Parameter-dependent Boundary Value Problems on Manifolds with Edges

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Abstract

As is known from Kondratyev's work, boundary value problems for elliptic operators on a manifold with conical singularities and boundary are controlled by a principal symbolic hierachy, where the conormal symbols belong to the typical new components, compared with the smooth case, with interior and boundary symbols. A similar picture may be expected on manifolds with corners when the base of the cone itself is a manifold with conical or edge singularities. This is a natural situation in a number of applications, though with essential new difficulties. We investigate here corresponding conormal symbols in terms of a calculus of holomorphic parameter-dependent edge boundary value problems on the base. We show that a certain kernel cut-off procedure generates all such holomorphic families, modulo smoothing elements, and we establish conormal symbols as an algebra as is necessary for parametrix constructions in the elliptic case.

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Contents

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Introduction

Boundary value problems on (pseudo-)manifolds with singularities (in particular, with edges and corners) can be studied in an iterative way, parallel to a successive procedure of geometrically generating cones and wedges, starting from a cone with a particularly simple base manifold X. In our case X is a compact C^∞ manifold with boundary. Some aspects of this program are still a great challenge, even for the analogous but simpler case of closed compact X. Given a class of typical differential operators (or differential boundary value problems) with a principal symbolic structure that determines ellipticity a basic question is to organize an algebra of pseudo-differential operators that contains the given operators together with the parametrices of elliptic elements. As is well-known, e.g., for the case of closed compact X, the answer very much depends on the specific context, in particular, on the nature of our manifold with singularities (e.g., whether we consider conical, edge, or "higher" singularities) and on assumptions about "regular" or "cuspidal" geometries.

In the present paper we study regular cases, where the strata of the configuration have transversal intersections; more precise conditions will be given below.

To illustrate some elements of the iterative approach we first consider the case of a closed compact manifold X. Let $L^\mu_{\mathrm{cl}}(X;\mathbb{R}^l)$ denote the space of classical parameter-dependent pseudo-differential operators of order μ on X, where the parameters $\lambda \in \mathbb{R}^l$ play the role of additional covariables in symbols $a(x,\xi,\lambda)$ of local representations, and $L^{-\infty}(X;\mathbb{R}^l) := \mathcal{S}(\mathbb{R}^l, L^{-\infty}(X))$ (with an identification $L^{-\infty}(X) \cong C^\infty(X \times X)$). More generally, we may consider spaces $L^\mu_{\mathrm{cl}}(X;U \times \mathbb{R}^l) := C^\infty(U, L^\mu_{\mathrm{cl}}(X;\mathbb{R}^l))$. A slight modification of such a definition allows us also to talk about the case $U = (\overline{\mathbb{R}}_+)^k \times U'$ for $k \in \mathbb{N}$, $U' \subseteq \mathbb{R}^{l'}$ open. Occasionally, when $l = 1$, it will be convenient to identify λ with $\Im z$ for a complex variable z varying on $\Gamma_\beta := \{z \in \mathbb{C} : \Re z = \beta\}$ for some $\beta \in \mathbb{R}$. We then write $L^\mu_{\mathrm{cl}}(X; U \times \Gamma_\beta)$ in such cases, or, more generally, $L^\mu_{\mathrm{cl}}(X; U \times \Gamma_\beta \times \mathbb{R}^{l'})$ when $z \in \Gamma_\beta$ is the first component of parameters (z, λ'), $\lambda' \in \mathbb{R}^{l'}$. In finitely many iteration steps we start from $L^\mu_{\mathrm{cl}}(X; U \times \mathbb{R}^l)$, with $U = U_1 \times \ldots \times U_m$ for open sets $U_j \subseteq \mathbb{R}^{p_j}$, and $\mathbb{R}^l = \mathbb{R}^{l_1} \times \ldots \times \mathbb{R}^{l_m}$, where we write $y = (y_1, \ldots, y_m)$, $\lambda = (\lambda_1, \ldots, \lambda_m)$. Freezing of variables $y_j = y^0_j$, $\lambda_j = \lambda^0_j$ for $j = 2 \ldots, m$ gives rise to a map

$$L^\mu_{\mathrm{cl}}(X; U \times \mathbb{R}^l) \longrightarrow L^\mu_{\mathrm{cl}}(X; U_1 \times \mathbb{R}^{l_1})$$

for every choice of $(y^0, \lambda^0) := (y^0_2, \ldots, y^0_m, \lambda^0_2, \ldots, \lambda^0_m)$. Then, $(y_1, \lambda_1) \in U_1 \times \mathbb{R}^{l_1}$ will be regarded as variables and covariables in operator-valued (i.e., $L^\mu_{\mathrm{cl}}(X)$-valued) amplitude functions in the first iteration step, while (y^0, λ^0) are treated as "sleeping" variables and covariables, activated in forthcoming iteration steps. As the first step we take the "conification" of the algebra of classical pseudo-differential operators on X, i.e., a construction of an operator algebra on an associated cone $X^\triangle := (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$ with base X. Here, close to the tip v (that corresponds to $\{0\} \times X$, collapsed to a point) we employ a splitting of variables $(r, x) \in X^\wedge := \mathbb{R}_+ \times X$. There is then a "cone algebra" of pseudo-differential operators on $X^\wedge$, consisting of a suitable sub-algebra of $\bigcup\limits_\mu L^\mu_{\mathrm{cl}}(X^\wedge)$, defined in terms of pseudo-differential operators on $\mathbb{R}_+ \ni r$, based on the Mellin transform $M_{r \to z}$ on $\mathbb{R}_+$, with amplitude functions $a(r, z) \in L^\mu_{\mathrm{cl}}(X; \mathbb{R}_+ \times \Gamma_{\frac{n+1}{2} - \gamma})$ of a specific behaviour in r near zero (and with a suitable weight $\gamma \in \mathbb{R}$; $n = \dim X$). There is no canonical choice for the cone algebra; different variants are motivated, e.g., by the index theory or by applications to asymptotic phenomena, cf. [12], [30], [17], [6]. "Higher" operator algebras as they follow by our iteration depend very much on the specific choice on the lowest singular level. Thus, in contrast to cone algebras in the beginning, where different authors essentially deal with "the same class" of Fuchs type operators, as far as it concerns the non-smoothing elements, the structures on manifols with higher geometric singularities, starting from edges, may be completely different; see, for instance, [19], [18], [21], [27], [15].

