On higher eta-invariants and metrics of positive scalar curvature

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Abstract : Let N be a closed connected spin manifold admitting one metric of positive scalar curvature. In this paper we use the higher eta invariant associated to the Dirac operator on N in order distinguish metrics of positive scalar curvature on N. In particular we give sufficient conditions, involving $\pi_1(N)$ and $\dim N$, for N to admit an infinite number of metrics of positive scalar curvature that are non-concordant or even non-bordant.

key words: Galois coverings, higher Atiyah-Patodi-Singer index theory, higher eta invariants, higher ρ-invariants, b-pseudodifferential calculus, bordism groups, positive scalar curvature metrics.

Contents.

0. Introduction.

Let N^n be a closed connected spin manifold of dimension n. In this Introduction we shall always *assume that N admits at least one metric of positive scalar curvature g.* One may ask how many such metrics there are on N. Since a small perturbation of g produces a different metric which is still of positive scalar curvature ($\equiv$ PSC), it is clear that in order to give a rigorous meaning to the above question one first needs to introduce a way for distinguishing two metrics of PSC g_0, g_1. There are three increasingly stronger conditions for distinguishing two metrics of PSC g_0 and g_1.

The first one is to say that g_0 and g_1 are not path-connected in $\mathcal{R}^+(N)$, the space of PSC metrics on N. Thus, in this case, we are interested in $\pi_0(\mathcal{R}^+(N))$, the set of arcwise connected components of $\mathcal{R}^+(N)$. Interesting papers by Hitchin [Hi], Gromov-Lawson [GL], Lawson-Michelson [LM], Botvinnik and Gilkey [Bo-Gi 1] and others show that under some assumptions on N the set $\pi_0(\mathcal{R}^+(N))$ is, if non-empty, of cardinality greater than one (as in [Hi]) or, in fact, of infinite cardinality (as in [GL], [LM], [Bo-Gi 1]). Sometime it is even possible to show, as in [Bo-Gi 1], [Kr-St 1], that

$$|\pi_0(\mathcal{R}^+(N)/\mathrm{Diff}(N))| = \infty\,.$$

The second way for distinguishing two PSC metrics employs the notion of *concordance*: g_0 and g_1 are concordant if there exists a metric of PSC on $N \times [0,1]$ extending g_0 on $N \times \{0\}$, g_1 on $N \times \{1\}$ and of

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product-type near the boundary. Notice that two metrics which are path connected are concordant; it is not known whether the converse is true. Thus, in this case one is interested in the set of concordance classes of PSC metrics on N, denoted by $\widetilde{\pi}_0(\mathcal{R}^+(N))$. One recent fundamental result concerning the set $\widetilde{\pi}_0(\mathcal{R}^+(N))$ is due to Stolz ([St 2]). Stolz shows that the relative bordism group $R^{\rm spin}_{n+1}(B\pi_1(N))$ introduced by Hajduk in [Ha] acts transitively and freely on $\widetilde{\pi}_0(\mathcal{R}^+(N))$. In particular there is a bijection

$$\widetilde{\pi}_0(\mathcal{R}^+(N^n)) \longleftrightarrow R^{\rm spin}_{n+1}(B\pi_1(N^n)).$$

The results by [GL], [LM], [Bo-Gi 1] show in fact that if N admits one metric of PSC then, under suitable assumptions, N admits an infinite number of *non-concordant* metrics of PSC. Equivalently, thanks to Stolz's theorem, they can be reinterpreted as giving sufficient conditions on Γ and ℓ ensuring that $R^{\rm spin}_{\ell}(B\Gamma)$ is non-trivial and, in fact, of infinite cardinality.

The third and more subtle way for distinguishing two PSC metrics g_0, g_1 on our spin manifold N employs the notion of *bordism*. Intuitively we allow ourself the freedom of taking *any* spin manifold W with boundary $\partial W = N \coprod -N$ and a PSC metric G on W which is of product-type near the boundary and restricts to $g_0 \coprod g_1$ on ∂W. See Section 1 for the precise definition. Clearly, two metrics that are concordant are also bordant; on the other hand there are example of non-concordant metrics that are bordant, see [LM, page 329]. In the already mentioned paper of Botvinnik and Gilkey, sufficient conditions are given on $\pi_1(N)$ and $\dim N$ ensuring the existence of an infinite number of *non-bordant* metrics. One of our results will considerably weaken these conditions (see Theorem 0.1 below).

It is important to remark that all these papers use index theory for Dirac operators in an essential way; the main point is to define invariants that can distinguigh two metrics of PSC. In the case of [GL], [LM], [Bo-Gi 1] and [Kr-St 1] these invariants involve, one way or another, the Atiyah-Patodi-Singer index theory on manifolds with boundary. It is therefore conceivable that by using a *higher* version of the Atiyah-Patodi-Singer index theory one should be able to define more general invariants having the property of distinguishing two metrics of PSC. This is the route we take here. The invariants we shall be using are defined in terms of Lott's higher eta invariant [L 2,3] and its delocalized part which is, by definition, the higher ρ-invariant. That the higher ρ-invariant could be used in connection with PSC problems was already observed in [L2] where it was proved that it is constant on the connected components of $\mathcal{R}^+(N)$. In this paper we elaborate on these ideas; the higher Atiyah-Patodi-Singer index formula proved in [LP 1,3] will play a fundamental role in our analysis. It should be added here that, in contrast with other invariants, the higher ρ-invariant is particularly suited for distinguishing *non-bordant* metrics.

Although we shall give in the paper rather general results, amenable to further applications, we prefer to state here the geometric outcome of our arguments. Let F be a finite group and let m be a odd positive integer. Botvinnik and Gilkey have introduced a non-negative integer $r_m(F)$, see [Bo-Gi 1]. We are interested in finite groups F and positive integers m such that $r_m(F) > 0$. In this direction we recall that if $m \equiv 3 \bmod 4$ or if $|F|$ is odd, then, automatically, $r_m(F) > 0$. If $m \equiv 1 \bmod 4$ then $r_m(F) > 0$ iff F contains an element which is not conjugate to its inverse.

Theorem 0.1. *Let $n \in \mathbb{N}$, $n \geq 5$. Let F be a finite group such that $r_m(F) > 0$ for m odd, $5 \leq m \leq n$. Let Γ, Γ', Δ be finitely generated groups. On these groups we make the following assumptions: (i) $\Gamma = \Gamma' \rtimes F$. (ii) Δ is virtually nilpotent or Gromov hyperbolic and there exists $i \in \mathbb{N}$ such that $H^{n-m-4i}(\Delta, \mathbb{Q}) \neq 0$. Let now N be a closed connected spin manifold of dimension n with fundamental group $\Gamma \times \Delta$ admitting one metric of positive scalar curvature, then N admits an infinite number of non-bordant metrics of PSC.*

The next corollary follows easily from Theorem 0.1 either by a direct argument or by an application of Stolz's theorem 1.2.

Corollary 0.2. *Let $n \in \mathbb{N}$, $n \geq 7$. Let F, Γ and Δ as in the assumptions of Theorem 0.1. Then*

$$|R^{\rm spin}_{n+1}(B(\Gamma \times \Delta))| = +\infty.$$

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1. Spin bordism groups and functoriality.

Let Γ be a finitely generated group. Let $B\Gamma$ be the associated classifying space and let $E\Gamma \to B\Gamma$ be the universal Γ-covering. We denote by $\Omega^{\rm spin}_n(B\Gamma)$ the bordism group consisting of triples (N, s, f) with N a *spin* n-dimensional closed compact manifold, s a spin structure on N and f a continuous map $N \to B\Gamma$.

We let $\Omega^{{\rm spin},+}_n(B\Gamma)$ be the subgroup of $\Omega^{\rm spin}_n(B\Gamma)$ consisting of triples (N, s, f) with N admitting a metric of positive scalar curvature (we shall briefly write PSC).

Finally let ${\rm Pos}^{\rm spin}_n(B\Gamma)$ be the bordism group consisting of quadruples (N, s, f, g) with g a metric of PSC on N. Two quadruples (N, s, f, g) and (N', s', f', g') are bordant in ${\rm Pos}^{\rm spin}_n(B\Gamma)$ if there exists a spin $(n+1)$-dimensional manifold with boundary W with a continuous map $F : W \to B\Gamma$, a spin structure S and a positive scalar curvature metric H on W such that (with obvious notation) $\partial W = N \coprod(-N')$, $F|_{\partial W} = f \coprod f'$, $S_{\partial W} = s \coprod s'$, $H|_{\partial W} = g \coprod g'$.

Definition 1.1. *Let N be a closed spin manifold of dimension n admitting one metric of PSC. Let $f : N \to B\pi_1(N)$ be the classifying map for the universal cover of N. We shall say that two metrics of PSC g_0 and g_1 are bordant if for any spin structure s we have*

$$[N, s, f, g_0] = [N, s, f, g_1] \ \text{ in } \ {\rm Pos}^{\rm spin}_n(B\pi_1(N))\,.$$

We now pass to the relative bordism groups. These were first introduced by Hajduk in [Ha]. They were successfully employed by Stolz in problems of classification of PSC metrics. See [St 1] (page 630) [St 2]. Following [St 1] we denote them by $R^{\rm spin}_n(B\Gamma)$. Thus we only consider the *spin* case; this corresponds to the choice of supergroup $\gamma = (\Gamma, 0, \Gamma \times \mathbb{Z}_2)$ in [St 2]. We recall that $R^{\rm spin}_n(B\Gamma)$ is the bordism group of quadruples (N, s, f, h) where N is a spin manifold of dimension n possibly with boundary, s is a spin structure, $f : N \to B\Gamma$ is a continuous map, h is a PSC metric on ∂N. Two such quadruples (N, s, f, h), (N', s', f', h') are said to be bordant if:

1) There is a bordism (V, S, F, H) between $(\partial N, s|_{\partial N}, f|_{\partial N}, h)$ and $(\partial N', s'|_{\partial N'}, f'|_{\partial N'}, h')$ viewed as elements in ${\rm Pos}^{\rm spin}_{n-1}(B\Gamma)$.

2) The closed (glued) spin manifold $N \cup_{\partial N} V \cup_{\partial N'} N'$ is the boundary of some spin manifold W. Moreover, there is a spin structure S_W on W and a continuous map $E_W : W \to B\Gamma$ restricting to s (resp. s', S) and f (resp. f', resp. F) on N (resp. N', resp. V).

We also recall that there exists a long exact sequence

$$\cdots \to \Omega^{\rm spin}_{n+1}(B\Gamma) \to R^{\rm spin}_{n+1}(B\Gamma) \xrightarrow{\partial} {\rm Pos}^{\rm spin}_n(B\Gamma) \to \Omega^{\rm spin}_n(B\Gamma) \to \cdots \tag{1.1}$$

In this setting, Stolz (see [St 2]) has proved the following fundamental result.

Theorem 1.2. *(Stolz) Let M be a smooth spin connected manifold with (possibly empty) boundary of dimension $n \geq 5$. Let $\Gamma = \pi_1(M)$ and let $f : M \to B\Gamma$ be the classifying map. Let s be a spin structure on M. Then:*

1] A positive scalar curvature metric h on ∂M extends to a positive scalar curvature metric on M which is a product near the boundary if and only the class $[M, s, f, h] \in R_n^{\rm spin}(B\Gamma)$ vanishes.

