of (analytically) constructible (complexes of) sheaves on X. A sheaf $\mathcal{F}$ of vector-spaces on X with finite dimensional stalks is (analytically) constructible, if there exists an analytic Whitney b-regular stratification as above such that the restriction of $\mathcal{F}$ to all strata is locally constant. Similarly, a bounded complex of sheaves is constructible, if all it cohomology sheaves have this property, and we denote by $D^b_c(X)$ the corresponding derived category of bounded constructible complexes on X. Then the Grothendieck group of the triangulated category $D^b_c(X)$ is isomorphic to the Grothendieck group of the abelian category of constructible sheaves (by associating to a constructible complex the alternating sum of its cohomology sheaves), and we denote it by $K_{const}(X)$.

Since we assume that all stalks of a constructible complex are finite dimensional, we get by taking stalkwise the Euler characteristic a natural group homomorphism $\chi_X : K_{const}(X) \to F(X)$ into the group $F(X)$ of constructible functions (on X). And it is very easy to show that χ_X is surjective.

If X is the analytic space associated to a seperated scheme of finite type over $spec(\mathbb{C})$, then one has the corresponding notation of an algebraically constructible (complex of) sheaves or function on X (by using an algebraic stratification of X), and in this case any algebraically constructible function is just a finite linear combination of functions 1_Z with Z a closed algebraic subset of X. As before one gets a corresponding group epimorphism χ_X, and in the following we treat the analytic and algebraic context (as far as possible) at the same time.

As is well known (and explained in detail in [Sch]), all the usual functors in sheaf theory, which respect the corresponding category of constructible complexes of sheaves, induce by the epimorphism χ_X well-defined group homomorphisms on the level of constructible functions. We just recall those, which are important for later applications or definitions. Let $f : X \to Y$ be a holomorphic (or algebraic) map of complex spaces (associated to seperated schemes of finite type over $spec(\mathbb{C})$). Then one has the following transformations:

- **pullback:** $f^* : F(Y) \to F(X)$; $\alpha \mapsto \alpha \circ f$, which corresponds to the usual pullback of sheaves.
- **Euler characteristic:** Suppose X is compact and $Y = \{pt\}$ is a point. Then one has $\chi : F(X) \to \mathbb{Z}$, corresponding to $R\Gamma(X, \cdot)$ on the level of constructible complexes of sheaves. By linearity it is characterized by the convention, that $\chi(1_Z)$ for a closed analytic (algebraic) subspace Z of X is just the usual Euler characteristic of the (finite dimensional) cohomology $H^*(Z)$ of Z.
- **proper pushdown:** Suppose f is proper. Then one has $f_* : F(X) \to F(Y)$, corresponding to Rf_* on the level of constructible complexes of sheaves. Explicitly it is given by $f_*(\alpha)(y) := \chi(\alpha|\{f = y\})$, and in this form it goes back to the paper [M] of MacPherson (Note that $\{f = y\}$ is compact, since f is proper so that we can use the Euler characteristic χ).
- **nearby cycles:** Assume $Y = \mathbb{C}$ and let $X_0 := \{f = 0\}$ be the zero fiber. Then one has $\psi_f : F(X) \to F(X_0)$, corresponding to Deligne's nearby cycle functor. This was first introduced in [CEP, Exp.7,prop.3.4] by using resolution of singularities (compare with [Sch] for another approach using stratification theory). By linearity, ψ_f is uniquely defined by the convention that for a closed analytic (algebraic) subspace $\psi_f(1_Z)(x)$ is just the Euler-characteristic of a local Milnor fiber $M_{f|Z,x}$ of $f|Z$ at x [CEP, Exp.7,prop.4.1]. Here this local Milnor fiber at x is given by $M_{f|Z,x} := Z \cap B_\epsilon(x) \cap \{f = y\}$, with $0 < y << \epsilon << 1$ and $B_\epsilon(x)$ an open (or closed) ball of radius ϵ around x (in some local coordinates).