Manifolds W with edges are locally described by wedges $X^\triangle \times \Omega$ for a model cone $X^\triangle$ and an open set $\Omega \subseteq \mathbb{R}^q$. The open stretched wedge $X^\wedge \times \Omega$ gives rise to a splitting of variables (r, x, y). Then an "edgification" of the cone algebra (to get a pseudo-differential algebra on W) starts from pseudo-differential operators on $\mathbb{R}_+ \times \Omega$ with amplitude functions $a(r, y, z, \eta) := \tilde{a}(r, y, z, \tilde{\eta})|_{\tilde{\eta} = r\eta}$ (i.e., with "edge degeneracy" in η), with $\tilde{a}(r, y, z, \tilde{\eta}) \in L^\mu_{\mathrm{cl}}(X; \mathbb{R}_+ \times \Omega \times \Gamma_{\frac{n+1}{2} - \gamma} \times \mathbb{R}^q_{\tilde{\eta}})$ of some specific structure, cf. [29], [31], [25], [10]. Also here, we may (and, in fact, will) have in mind sleeping parameters to be activated in the next conification step. The procedure leads to operators on a manifold with corners, locally modelled by $W^\triangle$, where the corner base W itself is a manifold with edges. Also the case of boundary value problems can be studied under the aspect of an iterative approach. Conifications and edgifications then start from boundary value problems on a compact C^∞ manifold X with boundary, based on the space $\mathcal{B}^{\mu,d}(X)$ of classical pseudo-differential boundary value problems of order $\mu \in \mathbb{Z}$ and type $d \in \mathbb{N}$, cf. [2], [20], or, more generally, on a parameter-dependent variant $\mathcal{B}^{\mu,d}(X; U \times \mathbb{R}^l)$ with parameters $(y, \lambda) \in U \times \mathbb{R}^l$ of similar meaning as above. A cone algebra of boundary value problems in that sense (also for non-classical operators) has been studied in [22], [23]. Basics of the symbolic structure for the edge algebra are given in [24], [26], [25]; the edge algebra itself may be found in [10], see also [5] or [4].

Corner and higher edge cases are interesting as well. For instance, it is a natural problem to study parabolicity of boundary value problems on configurations with edges. Then, even for differential operators, we need a parameter-dependent elliptic edge theory, see [1] for an analogous situation when the boundary is smooth. It has been observed in [13] that the space-time cylinder for $t \to \infty$ (with t being the time variable) behaves like a conical singularity (e.g., when the coefficients in a parabolic operator are constant for $t > T$ for some T, or when they are "smooth" up to ∞). It is then adequate to study a corresponding (anisotropic analogue of the) cone operator algebra for the tip at ∞. This yields invertibility of operators "up to ∞" and long-time asymptotics of solutions. Long-time (iterated) asymptotics for the case of spatial configurations with conical singularities (and without boundary) have been characterised in [14]

Parabolic boundary value problems with edge singularities require parameter-dependent operators on the corresponding spatial configuration.

Elements of such a calculus in the isotropic case belong to the program of the present paper. At the same time we establish necessary symbolic structures for parametrices of elliptic boundary value problems on manifolds with higher corners. This corresponds to configurations when the edge itself is not smooth but has, e.g., conical singuarities. In other words, we fulfill a specific part of the iterative approach for singular boundary value problems, see also [16] or [32] for an analogous situation in the boundaryless case, or [9] for the aspect of relative index formulas in certain cases with corners.

We start the iteration from parameter-dependent boundary value problems for the smooth case. Boundary value problems will be written themselves as elements of an "edgified" boundary symbolic calculus. Boundary symbols operate on a cone that is the inner normal, while the boundary plays the role of an edge. We then construct holomorphic families of order reducing elements by a kernel cut-off procedure, analogously to [28] for the case without boundary.

Parameter-dependent boundary value problems on a compact C^∞ manifold X with boundary will play the role of operator-valued amplitude functions for the edge calculus. Here, a kernel cut-off construction (in the covariable on the axial direction of the model cone) gives us a class of holomorphic amplitude functions.

We then establish a calculus of 2×2-block matrices of edge boundary value problems. Moreover, we add further entries that describe conditions of trace and potential type along the edge.
All this admits again parameters that we now activate to a final algebra of holomorphic edge-operator-valued functions, obtained by another kernel cut-off. Further we introduce a concept of parameter-dependent ellipticity with an extension to the complex plane. This gives rise to a space of meromorphic operator functions with a precise control of (operator-valued) Laurent coefficients.
The boundary of our manifold with edges is again a manifold with edges though without boundary (in the sense that the base X of the local model cone is closed). The case without boundary has been treated in [16] in a framework with continuous edge asymptotics. In our paper we concentrate on the effects from the boundary and do not touch the relatively complex behaviour in a larger calculus with asymptotics. The asymptotic part is also of interest (it is, in fact, automatically generated in parametrix constructions), though it can be regarded as a structure that is complementary to ours. This will be the main content of the paper [3].