2] If h extends to a positive scalar metric on M, then the group $R_{n+1}^{\rm spin}(B\Gamma)$ acts freely and transitively on the set of concordance classes of such metrics.

Thus the set, denoted $\widetilde{\pi}_0(\mathcal{R}^+(M \,{\rm rel}\, h))$, of concordance classes of positive scalar curvature metrics on M extending h, can be identified with the group $R_{n+1}^{\rm spin}(B\Gamma)$. If M is a closed connected $n-$dimensional spin manifold and admits a metric of positive scalar curvature then, similarly, there is a bijection between the set of concordance classes of PSC metrics on M and $R_{n+1}^{\rm spin}(B\Gamma)$, with $\Gamma = \pi_1(M)$. This means that if we wish to compute the cardinality of the set $\widetilde{\pi}_0(\mathcal{R}^+(M))$ of concordance classes of PSC metrics on M we can as well and more generally compute the cardinality of $R_{n+1}^{\rm spin}(B\Gamma)$.

The relative bordism groups satisfy interesting functoriality properties; as pointed out to us by Wolfgang Lück, these properties can sometimes be used in order to prove that $R_*^{\rm spin}(B\Gamma)$ is of *infinite cardinality.*

Lemma 1.3. *Let Γ and Γ' be two finitely generated groups. Let $p : \Gamma \mapsto \Gamma'$ be a group homomorphism, then the map $Bp : B\Gamma \mapsto B\Gamma'$ induces by composition, a group homomorphism $R_n(p) : R_n^{\rm spin}(B\Gamma) \mapsto R_n^{\rm spin}(B\Gamma')$ for any nonnegative integer n. If moreover there exists a group homomorphism $j : \Gamma' \mapsto \Gamma$ such that $p \circ j = {\rm Id}_{\Gamma'}$ then*

$$R_n(p) \circ R_n(j) = {\rm Id}_{R_n^{\rm spin}(B\Gamma')}$$

and $R_n^{\rm spin}(B\Gamma')$ may be identified to a subgroup of $R_n^{\rm spin}(B\Gamma)$. In particular $R_n^{\rm spin}(B\{1\})$ may be identified to a subgroup of $R_n^{\rm spin}(B\Gamma)$.

The easy proof is left to the reader. We shall now state some (mostly) known results about the cardinality of the relative bordism groups and some consequences of the functoriality lemma above; these results exist implicitly in the literature and we state them here both for future reference and to illustrate the power of Stolz's theorem. One obvious corollary of Lemma 1.3 is the following:

Proposition 1.4. *Let Γ_1 and Γ_2 be two finitely generated groups and assume that $|R_n(\Gamma_2)| = \infty$. Then*
1] $|R_n^{\rm spin}(\Gamma_1 * \Gamma_2)| = \infty$.
2] $|R_n^{\rm spin}(\Gamma_1 \times \Gamma_2)| = \infty$.
3] $|R_n^{\rm spin}(\Gamma_1 \rtimes \Gamma_2)| = \infty$.

Proposition 1.5. *For any discrete group Γ and any $k \geq 2$ we have:*

$$|R_{4k}^{\rm spin}(B\Gamma)| = \infty.$$

Proof. This follows immediatly from results of Gromov-Lawson and Stolz's theorem. More precisely, Lemma 1.3 shows that if $|R_{4k}^{\rm spin}(B\{1\})| = \infty$ then $|R_{4k}^{\rm spin}(B\Gamma)| = \infty$. So, using Theorem 1.2 2] for M with an empty boundary, it is enough to exhibit one smooth compact connected spin manifold N of dimension $4k - 1$ admitting an infinite number of non concordant metrics with positive scalar curvature. Such a manifold is given, for example, by S^{4k-1}, see [LM, page 329].

Proposition 1.6. *Let $m = 2n - 1$ be an odd integer, $m \geq 7$. Let F be a finite group satisfying the Botvinnik-Gilkey assumption $r_m(F) > 0$ [Bo-Gi 1]. Assume that there exists a split exact sequence*

$$1 \longrightarrow \Gamma' \longrightarrow \Gamma \xrightarrow{p} F \longrightarrow 1$$

Then $|R^{\rm spin}_{2n}(B\Gamma)| = \infty$.

Proof. This proposition follows easily from the results of Botvinnik and Gilkey. By assumption $\Gamma = \Gamma' \rtimes F$ thus, by Proposition 1.4, it suffices to show that $|R^{\rm spin}_m(BF)| = \infty$. Let Y be a spin closed manifold of dimension $m-2$ with fundamental group equal to F. Then $Y \times S^2$ has dimension m, fundamental group Γ and admits a metric of PSC since S^2 does. By [Bo-Gi 1, Theorem 0.3] there exists an infinite number of non-concordant metrics on Y; thus, by Stolz's Theorem 1.2 we get $|R^{\rm spin}_{2n}(BF)| = \infty$.

Remark. In fact, using a particular case of the index map introduced in the next section, see Definition 2.4, one could prove the last two propositions without using Stolz's Theorem

Remark. Let X be a closed connected spin manifold of dimension m with fundamental group Γ as in Proposition 1.6 and admitting a metric of PSC. The above argument shows that $\pi_0(\mathcal{R}^+(X))$ is infinite. Notice that m can be chosen ≥ 5 in this case (as we do not need to cross by S^2). We shall prove in the Section 4 (see Proposition 4.2) that, in fact,

$$|\pi_0(\mathcal{R}^+(X)/\mathrm{Diff}(X))| = \infty \ .$$

2. Spin bordism groups and higher APS index theory.

Given a quadruple (N^n, s, f, g) as in the definition of $\mathrm{Pos}^{\rm spin}_n(B\Gamma)$ in the previous section, we can consider the Dirac operator $\not{D}$ associated to s and g. We also have the Γ-covering $f^*E\Gamma$ with the Γ-invariant Riemannian metric $\widetilde{g}$ induced by g. We denote by $\widetilde{\not{D}}$ the lifted Dirac operator and by $\mathcal{D}$ the Dirac operator on N twisted by the flat bundle

$$\mathcal{V}_f = f^*E\Gamma \times_\Gamma \mathcal{B}^\infty_\Gamma \ ; \tag{2.1}$$

here $\mathcal{B}^\infty_\Gamma$ denotes the Connes-Moscovici algebra [CM]. Since g has PSC, the Dirac operator $\mathcal{D}$ is invertible in the $\mathcal{B}^\infty_\Gamma$-Mishchenko-Fomenko calculus. Thus, according to [L 2, Section 4.4], [L 3] the higher eta invariant $\widetilde{\eta}$ of $\mathcal{D}$ is well defined (see also [LP 3, Appendix]); it is an element in the space of non-commutative differential forms $\overline{\Omega}_*(\mathcal{B}^\infty_\Gamma)$ ([Ka, L1]. We shall denote the higher eta invariant associated to this Dirac operator by $\widetilde{\eta}(N, s, f, g)$; it is *not* defined on $\mathrm{Pos}^{\rm spin}_n(B\Gamma)$. The next Lemma gives a formula for the higher eta invariant of a product; it extends results by Atiyah-Patodi-Singer ([APS 3, page 84] and Lott ([L 2] page 205).

Lemma 2.1. *Let (N, s, f, g) be a quadruple as in the definition of $\mathrm{Pos}^{\rm spin}_m(B\Gamma)$ and let (M, σ, ϕ) be a triple as in the definition of $\Omega^{\rm spin}_n(B\Delta)$. Let h be a smooth metric on M, then for $\epsilon > 0$ small enough, $\mathcal{Q}(M \times N) = (M \times N, \sigma \times s, \phi \times f, \frac{1}{\epsilon}h \times g)$ defines a quadruple as in the definition of $\mathrm{Pos}^{\rm spin}_{m+n}(B(\Delta \times \Gamma))$ and we have :*

$$\widetilde{\eta}(\mathcal{Q}(M \times N)) \ = \ C(m,n) \int_M \widehat{A}(M) \wedge \omega_{\phi^*(E\Delta)} \widehat{\otimes}\, \widetilde{\eta}(N, s, f, g) \ \text{modulo} \ d\overline{\Omega}_*(\mathcal{B}^\infty_\Delta) \widehat{\otimes} \overline{\Omega}_*(\mathcal{B}^\infty_\Gamma) + \overline{\Omega}_*(\mathcal{B}^\infty_\Delta) \widehat{\otimes} d\overline{\Omega}_*(\mathcal{B}^\infty_\Gamma)$$

where $\omega_{\phi^*(E\Delta)}$ is the closed biform $\in \Omega^*(M) \widehat{\otimes}\, \overline{\Omega}_*(\mathcal{B}^\infty_\Delta)$ introduced by Lott ([L 1] page 436), and $C(m,n)$ is a nonzero constant depending only on m and n.

Proof. We will treat only the case where m and n are both odd. Let S_M, S_N denote the bundle of spinors over M and N respectively. Then the spinor bundle $S_{M \times N}$ over $M \times N$ can be identified with $S_M \otimes S_N \otimes \mathbb{C}^2$.

Let $\mathcal{Y}$ and $\mathcal{I}$ be the two anticommuting involutions of $\mathbb{C}^2$ defined by:

$$\mathcal{Y} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{I} = \begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}$$

Then for the product metric on $M \times N$ and any $\alpha \in T_x^* M$, $\beta \in T_y^* N$, the clifford action on $S_M \otimes S_N \otimes \mathbb{C}^2$ is given by:

$$\mathrm{cl}(\alpha) = \mathrm{cl}_M(\alpha) \otimes \mathrm{Id} \otimes \mathcal{Y}, \;\; \mathrm{cl}(\beta) = \mathrm{Id} \otimes \mathrm{cl}_N(\beta) \otimes \mathcal{I}$$

The (generalized) Dirac operator $\mathcal{D}_{M\times N}$ associated with $\mathcal{Q}(M \times N)$ acts on the sections of the $\mathcal{B}^\infty_{\Delta\times\Gamma}-$bundle $\mathcal{V}_\phi \otimes \mathcal{V}_f \otimes S_M \otimes S_N \otimes \mathbb{C}^2$ over $M \times N$; we shall endow this bundle with the $\mathbb{Z}_2-$grading induced by the decomposition $\mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}$. The operator $\mathcal{D}_{M\times N}$ may be identified with:

$$\mathcal{D}_{M\times N} = \begin{pmatrix} 0 & \mathcal{D}_M + \sqrt{-1}\mathcal{D}_N \\ \mathcal{D}_M - \sqrt{-1}\mathcal{D}_N & 0 \end{pmatrix} = \mathcal{D}_M \mathcal{Y} + \mathcal{D}_N \mathcal{I}.$$

Now let us recall that Lott has associated to $\mathcal{V}_\phi \otimes S_M$ ([L 1] page 436) a $\mathcal{B}^\infty_\Delta-$connection ∇_M on M with values in $\Omega_1(\mathcal{B}^\infty_\Delta)\widehat{\otimes}_{\mathcal{B}^\infty_\Delta} \mathcal{V}_\phi \otimes S_M$. One checks easily that the connection $\nabla_{M\times N}$ associated to $\mathcal{V}_\phi \otimes \mathcal{V}_f \otimes S_{M\times N}$ is given by

$$\nabla_{M\times N} = \nabla_M \otimes \mathrm{Id}_{\mathbb{C}^2} \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathrm{Id}_{\mathbb{C}^2} \otimes \nabla_N.$$

