Reduction of Orders in Boundary Value Problems without the Transmission Property [1]

G. Harutjunjan[2] and **B.-W. Schulze**[3]

Abstract. Given an algebra of pseudo-differential operators on a manifold, an elliptic element is said to be a reduction of orders, if it induces isomorphisms of Sobolev spaces with a corresponding shift of smoothness. Reductions of orders on a manifold with boundary refer to boundary value problems. We employ specific smooth symbols of arbitrary real orders and with parameters, and we show that the associated operators induce isomorphisms between Sobolev spaces on a given manifold with boundary. Such operators for integer orders have the transmission property and belong to the calculus of Boutet de Monvel [1], cf. also [8]. In general, they fit to the algebra of boundary value problems without the transmission property in the sense of [15] and [21]. Order reducing elements of the present kind are useful for constructing parametrices of mixed elliptic problems.
We show that order reducing symbols have the Volterra property and are parabolic of anisotropy 1; analogous relations are formulated for arbitrary anisotropies. We then investigate parameter-dependent operators, apply a kernel cut-off construction with respect to the parameter and show that corresponding holomorphic operator-valued Mellin symbols reduce orders in weighted Sobolev spaces on a cone with boundary. We finally construct order reducing operators on a compact manifold with conical singularities and boundary.

Key words and phrases: Boundary value problems, elliptic operators, order reduction, Volterra symbols.

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[1] Supported by the EU Research and Training Network "Geometric Analysis"
[2] Staatliche Universität Jerewan, Fakultät für Informatik, Alex Manukjan Str. 1, 375049 Jerewan, Armenien. E-mail: gohar@ysu.am
[3] Institut für Mathematik, Universität Potsdam, Postfach 601553, 14415 Potsdam, Germany. E-mail: schulze@math.uni-potsdam.de

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Introduction

Reductions of orders in problems for elliptic partial differential equations are useful for many purposes, e.g., for constructing parametrices, or in the index theory. The case of operators on a closed compact C^∞ manifold is standard and particularly simple. To reduce orders on a compact C^∞ manifold X with boundary Y, we have to take into account the specific influence of Y to the operations and to the choice of Sobolev spaces on X that we wish to reduce to $L^2(X)$. For pseudo-differential operators of integer order with the *transmission* property at the boundary there are order reducing operators that refer to the scale $H^s(X)$ of standard Sobolev spaces on X, cf. Boutet de Monvel [1], or Grubb [8] for specific constructions. Other variants of reductions have been used by numerous authors, in particular, by Eskin [4], or Schulze and Seiler [21]. While the construction in [4] as well as that in [8] reduces smoothness of standard Sobolev spaces to zero, the choice of [21] also works in Sobolev spaces with arbitrary weights at the boundary.

In the first part of the present paper we show that the analogues of symbols from [8] of arbitrary order $\mu \in \mathbb{R}$ give rise to operators that reduce the smoothness of standard Sobolev spaces by μ. In addition, we show that the order reducing symbols have the Volterra property and are parabolic of anisotropy 1, cf. Piriou [13], or Krainer [11]. Moreover, using a result of Burenkov, Schulze, and Tarkhanov [2] we establish a relation to operator-valued symbols on the boundary in the framework of twisted homogeneity, cf. [20].

In the second part we apply our order reducing operators in parameter-dependent form and obtain order reductions in weighted cone Sobolev spaces, based on the Mellin transform in axial direction. In addition, we apply the kernel cut-off construction from [18] and show that such reductions are possible in terms of holomorphic operator-valued Mellin symbols. These results also yield order reducing operators on a compact manifold with conical singularities and boundary.

Our investigations are motivated by the problem to express parametrices of general elliptic mixed problems within a complete pseudo-differential calculus. This concerns the case of smooth interfaces (where boundary conditions have their jump) as well as that of interfaces with conical singularities.

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1 Local constructions

1.1 Order reducing symbols

Let $U \subseteq \mathbb{R}^m$ be an open set, and let $S^\mu(U \times \mathbb{R}^n)$ denote the space of Hörmander's symbols of order $\mu \in \mathbb{R}$, i.e., the set of all $a(x,\xi) \in C^\infty(U \times \mathbb{R}^n)$

such that

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \leq c\langle\xi\rangle^{\mu-|\beta|} \tag{1}$$

for all $(x,\xi) \in K \times \mathbb{R}^n$ for arbitrary $K \subset\subset U$ and multi-indices $\alpha \in \mathbb{N}^m$, $\beta \in \mathbb{N}^n$, with constants $c = c(\alpha, \beta, K) > 0$; here, as usual, $\langle\xi\rangle := (1 + |\xi|^2)^{\frac{1}{2}}$. Moreover, let $S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^n)$ denote the subspace of all *classical* symbols of order μ, i.e., there are functions $a_{(\mu-j)}(x,\xi) \in C^\infty(U \times (\mathbb{R}^n \setminus \{0\}))$, $a_{(\mu-j)}(x, \lambda\xi) = \lambda^{\mu-j} a_{(\mu-j)}(x,\xi)$ for all $(x,\xi) \in U \times (\mathbb{R}^n \setminus \{0\})$ and all $\lambda \in \mathbb{R}_+$, $j \in \mathbb{N}$, such that

$$a(x,\xi) - \chi(\xi) \sum_{j=0}^{N} a_{(\mu-j)}(x,\xi) \in S^{\mu-(N+1)}(U \times \mathbb{R}^n)$$

for all $N \in \mathbb{N}$. Here, $\chi(\xi)$ is an arbitrary *excision* function, i.e., any $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ that equals 0 in a neighbourhood of $\xi = 0$ and 1 for $|\xi| > R$ for some $R > 0$. If a notation or a result is true both in the classical and the general case, we write "(cl)" as subscript.

We are interested in this section in a particular class of symbols of order μ that may be used for $\mu \in \mathbb{Z}$ in the calculus of pseudo-differential boundary value problems with the *transmission* property, cf. Boutet de Monvel [1], Rempel and Schulze [14], or Grubb [9]. We take, in particular, symbols from [8] of the following form. Set $x = (y,t)$ for $y = (y_1, \ldots, y_{n-1}) \in \mathbb{R}^{n-1}$, $t \in \mathbb{R}$, with covariables $\xi = (\eta, \tau)$. Choose an element $\varphi \in \mathcal{S}(\mathbb{R})$, such that $\varphi(0) = 1$ and $\operatorname{supp} F^{-1}\varphi \subset \mathbb{R}_-$ (where F is the Fourier transform on $\mathbb{R}$). It is easy to see that such functions φ exist. We now set

$$r^\mu_-(\eta,\tau) := \left(\varphi\left(\frac{\tau}{C\langle\eta\rangle}\right)\langle\eta\rangle - i\tau\right)^\mu, \tag{2}$$

$\mu \in \mathbb{R}$, for any constant $C > 0$. For our purposes we need the following properties:

Proposition 1.1 (i) $r^\mu_-(\eta,\tau) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^n)$,

(ii) *$r^\mu_-(\eta,\tau)$ is elliptic of order $\mu \in \mathbb{R}$ for a sufficiently large choice of $C > 0$ and extends with respect to τ to the upper complex half-plane $\tau + i\theta$, $\theta > 0$, as a holomorphic function that is C^∞ for $\theta \geq 0$, such that*

$$|r^\mu_-(\eta, \tau + i\theta)| \leq c(1 + |\eta| + |\tau| + \theta)^\mu \tag{3}$$

for all $(\eta,\tau,\theta) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \overline{\mathbb{R}}_+$ for a constant $c > 0$.

(iii) *The constant $C > 0$ in* (ii) *can be chosen in such a way that*

$$|r^\mu_-(\eta, \tau + i\theta)| \geq \tilde{c}(1 + |\eta| + |\tau| + \theta)^\mu \tag{4}$$

for all $(\eta,\tau,\theta) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \overline{\mathbb{R}}_+$ for a constant $\tilde{c} > 0$.

Proof. (i) Let us set $p(\xi) := \langle\xi\rangle^{-1}r_-(\xi)$. By virtue of $r_-(\xi) \in S^1_{\mathrm{cl}}(\mathbb{R}^n)$, cf. [16] or [20] we have $p(\xi) \in S^0_{\mathrm{cl}}(\mathbb{R}^n)$. Moreover, the symbol $p(\xi)$ is elliptic of order zero, and we have $p(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n$, cf. the arguments for assertions (ii), (iii) below. To show that $r^\mu_-(\xi)$ is classical we write $r^\mu_-(\xi) = \langle\xi\rangle^\mu p^\mu(\xi)$. Because of $\langle\xi\rangle^\mu \in S^\mu_{\mathrm{cl}}(\mathbb{R}^n)$ it suffices to show that $p^\mu(\xi) \in S^0_{\mathrm{cl}}(\mathbb{R}^n)$. For every fixed $\xi \in \mathbb{R}^n$ we have by Cauchy's theorem

$$p^\mu(\xi) = \frac{1}{2\pi i}\int_L \frac{\lambda^\mu}{(\lambda - p(\xi))}d\lambda \tag{5}$$

for any curve L in the complex plane, where $\lambda - p(\xi)$ does not vanish for all $\xi \in \mathbb{R}^n$ (such a curve always exists as we see from relation (6) below). Note that $\lambda \to (\lambda - p(\xi))^{-1}$ represents a continuous map $L \to S^0_{\mathrm{cl}}(\mathbb{R}^n)$. Formula (5) easily yields $p^\mu(\xi) \in S^0_{\mathrm{cl}}(\mathbb{R}^n)$. In fact, the integral can be written as a limit of finite integral sums of the form $\sum_{j=1}^N (2\pi i)^{-1}\lambda^\mu_{j,N}(\lambda_{j,N} - p(\xi))^{-1}\delta_{j,N}$ with points $\lambda_{j,N} \in L$ belonging to the i-th interval of the corresponding partition of the curve, where $\max\{|\delta_{j,N}|, j = 1, \dots, N\} \to 0$ as $N \to \infty$. We then get convergence in the Fréchet space $S^0_{\mathrm{cl}}(\mathbb{R}^n)$.

(ii), (iii) For the case $\mu = 1$ we first write

$$\varphi\left(\frac{\tau + i\theta}{C\langle\eta\rangle}\right) = \int_{-\infty}^0 e^{-i(\tau+i\theta)(C\langle\eta\rangle)^{-1}t}\psi(t)dt$$

for $\psi(t) \in \mathcal{S}(\mathbb{R})$, $\operatorname{supp}\psi \subset \mathbb{R}_-$, $\int_{-\infty}^0 \psi(t)dt = 1$. This shows that $r_-(\eta,\tau)$ extends to a holomorphic function in $\tau + i\theta$, $\theta > 0$. Moreover, we have

$$\left|\varphi\left(\frac{\tau + i\theta}{C\langle\eta\rangle}\right)\right| \leq c_1 \quad \text{for all } (\eta,\tau) \in \mathbb{R}^n,\, \theta \geq 0,$$

for some constant $c_1 > 0$. This yields

$$|r_-(\eta, \tau + i\theta)| \leq c_2(1 + |\eta| + |\tau| + \theta) \tag{6}$$

for all $(\eta,\tau) \in \mathbb{R}^n$, $\theta \geq 0$, for some $c_2 > 0$. In the proof below, cf. relation (7), we will show that $r_-(\eta, \tau + i\theta) \neq 0$ for all $(\eta,\tau) \in \mathbb{R}^n$, $\theta \geq 0$. Thus $\log(r_-(\eta, \tau + i\theta))$ is well-defined as a holomorphic function in $\tau + i\theta$ for $\theta > 0$ by the branch of the logarithm that is real for positive arguments. This gives us an extension of $r^\mu_-(\xi)$ in $\tau + i\theta$, $\theta > 0$, by

$$r^\mu_-(\eta, \tau + i\theta) = e^{\mu \log(r_-(\eta,\tau+i\theta))}.$$

Now relation (6) immediately implies estimate (3) for $\mu \geq 0$ for a suitable constant $c > 0$ and estimate (4) for $\mu \leq 0$ for a suitable constant $\tilde{c} > 0$.