- **vanishing cycles:** Assume $Y = \mathbb{C}$ and let $i : X_0 := \{f = 0\} \hookrightarrow X$ be the inclusion of the zero-fiber. Then one has $\phi_f : F(X) \to F(X_0)$; $\phi_f := \psi_f - i^*$, corresponding to Deligne's vanishing cycle functor.

Using these notations, we can now introduce the "constructible function version" of Verdier's specialization functor. Let $i : X \hookrightarrow Y$ be a closed embedding of complex analytic spaces (associated to seperated schemes of finite type over $spec(\mathbb{C})$). Then the deformation to the normal cone $C_X Y$ of X in Y is a complex analytic ("algebraic") space M together with two morphisms $\pi : M \to Y$ and $h : M \to \mathbb{C}$, with h flat so that one has a cartesian diagram

$$\begin{array}{ccccc} C_X Y & \xrightarrow{s} & M & \xleftarrow{j} & Y \times \mathbb{C}^* \\ \downarrow & & \downarrow h & & \downarrow pr \\ \{0\} & \longrightarrow & \mathbb{C} & \longleftarrow & \mathbb{C}^* \, . \end{array} \tag{17}$$

For details of its construction we refer to [Ful, Ch.5] and [GA, Exp.IX]. Let us only recall that $M = \tilde{M} \backslash \tilde{Y}$, with $\tilde{M}$ the blow-up of $Y \times \mathbb{C}$ along $X \times \{0\}$ and $\tilde{Y}$ the blow-up of Y along X (suitable embedded into $\tilde{M}$) so that π and h are induced from the corresponding projections by the blow-up map b:

$$M = \tilde{M} \backslash \tilde{Y} \hookrightarrow \tilde{M} \xrightarrow{b} Y \times \mathbb{C} \, . \tag{18}$$

Then $Sp_{X \backslash Y} : F(Y) \to F_{mon}(C_X Y)$ is defined as $Sp_{X \backslash Y} := \psi_h \circ \pi^*$. This corresponds exactly to Verdier's definition [V, p.352,p.358] for constructible sheaves, since $\psi_h(p^* \mathcal{F})$ depends by definition only on $j^* \pi^* \mathcal{F} = (\pi \circ j)^* \mathcal{F}$, and $\pi \circ j$ is by construction of the deformation just the projection $Y \times \mathbb{C}^* \to Y$. Note that the construction of (M, π, h) is functorial in the pair (Y, X).

By [V] we therefore get the following important properties of the functor $Sp_{X \backslash Y}$ (where the property (SP4) related to duality is trivial for constructible functions, so that we omit it):

(SP0) **localization:** $Sp_{X \backslash Y}$ commutes with restriction to open subsets.

(SP1) **monodromy:** $Sp_{X \backslash Y}$ maps to $F_{mon}(C_X Y)$.

(SP2) **proper direct image:** Consider a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{i'} & Y' \\ \downarrow & & \downarrow f \\ X & \xrightarrow[i]{} & Y \end{array}$$

with f proper. Then also the induced map $C(f) : C_{X'}Y' \to C_X Y$ is proper with

$$Sp_{X\backslash Y} \circ f_* = C(f)_* \circ Sp_{X'\backslash Y'} . \tag{19}$$

(SP3) **smooth base change:** Consider a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{i'} & Y' \\ \downarrow & & \downarrow f \\ X & \xrightarrow[i]{} & Y \end{array}$$

with f smooth. Then also the induced map $C(f) : C_{X'}Y' \to C_X Y$ is smooth with

$$C(f)^* \circ Sp_{X\backslash Y} = Sp_{X'\backslash Y'} \circ f^* . \tag{20}$$

(SP5) **restriction to the vertex:** $Sp_{X\backslash Y}(\alpha)|X = \alpha|X$ for all $\alpha \in F(Y)$.