Let us recall that the higher eta invariant $\widetilde{\eta}(\mathcal{Q}(M \times N))$ is defined to be equal to $\int_0^{+\infty} \eta(s)ds$ where the local eta-integrand $\eta(s)$ is defined to be:

$$\eta(s) = \frac{2}{\pi} \mathrm{STR}\, \mathcal{D}_{M\times N} \exp\left(-(s\mathcal{D}_{M\times N} + \nabla_{M\times N})^2\right).$$

Using supersign conventions, one checks easily that

$$(s\mathcal{D}_{M\times N} + \nabla_{M\times N})^2 = (s\mathcal{D}_M \mathcal{Y} + \nabla_M \mathrm{Id}_{\mathbb{C}^2})^2 + (s\mathcal{D}_N \mathcal{I} + \nabla_N \mathrm{Id}_{\mathbb{C}^2})^2 . \tag{2.2}$$

Moreover, using Duhamel's expansion formula we get:

$$\exp -(s\mathcal{D}_M \mathcal{Y} + \nabla_M \mathrm{Id}_{\mathbb{C}^2})^2 = \exp(-s^2\mathcal{D}_M^2)\mathrm{Id}_{\mathbb{C}^2} +$$

$$\sum_{k=1}^{+\infty} (-1)^k \int_0^1 \int_0^{1-u_1} \cdots \int_0^{1-u_1-\ldots-u_{k-1}} \exp(-u_1 s^2 \mathcal{D}_M^2)\mathcal{X}_s \ldots \exp(-(1-u_1-\ldots-u_{k-1})s^2\mathcal{D}_M^2) du_k \ldots du_1$$

where we have set:

$$\mathcal{X}_s = \nabla_M^2 \mathrm{Id}_{\mathbb{C}^2} + s(\mathcal{D}_M \nabla_M + \nabla_M \mathcal{D}_M)\mathcal{Y}.$$

Moreover we recall from [L 2 page 215] that $\mathcal{D}_M \nabla_M + \nabla_M \mathcal{D}_M$ and ∇_M^2 are operators that increase the degree of differential forms on $\mathcal{B}^\infty_\Delta$ by one and two respectively. Therefore, using the previous Duhamel's expansion and the fact that $\mathcal{Y}^2 = \mathrm{Id}_{\mathbb{C}^2}$, one sees easily that for any real $s > 0$:

$$\exp -(s\mathcal{D}_M \mathcal{Y} + \nabla_M \mathrm{Id}_{\mathbb{C}^2})^2 = \mathrm{Even}(s, M)\mathrm{Id}_{\mathbb{C}^2} + \mathrm{Odd}(s, M)\mathcal{Y} \tag{2.3}$$

where $\mathrm{Even}(s, M)$ (resp. $\mathrm{Odd}(s, M)$) is defined as the sum of a series of operators acting on the set of sections of the bundle $\Omega_*(\mathcal{B}^\infty_\Delta)\widehat{\otimes}_{\mathcal{B}^\infty_\Delta} \mathcal{V}_\phi \otimes S_M$ which increase the degree of differential form on $\mathcal{B}^\infty_\Delta$ by an even (resp. odd) number. Of course, $\mathrm{Even}(s, M)$ and $\mathrm{Odd}(s, M)$ are identified with diagonal operators when they act on the sections of the tensor product of the previous bundle with $\mathbb{C}^2$. Similarly, one can write for any real $s > 0$:

$$\exp -(s\mathcal{D}_N \mathcal{I} + \nabla_N \mathrm{Id}_{\mathbb{C}^2})^2 = \mathrm{Even}(s, N)\mathrm{Id}_{\mathbb{C}^2} + \mathrm{Odd}(s, N)\mathcal{I}. \tag{2.4}$$

For the proof of the Lemma we shall need the following intermediate result.

Sublemma 2.2. *For any real $s > 0$ one has:*

$$\mathrm{STR}\,\mathcal{D}_M\mathcal{Y}\exp\left(-(s\mathcal{D}_{M\times N} + \nabla_{M\times N})^2\right) =$$
$$-2\sqrt{-1}\mathrm{TR}_{\mathrm{even}}\,\mathcal{D}_M\,\mathrm{e}^{-(s\mathcal{D}_M+\nabla_M)^2}\,\widehat{\otimes}\,\mathrm{TR}_{\mathrm{odd}}\,\mathrm{e}^{-(s\mathcal{D}_N+\nabla_N)^2}$$

where the subscript even (odd) means that one projects onto the space of differential forms (on $\mathcal{B}^\infty_\Delta$ or $\mathcal{B}^\infty_\Gamma$) of even (odd) degrees.

Proof. Using the three formulas, (2.2), (2.3), (2.4), one sees that for any real $s > 0$:

$$\mathcal{D}_M\mathcal{Y}\exp\left(-(s\mathcal{D}_{M\times N} + \nabla_{M\times N})^2\right) =$$
$$(\mathcal{D}_M\mathrm{Even}(s,M)\mathcal{Y} + \mathcal{D}_M\mathrm{Odd}(s,M)\mathrm{Id}_{\mathbb{C}^2}) \circ (\mathrm{Even}(s,N)\mathrm{Id}_{\mathbb{C}^2} + \mathrm{Odd}(s,N)\mathcal{I}).$$

Since the supertraces of the matrixes $\mathcal{Y}, \mathcal{I}, \mathrm{Id}_{\mathbb{C}^2}$ vanish and the supertrace of $\mathcal{Y}\circ\mathcal{I}$ is equal to $-2\sqrt{-1}$, one then gets:

$$\mathrm{STR}\,\mathcal{D}_M\mathcal{Y}\exp\left(-(s\mathcal{D}_{M\times N} + \nabla_{M\times N})^2\right) = -2\sqrt{-1}\mathrm{TR}(\mathcal{D}_M\mathrm{Even}(s,M))\,\widehat{\otimes}\mathrm{TR}\,\mathrm{Odd}(s,N).$$

But analyzing Duhamel's expansion for $\mathrm{e}^{-(s\mathcal{D}_M+\nabla_M)^2}$ and $\mathrm{e}^{-(s\mathcal{D}_N+\nabla_N)^2}$, one recognizes that

$$\mathrm{TR}\,(\mathcal{D}_M\mathrm{Even}(s,M)) = \mathrm{TR}_{\mathrm{even}}\,\mathcal{D}_M\,\mathrm{e}^{-(s\mathcal{D}_M+\nabla_M)^2}$$
$$\mathrm{TR}\,\mathrm{Odd}(s,N) = \mathrm{TR}_{\mathrm{odd}}\,\mathrm{e}^{-(s\mathcal{D}_N+\nabla_N)^2}$$

which proves Sublemma 2.2.

Now we go back to the proof of Lemma 2.1, since $\mathcal{D}_N$ is invertible, we see that

$$\mathrm{TR}_{\mathrm{odd}}\,\mathrm{e}^{-(s\mathcal{D}_N+\nabla_N)^2} \tag{2.5}$$

tends to 0 as $s \to +\infty$; since (2.5) does not depend on s, modulo exact forms, Sublemma 2.2 shows that for each real $s > 0$

$$\mathrm{STR}\,\mathcal{D}_M\mathcal{Y}\exp\left(-(s\mathcal{D}_{M\times N} + \nabla_{M\times N})^2\right)$$

is an element of $\overline{\Omega}_*(\mathcal{B}^\infty_\Delta)\widehat{\otimes}d\overline{\Omega}_*(\mathcal{B}^\infty_\Gamma)$. Next, similarly, one proves that:

$$\mathrm{STR}\,\mathcal{D}_N\mathcal{I}\exp\left(-(s\mathcal{D}_{M\times N} + \nabla_{M\times N})^2\right) =$$
$$2\sqrt{-1}\mathrm{TR}_{\mathrm{odd}}\,\mathrm{e}^{-(s\mathcal{D}_M+\nabla_M)^2}\,\widehat{\otimes}\mathrm{TR}_{\mathrm{even}}\,\mathcal{D}_N\;\mathrm{e}^{-(s\mathcal{D}_N+\nabla_N)^2}.$$

Now we recall that

$$\widetilde{\eta}(N,s,f,g) = \mathrm{STR}_{\mathrm{Cl}(1)}\left(\mathcal{Y}\mathcal{D}_N\exp-(s\mathcal{Y}\mathcal{D}_N + \nabla_N\otimes\mathrm{Id}_{\mathbb{C}^2})^2\right).$$

Proceeding as in the proof of Sublemma 2.2, one checks easily that

$$\widetilde{\eta}(N,s,f,g) = \mathrm{TR}_{\mathrm{even}}\,\mathcal{D}_N\,\mathrm{e}^{-(s\mathcal{D}_N+\nabla_N)^2}.$$

Moreover the small time limit index theorem ([L 1] page 440 and also [L 2] page 219) shows that

$$\mathrm{TR}_{\mathrm{odd}}\;\mathrm{e}^{-(s\mathcal{D}_M+\nabla_M)^2}$$

goes to $\int_M \widehat{A}(M)\wedge\omega_{f_1{}^*(E\Delta)}$ modulo exact forms as $s \to 0^+$. One then obtains immediately the Lemma.

Next we consider an element $[M,s,f,g_\partial] \in R^{\mathrm{spin}}_n(B\Gamma)$. Let ${}^b g$ be a b-Riemannian metric [Me] on M extending g_∂. To such a quadruple (M,s,f,g_∂) we can associate the b-index class [LP 1, page 80, 88] $\mathrm{Ind}_b(\mathcal{D}) \in K_*(\mathcal{B}^\infty_\Gamma) \equiv K_*(C^*_r(\Gamma))$ of the Dirac operator defined by s and ${}^b g$ twisted by the flat bundle

$$\mathcal{V}_f = f^*E\Gamma\times_\Gamma\mathcal{B}^\infty_\Gamma.$$

This index class is well defined since, by definition of $R^{\mathrm{spin}}_n(B\Gamma)$, g_∂ is of PSC; thus the boundary operator is invertible in the Mishchenko-Fomenko calculus and the operator is $C^*_r(\Gamma)$-Fredholm according to the results in [LP 1, page 80]. The following proposition is a complex analogue of results of Bunke [Bu 2] and Stolz [St 2]:

Proposition 2.3. *The map ${}^b\Theta_{\mathbb{C}}$ gives a well defined morphism of groups*

$$ {}^b\Theta_{\mathbb{C}} : R_n^{\mathrm{spin}}(B\Gamma) \to K_*(C_r^*(\Gamma)) $$

It associates to a class $[M, s, f, g_\partial]$ the b-index of the $C_r^*(\Gamma)-$Dirac operator with coefficients in $\mathcal{V}_f$ defined by s and an extension of g_∂ to M.