We now show that $r^\mu_-(\eta,\tau)$ is elliptic for a sufficiently large $C > 0$. To this end, it suffices to consider the case $\mu = 1$. We have

$$\frac{r_-(\eta,\tau)}{\langle\eta\rangle - i\tau} = 1 + \frac{\varphi(\frac{\tau}{C\langle\eta\rangle}) - \varphi(0)}{\langle\eta\rangle - i\tau}\langle\eta\rangle = 1 + \frac{\varphi(\frac{\tau}{C\langle\eta\rangle}) - \varphi(0)}{\frac{\tau}{C\langle\eta\rangle}}\frac{\tau}{C(\langle\eta\rangle - i\tau)} = 1 + \alpha,$$

where $|\alpha| \le \frac{1}{C} \left| \frac{\varphi(\frac{\tau}{C\langle\eta\rangle}) - \varphi(0)}{\frac{\tau}{C\langle\eta\rangle}} \right|$. For fixed $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $|\alpha| \le \frac{1}{C}(|\varphi'(0)| + \varepsilon)$ for $\left|\frac{\tau}{\langle\eta\rangle}\right| < \delta(\varepsilon)$ and $|\alpha| \le \frac{1}{C}\frac{2c_1}{\delta(\varepsilon)}$ for $\left|\frac{\tau}{C\langle\eta\rangle}\right| \ge \delta(\varepsilon)$. Now it follows easily that $|\alpha| < q$ for a constant $q < 1$ for all $(\eta, \tau) \in \mathbb{R}^n$, when $C > 0$ is sufficiently large. We thus obtain

$$|r_-(\eta,\tau)| \ge (1-q)|\langle\eta\rangle - i\tau| \ge c_3\langle\xi\rangle \tag{7}$$

for some $c_3 > 0$. This yields estimate (3) for $\mu \le 0$ and $\theta = 0$ and estimate (4) for $\mu \ge 0$ and $\theta = 0$. Analogous calculations go through for $\tau + i\theta$, $\theta > 0$, where $|\tau|$ in the estimates is to be replaced by $|\tau| + \theta$. $\square$

Remark 1.2 *Let us set*

$$r_+^\mu(\eta,\tau) := \overline{r_-^\mu(\eta,\tau)} \tag{8}$$

(the complex conjugate) for every $\mu \in \mathbb{R}$. We then have an analogue of Proposition 1.1 *with the only exception that extensions with respect to τ concern the lower complex half-plane.*

Proposition 1.3 *For $\mu \in \mathbb{Z}$ the symbols $r_\mp^\mu(\xi)$ have the transmission property at $t = 0$.*

Proof. First recall that a symbol $a(\xi) \in S_{\mathrm{cl}}^\mu(\mathbb{R}^n)$ of integer order μ (here, with constant coefficients) is said to have the *transmission* property at $t = 0$ if

$$a(\eta, \langle\eta\rangle\tau) \in S_{\mathrm{cl}}^\mu(\mathbb{R}_\eta^{n-1})\hat{\otimes}_\pi H_\tau \quad \text{for} \quad \xi = (\eta,\tau),$$

where $H_\tau := H^+ \oplus H^- \oplus H'$, $H^\pm := \{F(e^\pm u) : u \in \mathcal{S}(\overline{\mathbb{R}}_\pm)\}$, with H' being the space of all polynomials in τ. In the present case the symbol

$$r_-^\mu(\eta, \langle\eta\rangle\tau) = \langle\eta\rangle^\mu \left(\varphi\left(\frac{\tau}{C}\right) - i\tau\right)^\mu$$

belongs to $S_{\mathrm{cl}}^\mu(\mathbb{R}_\eta^{n-1})\hat{\otimes}_\pi(H^- \oplus H')$ which is an immediate consequence of $(\varphi(\frac{\tau}{C}) - i\tau)^\mu \in H^- \oplus H'$ for any $\mu \in \mathbb{Z}$. $\square$

1.2 Actions in Sobolev spaces

We now turn to pseudo-differential actions between Sobolev spaces in the half-space $H^s(\mathbb{R}_\pm^n) := H^s(\mathbb{R}^n)|_{\mathbb{R}_\pm^n}$, where $\mathbb{R}_\pm^n := \{(y,t) \in \mathbb{R}^n : t \in \mathbb{R}_\pm\}$. Furthermore, we set $H_0^s(\overline{\mathbb{R}}_\pm^n) = \{u \in H^s(\mathbb{R}^n) : \operatorname{supp} u \subseteq \overline{\mathbb{R}}_\pm^n\}$. We use the fact, that for every $s \in \mathbb{R}$ there is a continuous extension operator

$$\mathrm{e}_s^\pm : H^s(\mathbb{R}_\pm^n) \to H^s(\mathbb{R}^n)$$

such that $\mathrm{r}^{\pm} \circ \mathrm{e}_s^{\pm} = \mathrm{id}$ on the space $H^s(\mathbb{R}^n_{\pm})$; here, $\mathrm{r}^{\pm} f := f|_{\mathbb{R}^n_{\pm}}$. If $p(x, x', \xi) \in S^{\mu}(U \times U \times \mathbb{R}^n)$, $U \subseteq \mathbb{R}^n$ open, is any symbol, we set

$$\mathrm{Op}(p)u(x) := \int\!\!\int e^{i(x-x')\xi} p(x, x', \xi) u(x') dx' đ\xi, \quad đ\xi := (2\pi)^{-n} d\xi$$

first for $u \in C_0^{\infty}(U)$, and then extended to Sobolev spaces.

The following lemma is standard:

Lemma 1.4 *Let $u \in \mathcal{S}(\mathbb{R}^n)$, such that* $\operatorname{supp} u \subseteq \overline{\mathbb{R}}^n_-$. *Then the Fourier transform $Fu(\eta, \tau)$ extends with respect to τ to a holomorphic function in $\tau + i\theta$ for $\theta > 0$ that is C^{∞} for $\theta \geq 0$, and for every $N \in \mathbb{N}$ there is a constant $c_N > 0$, such that*

$$(1 + |\eta| + |\tau| + \theta)^N |Fu(\eta, \tau + i\theta)| \leq c_N. \tag{9}$$

Lemma 1.5 *The operators* $\mathrm{Op}(r^{\mu}_{\pm})$, $\mu \in \mathbb{R}$, *induce continuous operators*

$$\mathrm{Op}(r^{\mu}_{\pm}) : H_0^s(\overline{\mathbb{R}}^n_{\pm}) \to H_0^{s-\mu}(\overline{\mathbb{R}}^n_{\pm})$$

for all $s \in \mathbb{R}$.

Proof. First, as a consequence of Proposition 1.1 and Remark 1.2, $r^{\mu}_{\pm}$ are standard symbols of order μ; then the operators $\mathrm{Op}(r^{\mu}_{\pm}) : H^s(\mathbb{R}^n) \to H^{s-\mu}(\mathbb{R}^n)$ are continuous for all $s \in \mathbb{R}$. Thus it remains to show that $\operatorname{supp} u \subseteq \overline{\mathbb{R}}^n_{\pm}$ implies $\operatorname{supp} \mathrm{Op}(r^{\mu}_{\pm})u \subseteq \overline{\mathbb{R}}^n_{\pm}$. Let us consider, for instance, minus symbols; the plus-case is analogous and will be dropped. The arguments are, in fact, the same as in Eskin's book, but for completeness we shall recall the main steps here. Because $\mathcal{S}(\overline{\mathbb{R}}^n_-) := \mathcal{S}(\mathbb{R}^n)|_{\overline{\mathbb{R}}^n_-}$ is dense in $H_0^s(\overline{\mathbb{R}}^n_-)$, it suffices to assume $u \in \mathcal{S}(\overline{\mathbb{R}}^n_-)$. By virtue of Proposition 1.1 and Lemma 1.4 the function $r^{\mu}_-(\eta, \tau + i\theta) Fu(\eta, \tau + i\theta)$ is holomorphic in $\theta > 0$ and continuous for $\theta \geq 0$. Applying Cauchy's Theorem we can write

$$\mathrm{Op}(r^{\mu}_-)u(y, t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{iy\eta + it(\tau + i\theta)} r^{\mu}_-(\eta, \tau + i\theta) Fu(\eta, \tau + i\theta) d\eta d\tau$$

for every $\theta \geq 0$. Using (3) and (9) we obtain

$$|\mathrm{Op}(r^{\mu}_-)u(y, t)| \leq c \int_{\mathbb{R}^n} e^{-t\theta} (1 + |\eta| + |\tau| + \theta)^{\mu} |Fu(\eta, \tau + i\theta)| d\eta d\tau \leq \tilde{c} e^{-t\theta} \tag{10}$$

for some constants $c, \tilde{c} > 0$. It follows that $\mathrm{Op}(r^{\mu}_-)u(y, t) = 0$ for $t > 0$ when we pass in (10) to the limit $\theta \to +\infty$. □

Proposition 1.6 *The operators* $R^{\mu}_{\mp, s} := \mathrm{r}^{\pm} \mathrm{Op}(r^{\mu}_{\mp}) \mathrm{e}_s^{\pm}$, $\mu \in \mathbb{R}$, *induce isomorphisms*

$$R^{\mu}_{\mp, s} : H^s(\mathbb{R}^n_{\pm}) \to H^{s-\mu}(\mathbb{R}^n_{\pm}) \tag{11}$$

for all $s \in \mathbb{R}$ (they do not depend on the choice of the extension operator $\mathrm{e}_s^{\pm}$), and we have $(R^{\mu}_{\mp, s})^{-1} = \mathrm{r}^{\pm} \mathrm{Op}(r^{-\mu}_{\mp}) \mathrm{e}^{\pm}_{s-\mu}$.

Proof. Let consider $R^{\mu}_{-,s}$; the case of plus-operators is analogous and will be omitted.