(SP6) **normalisation:** Suppose X is defined on Y by one equation $\{g = 0\}$. Then $\pi \times C(g) : C_X Y = N_X Y \to X \times \mathbb{C}$ is an isomorphism. Let $s : X \to N_X Y$ be the section corresponding to $id \times \{1\}$ under this isomorphism. Then $s^* \circ Sp_{X\backslash Y} = \psi_g$.

Recall the we introduced in (7) the "generalized vanishing cycles" Φ_i of i as the transformation

$$\Phi_i := Sp_{X\backslash Y} - \pi^* i^* : F(Y) \to F_{mon}(C_X Y) .$$

For a cartesian diagram as in (SP2,3) one gets also the commutative diagram (with k, k' the "zero-sections"):

$$\begin{array}{ccccc} X' & \xrightarrow{k'} & C_{X'}Y' & \xrightarrow{\pi'} & X' \\ f\downarrow & & \downarrow C(f) & & \downarrow f \\ X & \xrightarrow[k]{} & C_X Y & \xrightarrow[\pi]{} & X , \end{array} \tag{21}$$

which is cartesian for a flat morphism f.

Therefore one gets similar properties for Φ_i:

(gV0) **localization:** Φ_i commutes with restriction to open subsets.

(gV1) **monodromy:** Φ_i maps to $F_{mon}(C_X Y)$.

(gV2) **proper direct image:** Consider a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{i'} & Y' \\ \downarrow & & \downarrow f \\ X & \xrightarrow[i]{} & Y \end{array}$$

with f flat and proper (or more generally with f proper such that (21) is cartesian). Then also the induced map $C(f) : C_{X'}Y' \to C_X Y$ is proper with

$$\Phi_i \circ f_* = C(f)_* \circ \Phi_{i'} \ . \tag{22}$$

(gV3) **smooth base change:** Consider a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{i'} & Y' \\ \downarrow & & \downarrow f \\ X & \xrightarrow[i]{} & Y \end{array}$$

with f smooth. Then also the induced map $C(f) : C_{X'}Y' \to C_X Y$ is smooth with

$$C(f)^* \circ \Phi_i = \Phi_{i'} \circ f^* \ . \tag{23}$$

(gV5) **restriction to the vertex:** $\Phi_i(\alpha)|X = 0$ for all $\alpha \in F(Y)$.

(gV6) **normalisation:** Suppose X is defined on Y by one equation $\{g = 0\}$. Then $\pi \times C(g) : C_X Y = N_X Y \to X \times \mathbb{C}$ is an isomorphism. Let $s : X \to N_X Y$ be the section corresponding to $id \times \{1\}$ under this isomorphism. Then $s^* \circ \Phi_i = \phi_g$.

Let us now explain the definition of the corresponding specialization map sp in homology. First we consider the analytic context (compare [GA, p.217,218] and [CEP, p.149]). Since $\pi : M \backslash C_X Y = Y \times \mathbb{C}^* \to \mathbb{C}^*$ is just the projection, one gets that $(R\pi_! \mathbb{Z}_M)|\mathbb{C}^*$ is isomorphic to the constant complex $R_c(Y, \mathbb{Z})$. Therefore one gets a generalization morphism (independent of the choice of $t \in \mathbb{C}^*$):

$$R\Gamma_c(C_X Y, \mathbb{Z}) \simeq (R\pi_! \mathbb{Z}_M)_0 \simeq R(\mathbb{C}, R\pi_! \mathbb{Z}_M) \to (R\pi_! \mathbb{Z}_M)_t \simeq R\Gamma_c(Y, \mathbb{Z}) \ ,$$

which induces by duality a specialization homomorphism in Borel-Moore homology:

$$H_*^{BM}(C_X Y, \mathbb{Z}) \xleftarrow{sp} H_*^{BM}(Y, \mathbb{Z}) \ . \tag{24}$$

The definition in the algebraic context is simpler (compare [Ful, ch.5] and [GA, p.198]):

$$sp := s^* \circ \pi^* : A_*(Y, \mathbb{Z}) \to A_*(C_X Y, \mathbb{Z}) \ . \tag{25}$$

Here π^* is the flat pullback and s^* is the corresponding Gysin transformation for divisors. The algebraic and topological definitions commute with the cycle map $cl : A_*(\cdot) \to H_*^{BM}(\cdot, \mathbb{Z})$ (compare [GA, p.222,cor.8.10]).