1 Boundary value problems

1.1 Manifolds with edges and interior symbols

Let X be a compact C^∞ manifold with boundary, and let $2X := \widetilde{X}$ denote the double of X (that is a closed C^∞ manifold obtained by gluing together two copies $X_\pm$ of X along their boundaries in a canonical way); we then identify X with X_+. Let $X^\Delta := (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$ be the cone with base X and form analogously $\widetilde{X}^\Delta$; the tips v of the respective cones are represented by $\{0\} \times X$ $(\{0\} \times \widetilde{X})$, identified with a point. $X^\wedge := X^\Delta \setminus \{v\}$ is a C^∞ manifold with boundary, embedded in the C^∞ manifold $\widetilde{X}^\wedge := \widetilde{X}^\Delta \setminus \{v\}$. For the analysis on cones it will be convenient to fix splittings of variables (r, x), say, on $\mathbb{R}_+ \times \widetilde{X}$ and to impose what we call a cone structure. Two splittings of variables (r, x) and $(\tilde{r}, \tilde{x})$ are said to define the same cone structure on $\mathbb{R}_+ \times \widetilde{X}$, if $(r, x) \to (\tilde{r}, \tilde{x})$ is induced by a diffeomorphism $\overline{\mathbb{R}}_+ \times \widetilde{X} \to \overline{\mathbb{R}}_+ \times \widetilde{X}$. Up to this point, $\widetilde{X}$ may be an arbitrary closed C^∞ manifold. For the case $\widetilde{X} = 2X$ with a compact C^∞ manifold X with boundary we also consider (r, x), $(\tilde{r}, \tilde{x})$ on $\mathbb{R}_+ \times X$, and say that two such splittings define the same cone structure there, if $(r, x) \to (\tilde{r}, \tilde{x})$ defines a homeomorphism $\chi : X^\wedge \to X^\wedge$ induced by a diffeomorphism $\widetilde{\chi} : \overline{\mathbb{R}}_+ \times 2X \to \overline{\mathbb{R}}_+ \times 2X$, i.e. $\chi = \widetilde{\chi}\,|_{X^\wedge}$.
Given an open set $\Omega \subseteq \mathbb{R}^q$ and a closed C^∞ manifold $\widetilde{X}$ we now pass to (stretched) wedges $\widetilde{X}^\Delta \times \Omega$. Two splittings of variables (r, x, y) and $(\tilde{r}, \tilde{x}, \tilde{y})$ on $\widetilde{X}^\wedge \times \Omega$ are said to define the same wedge structure on $\widetilde{X}^\wedge \times \Omega$ if $(r, x, y) \longrightarrow (\tilde{r}, \tilde{x}, \tilde{y})$ is induced by a diffeomorphism $\overline{\mathbb{R}}_+ \times \widetilde{X} \times \Omega \to \overline{\mathbb{R}}_+ \times \widetilde{X} \times \Omega$, and, in addition, if we

write

$$(\tilde{r},\tilde{x},\tilde{y}) = (\tilde{r}(r,x,y),\tilde{x}(r,x,y),\tilde{y}(r,x,y)), \tag{1.1}$$

we have $\tilde{r}(0,x,y) = 0$; furthermore, $x \to \tilde{x}(0,x,y)$ for fixed y defines a diffeomorphism $\widetilde{X} \to \widetilde{X}$, and $\tilde{y}(0,x,y)$ is independent of x, where $y \to \tilde{y}$ at $r = 0$ induces a diffeomorphism $\Omega \to \Omega$. Moreover, for the case $\widetilde{X} = 2X$ with a compact C^∞ manifold X with boundary we also consider (r,x,y), $(\tilde{r},\tilde{x},\tilde{y})$ on $\mathbb{R}_+ \times X \times \Omega$, and say that two such splittings define the same wedge structures if $(r,x,y) \to (\tilde{r},\tilde{x},\tilde{y})$ defines a homeomorphism $\chi : X^\wedge \times \Omega \to X^\wedge \times \Omega$, induced by a diffeomorphism $\widetilde{\chi} : \overline{\mathbb{R}}_+ \times \widetilde{X} \times \Omega \to \overline{\mathbb{R}}_+ \times \widetilde{X} \times \Omega$ of the above kind, i.e. $\chi = \widetilde{\chi}\,|_{X^\wedge \times \Omega}$.

A topological space $\widetilde{W}$ (locally compact and paracompact) is said to be a manifold with edge $Y \subset \widetilde{W}$ (and without boundary), if $\widetilde{W} \setminus Y$ and Y are C^∞ manifolds of dimension $1+n+q$ and q, respectively, and every $y \in Y$ has a neighbourhood $\widetilde{V}$ in $\widetilde{W}$ that is homeomorphic to a wedge $\widetilde{X}^\Delta \times \Omega$ with a fixed wedge structure on $\widetilde{X}^\wedge \times \Omega$, where $\widetilde{X}$ is a certain closed C^∞ manifold, $n = \dim \widetilde{X}$.

For the case with boundary we apply the definition of $\widetilde{W}$ to the case of $\widetilde{X} = 2X$, where X is a compact C^∞ manifold with boundary.

A topological space W is said to be a manifold with edge $Y \subset W$ and boundary, if there is a manifold $\widetilde{W} := 2W$ with edge Y and without boundary ($\widetilde{W}$ will be as the double of W), such that the former neighbourhoods $\widetilde{V}$ can be regarded as doubles $2V$ of neighbourhoods V of points $y \in Y$ in W, where the wedge structures of $\widetilde{V} \setminus Y$ in the sense of representations as $(2X)^\wedge \times \Omega$ induce wedge structures on $X^\wedge \times \Omega$ in the above-mentioned sense.