Let $\mathrm{e}^+_s : H^s(\mathbb{R}^n_+) \to H^s(\mathbb{R}^n)$ be any continuous extension operator. Then the continuity of

$$R^{\mu}_{-,s} : H^s(\mathbb{R}^n_+) \to H^{s-\mu}(\mathbb{R}^n_+)$$

for every $s \in \mathbb{R}$ is evident. Let us show that $R^{-\mu}_{-,s-\mu}$ for any choice of $\mathrm{e}^+_{s-\mu}$ is a right inverse of $R^{\mu}_{-,s}$. In fact, we have for $u \in H^{s-\mu}(\mathbb{R}^n_+)$

$$\begin{aligned} &R^{\mu}_{-,s} R^{-\mu}_{-,s-\mu} u = \qquad (12) \\ &\mathrm{r}^+ \mathrm{Op}(r^{\mu}_-) \mathrm{e}^+_s \mathrm{r}^+ \mathrm{Op}(r^{-\mu}_-) \mathrm{e}^+_{s-\mu} u = \mathrm{r}^+ \mathrm{Op}(r^{\mu}_-) \mathrm{Op}(r^{-\mu}_-) \mathrm{e}^+_{s-\mu} u + \mathrm{r}^+ \mathrm{Op}(r^{\mu}_-) v, \end{aligned}$$

where $v = (\mathrm{e}^+_s \mathrm{r}^+ - 1) \mathrm{Op}(r^{-\mu}_-) \mathrm{e}^+_{s-\mu} u \in H^s_0(\overline{\mathbb{R}}^n_-)$. By Lemma 1.5 we have $\operatorname{supp} \mathrm{Op}(r^{\mu}_-) v \subset \overline{\mathbb{R}}^n_-$, i.e., $\mathrm{r}^+ \mathrm{Op}(r^{\mu}_-) v = 0$. The first summand on the right hand side of (12) equals $\mathrm{r}^+ \mathrm{e}^+_{s-\mu} u = u$. In an analogous manner we can show that $R^{\mu}_{-,s}$ has a left inverse, i.e., we have calculated the inverse $(R^{\mu}_{-,s})^{-1}$ as asserted. Finally, the action

$$\mathrm{r}^+ \mathrm{Op}(r^{\mu}_-) \mathrm{e}^+_s : H^s(\mathbb{R}^n_+) \to H^{s-\mu}(\mathbb{R}^n_+)$$

is independent of the choice of e^s_+, since for any other choice $\tilde{\mathrm{e}}^+_s$ we have

$$\mathrm{r}^+ \mathrm{Op}(r^{\mu}_-)(\mathrm{e}^+_s - \tilde{\mathrm{e}}^+_s) u = \mathrm{r}^+ \mathrm{Op}(r^{\mu}_-) v = 0$$

for $v = (\mathrm{e}^+_s - \tilde{\mathrm{e}}^+_s) u \in H^s_0(\overline{\mathbb{R}}^n_-)$. □

Let us define a linear map

$$\mathrm{e}^{\pm} : H^s(\mathbb{R}^n_{\pm}) \to \mathcal{S}'(\mathbb{R}^n)$$

for $s > -\frac{1}{2}$ by setting

$$\mathrm{e}^{\pm} f(x) := \begin{cases} f(x) & \text{for } t \in \mathbb{R}_{\pm} \\ 0 & \text{for } t \in \mathbb{R}_{\mp} \end{cases}, \quad x = (y, t), \quad f(x) \in H^s(\mathbb{R}^n_{\pm}).$$

This allows us to apply $\mathrm{Op}(r^{\mu}_{\mp})$ in $\mathbb{R}^n$ to $\mathrm{e}^{\pm} f$ in the distributional sense.

In the following we use the fact that operators $\mathrm{e}^{\pm} : H^s(\mathbb{R}^n_{\pm}) \to H^s(\mathbb{R}^n)$ (extensions by zero) are a possible choice of $\mathrm{e}^{\pm}_s$ for all $s \in \mathbb{R}$, $-\frac{1}{2} < s < \frac{1}{2}$.

Proposition 1.7 *The operators* $R^{\mu}_{\mp} := \mathrm{r}^{\pm} \mathrm{Op}(r^{\mu}_{\mp}) \mathrm{e}^{\pm}$, $\mu \in \mathbb{R}$, *induce isomorphisms*

$$R^{\mu}_{\mp} : H^s(\mathbb{R}^n_{\pm}) \to H^{s-\mu}(\mathbb{R}^n_{\pm}) \qquad (13)$$

for all $s \in \mathbb{R}$, $s > -\frac{1}{2}$, *and we have* $(R^{\mu}_{\mp})^{-1} = \mathrm{r}^{\pm} \mathrm{Op}(r^{-\mu}_{\mp}) \mathrm{e}^{\pm}_{s-\mu}$ *for* $s - \mu \leq -\frac{1}{2}$, $(R^{\mu}_{\mp})^{-1} = \mathrm{r}^{\pm} \mathrm{Op}(r^{-\mu}_{\mp}) \mathrm{e}^{\pm}$ *for* $s - \mu > -\frac{1}{2}$.

Proof. As noted before, by virtue of Proposition 1.6, it suffices to consider the case $s \geq \frac{1}{2}$. Let us discuss the case of R_-^μ; the plus-case is completely analogous. For $s - \mu \leq -\frac{1}{2}$ we have for $u \in H^{s-\mu}(\mathbb{R}_+^n)$

$$\begin{aligned}\mathrm{r}^+\mathrm{Op}(r_-^\mu)\mathrm{e}^+\mathrm{r}^+\mathrm{Op}(r_-^{-\mu})\mathrm{e}_{s-\mu}^+ u &= \mathrm{r}^+\mathrm{Op}(r_-^\mu)\mathrm{e}_s^+\mathrm{r}^+\mathrm{Op}(r_-^{-\mu})\mathrm{e}_{s-\mu}^+ u \\ &+ \mathrm{r}^+\mathrm{Op}(r_-^\mu)(\mathrm{e}^+ - \mathrm{e}_s^+)\mathrm{r}^+\mathrm{Op}(r_-^{-\mu})\mathrm{e}_{s-\mu}^+ u.\end{aligned}$$

Because of $s \geq \frac{1}{2}$ we have $v := \mathrm{r}^+\mathrm{Op}(r_-^{-\mu})\mathrm{e}_{s-\mu}^+ u \in H^0(\mathbb{R}_+^n)$ and hence $\mathrm{e}^+ v \in H^0(\mathbb{R}^n)$, $\mathrm{e}_s^+ v \in H^s(\mathbb{R}^n) \subset H^0(\mathbb{R}^n)$, i.e., $(\mathrm{e}^+ - \mathrm{e}_s^+)v \in H_0^0(\overline{\mathbb{R}}_-^n)$. This gives us $\mathrm{r}^+\mathrm{Op}(r_-^\mu)(\mathrm{e}^+ - \mathrm{e}_s^+)v = 0$, and we see from the proof Proposition 1.6 that $\mathrm{r}^+\mathrm{Op}(r_-^{-\mu})\mathrm{e}_{s-\mu}^+$ is a right inverse of R_-^μ. Moreover, for $f \in H^s(\mathbb{R}_+^n)$ we have

$$\begin{aligned}\mathrm{r}^+\mathrm{Op}(r_-^{-\mu})\mathrm{e}_{s-\mu}^+\mathrm{r}^+\mathrm{Op}(r_-^\mu)\mathrm{e}^+ f &= \mathrm{r}^+\mathrm{Op}(r_-^{-\mu})\mathrm{e}_{s-\mu}^+\mathrm{r}^+\mathrm{Op}(r_-^\mu)\mathrm{e}_s^+ f \\ &+ \mathrm{r}^+\mathrm{Op}(r_-^{-\mu})\mathrm{e}_{s-\mu}^+\mathrm{r}^+\mathrm{Op}(r_-^\mu)(\mathrm{e}^+ - \mathrm{e}_s^+)f.\end{aligned}$$

Because of $(\mathrm{e}^+ - \mathrm{e}_s^+)f \in H_0^0(\overline{\mathbb{R}}_-^n)$ we have as before $\mathrm{r}^+\mathrm{Op}(r_-^\mu)(\mathrm{e}^+ - \mathrm{e}_s^+)f = 0$, i.e., $\mathrm{r}^+\mathrm{Op}(r_-^{-\mu})\mathrm{e}_{s-\mu}^+$ is a left inverse of R_-^μ. It remains to consider the case $s - \mu \geq \frac{1}{2}$; because for $\frac{1}{2} > s - \mu > -\frac{1}{2}$ we may replace e^+ by $\mathrm{e}_{s-\mu}^+$ anyway. We have for $u \in H^{s-\mu}(\mathbb{R}_+^n)$

$$\mathrm{r}^+\mathrm{Op}(r_-^\mu)\mathrm{e}^+\mathrm{r}^+\mathrm{Op}(r_-^{-\mu})\mathrm{e}^+ u = \mathrm{r}^+\mathrm{Op}(r_-^\mu)\mathrm{e}^+\mathrm{r}^+\mathrm{Op}(r_-^{-\mu})\mathrm{e}_{s-\mu}^+ u$$

because $\mathrm{r}^+\mathrm{Op}(r_-^{-\mu})(\mathrm{e}_{s-\mu}^+ - \mathrm{e}^+)u = 0$ by the same arguments as before. Moreover, $v := \mathrm{r}^+\mathrm{Op}(r_-^{-\mu})\mathrm{e}_{s-\mu}^+ u \in H^s(\mathbb{R}_+^n)$, $s \geq \frac{1}{2}$, and we have again $\mathrm{r}^+\mathrm{Op}(r_-^\mu)(\mathrm{e}_s^+ - \mathrm{e}^+)v = 0$. It follows altogether

$$\mathrm{r}^+\mathrm{Op}(r_-^\mu)\mathrm{e}^+\mathrm{r}^+\mathrm{op}(r_-^{-\mu})\mathrm{e}^+ u = \mathrm{r}^+\mathrm{Op}(r_-^\mu)\mathrm{e}_s^+\mathrm{r}^+\mathrm{Op}(r_-^{-\mu})\mathrm{e}_{s-\mu}^+ u = u.$$

Thus the operator $\mathrm{r}^+\mathrm{Op}(r_-^{-\mu})\mathrm{e}^+$ is a right inverse of $\mathrm{r}^+\mathrm{op}(r_-^\mu)\mathrm{e}^+$. It is also a left inverse, because the consideration is now symmetric, due to $s \geq \frac{1}{2}$, $s - \mu \geq \frac{1}{2}$. □

Remark 1.8 *We will employ below symbols in parameter-dependent form, i.e., where* $\eta \in \mathbb{R}^{n-1}$ *is replaced by* $(\eta, \lambda) \in \mathbb{R}^{n-1} \times \mathbb{R}^l$ *for some* l. *According to Propositions* 1.6 *and* 1.7 *we then have parameter-dependent operators* $R_{\mp,s}^\mu(\lambda)$ *and* $R_\mp^\mu(\lambda)$ *that define isomorphisms* (11) *and* (13), *respectively, for every* $\lambda \in \mathbb{R}^l$.

1.3 A relation to classical Volterra symbols

If $U \subseteq \mathbb{C}^l$ is an open set and E a Fréchet space, $\mathcal{A}(U, E)$ denotes the space of all holomorphic functions in U with values in E (the space $\mathcal{A}(U, E)$ is endowed with the Fréchet topology that is immediate by the definition).

Definition 1.9 *Let us set* $\mathbb{H}_\pm := \{\tau + i\theta \in \mathbb{C} : \tau \in \mathbb{R}, \theta \in \mathbb{R}_\pm\}$. *We then define* $S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^{n-1} \times \overline{\mathbb{H}}_\pm)$ *for* $\mu \in \mathbb{R}$, $\Omega \subseteq \mathbb{R}^{n-1}$ *open, to be the space of all elements* $h(y, \eta, \tau + i\theta) \in C^\infty(\Omega \times \mathbb{R}^{n-1} \times \overline{\mathbb{H}}_\pm)$ *with the following properties:*

(i) $h(y, \eta, \tau + i\theta) \in \mathcal{A}(\mathbb{H}_\pm, C^\infty(\Omega \times \mathbb{R}^{n-1}_\eta))$,

(ii) $h(y, \eta, \tau + i\theta) \in S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^{n-1} \times \mathbb{R}_\tau \times \overline{\mathbb{R}}_{+,\theta})$, *i.e., h is a classical symbol of order* μ *in the covariables* (η, τ, θ) *for* (η, θ) *varying in* $\mathbb{R}^n$ *and* θ *in* $\overline{\mathbb{R}}_\pm$.