Now we can formulate the important specialization property for the Chern-Schwartz-MacPherson classes c_* (Compare with [Ful, p.352] and [GA, thm.6.5,p.211] for the corresponding specialization property of the "Todd-homology transformation" τ_* of Baum-Fulton-MacPherson).

Theorem 1.1. *Consider the complex algebraic context with algebraic constructible functions $F(\cdot)$ and $H_*(\cdot)$ one of the two possible homology theories (i.e. the Chow-homology $A_*(\cdot)$ or the Borel-Moore homology $H_*^{BM}(\cdot, \mathbb{Z})$), or the complex analytic context with Y compact. Then the following diagram is commutative:*

$$\begin{array}{ccc} F(Y) & \xrightarrow{Sp_{X \setminus Y}} & F_{mon}(C_X Y) \\ {\scriptstyle c_*}\downarrow & & \downarrow{\scriptstyle c_*} \\ H_*(Y) & \xrightarrow[sp]{} & H_*(C_X Y) \ . \end{array} \tag{26}$$

Proof. The analytic and algebraic case with Borel-Moore homology follows directly from a corresponding result of Verdier for the nearby cycle functor ψ_h [CEP, p.157,thm.5.1], with h the map of the deformation to the normal cone (17):

$$c_*\Big(\psi_h(\pi^*(\alpha))\Big) = sp\Big(c_*((\pi^*\alpha)|Y \times \{t\})\Big)$$

(for $0 < t << 1$). Of course $c_*((\pi^*\alpha)|Y \times \{t\})$ corresponds to $c_*(\alpha)$ under the identification $Y \times \{t\} \simeq Y$. Note that he assumes that the "situation is compactifiable", which in the algebraic context is automatically the case.

In the analytic context it follows from the compactness of Y and the construction of the deformation to the normal cone (e.g. $\pi^*(\alpha) = (\tilde{\pi}^*(\alpha))|M$, with $\tilde{\pi} : \tilde{M} \to Y$ the induced map, and the extension $\tilde{M} \to Y \times \mathbb{C} \to \mathbb{C}$ of h is proper for Y compact).

In the algebraic context with the Chow-group as our homology theory, the claim follows from a similar result of Kennedy for the functor ψ_h [Ken2, thm.2)] (or from the fact that the argument of Verdier also works on the level of Chow-groups). Note that Kennedy uses the Lagrangian approach so that he works with spaces that can be embedded into manifolds. Moreover one has to use the result of Sabbah [Sab] (compare also with [BMM, Fu, Gi, Sch3]), that the used specialization of Lagrangian cycles corresponds over the ground field $\mathbb{C}$ to the nearby cycle functor for constructible functions. For a general constructible function one can use resolution of singularities (and induction on the dimension of the support of the constructible function), to reduce to the case of a function on a smooth space. This works because all considered transformations commute with proper push down:

- For the Chern-Schwartz-MacPherson class c_* this is one of its defining properties.
- For the nearby cycle functor of constructible functions this follows from a similar statement for constructible sheaves (and this is exactly what is used for property (SP2) of the specialization functor): $p_* \circ \psi_{h \circ p} = \psi_h \circ p_*$ for a proper morphism p.

$\square$

Remark 1.1. *We believe that theorem 1.1 is true in the general analytic context, without any compactness assumptions, but for this one has to find a new proof. If this can be done, then all results of this paper extend without any modification to this more general analytic context (without any compactness assumptions). By the* **localization** *property (SP0) this is for example true, if one restricts the complex algebraic (or compact analytic) context to an open subset.*

Theorem 1.1 implies directly our important formula (11) and therefore (by the other arguments of the introduction) also our main theorem 0.1. This follows from the definition of the Gysin homomorphism $i^* : H_*(Y) \to H_*(X)$ for a regular embedding $i : X \hookrightarrow Y$, which by definition ([Ful, Ch.5,Ch.6] or [GA, Exp.IX]) is just the composition $i^* := k^* \circ sp$. Here $k : X \hookrightarrow C_X Y = N_X Y$ is the "zero-section" of the normal-bundle $\pi : N_X Y \to X$ (which is a vector-bundle for a regular embeddding) and k^* on homology is the inverse of the isomorphism $\pi^* : H_*(X) \to H_*(N_X Y)$.