Proof. We have to show that the map is well defined. Obviously the index class does not depend on the choice of the extending b-metric since the Dirac operators associated to two different extensions are $C_r^*(\Gamma)$-homotopic. Next we have to show that the class is the same for two bordant quadruples $(M, s, f, g_{\partial M})$, $(M', s', f', g_{\partial M'})$. This follows from a splitting formula for the b-index together with the cobordism invariance of the index. The splitting formula for the APS-index classes defined in [Wu 2] can be proved exactly as in [DZ 1, page 314], i.e. by employing Bunke's trick (see [Bu 1]). Since the APS-index classes equal the b-index classes (see [LLP, Theorem 10.1]), the same splitting formula holds in the b-context. The Proposition is proved.

Remark. We have defined a morphism of groups ${}^b\Theta_{\mathbb{C}} : R_n^{\mathrm{spin}}(B\Gamma) \to K_*(C_r^*(\Gamma))$; it should be interesting to compare this morphism with the composition of the morphism $\Theta : R_n^{\mathrm{spin}}(B\Gamma) \to KO_*(C_r^*(\Gamma)_{\mathbb{R}})$ defined by Bunke and Stolz (see [Bu 2], [St 2]) with the natural map $KO_*(C_r^*(\Gamma)_{\mathbb{R}}) \to K_*(C_r^*(\Gamma))$. See [L-M page 143] in the case of the trivial group.

Remark. In this paper we shall concentrate on the information we can get by taking the composition of ${}^b\Theta_{\mathbb{C}} : R_n^{\mathrm{spin}}(B\Gamma) \to K_*(C_r^*(\Gamma)) = K_*(\mathcal{B}_\Gamma^\infty)$ with a noncommutative Chern character (see below) and by then pairing the result with suitable cyclic cocycles. In other words, we want to consider *numerical* higher invariants associated to bordism classes; on the one hand this amounts to a loss of information, on the other hand these numerical invariants are sometimes easier to deal with than the K-theoretic counterparts. Notice that there is no real analogue of the Chern character map, not even in the case $\Gamma = \{1\}$ [AS V , LM].

Let $\overline{\mathrm{H}}_*(\mathcal{B}_\Gamma^\infty)$ be the (topological) noncommutative de Rham homology defined by the complex of noncommutative differential forms on the Connes-Moscovici algebra [Ka, Sect. 4.1]. Thanks to the work of Karoubi [Ka] we can take the Chern character $K_0(\mathcal{B}_\Gamma^\infty) \to \overline{\mathrm{H}}_{\mathrm{even}}(\mathcal{B}_\Gamma^\infty)$. In particular the Chern character of the b-index class associated to a quadruple $(M, s, f, g_{\partial M})$ is well defined when M has even dimension. In order to deal with the case in which M has odd dimension, we use suspension as in [MP], [LLP]. Observe that by Fourier transform, $C_r^*(\Gamma \times \mathbb{Z}) \cong C_r^*(\Gamma) \otimes C^0(S^1)$ and $\mathcal{B}_{\Gamma\times\mathbb{Z}^1}^\infty \cong \mathcal{B}_\Gamma^\infty \otimes C^\infty(S^1)$, where $\otimes$ denotes a projective tensor product. Let us consider the standard spin structure s_{S^1} on S^1, the standard metric g_{S^1} and $f_{S^1} : S^1 \to B\mathbb{Z}$ the classifying map for the universal cover of S^1. Let us consider the index class of the Dirac operator defined by the quadruple $[M \times S^1, f \times f_{S^1}, s \times s_{S^1}, g_{\partial M} \times g_{S^1}]$. Using the above isomorphisms we can think to this class as an element in $K_0(\mathcal{B}_\Gamma^\infty \otimes C^\infty(S^1))$. (This class is in fact the image of the index class associated to $(M, s, f, g_{\partial M})$ under of the map $j_* \circ \Sigma$ with Σ equal to the suspension map $K_1(\mathcal{B}_\Gamma^\infty) \equiv K_1(C_r^*(\Gamma)) \to K_0(C_r^*(\Gamma) \otimes C_0(]0, 2\pi[))$ and j_* the natural map $K_0(C_r^*(\Gamma) \otimes C_0(]0, 2\pi[)) \to K_0(C_r^*(\Gamma) \otimes C(S^1)) = K_0(\mathcal{B}_\Gamma^\infty \otimes C^\infty(S^1)))$. Let τ' be the generator for $\mathrm{H}^1(\mathbb{Z}; \mathbb{Z}) \subset \mathrm{H}^1(\mathbb{Z}; \mathbb{C})$. Put $\tau = \sqrt{-1}\,\tau'/2\pi \in \mathrm{H}^1(\mathbb{Z}; \mathbb{C})$. There is a natural desuspension map

$$ \langle \cdot \, , \tau \rangle : \overline{\mathrm{H}}_*(\mathcal{B}_{\Gamma\times\mathbb{Z}}^\infty) \to \overline{\mathrm{H}}_*(\mathcal{B}_\Gamma^\infty). $$

Definition 2.4. *We define a morphism of groups*

$$ {}^b\alpha_n : R_n^{\mathrm{spin}}(B\Gamma) \to \overline{\mathrm{H}}_*(\mathcal{B}_\Gamma^\infty) $$

as follows. If $n = 2m$ *then* ${}^b\alpha_{2m} := \mathrm{Ch} \circ {}^b\Theta_{\mathbb{C}}$. *If* $n = 2m - 1$ *then*

$$ {}^b\alpha_{2m-1}[M, s, f, g_{\partial M}] = \langle {}^b\alpha_{2m}[M \times S^1, f \times f_{S^1}, s \times s_{S^1}, g_{\partial M} \times g_{S^1}], \tau \rangle $$

If n *is even (resp. odd) the map* ${}^b\alpha$ *has values in* $\overline{\mathrm{H}}_{\mathrm{even}}(\mathcal{B}^\infty_\Gamma)$ *(resp.* $\overline{\mathrm{H}}_{\mathrm{odd}}(\mathcal{B}^\infty_\Gamma)$*).*

Using the higher APS index theorem in [LP 1] (Section 14) and Lemma 2.1 we immediatly obtain

Proposition 2.5. *The following formula holds*

$$ {}^b\alpha[M, s, f, g_\partial] = [\int_M \widehat{A}(M) \wedge \omega_{f^*(E\Gamma)} - \frac{1}{2}\widetilde{\eta}(\partial M, \partial s, f|_{\partial M}, g_{\partial M})] \quad \in \overline{H}_*(\mathcal{B}^\infty_\Gamma) $$

with $\omega_{f^*(E\Gamma)}$ *the bi-form introduced by Lott in [L 1, page 436].*

One use of the map ${}^b\alpha$ is the definition of a *higher Gromov-Lawson index* associated to a pair of PSC metrics on a closed spin manifold N of dimension n. Let g_0 and g_1 two metrics of PSC on N; let $f : N \to B\Gamma$ be a classifying map and s a spin structure on N. We consider the quadruple $[N \times [0, 1], s, f, g_0 \coprod g_1] \in R^{\mathrm{spin}}_n(B\Gamma)$.

Definition 2.6. *We define the higher Gromov-Lawson index associated to the pair* (g_0, g_1) *and to the triple* (N, s, f) *as the noncommutative de Rham class*

$$ \mathbf{hi}(N, s, f; g_0, g_1) := {}^b\alpha([N \times [0, 1], s, f, g_0 \coprod g_1]) \in \overline{\mathrm{H}}_{\mathrm{even/odd}}(\mathcal{B}^\infty_\Gamma) \; . $$

The higher Gromov-Lawson index is an homological version of the map

$$ i : \widetilde{\pi}_0(\mathcal{R}^+(N)) \times \widetilde{\pi}_0(\mathcal{R}^+(N)) \to R^{\mathrm{spin}}_{n+1}(B\Gamma) $$

used by Stolz ([St 2] Section 5) in order to prove theorem 1.2 2].

The next Proposition is a simple consequence of Lichnerowicz formula.

Proposition 2.7. *Let* N *be a spin closed manifold of dimension* n *and let* $g_0, g_1 \in \mathcal{R}^+(N)$*. If* g_0 *is concordant to* g_1 *then, for each spin structure* s *and each map* $f : N \to B\Gamma$ *we have* $\mathbf{hi}(N, s, f; g_0, g_1) = 0$.

Proposition 2.8. *Let* $(M, s, f, g_{\partial M})$ *be a representative of an element in* $R^{\mathrm{spin}}_n(B\Gamma)$*. Let* $g_0, g_1 \in \mathcal{R}^+(\partial M)$*. Then*

$$ {}^b\alpha[M, s, f, g_0] - {}^b\alpha[M, s, f, g_1] = \mathbf{hi}(\partial M, s|_{\partial M}, f|_{\partial M}; g_0, g_1) \text{ in } \overline{\mathrm{H}}_*(\mathcal{B}^\infty_\Gamma) \tag{2.6} $$

Proof. The formula already holds at the K-theory level and is an easy consequence of an additivity argument for APS-index classes.

Remark. The higher Gromov-Lawson index is in fact a *higher (noncommutative) spectral flow*, see [DZ 2], [Wu 1], [LP 2]. We shall not use this fact in this paper; we point out however that, this identification granted, formula (2.6) can also be proved by using a formula relating the difference of b-index classes associated to Dirac operators associated to two different metrics to the higher spectral flow of the resulting path of boundary-Dirac-operators. This formula is proved in [DZ 2, Sect. 5 b)] in the family case and can be easily extended to the noncommutative setting using the noncommutative relative index theorem established in [LP 4].

3. Higher spin bordism invariants.

We shall now define higher L^2 bordism invariants. We first deal with the case where Γ is *virtually nilpotent.* We follow closely [L 2, Sect. 4.6]. Under this additional assumption, the complex $\overline{\Omega}_*(\mathcal{B}^\infty_\Gamma)$ splits as the direct sum of sub-complexes labeled by the conjugacy classes $<\Gamma>$ of Γ. Thus $\widetilde{\eta}(N,s,f,g)$ splits as a direct sum

$$\widetilde{\eta}(N,s,f,g) = \bigoplus_{<x>\in<\Gamma>} \widetilde{\eta}(<x>) \tag{3.1}$$

The higher ρ-invariant associated to the quadruple (N,s,f,g) is, by definition,

$$\widetilde{\rho}(N,s,f,g)) := \bigoplus_{<x>\neq<e>} \widetilde{\eta}(<x>)$$

As pointed out in [L 2, page 222], the higher ρ-invariant has in fact values in $\overline{\mathrm{H}}_*(\mathcal{B}^\infty_\Gamma)$. Notice that $\overline{\mathrm{H}}_*(\mathcal{B}^\infty_\Gamma)$ splits as the direct sum $\overline{\mathrm{H}}_*(\mathcal{B}^\infty_\Gamma) = \overline{\mathrm{H}}^{\mathrm{t}}_*(\mathcal{B}^\infty_\Gamma) \oplus \overline{\mathrm{H}}^{\mathrm{nt}}_*(\mathcal{B}^\infty_\Gamma)$ with the first group on the right hand side associated to the subcomplex of $\overline{\Omega}_*(\mathcal{B}^\infty_\Gamma)$ labeled by the trivial conjugacy class and the second group associated to the subcomplexes labeled by the non-trivial conjugacy classes. Then $\widetilde{\rho}(N,s,f,g)$ has values in $\overline{\mathrm{H}}^{\mathrm{nt}}_*(\mathcal{B}^\infty_\Gamma)$, and more precisely, in $\overline{\mathrm{H}}^{\mathrm{nt}}_{\mathrm{even}}(\mathcal{B}^\infty_\Gamma)$ if N has odd dimension and $\overline{\mathrm{H}}^{\mathrm{nt}}_{\mathrm{odd}}(\mathcal{B}^\infty_\Gamma)$ if N has even dimension. For future reference we let π^{t} and π^{nt} be the natural projection maps.