The set

$$\begin{aligned} S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^n)_\pm \;:=\; & \{p(y, \eta, \tau) := h(y, \eta, \tau + i\theta)|_{\theta=0} : \\ & h(y, \eta, \tau + i\theta) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^{n-1} \times \overline{\mathbb{H}}_\mp)\} \end{aligned} \tag{14}$$

coincides with corresponding spaces of Volterra (for the case $\mathbb{H}_-$) and anti-Volterra (for the case $\mathbb{H}_+$) symbols of anisotropy 1 and order $\mu \in \mathbb{R}$, cf. Piriou [13], or Krainer [10]. Recall (to motivate the notation) that the inverse $(i\tau + |\xi|^2)^{-1}$ of the anisotropic homogeneous principal symbol of the heat operator $-\Delta + \partial_t$ (which is of anisotropy 2 and order 2) is Volterra in the classical sense; in particular, it extends to $\mathbb{R}^n \times \overline{\mathbb{H}}_-$.

Let $S^\mu_{\mathrm{cl}}(\mathbb{R}^{n-1} \times \overline{\mathbb{H}}_\pm)$ ($S^\mu_{\mathrm{cl}}(\mathbb{R}^n)_\pm$) denote the subspace of elements of $S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^{n-1} \times \overline{\mathbb{H}}_\pm)$ ($S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^n)_\pm$) that are independent of y.

The following theorem is valid for arbitrary $\mu \in \mathbb{R}$.

Theorem 1.10 (i) *We have*

$$r^\mu_\pm(\eta, \tau + i\theta) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^{n-1} \times \overline{\mathbb{H}}_\mp).$$

(ii) $r^\mu_\pm(\eta, \tau + i\theta)$ *is elliptic of order* μ *with respect to the covariables* (η, τ, θ), *i.e., for the homogeneous principal symbols*

$$\sigma_\psi(r^\mu_-)(\eta, \tau, \theta) = \varphi\left(\frac{\tau + i\theta}{C|\eta|}\right)|\eta| - i(\tau + i\theta), \quad \sigma_\psi(r^\mu_+)(\eta, \tau, \theta) = \overline{\sigma_\psi(r^\mu_-)(\eta, \tau, \theta)}$$

of $r^\mu_\mp$ *of order* μ *in* $(\eta, \tau, \theta) \in \mathbb{R}^{n-1} \times \overline{\mathbb{H}}_\mp \setminus \{0\}$ *we have*

$$\sigma_\psi(r^\mu_\pm)(\eta, \tau, \theta) \neq 0.$$

Proof. (i) Let us consider, for instance, the minus-case. First we verify that

$$r^\mu_-(\eta, \tau + i\theta) \in S^\mu(\mathbb{R}^{n-1}_\eta \times \mathbb{R}_\tau \times \overline{\mathbb{R}}_\theta)$$

(the space on the right of the latter relation is to be interpreted as a symbol space in the variables $(\eta, \tau, \theta) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \overline{\mathbb{R}}_+$ ignoring the aspect of holomorphy). It suffices to consider the case $\mu = 1$ for similar reasons as in the proof of Proposition 1.1 (here we use, in particular, that $r^\mu_-(\eta, \tau + i\theta) \neq 0$

for all $(\eta, \tau, \theta) \in \mathbb{R}^n \times \overline{\mathbb{R}}_+$ and for all μ, cf. Proposition 1.1 (iii). Because of $-i\tau + \theta \in S^1(\mathbb{R}^n \times \overline{\mathbb{R}}_+)$ it suffices to prove that

$$p(\eta, \tau, \theta) := \langle \eta \rangle \varphi \left(\frac{\tau + i\theta}{C\langle \eta \rangle} \right) \in S^1_{\mathrm{cl}}(\mathbb{R}^n \times \overline{\mathbb{R}}_+).$$

Since we have $\varphi = Ff$ for a function $f \in \mathcal{S}(\mathbb{R})$ supported in $\mathbb{R}_-$, we get $\varphi\left(\frac{\tau+i\theta}{C}\right) \in S^{-\infty}(\mathbb{R}\times\overline{\mathbb{R}}_+)$ by Lemma 1.4 for the case $n = 1$. From Proposition 2.2.1 of [21] the map

$$p(\eta, \tau, \theta) \to p(\eta, \langle \eta \rangle \tau, \langle \eta \rangle \theta)$$

induces isomorphisms

$$S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^{n-1} \times \mathbb{R} \times \overline{\mathbb{R}}_+) \to S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^{n-1}) \hat{\otimes}_{\pi} S^{\mu}_{(\mathrm{cl})}(\mathbb{R} \times \overline{\mathbb{R}}_+)$$

for all $\mu \in \mathbb{R}$, both for classical and general symbols.

In the present case we obviously have

$$p(\eta, \langle \eta \rangle \tau, \langle \eta \rangle \theta) = \langle \eta \rangle \varphi \left(\frac{\tau + i\theta}{C} \right) \in S^1_{\mathrm{cl}}(\mathbb{R}^{n-1}) \hat{\otimes}_{\pi} S^1_{\mathrm{cl}}(\mathbb{R} \times \overline{\mathbb{R}}_+).$$

(ii) For the proof that $r^{\mu}_{-}(\eta, \tau + i\theta)$ is elliptic of order μ it suffices again to consider the case $\mu = 1$. From Proposition 1.1 we know that there is a constant $c > 0$ such that

$$|r^{\mu}_{-}(\eta, \tau + i\theta)| \geq c(1 + |\eta| + |\tau| + \theta)^{\mu}$$

for all $(\eta, \tau, \theta) \in \mathbb{R}^n \times \overline{\mathbb{R}}_+$. Together with assertion (i) we conclude that $r^{\mu}_{-}(\eta, \tau + i\theta)$ is elliptic of order μ in the sense of symbols in $S^{\mu}_{\mathrm{cl}}(\mathbb{R}^n \times \overline{\mathbb{R}}_+)$. $\square$

Remark 1.11 *The considerations so far have a direct generalisation to anisotropic symbols of arbitrary anisotropy $l \in \mathbb{N}\setminus\{0\}$. Setting, for instance,*

$$r^{\mu}_{-}(\eta, \tau)_l := \left(\varphi \left(\frac{\tau}{C\langle \eta \rangle^l} \right) \langle \eta \rangle^l - i\tau \right)^{\mu}$$

we get a corresponding version of Proposition 1.1 *when we replace $|\eta|$ by $|\eta|^l$ in the estimates* (3) *and* (4), *respectively. The analogous plus-symbols $r^{\mu}_{+}(\eta, \tau)_l$ are then parabolic of order μ and anisotropy l in the sense of the work of Krainer* [11]. *Note that Piriou* [13] *required l to be an even number.*

1.4 Interpretation as operator-valued symbols

Given a symbol $a(x, \xi) \in S^{\mu}(U \times \mathbb{R}^n)$, $U \subseteq \mathbb{R}^n$ open, $U := \Omega \times \mathbb{R} \ni (y, t)$, we can carry out the pseudo-differential action in t (with the covariable τ) and then obtain a family of operators $\mathrm{Op}(a)(y, \eta) : C^{\infty}_0(\mathbb{R}) \to C^{\infty}(\mathbb{R})$. Let us assume that $a(y,t,\eta,\tau)$ is independent of t for $|t| > c$ for a constant $c > 0$. Then $\mathrm{op}(a)(y,\eta)$ extends to a family of continuous operators

$$\mathrm{Op}(a)(y,\eta) : H^s(\mathbb{R}) \to H^{s-\mu}(\mathbb{R}) \tag{15}$$

for every $s \in \mathbb{R}$, $(y,\eta) \in \Omega \times \mathbb{R}^{n-1}$. We now employ operator-valued symbols in the following sense:

Let E be a Hilbert space and $\{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$ be a strongly continuous group of isomorphisms $\kappa_\delta : E \to E$, $\kappa_\delta\kappa_\rho = \kappa_{\delta\rho}$ for all $\delta, \rho \in \mathbb{R}_+$. In particular, for $E := H^s(\mathbb{R})$, we set

$$(\kappa_\delta u)(t) := \delta^{\frac{1}{2}} u(\delta t), \quad \delta \in \mathbb{R}_+,$$

for arbitrary $s \in \mathbb{R}$.

If $(E, \{\kappa_\delta\}_{\delta\in\mathbb{R}_+})$, $(\tilde{E}, \{\tilde{\kappa}_\delta\}_{\delta\in\mathbb{R}_+})$ are Hilbert spaces with strongly continuous group actions in that sense, $S^\mu(\Omega \times \mathbb{R}^q; E, \tilde{E})$ for $\mu \in \mathbb{R}$, $\Omega \subseteq \mathbb{R}^p$ open, denotes the set of all $a(y,\eta) \in C^\infty(\Omega \times \mathbb{R}^q, \mathcal{L}(E,\tilde{E}))$ such that

$$||\tilde{\kappa}^{-1}_{\langle\eta\rangle}\{D^\alpha_y D^\beta_\eta a(y,\eta)\}\kappa_{\langle\eta\rangle}||_{\mathcal{L}(E,\tilde{E})} \leq c\langle\eta\rangle^{\mu-|\beta|}$$

for all $(y,\eta) \in K \times \mathbb{R}^q$ for arbitrary $K \subset\subset \Omega$ and multi-indices $\alpha \in \mathbb{N}^p$, $\beta \in \mathbb{N}^q$, with constants $c = c(\alpha,\beta,K) > 0$.

Further, let $S^{(\mu)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); E, \tilde{E})$ denote the set of all $f(y,\eta) \in C^\infty(\Omega \times (\mathbb{R}^q \setminus \{0\}), \mathcal{L}(E,\tilde{E}))$ such that

$$f(y,\delta\eta) = \delta^\mu \tilde{\kappa}_\delta f(y,\eta)\kappa^{-1}_\delta$$

for all $\delta \in \mathbb{R}_+, (y,\eta) \in \Omega \times (\mathbb{R}^q \setminus \{0\})$. Finally, $S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; E, \tilde{E})$ (the space of *classical* symbols) is defined to be the subspace of all $a(y,\eta) \in S^\mu(\Omega \times \mathbb{R}^q; E, \tilde{E})$ such that there are elements $a_{(\mu-j)}(y,\eta) \in S^{(\mu-j)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); E, \tilde{E})$ such that

$$a(y,\eta) - \chi(\eta)\sum_{j=0}^{N} a_{(\mu-j)}(y,\eta) \in S^{\mu-(N+1)}(\Omega \times \mathbb{R}^q; E, \tilde{E})$$

for all $N \in \mathbb{N}$. The subclass of elements of $S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; E, \tilde{E})$ that are independent of y will be denoted by $S^\mu_{(\mathrm{cl})}(\mathbb{R}^q; E, \tilde{E})$.

Example 1.12 *Let* $E := H^s(\mathbb{R})$, $\tilde{E} := H^s(\mathbb{R}_+)$, *both endowed with the groups* $\kappa_\delta : u(t) \to \delta^{\frac{1}{2}} u(\delta t)$, $\delta > 0$. *Then we have for the restriction operator* $\mathrm{r}^+ : H^s(\mathbb{R}) \to H^s(\mathbb{R}_+)$ *the homogeneity* $\mathrm{r}^+ = \kappa_\delta \mathrm{r}^+ \kappa^{-1}_\delta$ *for all* $\delta > 0$, *and hence,* $\mathrm{r}^+ \in S^0_{\mathrm{cl}}(\mathbb{R}^{n-1}; H^s(\mathbb{R}), H^s(\mathbb{R}_+))$.