2 Milnor-classes

In this last section, we explain some simple facts which follow from our generalization (15) of the formula of Parusiński-Pragacz for a general regular embedding $i : X \hookrightarrow Y$ of complex algebraic (or compact analytic) spaces, with Y smooth:

$$M_0(X) = c^*(N_X Y)^{-1} \cap k^*\Big(c_*(\Phi_i(1_Y))\Big) ,$$

with $k : X \to N_X Y$ the zero-section.

We already explained in the introduction how to get in the codimension one case the main result of Parusiński-Pragacz as formulated in [PP, thm.0.2]. Moreover, the definition of the **nearby cycles** for constructible functions in terms of local Milnor fibers and the normalization properties (SP6) and (gV6) show, that our constructible function $(-1)^{dim(Y)-1} \cdot \mu(1_Y)$ is exactly the function μ used in [PP]. Note that most authors include a similar sign-factor into their definition of the Milnor-class. But we prefer our definition because of its relation to the "(generalized) vanishing cycles" of constructible functions.

The next important fact is

$$supp(\Phi_i(1_Y)) \subset N_X Y | X_{sing} , \tag{27}$$

with X_{sing} the singular locus of X. This follows from the **localization** property (gV0) of Φ_i, and the fact that for a regular embedding $X \hookrightarrow Y$ of a smooth subspace the deformation to the normal cone M and the induced map $h : M \to \mathbb{C}$ are also smooth! This can be easily checked in local coordinates, and is of course very important for the D-module approach to the specialization functor (compare [La]). But for a smooth morphism h one has $\psi_h(1_M) = 1_{N_X Y}$ and therefore $\Phi_i(1_M) = 0$.

Let us denote by $i_\alpha : S_\alpha \hookrightarrow X$ the (closed) inclusions of the connected components of X_{sing}, and by $n_\alpha : N_X Y | S_\alpha \to N_X Y$ the corresponding inclusions. Then we get $\Phi_i(1_Y) = \sum_\alpha n_{\alpha *}(\mu_\alpha)$, with $\mu_\alpha := n_\alpha^* \Phi_i(1_Y)$ and

$$c_*(\Phi_i(1_Y)) = \sum_\alpha n_{\alpha *}(c_*(\mu_\alpha)) .$$

¿From the cartesian diagram (with k the zero-section)

$$\begin{array}{ccc} S_\alpha & \xrightarrow{s_\alpha} & X \\ {\scriptstyle k_\alpha}\downarrow & & \downarrow{\scriptstyle k} \\ N_XY|S_\alpha & \xrightarrow[n_\alpha]{} & N_XY \end{array}$$

one gets by the base-change property $k^* \circ n_{\alpha *} = s_{\alpha *} \circ k_\alpha^*$ the following localization formula for the Milnor-class:

$$M_0(X) = \sum_\alpha c^*(N_XY)^{-1} \cap s_{\alpha *} k_\alpha^* \Big(c_*(\mu_\alpha) \Big)$$

$$= \sum_\alpha s_{\alpha *} \Big(c^*(N_XY|S_\alpha)^{-1} \cap k_\alpha^*(c_*(\mu_\alpha)) \Big) . \tag{28}$$

So we get a localization result similar to [BLSS, thm. A] or [BLSS2, thm.5.2], and it would be very interesting to compare our "localized Milnor-class"

$$M_0(X, S_\alpha) := c^*(N_XY|S_\alpha)^{-1} \cap k_\alpha^*(\ c_*(\mu_\alpha)\ ) \in\ H_*(S_\alpha) \tag{29}$$

with the corresponding notation introduced in [BLSS, BLSS2] with the help of obstruction theory.