That the higher ρ-invariant could be used in questions of PSC was already observed in [L 2]. First of all it is proved in [L 2, page 221] that the higher ρ-invariants is a deformation invariant with respect to the metric g. This means that the higher ρ-invariant is *locally* constant on the space of PSC metrics on the spin manifold N. Secondly, it was remarked in [L2] that *if* the higher eta invariant was the boundary correction term in a to-be-proved higher Atiyah-Patodi-Singer index theorem, *then* the higher ρ invariant would be an obstruction to extending a PSC metric from the boundary ∂M of a compact spin manifold to the entire manifold M, so as to have a product metric near the boundary (see [L 2] page 228). The higher Atiyah-Patodi-Singer index formula conjectured in [L 2] was proved in [LP 1, page 97] and a precise form of Lott's remark is now stated in Section 15 there. Fundamental in both these applications of the higher ρ-invariant is the fact that Lott's biform is concentrated in the trivial conjugacy class. We shall now use once again this property in order to prove the following

Proposition 3.1. *Let Γ be virtually nilpotent. The higher ρ-invariant defines a group homomorphism*

$$\widetilde{\rho} : \mathrm{Pos}^{\mathrm{spin}}_n(B\Gamma) \to \overline{H}^{\mathrm{nt}}_*(\mathcal{B}^\infty_\Gamma) \subset \overline{H}_*(\mathcal{B}^\infty_\Gamma) \tag{3.2}$$

Moreover the following diagram is commutative:

$$\begin{array}{ccc} R^{\mathrm{spin}}_{n+1}(B\Gamma) & \xrightarrow{\partial} & \mathrm{Pos}^{\mathrm{spin}}_n(B\Gamma) \\ \Big\downarrow {}^{b}\alpha_{n+1} & & \Big\downarrow -\frac{1}{2}\widetilde{\rho} \\ \overline{H}_*(\mathcal{B}^\infty_\Gamma) & \xrightarrow{\pi^{\mathrm{nt}}} & \overline{H}^{\mathrm{nt}}_*(\mathcal{B}^\infty_\Gamma) \end{array}$$

Proof. We need to prove that if (N,s,f,g) is bordant to (N',s',f',g') in $\mathrm{Pos}^{\mathrm{spin}}_n(B\Gamma)$ then $\widetilde{\rho}(N,s,f,g) = \widetilde{\rho}(N',s',f',g')$. Let (W,S,F,H) be a bordism between (N,s,f,g) and (N',s',f',g'); then using the fact that H is of PSC and the index formula of [LP 1, page 97][LP 3] we obtain

$$0 = \int_W \widehat{A}(W,\nabla^W) \wedge \omega_{F^*(E\Gamma)} - \frac{1}{2}\widetilde{\eta}(N,s,f,g) + \frac{1}{2}\widetilde{\eta}(N',s',f',g') \; ;$$

since the first summand on the right hand side is concentrated on the trivial conjugacy class of Γ (see [L1] page 441) we obtain immediately the equality $0 = \widetilde{\rho}(N,s,f,g) - \widetilde{\rho}(N',s',f',g')$. The commutativity of the diagram is an easy consequence of Proposition 2.5 and of the definition of $\widetilde{\rho}$.

Remark. Let $\Xi = [(M,s,f,g_{\partial M})] \in R^{\mathrm{spin}}_{n+1}(B\Gamma)$, with Γ virtually nilpotent. If $\widetilde{\rho}(\partial\Xi) \neq 0$ in $\overline{\mathrm{H}}_*(\mathcal{B}^\infty_\Gamma)$ then $g_{\partial M}$ does not extend to a PSC metric on M. As already remarked (see [L 2] and then [LP 1, Sect 15]) this follows from the definition of $\widetilde{\rho}$ and the higher APS index formula. Of course, this is now also a consequence of Stolz's Theorem 1.2 1], as $\Xi \neq 0$.

For an arbitrary finitely generated group the existence of the splitting (3.1) seems to be unknown (and possibly false). However we can still consider interesting bordism invariants defined in terms of Lott's higher eta invariants. So let Γ be an arbitrary finitely generated group and let $\mathcal{B}^\infty_\Gamma$ be the Connes-Moscovici algebra. Consider the higher eta invariant $\widetilde{\eta}(N,s,f,g)$. Recall (see [L 2, page 209]) that given $x \in \Gamma$, $x \neq e$, it is possible to define a cochain complex C^k_x with $H^k_x \cong H^k(N_{<x>})$; here $N_{<x>}$ is the quotient $Z_x \backslash \{x\}$, with $\{x\}$ equal to the cyclic group generated by x and Z_x the centralizer of x in Γ. The cyclic cohomology of $\mathbb{C}\Gamma$ can be expressed in terms of $H^k(N_{<x>})$ ([Bur]). Given $\tau \in ZC^k_x \subset C^k_x$ there is a cyclic cocycle $\mathcal{S}_\tau \in ZC^k(\mathbb{C}\Gamma)$ defined as

$$\mathcal{S}_\tau(\gamma_0,\dots,\gamma_k) = \begin{cases} 0 & \text{if } \gamma_0\cdots\gamma_k \notin <x> \\ \tau(g, g\gamma_0,\dots,g\gamma_0\dots\gamma_{k-1}) & \text{if } \gamma_0\cdots\gamma_k = g^{-1}xg. \end{cases} \tag{3.3}$$

Let us assume that $\mathcal{S}_\tau$ *extends* to a cyclic cocycle of $\mathcal{B}^\infty_\Gamma$, then we consider

$$\widetilde{\rho}(\mathcal{S}_\tau)[N,s,f,g] := \langle \widetilde{\eta}[N,s,f,g]; \mathcal{S}_\tau\rangle \in \mathbb{C} \tag{3.4}$$

with $\langle\cdot;\cdot\rangle$ the pairing between the noncommutative differential forms and the cyclic cocycles of $\mathcal{B}^\infty_\Gamma$. It is easy to show, proceeding as in Proposition 3.1, that $\widetilde{\rho}(\mathcal{S}_\tau)$ is well defined as a map $\mathrm{Pos}^{\mathrm{spin}}_n(B\Gamma) \to \mathbb{C}$. More generally, we define in the next Proposition a kind of higher $\rho-$invariant in the case of groups of the form $\Gamma \times \Delta$.

Proposition 3.2. *Let $\mathcal{S}_\tau$ be a cyclic cocycle of $\mathbb{C}\Gamma$ constructed as in (3.3). Assume that $\mathcal{S}_\tau$ extends to a cyclic cocycle of $\mathcal{B}^\infty_\Gamma$; then the application $\widetilde{\rho}(\mathcal{S}_\tau) : \mathrm{Pos}^{\mathrm{spin}}_n(B(\Gamma\times\Delta)) \to \overline{\Omega}_*(\mathcal{B}^\infty_\Delta)$ given by*

$$\mathrm{Pos}^{\mathrm{spin}}_n(B(\Gamma\times\Delta)) \ni [N,s,f,g] \to \langle \widetilde{\eta}[N,s,f,g]; \mathcal{S}_\tau\rangle \in \overline{\Omega}_*(\mathcal{B}^\infty_\Delta)$$

is well defined.

Proof. One observes that

$$\overline{\Omega}_*(\mathcal{B}^\infty_{\Gamma\times\Delta}) \simeq \overline{\Omega}_*(\mathcal{B}^\infty_\Gamma)\widehat{\otimes}\overline{\Omega}_*(\mathcal{B}^\infty_\Delta).$$

The Proposition is then proved once again using the higher APS index theorem of [LP 1,3], exactly as in the proof of Proposition 3.1.

4. Distinguishing metrics of PSC.

We start with a Proposition that is directly inspired by the work of Botvinnik and Gilkey [Bo-Gi 1].

Proposition 4.1. *Let N be a connected spin closed manifold of dimension $n \geq 5$ admitting one metric of PSC. Assume that $\pi_1(N) = \Gamma\times\Delta$ and that there exists $[(N_1,s_1,f_1,g_1)] \in \mathrm{Pos}^{\mathrm{spin}}_n(B\pi_1(N))$ and a cyclic cocycle $\mathcal{S}_\tau$ as in (3.3) such that $\langle\widetilde{\eta}([(N_1,s_1,f_1,g_1)]); \mathcal{S}_\tau\rangle \neq 0$ and such that $[N_1,f_1,s_1]$ is a torsion element* *in $\Omega_n^{\rm spin}(B\pi_1(N))$. Then the space of PSC metrics on N has an infinite number of pairwise non-bordant metrics.*

Proof. By assumption there exists a $k \in \mathbb{N}^*$ such that $k[(N_1, s_1, f_1)] = 0$ in $\Omega_n^{\rm spin}(B\Gamma)$. Let $f : N \to B\pi_1(N) \equiv B\Gamma$ be the classifying map of the universal cover; then for each $j \in \mathbb{N}$, $j \geq 2$

$$[(N, s, f)] = [(N, s, f)] + jk[(N_1, s_1, f_1)]\,. \tag{4.1}$$

By assumption there exists a metric of PSC on N, g. Thus $[(N, s, f, g)] + jk[(N_1, s_1, f_1, g_1)]$ is well defined in $\mathrm{Pos}_n^{\rm spin}(B\pi_1(N))$. By a well known result of Miyazaki and Rosenberg (see, for example, Lemma 3.1 in [Bo-Gi]) we infer that $\forall j \in \mathbb{N} \setminus \{0, 1\}$, there exists a metric g_j of PSC such that:

$$[(N, s, f, g_j)] = [(N, s, f, g)] + jk[(N_1, s_1, f_1, g_1)] \quad \text{in} \quad \mathrm{Pos}_n^{\rm spin}(B\pi_1(N))\,. \tag{4.2}$$

Here it is crucial to work in the spin category, see [St 1, page 629]. Using the assumption

$$\langle \widetilde{\eta}([(N_1, s_1, f_1, g_1)]); \mathcal{S}_\tau \rangle \neq 0$$

and Proposition 3.2 we immediately obtain

$$\langle \widetilde{\eta}([(N, s, f, g_j)]); \mathcal{S}_\tau \rangle \neq \langle \widetilde{\eta}([(N, s, f, g_\ell)]); \mathcal{S}_\tau \rangle \quad \text{if} \quad \ell \neq j\,.$$

This implies that g_j and g_ℓ, for $j \neq \ell$, are not bordant (and thus not concordant). The proof of the Proposition is complete.

We have seen in Proposition 1.6 that $|R_{2n}^{\rm spin}(B\Gamma)| = \infty$ if $n \geq 4$ and $\Gamma = \Gamma' \rtimes F$ with F finite and $r_{2n-1}(F) > 0$. The next Proposition sharpens this result; moreover, its proof allows us to distinguish non-bordant metrics through suitable eta-invariants.