Proposition 1.13 ([20]) *Let* $a(x,\xi) \in S^\mu(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ *be independent of* t *for* $|t| > c$ *for some* $c > 0$. *Then we have* $\mathrm{Op}(a)(y,\eta) \in S^\mu(\Omega \times \mathbb{R}^{n-1}; H^s(\mathbb{R}), H^{s-\mu}(\mathbb{R}))$ *for all* $s \in \mathbb{R}$. *In addition, if* a *is independent of* t, $\mathrm{op}(a)(y,\eta)$ *is classical.*

To apply such an observation analogously to $\mathbb{R}_+$, we need a specific choice of our extension operators $\mathrm{e}_s^{\pm}$ that are compatible with the group action $\{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$. To this end we employ a result of [2] that says that $\mathrm{e}_s^{\pm}$ can be chosen in such a way that

$$\kappa_\delta \mathrm{e}_s^{\pm} = \mathrm{e}_s^{\pm}\kappa_\delta \tag{16}$$

for all $\delta \in \mathbb{R}_+$; the action of κ_δ on $H^s(\mathbb{R}_+)$ is also defined by $\kappa_\delta u(t) = \delta^{\frac{1}{2}}u(\delta t)$.

Theorem 1.14 *If e_s^+ is an extension operator with the property* (16)*, for every $p(y,\eta,\tau) \in S^\mu_{\mathrm{cl}}(\Omega\times\mathbb{R}^n)$ we have*

$$\mathrm{r}^+\mathrm{Op}(p)(y,\eta)\mathrm{e}_s^+ \in S^\mu_{\mathrm{cl}}(\Omega\times\mathbb{R}^{n-1};H^s(\mathbb{R}_+),H^{s-\mu}(\mathbb{R}_+))$$

for every $s,\mu\in\mathbb{R}$. For $s>-\frac{1}{2}$ and $p(y,\eta,\tau)\in S^\mu_{\mathrm{cl}}(\Omega\times\mathbb{R}^n)_-$ we have an analogous relation when we replace e_s^+ by e^+.

Proof. The symbol $\mathrm{r}^+\mathrm{Op}(p)(y,\eta)\mathrm{e}_s^+$ is a composition of the operator-valued symbols $\mathrm{e}_s^+ \in S^0_{\mathrm{cl}}(\mathbb{R}^{n-1}_\eta;H^s(\mathbb{R}_+),H^s(\mathbb{R}))$, cf. relation (16), $\mathrm{Op}(p)(y,\eta)\in S^\mu_{\mathrm{cl}}(\Omega\times\mathbb{R}^{n-1};H^s(\mathbb{R}),H^{s-\mu}(\mathbb{R}))$, cf. Proposition 1.13, and the restriction r^+ may be interpreted as $\mathrm{r}^+\in S^0_{\mathrm{cl}}(\mathbb{R}^{n-1};H^{s-\mu}(\mathbb{R}),H^{s-\mu}(\mathbb{R}_+))$, cf. Example 1.12. The second assertion follows similarly to the proof of Proposition 1.7. $\square$

Remark 1.15 *The pseudo-differential operator $\tilde{\mathrm{e}}_s^+ := \mathrm{Op}_y(\mathrm{e}_s^+) : H^s(\mathbb{R}^n_+) \to H^s(\mathbb{R}^n)$ is an extension operator in $\mathbb{R}^n_+$, i.e., $\mathrm{r}^+\tilde{\mathrm{e}}_s^+ = \mathrm{id}$ on $H^s(\mathbb{R}^n_+)$, and for $p(y,\eta,\tau)\in S^\mu_{\mathrm{cl}}(\Omega\times\mathbb{R}^n)$ we have*

$$\mathrm{Op}_y(\mathrm{r}^+\mathrm{Op}_t(p)(y,\eta)\mathrm{e}_s^+) = \mathrm{r}^+\mathrm{Op}_{y,t}(p)\tilde{\mathrm{e}}_s^+.$$

1.5 Further elements of the local calculus

We now apply Definition 1.9 and notation (14) in the variants

$$S^\mu_{(\mathrm{cl})}(\Omega\times\mathbb{R}^{n-1+l}\times\overline{\mathbb{H}}_\pm) \quad\text{and}\quad S^\mu_{(\mathrm{cl})}(\Omega\times\mathbb{R}^{n+l})_\pm,$$

i.e., where the covariable $\eta\in\mathbb{R}^{n-1}$ is replaced by $(\eta,\lambda)\in\mathbb{R}^{n-1+l}$. In particular, we have the (η,λ)- dependent versions $r^\mu_\pm(\eta,\lambda,\tau)$ of the symbols (2) and (8), respectively.

Example 1.16 *Let $J(y)\in C^\infty(\Omega)\otimes\mathbb{R}^{n-1}\otimes\mathbb{R}^{n-1}$ be an $(n-1)\times(n-1)$-matrix function on Ω with real-valued entries. Then we have*

$$p_\pm(y,\eta,\lambda,\tau) := r^\mu_\pm(J(y)\eta,\lambda,\tau)\in S^\mu_{\mathrm{cl}}(\Omega\times\mathbb{R}^{n+l})_\pm.$$

Theorem 1.17 ([10]) *Let* $p_j(y,\eta,\tau,\lambda) \in S^{\mu-j}_{(\mathrm{cl})}(\Omega\times\mathbb{R}^{n+l})_\pm$, $j \in \mathbb{N}$, *be an arbitrary sequence. Then there is a* $p(y,\eta,\tau,\lambda) \in S^{\mu}_{(\mathrm{cl})}(\Omega\times\mathbb{R}^{n+l})_\pm$ *such that*

$$p - \sum_{j=0}^{N} p_j \in S^{\mu-(N+1)}(\Omega\times\mathbb{R}^{n+l})_\pm,$$

for every $N \in \mathbb{N}$, *and* p *is unique modulo a symbol in the* $\pm$*-class of order* $-\infty$.

Example 1.18 *Let* $\chi : \Omega \to \tilde{\Omega}$ *be a diffeomorphism. Then the asymptotic summation for the symbol push-forward (belonging to the push- forward of associated pseudo-differential operators) can be carried out in* $S^{\mu}_{\mathrm{cl}}(\Omega\times\mathbb{R}^{n+l})_\pm$. *In fact, according to the standard formula in coordinate substitutions for pseudo-differential operators, the sum has the form*

$$\tilde{p}(\tilde{y},\tilde{\eta},\lambda,\tau)|_{\tilde{y}=\chi(y)} \sim \sum_{\alpha\in\mathbb{N}^{n-1}} \frac{1}{\alpha!}(\partial^{\alpha}_{\eta})p(y,{}^{t}d\chi(y)\tilde{\eta},\lambda,\tau)\Phi_{\alpha}(y,\tilde{\eta}),$$

where $\Phi_\alpha(y,\tilde{\eta}) = D^{\alpha}_{y} e^{i\delta(y,z)\tilde{\eta}}|_{z=y}$ *for* $\delta(y,z) = \chi(z)-\chi(y)-d\chi(y)(z-y)$ *are polynomials in* $\tilde{\eta}$ *of degree* $\leq \frac{|\alpha|}{2}$.

Remark 1.19 *Let* $\chi : \Omega \to \tilde{\Omega}$ *be a diffeomorphism, and assume that* $\det d\chi(y) = 1$ *for all* $y \in \Omega$. *Then, applying the symbol push-forward of Example* 1.18 *to* $p(y,\eta,\lambda,\tau) := r^{\mu}_{-}(\eta,\lambda,\tau)$ *(in the* y*- independent case) we have*

$$\tilde{p}(\tilde{y},\tilde{\eta},\lambda,\tau) = r^{\mu}_{-}(\tilde{\eta},\lambda,\tau) \mod S^{\mu-1}_{\mathrm{cl}}(\tilde{\Omega}\times\mathbb{R}^{n+l})_{-}.$$

Let $L^{-\infty}(\Omega\times\mathbb{R}_+)_-$ denote the space of all integral operators

$$Cu(y,t) = \int_{\Omega}\int_{-\infty}^{\infty} c(y,t,y',t')u(y',t')dt'dy',$$

$u \in C^{\infty}_{0}(\Omega\times\mathbb{R})$, the kernel of which belongs to $C^{\infty}(\Omega\times\mathbb{R}\times\Omega\times\mathbb{R})$, where $c(y,t,y',t')$ has the Volterra property, i.e., $c(y,t,y',t') = 0$ whenever $t \leq t'$. The space of these operators is Fréchet in a natural way, and we can form $L^{-\infty}(\Omega\times\mathbb{R}_+;\mathbb{R}^l)_- := \mathcal{S}(\mathbb{R}^l, L^{-\infty}(\Omega\times\mathbb{R}_+)_-)$. We now define the space

$$\begin{aligned} L^{\mu}_{\mathrm{cl}}(\Omega\times\mathbb{R}_+;\mathbb{R}^l)_- := \quad & \{\mathrm{Op}(p) + C : p(y,t,\tau) \in S^{\mu}_{\mathrm{cl}}(\Omega\times\mathbb{R}^{n+l})_-, \\ & C \in L^{-\infty}(\Omega\times\mathbb{R}_+;\mathbb{R}^l)_-\}. \end{aligned}$$

In an analogous manner we can define $L^{\mu}_{\mathrm{cl}}(\Omega\times\mathbb{R}_+;\mathbb{R}^l)_+$. Note that the elements of $L^{\mu}_{\mathrm{cl}}(\Omega\times\mathbb{R}_+;\mathbb{R}^l)_\mp$ correspond to parameter-dependent Volterra and anti-Volterra operators that are (modulo smoothing operators) translation invariant with respect to t. We could have defined analogous operators with smooth dependence on t also in general; more details may be found in Krainer [10].