Notice that the definition of $\widetilde{W}$ allows us to form a C^∞ manifold $\widetilde{\mathbb{W}}$ with boundary $\partial\widetilde{\mathbb{W}}$ where a neighbourhood of $\partial\widetilde{\mathbb{W}}$ is modelled by the sets $\overline{\mathbb{R}}_+ \times \widetilde{X} \times \Omega$, where $\partial\widetilde{\mathbb{W}}$ locally corresponds to $\{0\} \times \widetilde{X} \times \Omega$. By virtue of the nature of transition maps (i.e., other admitted splittings of variables in the wedge structures on $\widetilde{X}^\wedge \times \Omega$) the definition of $\widetilde{\mathbb{W}}$ is invariant, and $\partial\widetilde{\mathbb{W}}$ is an $\widetilde{X}$-bundle on Y. We call $\widetilde{\mathbb{W}}$ the stretched manifold with edges associated with $\widetilde{W}$. For purposes below we also set $\widetilde{\mathbb{W}}_{\text{sing}} := \partial\widetilde{\mathbb{W}}$, $\widetilde{\mathbb{W}}_{\text{reg}} := \widetilde{\mathbb{W}} \setminus \partial\widetilde{\mathbb{W}}$. Moreover, if $\widetilde{W} = 2W$ where W is a manifold with edge and boundary, we can also form the associated stretched manifold $\mathbb{W}$, locally being of the form $\overline{\mathbb{R}}_+ \times X \times \Omega$, where $\widetilde{\mathbb{W}} = 2\mathbb{W}$ is the double of $\mathbb{W}$. Similarly to the definition of doubles of smooth manifolds with boundary we represent $2\mathbb{W}$ as the union of two copies $\mathbb{W}_\pm$ of $\mathbb{W}$ glued together in a natural way. If we identify $\mathbb{W}$ with $\mathbb{W}_+$, we then set $\mathbb{W}_{\text{sing}} := \widetilde{\mathbb{W}}_{\text{sing}} \cap \mathbb{W}_+$ and $\mathbb{W}_{\text{reg}} := \widetilde{\mathbb{W}}_{\text{reg}} \cap \mathbb{W}_+$. Note that $\mathbb{W}_{\text{reg}}$ is a (in general, non-compact) C^∞ manifold with boundary.

Example 1.1. *Let X be a compact C^∞ manifold with boundary ∂X and let X^Δ be the cone with base X, where $X^\wedge$ is endowed with a cone strucure. Then $W := X^\Delta \times \Omega$ for an open set $\Omega \subseteq \mathbb{R}^q$ is a manifold with edge Ω and boundary. We then have*

$$\mathbb{W} = \overline{\mathbb{R}}_+ \times X \times \Omega,$$

$$\mathbb{W}_{\text{sing}} = \{0\} \times X \times \Omega,\ \mathbb{W}_{\text{reg}} = \mathbb{R}_+ \times X \times \Omega.$$

Moreover, $\widetilde{W} := 2W := (2X)^{\Delta} \times \Omega$ *is a closed manifold with edge* Ω*, where* $\widetilde{\mathbb{W}} = \overline{\mathbb{R}}_+ \times (2X) \times \Omega$.

For notational convenience we impose some assumptions on the nature of our manifolds $\widetilde{W}$ and W with edge Y. We assume that Y has a neighbourhood $\widetilde{V}$ in $\widetilde{W}$ such that $\widetilde{V} \setminus Y$ is homeomorphic to $\widetilde{X}^{\wedge} \times Y$ with a global wedge structure where the transition diffeomorphisms $\widetilde{X}^{\wedge} \times \Omega \to \widetilde{X}^{\wedge} \times \Omega$ in a neighbourhood of $r = 0$ only depend on y (not on r or x). For W with boundary we impose a similar condition. These assumptions are not really essential for our results, but we intend to concentrate on analytic effects that become more transparent under such precautions. On $\widetilde{\mathbb{W}}$, Y and $\widetilde{X}$ we fix Riemannian metrics such that the metric on $\widetilde{\mathbb{W}}$ corresponds to the product metric on $\overline{\mathbb{R}}_+ \times \widetilde{X} \times Y$ in a neighbourhood of $r = 0$, and on $\mathbb{W}$ and X we take the metrics induced by the ones on $\widetilde{\mathbb{W}}$ and $\widetilde{X}$, respectively. Finally, for our manifold X with boundary we assume that the Riemannian metric is the product metric in a collar neigbourhood $\cong \partial X \times [0, 1)$ of the boundary. Transition diffeomorphisms to charts near the boundary will be assumed to be independent of the normal variable x_n for small x_n.
By $\text{Vect}\,(\cdot)$ we denote the set of all smooth complex vector bundles on the manifold in the brackets, i.e., we have the sets $\text{Vect}\,(\widetilde{\mathbb{W}})$, $\text{Vect}\,(\widetilde{X})$, etc., and we set $\text{Vect}\,(\mathbb{W}) := \{\widetilde{E}|_{\mathbb{W}} : \widetilde{E} \in \text{Vect}\,(\widetilde{\mathbb{W}})\}$, $\text{Vect}\,(X) := \{\widetilde{G}|_X : \widetilde{G} \in \text{Vect}\,(\widetilde{X})\}$. In the bundles in consideration we fix Hermitian metrics. Moreover, for $\widetilde{E} \in \text{Vect}\,(\widetilde{\mathbb{W}})$ the restriction of $\widetilde{E}$ to $\partial\widetilde{\mathbb{W}} = \widetilde{X} \times Y$ can be lifted to a bundle on $\mathbb{R}_+ \times \widetilde{X} \times Y$ that we call $\widetilde{E}^{\wedge}$, where the Hermitian metric is also assumed to be the lifting of the metric on $\widetilde{E}|_{\partial\mathbb{W}}$. Similar notation is used in connection with bundles $E \in \text{Vect}\,(\mathbb{W})$ and $E^{\wedge} \in \text{Vect}\,(\mathbb{R}_+ \times X \times Y)$.
To motivate the choice of interior (pseudo-differential) symbols in local coordinates $(r, x, y) \in \mathbb{R}_+ \times \Sigma \times \Omega$, where $\Sigma \subseteq \mathbb{R}^n$ is an open set (belonging to a chart on X) we first describe the form of typical differential operators on an open stretched wedge $X^{\wedge} \times \Omega$. Operators are assumed to be edge-degenerate, i.e., they have the form

$$A = r^{-\mu} \sum_{j+|\alpha| \le \mu} a_{j\alpha}(r, y) \Big(-r\frac{\partial}{\partial r}\Big)^j (rD_y)^{\alpha} \tag{1.2}$$

where $a_{j\alpha}(r, y) \in C^{\infty}(\overline{\mathbb{R}}_+ \times \Omega, \text{Diff}^{\mu-(j+|\alpha|)}(X))$. Here, $\text{Diff}^{\nu}(X)$ denotes the space of all differential operators on X with smooth coefficients up to the boundary, endowed with a natural Fréchet topology. Operators like (1.2) appears (for $\mu = 2$) as Laplace-Beltrami operators for wedge metrics on $X^{\wedge} \times \Omega$ and they are also induced by substituing polar coordinates $\tilde{x} \to (r, x)$ in operators with smooth coefficients in $\mathbb{R}^{n+1}_{\tilde{x}} \times \Omega_y$. Observe that operators (1.2) behave invariant under the above-mentioned transition maps.