Proposition 4.2. *Let Γ be a finitely generated group such that there exists a split exact sequence*

$$1 \to \Gamma' \to \Gamma \to F \to 1 \tag{4.3}$$

with F a finite group satisfying $r_{2n-1}(F) > 0$ with $n \geq 3$. Let N be a connected closed spin manifold of dimension $2n - 1$ admitting one metric of PSC and such that $\pi_1(N) = \Gamma$. Then

1] N admits an infinite number of pairwise non-bordant metrics.

2] The moduli space $\mathcal{M}(N) = \mathcal{R}^+(N)/\mathrm{Diff}(N)$ has an infinite number of arc-components:

$$|\pi_0(\mathcal{M}(N))| = \infty\,. \tag{4.4}$$

Proof. 1] This is in fact a direct consequence of [Bo-Gi 1] and of some functorial arguments. Recall that in [Bo-Gi 1] is proved the existence of an element $[N_1, s_1, f_1, g_1] \in \mathrm{Pos}_{2n-1}^{\rm spin}(BF)$ and a virtual representation $\phi \in R_0(F)$ such that the twisted eta invariant (i.e. the eta invariant of the Dirac operator twisted by the flat bundle associated to ϕ) is different from zero: $\eta[N_1, s_1, f_1, g_1](\phi) \neq 0$. Since $\Omega_{2n-1}^{\rm spin}(BF)$ is torsion there exists a $k \in \mathbb{N}^*$, such that $k[N_1, s_1, f_1] = 0$. We let i be the splitting map $i : F \to \Gamma$ and p be the surjection $\Gamma \to F$ in the exact sequence (4.3). We have $p \circ i = \mathrm{Id}$ by assumption and thus $Bp \circ Bi = \mathrm{Id}$. Notice that $k[N_1, s_1, Bi \circ f_1] = 0$. By applying p_* to (4.2) above with $[(N_1, s_1, Bi \circ f_1, g_1)]$ replacing $[(N_1, s_1, f_1, g_1)]$ and using $Bp \circ Bi = \mathrm{Id}$, we obtain a sequence of PSC metrics $\{g_j\}$, $j \geq 2$ such that :

$$[(N, s, Bp \circ f, g_j)] = [(N, s, Bp \circ f, g)] + jk[(N_1, s_1, f_1, g_1)] \quad \text{in} \quad \mathrm{Pos}_n^{\rm spin}(BF).$$

Thus

$$\eta(N,s,Bp\circ f,g_j)(\phi)\neq\eta(N,s,Bp\circ f,g_\ell)(\phi)\quad\text{if}\quad \ell\neq j\,. \tag{4.5}$$

This implies that g_j and g_ℓ are not bordant (and thus, in particular, belong to different arc-components in $\mathcal{R}^+(N)$). (To see that the metrics g_j and g_ℓ are not bordant remark that if they were then, by definition, $[N,s,f,g_j]=[N,s,f,g_\ell]$, which would imply, by functoriality, that $[(N,s,Bp\circ f,g_j)]=[(N,s,Bp\circ f,g_\ell)]$ which, in turn, would contradict (4.5).)
2] This proceeds as in [Bo-Gi 1, page 516]. More precisely we consider the F-covering of N obtained by taking the quotient of $\widetilde{N}$ by the action of Γ'. We denote by $\widetilde{N}^F$ this Galois covering. This F-covering is classified by the map $Bp\circ f$ where, as in the proof of Proposition 4.1, f is the classifying map of the universal covering of N. Let now $H\subset \mathrm{Diff}(N)$ be the finite-index subgroup of $\mathrm{Diff}(N)$ consisting of all orientation preserving and spin preserving diffeomorphisms of N inducing the identity on F. If $\chi\in H$ then $(Bp\circ f\circ\chi)^*(EF)\cong(Bp\circ f)^*(EF)$ and this implies $\eta(N,s,Bp\circ f,g_j)(\phi)=\eta(N,s,Bp\circ f,\chi^*(g_j))(\phi)$. This completes the proof of the Proposition.

Proposition 4.3. *Let N be a connected closed spin manifold of dimension n such that $\pi_1(N)=\Gamma\times\Delta$ with $\Gamma=\Gamma'\rtimes F$ where F is finite and $r_m(F)>0$ for some odd integer m, $5\leq m<n$. We assume that $\mathcal{R}^+(N)\neq\emptyset$ and that there exists $\mathcal{X}=[X,s,f]\in\Omega^{\mathrm{spin}}_{n-m}(B\Delta)/\Omega^{\mathrm{spin},+}_{n-m}(B\Delta)$ such that $\int_X\widehat{A}(X,\nabla^X)\wedge\omega_{f^*(E\Delta)}\neq 0$ in $\overline{\Omega}_*(\mathcal{B}^\infty_\Delta)$. Then there exists an infinite number of pairwise non-bordant PSC metrics on N. In particular $|\widetilde{\pi}_0(\mathcal{R}^+(N))|=\infty$.*

Remark. This proposition is particularly interesting in the *even* dimensional case, where the ordinary eta invariant is zero. Notice that in the case $n=4k+2$ there are analogous results for *non-orientable* $^-$Pin manifolds with finite fundamental group admitting a surjective morphism onto $\mathbb{Z}/2\mathbb{Z}$. See [Bo-Gi 2]. The assumption that the fundamental group is finite seems to be crucial in their proof (a delicate point being, for instance, the Miyazaki-Rosenberg lemma in the non-spin category).

Proof. We will prove the Proposition when $\Gamma=F$ and leave the extension to the semidirect product case to the reader (the steps are the same as in the proof of Proposition 4.2 1]). Using again the results of [Bo-Gi 1] we know that there exists a bordism class $[L,s_L,f_L,g_L]\in\mathrm{Pos}_m(BF)$ and a virtual representation $\phi\in R_0(F)$ such that

$$\eta[L,s_L,f_L,g_L](\phi)\neq 0. \tag{4.6}$$

We also know that $BF\times B\Delta=B(F\times\Delta)\equiv B(\pi_1(N))$. Let X and $f:X\to B\Delta$ be as in the hypothesis; let s_X be a spin structure on X and g_X a riemannian metric. We consider $[L\times X,s_L\times s_X,f_L\times f,g_L\times\frac{1}{\epsilon}g_X]\in\mathrm{Pos}_n(B\pi_1(N))$ with $\epsilon>0$ small enough so that $g_L\times\frac{1}{\epsilon}g_X$ have PSC. Then Lemma 2.1 shows that:

$$\widetilde{\eta}[L\times X,s_L\times s_X,f_L\times f,g_L\times\frac{1}{\epsilon}g_X](\phi)=C\cdot\eta(L,s_L,f_L,g_L)(\phi)\cdot\int_X\widehat{A}(X,\nabla^X)\wedge\omega_{f^*(E\Delta)} \tag{4.7}$$

where C is a nonzero constant. Recall from [APS 2], [L 2, page 195] that

$$\eta(L,s_L,f_L,g_L)(\phi)\;=\;\sum_{\gamma\in F}\frac{1}{N}\chi_\phi(\gamma)\eta(\gamma) \tag{4.8}$$

where N is the common dimension of the two representations ϕ_0,ϕ_1 of F such that $\phi=\phi_1-\phi_0$, and the higher eta invariant of the Dirac operator of $[L,s_L,f_L,g_L]$ associated to $\mathcal{V}_{f_L}$ (see (2.1)) is given by:

$$\widetilde{\eta}([L,s_L,f_L,g_L])=\sum_{\gamma\in F}\eta(\gamma)\gamma\in\mathbb{C}F.$$

Recall that for any non trivial conjugacy class $< x >$ of F, one defines a cyclic cocycle $\mathcal{T}_{<x>}$ on $\mathbb{C}F$ of the type (3.3) by the following formula:

$$\mathcal{T}_{<x>}(\sum_{\gamma \in F} a_\gamma \gamma) = \sum_{\gamma \in <x>} a_\gamma$$

From Equations (4.6) and (4.8) one easily deduces that there exists a non trivial conjugacy class $< x >$ of F such that: $\mathcal{T}_{<x>}(\sum_{\gamma \in F} \eta(\gamma)\gamma) \neq 0$. Thus from equation (4.7) and the fact that $\int_X \widehat{A}(X, \nabla^X) \wedge \omega_{f^*(E\Delta)} \neq 0$ one obtains:

$$\langle \widetilde{\eta}([L \times X, s_L \times s_X, f_L \times f, g_L \times \frac{1}{\epsilon} g_X]); \mathcal{T}_{<x>} \rangle \neq 0.$$

We now remark that $[L, s_L, f_L]$ is a torsion element in $\Omega_m^{\rm spin}(BF)$; thus $[L \times X, s_L \times s_X, f_L \times f_X]$ is *also* a torsion element in $\Omega_n^{\rm spin}(B(F \times \Delta))$. Since all the hypothesis of Proposition 4.1 are fullfilled for $\Gamma \times \Delta = F \times \Delta$, the proposition follows.

Recall, see [St 2], that one can define a natural pairing

$$\mu : \Omega_\ell^{\rm spin}(B\Delta)/\Omega_\ell^{{\rm spin},+}(B\Delta) \otimes R_m^{\rm spin}(B\Gamma) \longrightarrow R_{\ell+m}^{\rm spin}(B(\Delta \times \Gamma))$$

which associates to $\mathcal{X} = [X, s_X, f_X] \in \Omega_\ell^{\rm spin}(B\Delta)/\Omega_\ell^{{\rm spin},+}(B\Delta)$ and $\Xi = [M, s_M, f_M, g_{\partial M}] \in R_m^{\rm spin}(B\Gamma)$ the element

$$\mu(\mathcal{X}, \Xi) := [X \times M, s_X \times s_M, f_X \times f_M, \frac{1}{\epsilon} g_X \times g_{\partial M}] \in R_{\ell+m}^{\rm spin}(B(\Delta \times \Gamma))$$

with g_X an arbitrary Riemannian metric and ϵ small enough so that $\frac{1}{\epsilon} g_X \times g_{\partial M}$ is of PSC on $\partial(X \times M)$.