Of course these "localized Milnor-classes" $M_0(X, S_\alpha)$ are very difficult to calculate in general. But by dimension reasons, the corresponding multiplicity in $\mathbb{Z}\cdot[S'_\alpha] \subset H_*(S_\alpha)$, with $[S'_\alpha]$ the fundamental-class of an irreducible component S'_α of S_α is nothing but the "generic value" of μ_α (or $\Phi_i(1_Y)$) on $N_XY|S'_\alpha$. Note that the restriction of the constructible function μ_α to the irreducible space $N' := N_XY|S'_\alpha$ has a "generic value" (i.e. the coefficient of $1_{N'}$ in a description of this constructible function as a linear combination of indicator-functions of closed irreducible subsets of N').

Especially if $S_\alpha = \{x_\alpha\}$ is just a point, then it follows from (the corresponding local Milnor-fibration and) the definition of $\Phi_i(1_Y)$ in terms of nearby-cycles that this "generic value" is just $\chi(M_{x_\alpha}) - 1$, with M_{x_α} a local Milnor-fiber of the local complete intersection X at the isolated singular point x_α. But this Milnor-fiber is wedge of spheres of dimension $d := dim_{x_\alpha}(X)$ so that $(-1)^d \cdot (\chi(M_{x_\alpha}) - 1)$ is just the usual Milnor-number of this isolated local complete intersection singularity (i.e. the corresponding number of spheres). So for a complete intersection X with only isolated singularities we get back a result of Suwa [Su, thm.,p.68] that the Milnor-class $M_0(X)$ is just the sum

$$M_0(X) = \sum (\chi(M_{x_\alpha}) - 1) \cdot s_{\alpha *}([\{x_\alpha\}]) \tag{30}$$

of the classes of the singular points x_α weighted by the corresponding Milnor-numbers (up to suitable signs).

There is also another special case, where one can give a simpler formula for the "localized Milnor-class" $M_0(X, S_\alpha)$. Suppose $\mu_\alpha = \sum m_j \cdot 1_{V_j}$, with $p_j : V_j \to S_j$ a closed subvector-bundle of $N_X Y|S_j$ for a closed subspace $S_j \subset S_\alpha$. By linearity (and to simplify the notations) we can assume $\mu_\alpha = 1_V$ with $p : V \to S$ a closed subvector-bundle of $N_X Y|S$ for a closed subspace $S \subset S_\alpha$. Consider the commutative diagram

$$\begin{array}{ccccc} V & \xrightarrow{j} & N_X Y|S & \xrightarrow{i'} & N_X Y|S_\alpha \\ {\scriptstyle\kappa}\uparrow & & \uparrow{\scriptstyle\kappa'} & & \uparrow{\scriptstyle k_\alpha} \\ S & = & S & \xrightarrow[i]{} & S_\alpha \ , \end{array}$$

where the vertical maps are the "zero-sections". Then one gets:

$$k_\alpha^*(c_*(1_V)) = k_\alpha^* i'_*(c_*(1_V)) = i_* \kappa'^*(c_*(1_V))$$

$$= i_* \kappa^* j^* j_*(c_*(1_V)) = i_* \kappa^*(c^d(N) \cap c_*(1_V)) = i_* \Big(c^d(\kappa^* N) \cap \kappa^*(c_*(1_V)) \Big) \ .$$

Here we consider the constructible function 1_V and its chern-class $c_*(1_V)$ on the corresponding spaces. Moreover, we use for the inclusion j the "self-intersection formula" with N the corresponding normal-bundle of rang d so that $\kappa^* N$ is isomorphic to the quotient-bundle $(N_X Y|S)/V$. Since $\kappa^*(c_*(1_V)) = c^*(V) \cap c_*(1_S)$ (by the Verdier-formula for the smooth projection p), we get altogether

$$k_\alpha^*(c_*(1_V)) = i_* \Big(\ c^d((N_X Y|S)/V) \cap c^*(V) \cap c_*(1_S) \ \Big) \ .$$

By (29) we get a similar formula for the "localized Milnor-class", and arguments of this type can be used to relate our general formula (15) to the product-formula for Milnor-classes obtained in [OY].