Definition 1.20 (i) *Let $H^s_{\mathrm{loc}(y)}(\Omega \times \mathbb{R})$, $s \in \mathbb{R}$, defined to be the set of all $u \in \mathcal{D}'(\Omega \times \mathbb{R})$ such that $\varphi u \in H^s(\mathbb{R}^n)$ for every $\varphi \in C_0^\infty(\Omega)$. Moreover, let $H^s_{\mathrm{comp}(y)}(\Omega \times \mathbb{R})$, $s \in \mathbb{R}$, denote the subspace of all $H^s_{\mathrm{loc}(y)}(\Omega \times \mathbb{R})$ such that $u(y,t) = 0$ for all $y \in (\Omega \setminus K) \times \mathbb{R}$ for some $K \subset\subset \Omega$.*

(ii) *Set*

$$H^s_{0,\mathrm{loc}(y)}(\Omega \times \overline{\mathbb{R}}_\pm) := \{u \in H^s_{\mathrm{loc}(y)}(\Omega \times \mathbb{R}) : \operatorname{supp} u \subseteq \Omega \times \overline{\mathbb{R}}_\pm\},$$
$$H^s_{0,\mathrm{comp}(y)}(\Omega \times \overline{\mathbb{R}}_\pm) := H^s_{\mathrm{comp}(y)}(\Omega \times \mathbb{R}) \cap H^s_{0,\mathrm{loc}(y)}(\Omega \times \overline{\mathbb{R}}_\pm).$$

Moreover, let

$$H^s_{\mathrm{loc}(y)}(\Omega \times \mathbb{R}_\pm) := H^s_{\mathrm{loc}(y)}(\Omega \times \mathbb{R})|_{\Omega \times \mathbb{R}_\pm},$$
$$H^s_{\mathrm{comp}(y)}(\Omega \times \mathbb{R}_\pm) := H^s_{\mathrm{comp}(y)}(\Omega \times \mathbb{R})|_{\Omega \times \mathbb{R}_\pm}.$$

For every $p(y, \eta, \tau, \lambda) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^{n+l})$ we have families of continuous operators

$$\mathrm{Op}(p)(\lambda) : H^s_{\mathrm{comp}(y)}(\Omega \times \mathbb{R}) \to H^{s-\mu}_{\mathrm{loc}(y)}(\Omega \times \mathbb{R})$$

for all $s \in \mathbb{R}$. There are canonical embeddings

$$H^s_{\mathrm{comp}(y)}(\Omega \times \mathbb{R}) \hookrightarrow H^s(\mathbb{R}^n), \quad H^s_{\mathrm{comp}(y)}(\Omega \times \mathbb{R}_\pm) \hookrightarrow H^s(\mathbb{R}^n_\pm).$$

Thus, to $u \in H^s_{\mathrm{comp}(y)}(\Omega \times \mathbb{R}_\pm)$ we may apply extension operators $\tilde{\mathrm{e}}^\pm_s$. In particular, we get well-defined families of continuous operators

$$\mathrm{r}^+\mathrm{Op}(p)(\lambda)\tilde{\mathrm{e}}^+_s : H^s_{\mathrm{comp}(y)}(\Omega \times \mathbb{R}_+) \to H^{s-\mu}_{\mathrm{loc}(y)}(\Omega \times \mathbb{R}_+), \tag{17}$$

$s \in \mathbb{R}$, for every $p \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^{n+l})$. Similar mappings can be considered for the opposide side.

As before, we mainly consider minus-symbols. The plus-case will be analogous.

Proposition 1.21 *Let $p \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^{n+l})_-$; then the operator* (17) *is independent of the specific choice of the extension operator $\tilde{\mathrm{e}}^+_s$.*

The arguments are completely analogous to those in Proposition 1.6.

Remark 1.22 *Applying Theorem* 1.14 *to a symbol $p(y, \eta, \tau, \lambda) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^{n+l})$ we get an operator-valued symbol*

$$\mathrm{r}^+\mathrm{op}(p)(y, \eta, \lambda)\mathrm{e}^+_s \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^{n-1+l}; H^s(\mathbb{R}_+), H^{s-\mu}(\mathbb{R}_+)).$$

For $p(y, \eta, \tau, \lambda) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^{n+l})_-$ and $s > -\frac{1}{2}$ we have an analogous relation, when we replace e^+_s by e^+, the corresponding extensions by zero. This is a parameter-dependent analogue of Theorem 1.14.

2 Operators on a manifold with boundary

2.1 Global reduction of orders

Let X be an oriented compact C^∞ manifold with boundary Y, and let $2X$ denote the double of X, obtained by gluing together two copies X_+, X_- of X along their common boundary Y (we then identify X with X_+). Choose a collar neighbourhood V of Y in X with a global splitting of variables into (y,t) for $y \in Y$, $t \in [0,1)$, and fix a system of charts

$$\chi_j : U_j \to \overline{\mathbb{R}}^n_+, \quad j = 1, \dots, L, \tag{18}$$

$$\chi_j : U_j \to \mathbb{R}^n, \quad j = L+1, \dots, N \tag{19}$$

on X with coordinate neighbourhoods U_j on X, such that $U_j \cap Y \neq \emptyset$ for $j = 1, \dots, L$, and $U_j \cap Y = \emptyset$ for $j = L+1, \dots, N$, where $U_j = U'_j \times [0,1)$, $j = 1, \dots, L$, for an open covering $\{U'_1, \dots, U'_L\}$ of Y by coordinate neighbourhoods. Assume for convenience that the functions $\tilde{y}(y,t)$ and $\tilde{t}(y,t)$ in the transition diffeomorphisms $\chi_j \chi_k^{-1} : \overline{\mathbb{R}}^n_+ \to \overline{\mathbb{R}}^n_+$, $(y,t) \to (\tilde{y}(y,t), \tilde{t}(y,t))$, are independent of t for small t for $j = 1, \dots, L$. Moreover, without loss of generality, we assume that the transition diffeomorphisms $\chi'_j \chi'^{-1}_k : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ induced by the charts $\chi'_j := \chi_j|_{U_j \cap Y} : U'_j \to \mathbb{R}^{n-1}$ for $j, k = 1, \dots, L$ have Jacobians of determinant 1. We now consider local parameter-dependent symbols

$$\tilde{r}^\mu_-(t, \xi, \lambda) := r^\mu_-(\xi, \lambda)^{\omega(t)} \langle \xi, \lambda \rangle^{\mu(1-\omega(t))}$$

on $\mathbb{R}^n$, where $\omega(t)$ is a cut-off function (i.e., $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$, $\omega \equiv 1$ near $t = 0$). Here, $\mathbb{R}^n$ is regarded as the double of $\overline{\mathbb{R}}^n_+$ in connection with charts (18). Moreover, for the charts (19) we take symbols $\langle \xi, \lambda \rangle^\mu$.

Let $\{\varphi_1, \dots, \varphi_N\}$ be a partition of unity on X, subordinate to $\{U_1, \dots, U_N\}$, and let $\{\psi_1, \dots, \psi_N\}$ be a system of functions $\psi_j \in C_0^\infty(U_j)$ that equal 1 on $\operatorname{supp} \varphi_j$. The charts (18) near the boundary will be chosen as restrictions of charts $\tilde{\chi}_j : \tilde{U}_j \to \mathbb{R}^n$ for the double $\tilde{U}_j := 2U_j$ to $U_j, j = 1, \dots, L$. Then the sets $\tilde{U}_1, \dots, \tilde{U}_L$ cover a tubular neighbourhood of Y of the form $Y \times (-1,1)$; let $\tilde{X}$ denote the union of X with that tubular neighbourhood. Moreover, let $\tilde{\varphi}_j \in C_0^\infty(\tilde{U}_j)$ be functions such that $\tilde{\varphi}_j|_{U_j} = \varphi_j$ for $j = 1, \dots, L$, and let $\tilde{\psi}_j \in C_0^\infty(\tilde{U}_j)$ be functions that equal 1 on $\operatorname{supp} \tilde{\varphi}_j$, and satisfy $\tilde{\psi}_j|_{U_j} = \psi_j$, $j = 1, \dots, L$. In addition we assume the functions $\tilde{\varphi}_j$ and $\tilde{\psi}_j$ to be independent of t for $|t| < \varepsilon$ for some $\varepsilon > 0$. We now form global parameter-dependent pseudo-differential operators on $\tilde{X}$ by

$$\tilde{R}^\mu(\lambda) := \sum_{j=1}^{L} \tilde{\varphi}_j (\tilde{\chi}_j^{-1})_* \mathrm{Op}(\tilde{r}^\mu_-)(\lambda) \tilde{\psi}_j + \sum_{j=L+1}^{N} \varphi_j (\chi_j^{-1})_* \mathrm{Op}(\langle \xi, \lambda \rangle^\mu) \psi_j. \tag{20}$$

The operator family (20) (extended by zero to $2X \setminus \tilde{X}$) then belongs to $L^\mu_{\mathrm{cl}}(2X; \mathbb{R}^l)$. Concerning terminology, in particular, for the space $L^\mu_{\mathrm{cl}}(M; \mathbb{R}^l)$

of *classical parameter-dependent pseudo-differential operators* of order $\mu \in \mathbb{R}$ on a C^∞ manifold M, we refer to [20].

If M is a closed compact C^∞ manifold, $H^s(M)$ denotes the standard Sobolev space on M of smoothness $s \in \mathbb{R}$. Set $H^s(X) := \mathrm{r}^+ H^s(2X)$ with r^+ being the restriction to $\operatorname{int} X$, and let $\mathrm{e}_s^+ : H^s(X) \to H^s(2X)$ denote any continuous extension operator (i.e., $\mathrm{r}^+ \circ \mathrm{e}_s^+ = \mathrm{id}$ on the space $H^s(X)$). Moreover, for $s > -\frac{1}{2}$ we define e^+ to be the extension from $\operatorname{int} X$ to $2X$ by zero.

The operator

$$R^\mu(\lambda) := \mathrm{r}^+ \tilde{R}^\mu(\lambda) \mathrm{e}_s^+ : H^s(X) \to H^{s-\mu}(X) \tag{21}$$

is continuous for all $s \in \mathbb{R}$ (and every fixed λ) and does not depend on the choice of e_s^+. Moreover, because of Proposition 1.7 we have $R^\mu(\lambda) = \mathrm{r}^+ \tilde{R}^\mu(\lambda) \mathrm{e}^+$ for $s > -\frac{1}{2}$.

Theorem 2.1 *There exists a constant $c > 0$ such that operator* (21) *induces isomorphisms for all $|\lambda| > c$, $s \in \mathbb{R}$.*

Proof. Because of our assumptions on the charts and the localising functions $\tilde{\varphi}_j$ and $\tilde{\psi}_j$ in (20) we may apply Remark 1.19; then the operators of the family $R^\mu(\lambda)$ have the following properties: For $j = 1, \dots, L$ the operators

$$R_j^\mu(\lambda) := (\chi_j)_* R^\mu(\lambda) \ \text{ in } \mathbb{R}_+^n$$

have the form

$$R_j^\mu(\lambda) u = \mathrm{r}^+ \mathrm{Op}(r_-^\mu)(\lambda) \mathrm{e}_s^+ u + T_j(\lambda) u$$

on functions $u \in H^s(\mathbb{R}_+^n)$ that vanish for $(y, t) \notin K \times (0, \varepsilon)$ for some $K \subset\subset \mathbb{R}^{n-1}$ and $\varepsilon > 0$ sufficiently small, where $T_j(\lambda) \in L_{\mathrm{cl}}^{\mu-1}(\Omega \times \mathbb{R}_+; \mathbb{R}^l)_-$ is a parameter-dependent family of order $\mu - 1$. Moreover, $R_j^\mu(\lambda) := (\chi_j)_* R^\mu(\lambda)$ for arbitrary $j = 1, \dots, N$ acts on functions $u \in H^s(\mathbb{R}_+^n)$ for $j = 1, \dots, L$ and on $u \in H^s(\mathbb{R}^n)$ for $j = L+1, \dots, N$ with compact support as standard classical parameter-dependent elliptic operators of the class $L_{\mathrm{cl}}^\mu(\mathbb{R}_+^n; \mathbb{R}^l)$ and $L_{\mathrm{cl}}^\mu(\mathbb{R}^n; \mathbb{R}^l)$, respectively. We now define the system of Leibniz inverses of the local parameter-dependent symbols of the operators $R_j^\mu(\lambda)$ and pass to associated operators $P_j^{-\mu}(\lambda)$ in $\mathbb{R}_+^n$ or $\mathbb{R}^n$, according to the cases $1 \le j \le L$ and $L+1 \le j \le N$, respectively. For $1 \le j \le L$ we can choose $P_j(\lambda)$ in such a way that

$$P_j^{-\mu}(\lambda) u = \mathrm{r}^+ \mathrm{Op}(r_-^{-\mu})(\lambda) \mathrm{e}_{s-\mu}^+ + S_j(\lambda) u$$