In our theory we will be interested in fact in parameter-dependent operators with extra covariables $\lambda \in \mathbb{R}^l$ that formally play the same role as $\eta \in \mathbb{R}^q$, the covariable to $y \in \Omega$. Pseudo-differential symbols of order $\mu \in \mathbb{R}$ with parameters λ are then assumed to be of the form

$$r^{-\mu} p(r, x, y, \rho, \xi, \eta, \lambda)$$

where

$$p(r, x, y, \rho, \xi, \eta, \lambda) = \tilde{p}(r, x, y, \tilde{\rho}, \xi, \tilde{\eta}, \tilde{\lambda})|_{\tilde{\rho}=r\rho,\, \tilde{\eta}=r\eta,\, \tilde{\lambda}=r\lambda} \tag{1.3}$$

for a symbol $\tilde{p}(r, x, y, \tilde{\rho}, \xi, \tilde{\eta}, \tilde{\lambda})$ in the standard Hörmander symbol class $S^{\mu}_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{R}^{1+n+q+l}_{\tilde{\rho}, \xi, \tilde{\eta}, \tilde{\lambda}})$. Here, subscript "(cl)" means that we are talking about classical or non-classical symbols in $(\tilde{\rho}, \xi, \tilde{\eta}, \tilde{\lambda})$. Below we mainly look at classical symbols, indicated by "cl". Since we are interested in boundary value problems we do not only consider open sets $\Sigma \subseteq \mathbb{R}^n$ but sets of the form $\Sigma := \Sigma' \times \overline{\mathbb{R}}_+$ where $\Sigma' \subseteq \mathbb{R}^{n-1}$ corresponds to a chart on the boundary ∂X while $\overline{\mathbb{R}}_+$ corresponds to the inner normal to ∂X (with respect to a choosen Riemannian metric on X that is supposed to be the product metric of $\partial X \times [0, 1)$ on a collar neighbourhood of the boundary). Near the boundary we write $x = (x', x_n)$ with covariables $\xi = (\xi', \xi_n)$.

Definition 1.2. *A symbol $p(r, x, y, \rho, \xi, \eta, \lambda)$ written in the form (1.3) for $\tilde{p}(r, x, y, \tilde{\rho}, \xi, \tilde{\eta}, \tilde{\lambda}) \in S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times (\Sigma' \times \overline{\mathbb{R}}_+) \times \Omega \times \mathbb{R}^{1+n+q+l})$, $\mu \in \mathbb{Z}$, is said to have the transmission property with respect to $x_n = 0$ if the homogeneous components $\tilde{p}_{(\mu-j)}$ of $\tilde{p}$ of order $\mu - j$ in $(\tilde{\rho}, \xi, \tilde{\eta}, \tilde{\lambda}) \neq 0$, $j \in \mathbb{N}$, satisfy the condition*

$$\begin{aligned} D^k_{x_n} D^{\alpha}_{\tilde{\rho}, \xi', \tilde{\eta}, \tilde{\lambda}} \{ & \tilde{p}_{(\mu-j)}(r, x', x_n, y, \tilde{\rho}, \xi', \xi_n, \tilde{\eta}, \tilde{\lambda}) \\ & - (-1)^{\mu-j} \tilde{p}_{(\mu-j)}(r, x', x_n, y, -\tilde{\rho}, -\xi', -\xi_n, -\tilde{\eta}, -\tilde{\lambda}) \} = 0 \end{aligned}$$

on the set $\{(r, x', x_n, y, \tilde{\rho}, \xi', \xi_n, \tilde{\eta}, \tilde{\lambda}) : (r, x', y) \in \overline{\mathbb{R}}_+ \times \Sigma' \times \Omega,\ x_n = 0,\ (\tilde{\rho}, \xi', \tilde{\eta}, \tilde{\lambda}) = 0,\ \xi_n \in \mathbb{R} \setminus \{0\}\}$ for all $k \in \mathbb{N}$, $\alpha \in \mathbb{N}^{n+q+l}$.

Symbols relying on variables in a set like $\overline{\mathbb{R}}_+ \times \Sigma' \times \overline{\mathbb{R}}_+ \times \Omega$ are interpreted as restrictions of symbols in $\mathbb{R} \times \Sigma' \times \mathbb{R} \times \Omega \ni (r, x', x_n, y)$ to $\overline{\mathbb{R}}_+ \times \Sigma' \times \overline{\mathbb{R}}_+ \times \Omega$. In particular, we can also define the transmission property for symbols defined in $\mathbb{R} \times \Sigma' \times \mathbb{R} \times \Omega$ (or also $\overline{\mathbb{R}}_+ \times \Sigma' \times \mathbb{R} \times \Omega$) by requiring an analogue of the conditions of Definition 1.2 for (r, x', x_n, y) in the respective larger sets. Let $S^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times (\Sigma' \times \overline{\mathbb{R}}_+) \times \Omega \times \mathbb{R}^{1+n+q+l})_{\mathrm{tr}}$ denote the subspace of all symbols with the transmission property. Similarly, we write the subscript "tr" for spaces of symbols with the transmission property when the spatial variables run over $\mathbb{R}$ instead of $\overline{\mathbb{R}}_+$ with respect to r or x_n.