Let us end this paper with some functoriality results for the "Milnor-class", which are easy applications of our formula (15), the general results about our "generalized vanishing cycles" Φ_i and known properties of the Chern-Schwartz-MacPherson classes c_*.

First we explain the behaviour of $M_0(X)$ under smooth pullback (compare [Y3, thm.2.2]). Consider a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{i'} & Y' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ X & \xrightarrow[i]{} & Y \end{array},$$

with f' smooth and $i : X \hookrightarrow Y$ a closed embedding. Then the corresponding diagram (21) is cartesian, with $f, C(f)$ smooth and $i' : X' \hookrightarrow Y'$ is also a closed embedding with $f^*(C_X Y) \simeq C_{X'}Y'$. Then we get by (gV3) and the Verdier-formula (3) for the smooth map $C(f)$:

$$k'^*\Big(\ c_*(\Phi_{i'}(1_{Y'}))\ \Big) = k'^*\Big(\ c_*(\ (\Phi_{i'} \circ f^*)(1_Y)\ )\ \Big) =$$

$$k'^*\Big(\ c^*(T_{C(f)}) \cap\ C(f)^*(\ c_*(\Phi_i(1_Y))\ )\ \Big) = c^*(T_f) \cap\ f^*\Big(\ k^*(c_*(\Phi_i(1_Y)))\ \Big)\ .$$

Assume we consider the algebraic (or analytic) context with Y, Y' (compact) manifolds and i a regular embedding. Then i' is also a regular embedding with $f^*(N_X Y) \simeq N_{X'}Y'$. Then we get by this calculation and our formula (15) the following relation between the corresponding "Milnor-classes":

$$M_0(X') = c^*(T_f) \cap\ f^* M_0(X)\ . \tag{31}$$

A similar argument applies also to the "proper pushdown" of "Milnor-classes" studied in [Y3]. Consider a cartesian diagram as before, but this time with f' proper (but not necessarily smooth). Assume the closed embeddings i, i' are regular of the same (local) codimension. Then one has $f^*(N_X Y) \simeq N_{X'}Y'$ (since $f^*(N_X Y)$ is a subbundle of $N_{X'}Y'$ of the same rang) so that the diagram (21) is also cartesian in this case, and $C(f)$ is a proper by "base-change". By (gV2) and the functoriality of c_* with respect to proper morphisms one gets

$$f_* k'^*\Big(\ c_*(\Phi_{i'}(1_{Y'}))\ \Big) = k^* C(f)_*\Big(\ c_*(\Phi_{i'}(1_{Y'}))\ \Big) = k^*\Big(c_*(\ \Phi_i(f'_*(1_{Y'}))\ )\Big)\ .$$

Assume we consider the algebraic (or analytic) context with Y, Y' (compact) manifolds and (for simplicity) Y connected. Let χ be the "generic value" of the constructible function $f'_*(1_{Y'})$ (i.e. the Euler-characteristic of a "generic" fiber of f') so that $\alpha := f'_*(1_{Y'}) - \chi \cdot 1_Y$ is a constructible function of smaller support. Then one gets by this calculation, the projection-formula and our formula (15) the following relation between the corresponding "Milnor-classes":

$$f_* M_0(X') = \chi \cdot M_0(X) + c^*(N_X Y)^{-1} \cap k^* \Big(c_*(\Phi_i(\alpha)) \Big) \ . \tag{32}$$

Especially, if all fibers of f' have the same Euler-characteristic (e.g. f' is smooth or more generally an "Euler-morphism" as in [Y3, cor.3.8]), then one gets

$$f_* M_0(X') = \chi \cdot M_0(X) \ .$$

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