on functions $u \in H^{s-\mu}(\mathbb{R}_+^n)$ with support in $K \times [0, \varepsilon)$ for some $K \subset\subset \mathbb{R}^{n-1}$ and $\varepsilon > 0$ sufficiently small, and an element $S_j(\lambda) \in L_{\mathrm{cl}}^{-\mu-1}(\Omega \times \mathbb{R}_+; \mathbb{R}^l)_-$. Globally, we form the operator family

$$P^{-\mu}(\lambda) := \sum_{j=1}^N \varphi_j (\chi_j^{-1})_* P_j^{-\mu}(\lambda) \psi_j$$

and obtain

$$P^{-\mu}(\lambda)R^{\mu}(\lambda) = I - C_l(\lambda), \tag{22}$$

$$R^{\mu}(\lambda)P^{-\mu}(\lambda) = I - C_r(\lambda), \tag{23}$$

where $C_l(\lambda)$ and $C_r(\lambda)$ are operator families in $\mathcal{S}(\mathbb{R}^l, \mathcal{L}(H^s(X), C^\infty(X)))$ for all $s \in \mathbb{R}$. To see the invertibility of $R^\mu(\lambda)$ for large $|\lambda|$ we consider, for instance, relation (22). We have

$$||C_l(\lambda)||_{\mathcal{L}(H^s(X), H^s(X))} \leq b\langle\lambda\rangle^{-N}$$

for every $N \in \mathbb{N}$, where $b = b(N) > 0$ is a suitable constant. We then conclude by a Neumann series argument that

$$R^\mu(\lambda) : H^s(X) \to H^{s-\mu}(X) \tag{24}$$

has a left inverse for $|\lambda| \geq c_1$. Analogously, using relation (23), we also have a right inverse of (24) for $|\lambda| \geq c_2$. Thus (24) is invertible for $|\lambda| \geq c = \max(c_1, c_2)$. Moreover, a simple argument in terms of elliptic regularity shows that $\ker R^\mu(\lambda)$ and $\operatorname{coker} R^\mu(\lambda)$ are independent of the choice of s. Thus, the constant c is independent of s. $\square$

2.2 Holomorphic families of order reducing operators

We now turn to a construction that is of importance for the analysis of boundary value problems (with or without the transmission property) on a manifold with conical singularities. We consider order reducing symbols $r_-^\mu(\xi, \lambda)$ with parameter $\lambda \in \mathbb{R}^l$.

Definition 2.2 *Let $S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^n_\xi \times \mathbb{C}^l_z)$, $\mu \in \mathbb{R}$, $U \subseteq \mathbb{R}^m$ open, denote the set of all $a(x, \xi, z) \in \mathcal{A}(\mathbb{C}^l, S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^n))$ such that*

$$a(x, \xi, \lambda + i\beta) \in S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^{n+l}_{\xi,\lambda})$$

for every $\beta \in \mathbb{R}^l$, uniformly in $\beta \in K$ for every $K \subset\subset \mathbb{R}^l$. By $S^\mu_{(\mathrm{cl})}(\mathbb{R}^n \times \mathbb{C}^l)$ we denote the subspace of elements that are independent of x.

Here we use the natural Fréchet topologies in the spaces $S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^n)$.

The symbol spaces $S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^n \times \mathbb{C}^l)$ have many properties as they are known in analogous form from the spaces $S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^{n+l})$.

We now recall a kernel cut-off construction for symbols $a(x, \xi, \lambda) \in S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^{n+l})$ which we specify below for our order reducing symbols.

Set

$$k(a)(x, \xi, \zeta) := \int e^{i\zeta\lambda} a(x, \xi, \lambda) d\lambda;$$

here, the correspondence $a(x, \xi, \lambda) \to k(a)(x, \xi, \zeta)$ is first considered for fixed (x, ξ) as a map $\mathcal{S}'(\mathbb{R}^l) \to \mathcal{S}'(\mathbb{R}^l)$.

Theorem 2.3 *Let $a(x,\xi,\lambda) \in S^{\mu}_{(\mathrm{cl})}(U \times \mathbb{R}^{n+l}_{\xi,\lambda})$, $\mu \in \mathbb{R}$, and let $\varphi(\zeta) \in C_0^\infty(\mathbb{R}^l_\zeta)$. Then the expression*

$$H(\varphi)a(x,\xi,z) := \int e^{-iz\zeta}\varphi(\zeta)k(a)(x,\xi,\zeta)d\zeta, \tag{25}$$

$(x,\xi,z) \in U \times \mathbb{R}^n \times \mathbb{C}^l$ defines an element in $S^{\mu}_{(\mathrm{cl})}(U \times \mathbb{R}^n \times \mathbb{C}^l)$, and the corresponding map $\varphi \to H(\varphi)a$ for fixed a represents a continuous operator

$$H(\cdot) : C_0^\infty(\mathbb{R}^l) \to S^{\mu}_{(\mathrm{cl})}(U \times \mathbb{R}^n \times \mathbb{C}^l).$$

In particular, if $\psi(\zeta) \in C_0^\infty(\mathbb{R}^l_\zeta)$ is a cut-off function (i.e., $\psi \equiv 1$ in a neighbourhood of $\zeta = 0$) we have

$$a(x,\xi,\lambda) = H(\psi)a(x,\xi,z)|_{\mathrm{Im}\, z=0} \mod S^{-\infty}(U \times \mathbb{R}^{n+l}).$$

A proof of this result may be found in [18], see also [20], or Dorschfeldt [3]; alternative arguments are given in Gil, Schulze, Seiler [7].

Notice that the kernel cut-off operators $H(\varphi)$ only act on the covariables $\lambda \in \mathbb{R}^l$, while the other variables remain untouched. An inspection of the proof of Theorem 2.3 shows that $H(\varphi)$ preserves specific subspaces of symbols. In particular, we have the following result:

Proposition 2.4 *Let $a(y,\eta,\lambda,\tau+i\theta) \in S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^{n-1+l} \times \overline{\mathbb{H}}_\pm)$ for $\mu \in \mathbb{R}$, $\Omega \subseteq \mathbb{R}^{n-1}$ open, and let $\varphi(\zeta) \in C_0^\infty(\mathbb{R}^l_\zeta)$. Then we have*

$$H(\varphi)a(y,\eta,z,\tau+i\theta) \in S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^{n-1} \times \mathbb{C}^l \times \overline{\mathbb{H}}_\pm)$$

(where the symbol space in the latter relation is defined in a similar manner as that in Definition 2.2), and $\varphi \to H(\varphi)a$ is continuous as a map

$$C_0^\infty(\mathbb{R}^l) \to S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^{n-1} \times \mathbb{C}^l \times \overline{\mathbb{H}}_\pm).$$

If $\psi(\zeta) \in C_0^\infty(\mathbb{R}^l_\zeta)$ is a cut-off function, we have

$$a(y,\eta,\lambda,\tau+i\theta) = H(\psi)a(y,\eta,z,\tau+i\theta)|_{\mathrm{Im}\, z=0} \mod S^{-\infty}(\Omega\times\mathbb{R}^{n-1}\times\mathbb{R}^l\times\overline{\mathbb{H}}_\pm).$$

Remark 2.5 *The kernel cut-off operators can alternatively be applied to symbols $a(y,\eta,\lambda,\tau) \in S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^{n+l})_\pm$; then $H(\varphi)a(y,\eta,z,\tau)$ again belongs to $S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^{n-1} \times \mathbb{C}^l)_\pm$.*

Remark 2.6 *As is known in connection with the proof of Theorem 2.3, the operator $H(\psi)$ for a cut-off function ψ preserves ellipticity also in the variable $z \in \mathbb{C}^l$, i.e., ellipticity in the variables $(\xi,\lambda) \in \mathbb{R}^{n+l}$ implies ellipticity in $(\xi,z) \in \mathbb{R}^n \times \mathbb{C}^l$, uniformly in $\mathrm{Im}\, z$ in compact subsets of $\mathbb{R}^l$. The same is true of symbols in $S^{\mu}_{(\mathrm{cl})}(\Omega\times\mathbb{R}^{n-1+l}\times\overline{\mathbb{H}}_\pm)$. In particular, for $a(y,\eta,\lambda,\tau) := r^{\mu}_{-}(\eta,\lambda,\tau)$ the symbol $H(\psi)r^{\mu}_{-}(\eta,z,\tau)$ is elliptic in that sense. More precisely, to every $K \subset\subset \mathbb{R}^l$ there is a $C > 0$ such that $H(\psi)r^{\mu}_{-}(\eta,z,\tau+i\theta)$ is invertible for all $(\eta,z,\tau+i\theta) \in \mathbb{R}^{n-1} \times \mathbb{R}^l \times K \times \overline{\mathbb{H}}_+$ and fulfills estimates similarly to those in Proposition 1.1.*

2.3 Operators on manifolds with conical singularities

Another consequence of the kernel cut-off construction is that we can apply $H(\varphi)$, $\varphi \in C_0^\infty(\mathbb{R}^l_\lambda)$, to operator families $a(\lambda) \in L^\mu_{(\mathrm{cl})}(M;\mathbb{R}^l_\lambda)$, where M is a closed compact C^∞ manifold. It suffices to apply $H(\varphi)$ to the corresponding local amplitude functions, cf. [17], [20]. This gives us holomorphic functions $h(z) \in \mathcal{A}(\mathbb{C}^l, L^\mu_{(\mathrm{cl})}(M))$, where $h(\lambda + i\beta) \in L^\mu_{(\mathrm{cl})}(M;\mathbb{R}^l_\lambda)$ for every $\beta \in \mathbb{R}^l$, uniformly in $\beta \in K$ for any $K \subset\subset \mathbb{R}^l$. Recall that such constructions belong to the Mellin quantisation procedures for pseudo-differential operators on (closed) manifolds with conical singularities without boundary. In the present section we want to apply our order reducing results for analogous constructions on manifolds with conical singularities where the base is a compact C^∞ manifold X with boundary.

Consider an operator family

$$R^\mu(\lambda, \tilde{\lambda}), \tag{26}$$

where the operators $R^\mu(\lambda, \tilde{\lambda})$ for $(\lambda, \tilde{\lambda}) \in \mathbb{R}^{l+\tilde{l}}$ are constructed in an analogous manner as the order reducing elements of Theorem 2.1 that are of the form (21) (with λ replaced by $(\lambda, \tilde{\lambda})$). Then, as a corollary of Theorem 2.1 we see that

$$R^\mu(\lambda, \tilde{\lambda}) : H^s(X) \to H^{s-\mu}(X)$$

consists of isomorphisms for all $s \in \mathbb{R}$ and all $\lambda \in \mathbb{R}^l$, when the absolute value of $\tilde{\lambda} \in \mathbb{R}^{\tilde{l}}$ is sufficiently large.

Theorem 2.7 ([17]) *For every $K \subset\subset \mathbb{R}^l$ there exists a $\tilde{C} = \tilde{C}(K) > 0$ such that*

$$h^\mu(z, \tilde{\lambda}) := H(\psi) R^\mu(z, \tilde{\lambda})$$

(with $H(\psi)$ acting on the variable $\lambda \in \mathbb{R}^l$ as before) is a holomorphic (in $z = \lambda + i\beta \in \mathbb{C}^l$) family of continuous operators

$$h^\mu(\lambda + i\beta, \tilde{\lambda}) : H^s(X) \to H^{s-\mu}(X),$$

that consists of isomorphisms for all $s \in \mathbb{R}$, for all $z = \lambda + i\beta$ for arbitrary $\beta \in K$, provided $|\tilde{\lambda}| \geq \tilde{C}$.