Proposition 4.4. *Let Γ be a finitely generated group. We make the following two assumptions:*

1] *There exists a sequence $(\Xi_k)_{k \in \mathbb{N}}$ of elements Ξ_k in $R_m^{\rm spin}(B\Gamma)$ such that ${}^b\alpha_m(\Xi_k) \neq {}^b\alpha_m(\Xi_j)$ for $k \neq j$.*

2] *There exists $\mathcal{X} = [X, s, f] \in \Omega_\ell^{\rm spin}(B\Delta)/\Omega_\ell^{{\rm spin},+}(B\Delta)$ and a cyclic cocycle $\mathcal{T}$ of $\mathcal{B}_\Delta^\infty$ with the property that $\langle \int_X \widehat{A}(X, \nabla^X) \wedge \omega_{f^*(E\Delta)}\, ; \mathcal{T} \rangle \neq 0$. Then*

$$ {}^b\alpha_{m+\ell}(\mu(\mathcal{X}, \Xi_k)) \neq {}^b\alpha_{m+\ell}(\mu(\mathcal{X}, \Xi_j)), \quad \text{if} \quad k \neq j$$

Proof. For each $k \in \mathbb{N}$, let $[M_k, s_k, f_k, g_k]$ be a representative of Ξ_k. Consider

$$\mu(\mathcal{X}, \Xi_k) = [X \times M_k, s_X \times s_k, f_X \times f_k, \frac{1}{\epsilon(k)} g_X \times g_k] \in R_{m+\ell}^{\rm spin}(B(\Delta \times \Gamma)$$

where $\epsilon(k) > 0$ is small enough so that $\frac{1}{\epsilon(k)} g_X \times g_k$ is of PSC on the boundary of $X \times M_k$. Then using Proposition 2.5 and the arguments of the proof of Lemma 2.1 it is easy to show that:

$$ {}^b\alpha(\mu(\mathcal{X}, \Xi_k)) = (\int_X \widehat{A}(X, \nabla^X) \wedge \omega_{f^*(E\Delta)}) \widehat{\otimes}\, {}^b\alpha(\Xi_k)\,, \ \text{modulo}\ \overline{\Omega}_*(\mathcal{B}_\Delta^\infty) \widehat{\otimes} d\overline{\Omega}_*(\mathcal{B}_\Gamma^\infty) + d\overline{\Omega}_*(\mathcal{B}_\Delta^\infty) \widehat{\otimes} \overline{\Omega}_*(\mathcal{B}_\Gamma^\infty). \tag{4.9}$$

Moreover, we observe that $\mathcal{T}$ induces, via a natural pairing, a map:

$$M(\mathcal{T}) : \overline{\Omega}_*(\mathcal{B}_\Delta^\infty) \widehat{\otimes}\, \overline{\Omega}_*(\mathcal{B}_\Gamma^\infty) \to \overline{\Omega}_*(\mathcal{B}_\Gamma^\infty)$$

By using (4.9), we see that $M(\mathcal{T})({}^b\alpha(\mu(\mathcal{X}, \Xi_k))$ is well defined as an element of $\overline{H}_*(\mathcal{B}_\Gamma^\infty)$ and equal to

$$\langle \int_X \widehat{A}(X, \nabla^X) \wedge \omega_{f^*(E\Delta)}\, ; \mathcal{T} \rangle \cdot {}^b\alpha(\Xi_k).$$

Since, by assumption, $\langle \int_X \widehat{A}(X,\nabla^X)\wedge\omega_{f^*(E\Delta)}\, ;\, \mathcal{T}\rangle$ is a non zero complex number and the ${}^b\alpha(\Xi_k)$ are pairwise distinct, we see that the ${}^b\alpha(\mu(\mathcal{X},\Xi_k))$ are also pairwise distinct for $k\in\mathbb{N}$. Proposition 4.4 is proved.

Remark. Suppose that Y is a closed spin manifold and that g_0 and g_1 are two *non-concordant* metrics of positive scalar curvature. We assume that we can distinguish these two metrics through a higher Gromov-Lawson index; namely, we assume that there exists $f_Y : Y \to B\Gamma$ and a spin structure s_Y such that

$$\mathbf{hi}(Y, s_Y, f_Y; g_0, g_1) \neq 0 \ \text{ in } \ \overline{\mathrm{H}}_*(\mathcal{B}_\Gamma^\infty).$$

If $[X,s,f]$ is as in the above Proposition, then for any metric g_X on X, the PSC metrics $g_0 \times \frac{1}{\epsilon} g_X$ and $g_1 \times \frac{1}{\epsilon} g_X$ on $Y\times X$ *remain non-concordant.* In fact proceeding as in the previous Proposition one checks that

$$\mathbf{hi}(Y\times X, s_Y\times s, f_Y\times f\, ;\, g_0\times \frac{1}{\epsilon} g_X\ ,\ g_1 \times \frac{1}{\epsilon} g_X) \neq 0.$$

The above Proposition and this Remark give a partial answer to a question raised in [We, page 318].

Suppose now that Δ is a Gromov hyperbolic or virtually nilpotent group. Then we can give a simple sufficient condition for the existence of a manifold X as in the statement of the above two Propositions. We make a preliminary remark. Let X be a closed spin manifold and $f: X\to B\Delta$ a continuous map. By [L 1] we know that

$$< \int_X \widehat{A}(X,\nabla^X)\wedge \omega_{f^*(E\Delta)}\, ,\, \tau_c >=< [\widehat{A}(X,\nabla^X)]\cup f^*[c]\, ;\, [X] > \quad \forall [c]\in H^*(\Delta,\mathbb{Q}) \qquad (4.10).$$

This means that X and $f: X\to B\Delta$ are such that $\int_X \widehat{A}(X,\nabla^X)\wedge\omega_{f^*(E\Delta)}\neq 0$ provided there exists at least *one* higher $\widehat{A}$-genus of X with respect to f which is different from 0. The next Lemma gives a sufficient condition for the existence of such a manifold.

Lemma 4.5. *Let $n,m\in\mathbb{N}$. We assume that Δ is Gromov hyperbolic or virtually nilpotent and that there $\exists\ k\in\mathbb{N}$, and $[c]\neq 0\ \in\ H^{(n-m)-4k}(\Delta;\mathbb{C})$. Then there exists a closed spin manifold of dimension $n-m$, X, and a continuous map $f: X\to B\Delta$ such that $\int_X \widehat{A}(X,\nabla^X)\wedge\omega_{f^*(E\Delta)}\neq 0$.*

Proof. We first recall a fundamental result in bordism theory. For any integer $k\geq 0$ we denote by $\{i_1,i_2,\dots,i_r\}$ a partition of k (see [Mi-Sta]). One can define a natural map χ :

$$\Omega_n^{\rm spin}(B\Delta)\otimes_{\mathbb{Z}}\mathbb{Q}\longrightarrow \oplus_{k\geq 0}\oplus_{\{i_1,i_2,\dots,i_r\}} H_{n-4(i_1+\cdots+i_r)}(B\Delta,\mathbb{Q})$$

by

$$\chi : \,[N,s,f]\longrightarrow \oplus_{k\geq 0}\oplus_{\{i_1,i_2,\dots,i_r\}} f_*\Big(PD(p_{i_1}(N)\wedge\cdots p_{i_r}(N))\Big) \qquad (4.11)$$

with PD denoting Poincaré duality and $\{p_j\,(N)\}$ denoting the Pontrjagin classes of N. It is shown in [Co-Fl] that this map is an isomorphism: that the two spaces are isomorphic follows from Theorem (14.2) page 41 in [Co-Fl]. That the map χ implements this isomorphism follows from Theorem (17.4) page 48 in [Co-Fl]. Notice that in [Co-Fl] these results are established for the oriented bordism groups; since we are tensoring with the rationals the same results hold for the spin bordism groups (see [Sto, Chapter XI, page 336]). Recall now that the $\widehat{A}-$genus may be written as:

$$\widehat{A} = 1+\widehat{A}_1(p_1)+\widehat{A}_2(p_1,p_2)\cdots+\widehat{A}_k(p_1,p_2,\dots,p_k)+\cdots$$

Let us fix a partition $\{i_1, i_2, \dots, i_r\}$ of k such that the coefficient of $p_{i_1} \cdots p_{i_r}$ in $\widehat{A}_k(p_1, p_2, \dots, p_k)$ is not zero. Let us fix $\delta \in H_{n-m-4k}(B\Delta, \mathbb{Q})$ such that $< \delta\,; [c] > \neq 0$. Since the above map (4.11) is an isomorphism with $n-m$ in place of n, there exists

$$\sum_{j=1}^{N} d_j [M_j, s_j, f_j] \in \Omega^{\rm spin}_{n-m}(B\Delta) \otimes_{\mathbb{Z}} \mathbb{Q}$$

where the $d_j \in \mathbb{Q}$, such that $\sum_{j=1}^{N} d_j {f_j}_* (PD(p_{i_1}(M_j) \wedge \dots p_{i_r}(M_j))) = \delta$ and such that for any other partitition $\{j_1, j_2, \dots, j_\ell\}$ of k:

$$\sum_{j=1}^{N} d_j {f_j}_* (PD(p_{j_1}(M_j) \wedge \dots p_{j_\ell}(M_j))) = 0.$$

This implies that $< \sum_{j=1}^{N} d_j {f_j}_* (PD(\widehat{A}_k(M_j)))\,; [c] > \neq 0$ and so there exists $j \in \{1, \dots, N\}$ such that

$$< {f_j}_* (PD(\widehat{A}_k(M_j)))\,; [c] > = < \widehat{A}(M_j) \cup {f_j}^*[c]\,; [M_j] > \neq 0.$$

Using (4.10), one then gets immediately the Lemma.

We can now prove the two statements given in the Introduction.

Proof of Theorem 0.1. It follows directly from Proposition 4.3 and Lemma 4.5.

Proof of Corollary 0.2. It is well known [Ros, page 343] that one can find (after a suitable surgery process) a closed connected spin manifold Y of dimension $n-2 \geq 5$ such that $\pi_1(Y) = \Gamma \times \Delta$. We observe that the spin manifold $N := Y \times S^2$ has dimension n, admits a PSC metric g (since S^2 does) and has $\Gamma \times \Delta$ as fundamental group; we denote by $f : N \to B(\Gamma \times \Delta)$ the classifying map of the universal covering of N and by $p : \Gamma \to F$ the surjective group homomorphism associated with the decomposition: $\Gamma = \Gamma' \rtimes F$. Then, Lemma 4.5 and the proofs of Propositions 4.3 and 4.1 show that $N := Y \times S^2$ admits an infinite number of non-bordant and thus non-concordant metrics g_k ($k \in \mathbb{N}$) of PSC and that there exists a cyclic cocycle $\mathcal{T}_{<x>}$ of $\mathbb{C}F$ such that:

$$\langle \widetilde{\eta}([N, s, Bp \circ f, g_j]); \mathcal{T}_{<x>} \rangle \neq \langle \widetilde{\eta}([N, s, Bp \circ f, g_k]); \mathcal{T}_{<x>} \rangle \quad \text{for} \quad j \neq k\,.$$

Applying Stolz's Theorem 1.2 the Corollary follows. Alternatively, using Proposition 2.5, the fact that Lott's biform $\omega_{f^*(E(F \times \Delta))}$ is concentrated in the trivial conjugacy class of $F \times \Delta$ and the fact that the conjugacy class $< x >$ of F in $\mathcal{T}_{<x>}$ is non trivial, one obtains (as in the proof of Proposition 3.1) that the

$${}^b\alpha([N \times [0,1], s \times s_{[0,1]}, Bp \circ f, g_0 \coprod g_k])$$

are pairwise distinct as k runs over $\mathbb{N}$. This implies that the

$$\Theta_k = [N \times [0,1], s \times s_{[0,1]}, Bp \circ f, g_0 \coprod g_k]$$

are pairwise distincts elements of $R^{\rm spin}_{n+1}(B(F \times \Delta))$ as k runs over $\mathbb{N}$. Since, using the notation in the statement of Lemma 1.3, $\Theta_k = R_{n+1}(p)(\Xi_k)$ with

$$\Xi_k = [N \times [0,1], s \times s_{[0,1]}, f, g_0 \coprod g_k] \in R^{\rm spin}_{n+1}(B(\Gamma \times \Delta))\,,$$

the Corollary is proved (but without using Stolz's theorem) as we have found an infinite number of pairwise distinct elements in $R^{\rm spin}_{n+1}(B(\Gamma \times \Delta))$.