1.2 Boundary symbols associated with interior symbols

Given a symbol $p(r,x,y,\rho,\xi,\eta,\lambda)$ with the transmission property at $x_n = 0$ in the sense of Definition 1.2 we now pass to an associated boundary symbol that is operator-valued, acting on the x_n-half-axis. The formalities will not depend on the specific dimensions of variables and covariables. Therefore, we shall ignore the covariable λ for a while (it is, in fact, a "sleeping" covariable, to make it active in a higher floor of our calculus). In other words, for convenience we now speak about symbols $r^{-\mu}p(r,x,y,\rho,\xi,\eta)$ where

$$p(r,x,y,\rho,\xi,\eta) = \tilde{p}(r,x,y,r\rho,\xi,r\eta)$$

with a symbol $\tilde{p}$ that is smooth up to $r = 0$ and has the transmission property (in the notation with covariables $(\tilde{\rho},\xi,\tilde{\eta})$, $\xi = (\xi',\xi_n)$). Also the dimension of y- and η-variables may now be independent; we simply assume both dimension to be q. Let $\mathcal{S}(\overline{\mathbb{R}}_+) := \{u|_{\overline{\mathbb{R}}_+} : u \in \mathcal{S}(\mathbb{R})\}$ and set

$$(\mathrm{e}^+u)(x_n) = \begin{cases} u(x_n) & \text{for } x_n > 0 \\ 0 & \text{for } x_n \leq 0 \end{cases}.$$

In addition, let $\mathrm{r}^+ : \mathcal{D}'(\mathbb{R}) \to \mathcal{D}'(\mathbb{R}_+)$ denote the operator of restriction of distributions from $\mathbb{R}$ to $\mathbb{R}_+$. In particular, if $H^s(\mathbb{R})$ is the standard Sobolev space of smoothness $s \in \mathbb{R}$ on $\mathbb{R}$, we set $H^s(\mathbb{R}_+) = \{u|_{\mathbb{R}_+} : u \in H^s(\mathbb{R})\}$ and use e^+ in the sense $\mathrm{e}^+ : H^s(\mathbb{R}_+) \to \mathcal{S}'(\mathbb{R})$ for $s > -\frac{1}{2}$. Given a symbol $p(r,x,y,\rho,\xi,\eta)$ (of order $\mu \in \mathbb{Z}$) with the transmission property we set $\sigma_\psi(p)(r,x,y,\rho,\xi,\eta) = p_{(\mu)}(r,x,y,\rho,\xi,\eta)$ and define its boundary symbol as

$$\sigma_\partial(p)(r,x',y,\rho,\xi',\eta) := \mathrm{r}^+\mathrm{op}(p_{(\mu)}|_{x_n=0})\mathrm{e}^+(r,x',y,\rho,\xi',\eta)$$

for $(\rho,\xi',\eta) \neq 0$, regarded as a family of continuous operators $H^s(\mathbb{R}_+) \longrightarrow H^{s-\mu}(\mathbb{R}_+)$ for $s > -\frac{1}{2}$, or, alternatively, $\mathcal{S}(\overline{\mathbb{R}}_+) \to \mathcal{S}(\overline{\mathbb{R}}_+)$. Here, $\mathrm{op}(a)u(x_n) := \int\int e^{i(x_n-x_n')\xi_n} a(x_n,x_n',\xi_n)u(x_n')\,dx_n'\,đ\xi_n$ for a symbol $a(x_n,x_n',\xi_n)$ in variables (x_n,x_n') and covariable ξ_n.

Notice that when we set $\kappa_\delta(u)(x_n) := \delta^{\frac{1}{2}}u(\delta x)$, $\delta \in \mathbb{R}_+$ we have a strongly (in $\delta \in \mathbb{R}_+$) continuous group of isomorphisms $\kappa_\delta : H^s(\mathbb{R}_+) \to H^s(\mathbb{R}_+)$ for all s, and

$$\sigma_\partial(p)(r,x',y,\delta\rho,\delta\xi',\delta\eta) = \delta^\mu\kappa_\delta\sigma_\partial(p)(r,x',y,\rho,\xi',\eta)\kappa_\delta^{-1}$$

for all r, x', y, $(\rho,\xi',\eta) \neq 0$, $\delta \in \mathbb{R}_+$.

For purposes below we also note the fact that, provided the symbol $p(r,x,y,\rho,\xi,\eta)$ is independent of x_n for $|x_n| \geq c$ for a constant $c > 0$, the operator functions

$$\mathrm{op}(p)(r,x',y,\rho,\xi',\eta) : H^s(\mathbb{R}) \longrightarrow H^{s-\mu}(\mathbb{R}) \tag{1.4}$$

and

$$\mathrm{op}^+(p)(r,x',y,\rho,\xi',\eta)\ :\ H^s(\mathbb{R}_+)\longrightarrow H^{s-\mu}(\mathbb{R}_+) \tag{1.5}$$

for $\mathrm{op}^+(\cdot):=\mathrm{r}^+\mathrm{op}(\cdot)\mathrm{e}^+$ and $s>-\frac{1}{2}$ are operator-valued symbols in the following sense.