We now apply this result for the case $l = 1$. A slight modification of the constructions allows us to interpret $\lambda \in \mathbb{R}$ as $\mathrm{Im}\, z$ for $z \in \mathbb{C}$, running on a line

$$\Gamma_\beta := \{z \in \mathbb{C} : \mathrm{Re}\, z = \beta\}$$

for some $\beta \in \mathbb{R}$. There is then a simple modification of Theorem 2.7 with holomorphy in the variable $z = \beta + i\lambda$. Instead of the compact set K we now take an interval $d \leq \mathrm{Re}\, z \leq d'$ for some given $d \leq d'$. Choosing the above $\tilde{C}$ sufficiently large, we find a family $h^\mu(z, \tilde{\lambda})$ that is holomorphic in $z \in \mathbb{C}$

and parameter-dependent with parameter $(z, \tilde{\lambda}) \in \Gamma_\beta \times \mathbb{R}^{\tilde{l}}$ for every $\beta \in \mathbb{R}$, such that

$$h^\mu(z, \tilde{\lambda}) : H^s(X) \to H^{s-\mu}(X) \tag{27}$$

is a family of isomorphisms for all $z \in \mathbb{C}$ such that $d \leq \beta \leq d'$, provided $|\tilde{\lambda}| \geq \tilde{C}$ for sufficiently large $\tilde{C} = \tilde{C}(d, d') > 0$. Let us now insert $\beta = \frac{n+1}{2} - \gamma$ for $n = \dim X$.

Definition 2.8 *The space $\mathcal{H}^{s,\gamma}(X^\wedge)$ for $s, \gamma \in \mathbb{R}$ and $X^\wedge := \mathbb{R}_+ \times X$ is defined to be the completion of $C_0^\infty(\mathbb{R}_+, C^\infty(X))$ with respect to the norm*

$$\left\{ \frac{1}{2\pi i} \int_{\Gamma_{\frac{n+1}{2}-\gamma}} ||h^s(z, \tilde{\lambda})(Mu)(z)||^2_{L^2(X)} dz \right\}^{\frac{1}{2}}$$

for some fixed $\tilde{\lambda} \in \mathbb{R}^{\tilde{l}}$, $|\tilde{\lambda}| \geq \tilde{C}$.

Remark 2.9 *The space $\mathcal{H}^{s,\gamma}(X^\wedge)$ is independent of the specific $\tilde{\lambda}$ and of the other involved data such as the cut-off function ψ or the other ingredients of the family $R^s(\lambda, \tilde{\lambda})$ from Section* 2.1.

Now, as in the operator calculus for conical singularities on an open stretched cone $X^\wedge$, here for a base X that is a smooth compact manifold with boundary, we have reductions of orders in terms of Mellin pseudo-differential operators as follows: Set

$$\mathrm{op}_M^\delta(h^\mu)(\tilde{\lambda})u(r) := \frac{1}{2\pi i} \int_{\Gamma_{\frac{1}{2}-\delta}} \int_0^\infty \left(\frac{r}{r'}\right)^{-z} h^\mu(z, \tilde{\lambda})u(r') \frac{dr'}{r'} dz,$$

$\delta \in \mathbb{R}$, first on $u \in C_0^\infty(\mathbb{R}_+, C^\infty(X))$, and then extended to our Sobolev spaces. We then have the following result:

Theorem 2.10 *For every $\mu \in \mathbb{R}$ and every $d \leq d'$ there is an operator family $h^\mu(z, \tilde{\lambda})$ with the above-mentioned properties such that*

$$\mathrm{op}_M^{\gamma-\frac{n}{2}}(h^\mu)(\tilde{\lambda}) : \mathcal{H}^{s,\gamma}(X^\wedge) \to \mathcal{H}^{s-\mu,\gamma}(X^\wedge) \tag{28}$$

is a family of isomorphisms for all $|\tilde{\lambda}| \geq \tilde{C} = \tilde{C}(d, d')$, for all $s \in \mathbb{R}$ and for all $\gamma \in \mathbb{R}$ in the interval $[\frac{n+1}{2} - d', \frac{n+1}{2} - d]$.

Proof. By construction, the operators $h^\mu(z, \tilde{\lambda})$ define isomorphisms (27) for all z in a sufficiently wide strip $d \leq \beta \leq d'$ for any given $d \leq d'$, provided $|\tilde{\lambda}|$ is sufficiently large. At the same time, $h^\mu(z, \tilde{\lambda})$ is an operator-valued Mellin symbol of order μ with constant coefficients, acting between Sobolev spaces on X. This shows that $\mathrm{op}_M^{\gamma-\frac{n}{2}}(h^\mu)(\tilde{\lambda}) = r^{\gamma-\frac{n}{2}} \mathrm{op}_M^0(h_0^\mu)(\tilde{\lambda}) r^{-\gamma+\frac{n}{2}}$ for $h_0^\mu(z, \tilde{\lambda}) := h^\mu(z - \gamma + \frac{n}{2}, \tilde{\lambda})$ defines an invertible family of operators (28) for all γ such that $d \leq \mathrm{Re}\,(z-\gamma+\frac{n}{2}) \leq d'$. The Mellin operator op_M^0 refers to $\Gamma_{\frac{1}{2}} = \{z : \mathrm{Re}\, z = \frac{1}{2}\}$ and hence we get isomorphisms for all weights γ in the interval $\left[\frac{n+1}{2} - d', \frac{n+1}{2} - d\right]$. $\square$

Let B be a compact manifold with conical singularities and boundary, and let $\mathbb{B}$ denote the associated stretched manifold, cf. [5] or [16] for the terminology. In particular, we may double up B to a closed manifold $2B$ with conical singularities, and then the associated stretched manifold $2\mathbb{B}$ is a compact C^∞ manifold with boundary. If X is the base of the conical singularity of B which is a compact C^∞ manifold with boundary, then we have $2X = \partial(2\mathbb{B})$. There is then a subset $(2\mathbb{B})_{\mathrm{sing}} := \partial(2\mathbb{B})$ of $2\mathbb{B}$, and we set $\mathbb{B}_{\mathrm{sing}} := (2\mathbb{B})_{\mathrm{sing}} \cap \mathbb{B}$.

Let $\mathcal{H}^{s,\gamma}(2\mathbb{B}), s, \gamma \in \mathbb{R}$, denote the scale of weighted Sobolev spaces on $2\mathbb{B}$, locally modelled by $\mathcal{H}^{s,\gamma}((2X)^\wedge)$ near $\partial(2\mathbb{B}) \cong 2X$ and by $H^s_{\mathrm{loc}}(2\mathbb{B} \setminus \partial(2\mathbb{B}))$ outside $\partial(2\mathbb{B})$. Then we get similar spaces $\mathcal{H}^{s,\gamma}(\mathbb{B}) := \mathcal{H}^{s,\gamma}(2\mathbb{B})|_{\mathbb{B}}$ on $\mathbb{B}$.

Let $\omega_0, \omega_1, \omega_2$ be cut-off functions on $\mathbb{B}$, that are restrictions of corresponding cut-off functions on $2\mathbb{B}$ to $\mathbb{B}$ that equal 1 near $\partial(2\mathbb{B})$ and are supported in a collar neighbourhood $\cong [0,1) \times \partial(2\mathbb{B})$ of $\partial(2\mathbb{B})$. Assume that $\omega_0\omega_1 = \omega_0, \omega_0\omega_2 = \omega_2$. Moreover, let $\mathbb{D}$ denote a compact C^∞ manifold with boundary, obtained by gluing together $\mathbb{B} =: \mathbb{B}_+$ and another (stretched) manifold $\mathbb{B}_-$ with conical singularity with boundary such that $\mathbb{B}_{+,\mathrm{sing}} \cong \mathbb{B}_{-,\mathrm{sing}}$, by identifying corresponding points of the singular subsets. An example for such a construction (though, for simplicity, with non-compact manifolds) is $\mathbb{B}_+ := [0,\infty) \times X$ and $\mathbb{B}_- := [-\infty, 0] \times X$; then $\mathbb{D} := \mathbb{R} \times X$.

We now apply Theorem 2.1 to $\mathbb{D}$ in place of X and $(\lambda, \tilde{\lambda}) \in \mathbb{R}^{l+\tilde{l}}$ instead of λ. We then get an order reducing family

$$R^\mu(\lambda, \tilde{\lambda}) : H^s(\mathbb{D}) \to H^{s-\mu}(\mathbb{D}).$$

Moreover, let us apply Theorem 2.10 for $(\lambda, \tilde{\lambda})$ instead of $\tilde{\lambda}$. Then we can form a $(\lambda, \tilde{\lambda})$ - dependent family of continuous operators

$$\begin{aligned} &\omega_0 \mathrm{op}_M^{\gamma-\frac{n}{2}}(h^\mu)(\lambda,\tilde{\lambda})\omega_1 + (1-\omega_0)R^\mu(\lambda,\tilde{\lambda})(1-\omega_2) \\ =:\ &S^\mu(\lambda,\tilde{\lambda}) : \mathcal{H}^{s,\gamma}(\mathbb{B}) \to \mathcal{H}^{s-\mu,\gamma}(\mathbb{B}) \end{aligned} \tag{29}$$

(the second summand on the left hand side is interpreted as an operator on $\mathbb{B} = \mathbb{B}_+$ that vanishes in a neighbourhood of $\mathbb{B}_{\mathrm{sing}}$).

Theorem 2.11 *For every $\mu \in \mathbb{R}$ and every $d \leq d'$ the operators* (29) *induce isomorphisms for all $\tilde{\lambda} \in \mathbb{R}^{\tilde{l}}, |\tilde{\lambda}| \geq \tilde{C}$ for a constant $\tilde{C} > 0$, for all $\lambda \in \mathbb{R}^l, s \in \mathbb{R}$ and all $\gamma \in \mathbb{R}$ in the interval $[\frac{n+1}{2} - d', \frac{n+1}{2} - d]$.*

This result is a corollary of Theorems 2.1 and 2.10 and of the technique of the proof of Theorem 2.1 in connection of remainders in parameter-dependent parametrices that behave as Schwartz functions in $(\lambda, \tilde{\lambda})$, cf. formulas (22), (23).

Remark 2.12 *A manifold B with conical singularities and boundary is a corner manifold with two independent axial directions. In direction normal to the boundary our result concerns standard Sobolev spaces. If we ask a similar construction for spaces with weights in normal direction, see* [21], *then we have, in fact, double weighted spaces. Order reducing results in this framework are also desirable, though such constructions are more voluminous. Corner operators of this type are then elliptic in a respective corner pseudo-differential algebra, see* [19] *or* [6].

Let us finally note that reductions of orders on an infinite stretched cone $X^\wedge$ with boundary are also of interest in another scale of weighted Sobolev spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$ instead of the ones in Theorem 2.10, defined by $\omega\mathcal{K}^{s,\gamma}(X^\wedge) = \omega\mathcal{H}^{s,\gamma}(X^\wedge)$ for any cut-off function $\omega(r)$ and by the standard Sobolev spaces for large r. In this case, similarly to the construction for Theorem 2.11, we glue together the operator (28) near $r = 0$ with another order reducing operator for the standard Sobolev spaces for large r. For the latter part a variant of the calculus of boundary value problems without the transmission property on a manifold with exits to infinity is to be applied (locally, in the half-space, this corresponds to a refinement of Kumano-go's calculus [12] in the variant of boundary value problems).

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