The following result is an immediate consequence of Proposition 4.4 and of Lemma 4.5

Proposition 4.6. *Let Γ be a finitely generated group and $m \in \mathbb{N}$. Assume that there exists a sequence $(\Xi_k)_{k\in\mathbb{N}}$ of elements Ξ_k in $R^{\rm spin}_m(B\Gamma)$ such that ${}^b\alpha(\Xi_k) \neq {}^b\alpha(\Xi_j)$ for $k \neq j$. Then, for Δ a Gromov hyperbolic or virtually nilpotent group for which there exists $(i,\ell) \in \mathbb{N}\times\mathbb{N}$ such that $H^{\ell-4i}(\Delta,\mathbb{Q}) \neq 0$ one has*

$$|R^{\rm spin}_{m+\ell}(B(\Gamma\times\Delta)| = \infty\,.$$

Remark. One can use the above Proposition and the product manifolds employed in the proof of Proposition 4.3 to give yet another proof of Corollary 0.2.

Remark. If we apply Propositions 4.6 and 4.4 with $\Gamma = 1$ and $m = 4p$ we obtain immediately from Lemma 1.5 the following result

$$|R^{\rm spin}_{4p+\ell}(B\Delta)| = \infty\,, \tag{4.12}$$

for a group Δ as in the statement of Proposition 4.6.

One can in fact prove a more general result than (4.12) by working directly at the K-theory level:

Proposition 4.7. *Let $\ell, p \in \mathbb{N}$, $p \geq 2$. Let Δ be a finitely generated group satisfying the Strong Novikov conjecture and such that $KO_\ell(B\Delta)\otimes\mathbb{Q} \neq 0$. Then*

$$|R^{\rm spin}_{4p+\ell}(B\Delta)| = +\infty\,.$$

Remark. Let Δ be a virtually nilpotent or a Gromov hyperbolic group for which there exists $i \in \mathbb{N}$ such that $H^{\ell-4i}(\Delta,\mathbb{Q}) \neq 0$. Thanks to Yu's result [Yu], we know that Δ satisfies the Strong Novikov conjecture. Then, Lemma 4.5 implies immediatly that $KO_\ell(B\Delta)\otimes\mathbb{Q} \neq 0$ and we recover (4.12).

Proof of Proposition 4.7. Since the group Δ satisfies the Strong Nonikov conjecture, the assembly map $A : KO_\ell(B\Delta) \to KO_\ell(\,C^*_r(\Delta)_{\mathbb{R}}\,)$ is rationally injective. Moreover, Kreck and Stolz ([Kr-St 2]) have constructed an epimorphism $p : \Omega^{\rm spin}_\ell(B\Delta) \to KO_\ell(B\Delta)$. So, by assumption, there exists a class $\mathcal{Y} \in \Omega^{\rm spin}_\ell(B\Delta)$ such that $A\circ p(\mathcal{Y})$ is not zero in $KO_\ell(\,C^*_r(\Delta)_{\mathbb{R}}\,)\otimes\mathbb{Q}$. We now consider the index map $\Theta : R^{\rm spin}_n(B\Gamma) \to KO_n(\,C^*_r(\Gamma)_{\mathbb{R}}\,)$ defined in [St 1,2]. The proof of Proposition 1.5 shows that there exists a sequence $(\mathcal{X}_k)_{k\in\mathbb{N}}$ of elements of $R^{\rm spin}_{4p}(B\{1\})$ such that the $\Theta(\mathcal{X}_k)$ are pairwise distinct in $KO_{4p}(\,C^*_r(\{1\})_{\mathbb{R}}\,) \simeq \mathbb{Z}$. We consider once again the natural product map $\mu : R^{\rm spin}_{4p}(B\{1\}) \times \Omega^{\rm spin}_\ell(B\Delta) \to R^{\rm spin}_{4p+\ell}(B\Delta)$. Since $\Theta(\mathcal{Y})$ is not zero in $KO_\ell(\,C^*_r(\Delta)_{\mathbb{R}}\,)\otimes\mathbb{Q}$, and since one can check directly that

$$\Theta(\mu(\mathcal{X}_k\,,\,\mathcal{Y}\,)) = \Theta(\mathcal{X}_k)\Theta(\mathcal{Y}), \tag{4.13}$$

we see that $\Theta(\mu(\mathcal{X}_k\,,\,\mathcal{Y}\,))$ are pairwise distinct in $KO_{4p+\ell}(\,C^*_r(\Delta)_{\mathbb{R}}\,)\otimes\mathbb{Q}$ as k runs over $\mathbb{N}$. The Proposition is therefore proved.

Remark. The proof of Proposition 4.7 rests on the product formula (4.13). This is the same idea used in the proof of Proposition 4.4, see formula (4.9), but at the K-theory level and for $\Gamma = \{1\}$; in the general case addressed by Proposition 4.4, i.e. when both groups are non-trivial, there is a beautiful K-theoretic formula, due to Stolz (see [St 2, Theorem 1.3]), which reads as formula (4.13) but with the Kasparov product appearing on the right-hand side. One might then try to find sufficient conditions on Γ and Δ ensuring that results such as Proposition 4.6 hold. Obviously, this is a purely K-theoretic problem. As the main goal of this paper was to show how to use Lott's higher eta invariant in problems of PSC, we shall not pursue this line of research here.

References.

[APS 1,2,3] Atiyah M.F., Patodi V. and Singer I 1975. Spectral Asymmetry and Riemannian Geometry I, *Math. Proc. Cambridge Phil. Soc.*, **77**, page 43; II **78**, page 405; III **79**, page 71.

[AS V] Atiyah M.F. and Singer I. 1971. The index of elliptic operators V, *Ann. Math*, **93**, page 139.

[Bo-Gi 1] Botvinnik B.and Gilkey P. 1995. The eta invariant and metrics of positive scalar curvature, *Math. Ann.*, **302**, page 507.

[Bo-Gi 2] Botvinnik B.and Gilkey P. 1996. Metrics of positive scalar curvature on spherical space forms, *Canadian Math. Journal*, **48**, page 64.

[Bu 1] Bunke U. 1995. On the gluing problem for the η-invariant. *J. Diff. Geom.* **41**, page 397.

[Bu 2] Bunke U. 1995. A K-theoretic relative index theorem and Callias-type Dirac operators. *Math. Ann.*, **303**, page 241.

[Bur] Burghelea, D. 1985. The cyclic homology of the group rings, *Comment Math. Helv.* , **60**, page 354.

[Co-Fl] Conner P. and Floyd E. 1964. *Differentiable periodic maps.* Springer-Verlag.

[C-M] Connes A. and Moscovici H. 1990. Cyclic Cohomology, the Novikov Conjecture and Hyperbolic Groups, *Topology* **29:3**, page 345.

[DZ 1] Dai X. and Zhang W. 1996. Splitting of the family index, *Comm. Math. Phys.*, **182**, page 303.

[DZ 2] Dai X. and Zhang W. 1998. Higher spectral flow, *Jour. Funct. Anal.*, **157**, page 432.

[G-L] Gromov M. and Lawson B. 1983. Positive scalar curvature and the Dirac operator on complete Riemannian manifolds, *Publ. Math. IHES*, No **58** page 83.

[Ha] Hajduk B. 1991. On the obstruction group to the existence of Riemannian metrics of positive scalar curvature, *Lecture Notes in Mathematics, Springer Verlag* **1481**, Proc. International Conf. on Algebraic Topology, Poznan, page 62.

[Hi] Hitchin N. 1974. Harmonic spinors, *Advances in Math.*, **14**, page 1.

[Ka] Karoubi M. 1987. Homologie cyclique et K-théorie, *Astérisque* **149**.

[Kr-St 1] Kreck M. and Stolz S. 1993. Nonconnected moduli spaces of positive sectional curvature metrics, *Journal of the American Mathematical Society*, Volume **6**, Number 4, page 825.

[Kr-St 2] Kreck M. and Stolz S. 1993. HP^2-bundles and elliptic homology, *Acta. Math.*, **171**, page 231.

[LM] Lawson B. and Michelson M-L. 1989. *Spin Geometry*, Princeton Math. Ser. **38**, Princeton University Press.

[LLP] Leichtnam E., Lott J. and Piazza P. 1999. On the homotopy invariance of higher signatures on manifolds with boundary. To appear in *J. Diff. Geom.* (see `http://www.math.lsa.umich.edu/~lott`).

[LP 1] Leichtnam E. and Piazza P. 1997. The b-pseudodifferential calculus on Galois coverings and a higher Atiyah-Patodi-Singer index theorem, *Mémoires. S. M. F.*, **68**, 121 pages.

[LP 2] Leichtnam E. and Piazza P. 1998. Spectral sections and higher Atiyah-Patodi-Singer index theory on Galois coverings, *GAFA*, **8** , page 17.

[LP 3] Leichtnam E. and Piazza P. 1999. Homotopy invariance of twisted higher signatures on manifolds with boundary. *Bull. Soc. Math. France*, **127**, page 307.

[LP 4] Leichtnam E. and Piazza P. 2000. A higher APS index theorem for the signature operator on Galois coverings. *Ann. Glob. Anal. and Geom.* **18**, page 171.

[L1] Lott J. 1992. Superconnections and higher index theory, *GAFA*, **2**, page 421.

[L2] Lott J. 1992. Higher eta invariants, *K-Theory* **6**, page 191.

[L3] Lott J. 1997. Diffeomorphisms, Analytic Torsion and Non Commutative Geometry. *Mem. of the Amer. Math. Soc*, **141**, No. 673. viii + 56 pp.

[Me] Melrose R. 1993. The Atiyah-Patodi-Singer index theorem. A. and K. Peters. Wellesley, MA.

[MP] Melrose R. and Piazza P. 1997. An index theorem for families of Dirac operators on odd dimensional manifolds with boundary, *J. Diff. Geom.*, **46**, No. 2, page 287.

[Mi-Sta] Milnor J. and Stasheff J. 1974. Characteristic classes, *Annals of Math. Studies*, Number 76, Princeton University Press.

[Ros] Rosenberg J. 1986. C^*-algebras, positive scalar curvature, and the Novikov conjecture II, *Geometric Methods in Operator Algebras, Pitman Res. Notes*, **123**, page 341.

[St 1] Stolz S. 1995. Positive Scalar Curvature Metrics. Existence and Classification Questions *Proceeding of I.C.M. Zurich 1994*, Birkhauser Verlag, Basel, Switzerland, page 626.

[St 2] Stolz S. 1999. Concordance classes of positive scalar curvature metrics. Preprint.
(See `http://www.nd.edu/ stolz/preprint.html`).

[Sto] Stong R. 1968. *Notes on cobordism theory.* Princeton University Press.

[We] Weinberger S. 1999. Higher ρ-invariants, in *Tel Aviv conference, Rothenberg Festschrift*, M. Farber, W. Lück and S. Weinberger editors, Cont. Math. **231**.

[Wu 1] Wu F. 1997. The noncommutative spectral flow. Unpublished.

[Wu 2] Wu F. 1997. The higher Γ-index for coverings of manifolds with boundaries. *Fields Inst. Commun.* **17**, *Cyclic cohomology and noncommutative geometry.* page 169.

[Yu] Yu G. 2000. The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into a Hilbert space. *Invent. Math.* **139**, No. 1, page 201.