To reduce technicalities we now employ for a moment variables and covariables $(y,\eta)\in U\times\mathbb{R}^q$ independently of the specific meaning before, where $U\subseteq\mathbb{R}^p$ is an open set and p, $q\in\mathbb{N}$ are arbitrary. Let E be a Hilbert space endowed with a strongly continuous group $\{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$ of isomorphisms $\kappa_\delta\ :\ E\to E$, where $\kappa_\delta\kappa_{\delta'}=\kappa_{\delta\delta'}$ for all $\delta,\delta'\in\mathbb{R}_+$ (in such a case we simply talk about a group action on E). Moreover, let $\widetilde{E}$ be another Hilbert space with such a group action $\{\tilde{\kappa}_\delta\}_{\delta\in\mathbb{R}_+}$. Then $S^\mu(U\times\mathbb{R}^q;E,\widetilde{E})$ is defined to be the subspace of all $a(y,\eta)\in C^\infty(U\times\mathbb{R}^q,\mathcal{L}(E,\widetilde{E}))$ such that

$$\|\tilde{\kappa}^{-1}_{\langle\eta\rangle}\{D^\alpha_y D^\beta_\eta a(y,\eta)\}\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(E,\widetilde{E})}\leq c\langle\eta\rangle^{\mu-|\beta|} \tag{1.6}$$

for all multi-indices $\alpha\in\mathbb{N}^p$, $\beta\in\mathbb{N}^q$, and all $y\in K$, $\eta\in\mathbb{R}^q$, for arbitrary $K\subset\subset U$, with constants $c=c(\alpha,\beta,K)>0$. Here, as usual, $\langle\eta\rangle=(1+|\eta|^2)^{\frac{1}{2}}$. Equivalently, we may replace the function $\langle\eta\rangle$ by another strictly positive C^∞ function $[\eta]$ in $\mathbb{R}^q$ such that $[\eta]=|\eta|$ for $|\eta|\geq c$ for some $c>0$. The last constant in the symbol estimates (1.6) turn $S^\mu(U\times\mathbb{R}^q;E,\widetilde{E})$ to a Fréchet space. It is also necessary to generalize this definition to the case of Fréchet spaces, e.g., if $\widetilde{E}=\varprojlim_{k\in\mathbb{N}}\widetilde{E}^k$ is a Fréchet space, written as a projective limit of Hilbert spaces $\widetilde{E}^k$ with continuous embeddings $\widetilde{E}^{k+1}\hookrightarrow\widetilde{E}^k$ for all k, where $\widetilde{E}$ is endowed with a group action $\{\tilde{\kappa}_\delta\}_{\delta\in\mathbb{R}_+}$ that restricts to group actions on $\widetilde{E}^k$ for every k. Then, we have the spaces $S^\mu(U\times\mathbb{R}^q;E,\widetilde{E}^k)$ for all k that are continuously embedded into corresponding spaces referring to $\widetilde{E}^l$, for all $l\leq k$, and we then set

$$S^\mu(U\times\mathbb{R}^q;E,\widetilde{E}):=\varprojlim_{k\in\mathbb{N}}S^\mu(U\times\mathbb{R}^q;E,\widetilde{E}^k)$$

in the Fréchet topology of the projective limit. We also may admit E to be a Fréchet space with similar assumptions. We then get $S^\mu(U\times\mathbb{R}^q;E,\widetilde{E})$ also in such cases; explicit definitions may be found in [31], Section 1.3.1. We also employ corresponding subspaces of classical symbols, indicated by subscript "cl", that are based on components with "twisted" homogeneity in the sense of identities of the kind

$$f(y,\delta\eta)=\delta^{\mu-j}\tilde{\kappa}_\delta f(y,\eta)\kappa^{-1}_\delta,\ \ \delta\in\mathbb{R}_+,$$

when $f(y,\eta)\in C^\infty(U\times(\mathbb{R}^q\setminus\{0\}),\mathcal{L}(E,\widetilde{E}))$, $j\in\mathbb{N}$. Then, e.g., when E is Hilbert, $\widetilde{E}$ a Fréchet space, $S^\mu_{\mathrm{cl}}(U\times\mathbb{R}^q;E,\widetilde{E})$ is Fréchet in a natural way (where the topology is stronger than that induced by $S^\mu(U\times\mathbb{R}^q;E,\widetilde{E})$), while for the case that both E and $\widetilde{E}$ are Fréchet we have corresponding inductive limit topologies both in $S^\mu(U\times\mathbb{R}^q;E,\widetilde{E})$ as well as in $S^\mu_{\mathrm{cl}}(U\times\mathbb{R}^q;E,\widetilde{E})$, cf. [31], Section

1.3.1. Incidentally, U is to be replaced by $(\overline{\mathbb{R}}_+)^k \times U'$ for $k \in \mathbb{N}$, $U' \subseteq \mathbb{R}^{p'}$ open; then there is a straightforward extension of symbol spaces to $(\overline{\mathbb{R}}_+)^k \times U'$ in place of U (e.g., by restrictions of symbol spaces from larger open sets to $(\overline{\mathbb{R}}_+)^k \times U'$). The choice of the actions $\{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$ and $\{\tilde{\kappa}_\delta\}_{\delta\in\mathbb{R}_+}$ on E and $\widetilde{E}$, respectively, is assumed to be known and fixed. In the case $E = H^s(\mathbb{R}_+)$ or $H^s(\mathbb{R})$ we always take $\kappa_\delta : u(x_n) \to \delta^{\frac{1}{2}} u(\delta x_n)$, $\delta \in \mathbb{R}_+$. For $E = \mathbb{C}^N$ we usually set $\kappa_\delta = \mathrm{id}_E$, $\delta \in \mathbb{R}_+$.
Clearly, our symbol spaces depend on the choice of the group actions $\kappa := \{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$ and $\tilde{\kappa} := \{\tilde{\kappa}_\delta\}_{\delta\in\mathbb{R}_+}$ on the spaces E and $\widetilde{E}$, respectively. Incidentally, it will be necessary to indicate that explicitly; in this case we write $S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^q; E, \widetilde{E})_{\kappa,\tilde{\kappa}}$ in place of $S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^q; E, \widetilde{E})$.
Observe that $S^{-\infty}(U \times \mathbb{R}^q; E, \widetilde{E})$ (the intersection of all $S^\mu(U \times \mathbb{R}^q; E, \widetilde{E})$ over $\mu \in \mathbb{R}$) does not depend on the choice of κ, $\tilde{\kappa}$ (i.e., we may take for κ and $\tilde{\kappa}$ the identities), and we have $S^{-\infty}(U \times \mathbb{R}^q; E, \widetilde{E}) = C^\infty(U, \mathcal{S}(\mathbb{R}^q, \mathcal{L}(E, \widetilde{E})))$.

Remark 1.3. *The operator family* (1.4) *belongs to* $S^\mu(U\times\mathbb{R}^{n+q}; H^s(\mathbb{R}), H^{s-\mu}(\mathbb{R}))$ *for all* $s \in \mathbb{R}$ *while* (1.5) *is an element of* $S^\mu(U \times \mathbb{R}^{n+q}; H^s(\mathbb{R}_+), H^{s-\mu}(\mathbb{R}_+))$ *for all* $s > -\frac{1}{2}$*; here,* $U = \overline{\mathbb{R}}_+ \times \Sigma' \times \Omega$. *For the case that* p *is independent of* x_n *we get classical symbols. In addition,* (1.4) *and* (1.5) *belong to* $S^\mu(U\times\mathbb{R}^{n+q}; \mathcal{S}(\mathbb{R}), \mathcal{S}(\mathbb{R}))$ *and* $S^\mu(U \times \mathbb{R}^{n+q}; \mathcal{S}(\overline{\mathbb{R}}_+), \mathcal{S}(\overline{\mathbb{R}}_+))$, *respectively* (*and they are again classical for* x_n*-independent* p).
In the latter description we employ the representation

$$\mathcal{S}(\overline{\mathbb{R}}_+) = \varprojlim_{k\in\mathbb{N}} \langle x_n\rangle^{-k} H^k(\mathbb{R}_+).$$

For purposes below we want to add some more information on operator-valued symbols and associated operators. Let $\Omega \subseteq \mathbb{R}^q$ be an open set and consider the space $S^\mu_{(\mathrm{cl})}(\Omega \times \Omega \times \mathbb{R}^{q+l}; E, \widetilde{E}) \ni a(y, y', \eta, \lambda)$ of operator-valued symbols and associated parameter-dependent pseudo-differential operators

$$L^\mu_{(\mathrm{cl})}(\Omega; E, \widetilde{E}; \mathbb{R}^l) := \{\mathrm{Op}\,(a)(\lambda) : a(y, y', \eta, \lambda) \in S^\mu_{(\mathrm{cl})}(\Omega\times\Omega\times\mathbb{R}^{q+l}; E, \widetilde{E})\}. \quad (1.7)$$

Then, similarly to scalar pseudo-differential, we have a decomposition

$$L^\mu_{(\mathrm{cl})}(\Omega; E, \widetilde{E}; \mathbb{R}^l) = L^\mu_{(\mathrm{cl})}(\Omega; E, \widetilde{E}; \mathbb{R}^l)_K + L^{-\infty}(\Omega; E, \widetilde{E}; \mathbb{R}^l), \quad (1.8)$$

where $L^{-\infty}(\Omega; E, \widetilde{E}; \mathbb{R}^l) = \bigcap_{\mu\in\mathbb{R}} L^\mu(\Omega; E, \widetilde{E}; \mathbb{R}^l)$ is the space of all smoothing operator families (that equals $\mathcal{S}(\mathbb{R}^l, L^{-\infty}(\Omega; E, \widetilde{E}))$), and $K \subset \Omega \times \Omega$ is any proper compact set that contains $\mathrm{diag}\,(\Omega \times \Omega)$ in its interior, and $L^\mu_{(\mathrm{cl})}(\Omega; E, \widetilde{E}; \mathbb{R}^l)_K$ denotes the space of all elements in (1.7) the (operator-valued) distributional kernel of which is supported by K. Then, forming $\sigma(A)(y, \eta, \lambda)f := e^{-iy\eta} A(\lambda) e^{iy\eta} f$, $f \in E$, for $A(\lambda) \in L^\mu_{(\mathrm{cl})}(\Omega; E, \widetilde{E}; \mathbb{R}^l)_K$ gives us an element $\sigma(A)(y, \eta, \lambda) \in S^\mu_{(\mathrm{cl})}(\Omega\times$

$\mathbb{R}^{q+l}; E, \widetilde{E})$ where $A(\lambda) = \mathrm{Op}\,(\sigma(A))(\lambda)$. The space of such $\sigma(A)(y,\eta,\lambda)$ is a closed subspace $S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^{q+l}; E, \widetilde{E})_K$ of $S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^{q+l}; E, \widetilde{E})$ and has as such a Fréchet topology (the induced one from the larger space). The bijection

$$\mathrm{Op} : S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^{q+l}; E, \widetilde{E})_K \longrightarrow L^{\mu}_{(\mathrm{cl})}(\Omega; E, \widetilde{E}; \mathbb{R}^l)_K$$

then gives us a Fréchet topology also in $L^{\mu}_{(\mathrm{cl})}(\Omega; E, \widetilde{E}; \mathbb{R}^l)_K$, and (1.8) yields a Fréchet topology in the space (1.7) itself, via the corresponding non-direct sum (concerning non-direct sums, cf. notation in Section 2.2 below). A similar construction holds for the case of Fréchet space $\widetilde{E}$ with group action.

1.3 Green, trace and potential symbols

Boundary symbols of boundary value problems with the transmission property also contain Green, trace and potential entries. They may be obtained by specifying the above abstract operator-valued symbols for the case

$$E := \begin{array}{c} L^2(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{j_-} \end{array}, \qquad \widetilde{E} := \begin{array}{c} \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{j_+} \end{array} \tag{1.9}$$

for certain j_-, $j_+ \in \mathbb{N}$. We then have symbols of the kind

$$g(x', \xi') \in S^{\mu}_{\mathrm{cl}}(U \times \mathbb{R}^q; E, \widetilde{E}),$$

$U \subseteq \mathbb{R}^p$ open, with $\mathrm{diag}\,(\{\kappa_\delta\}_{\delta \in \mathbb{R}_+}, \mathrm{id})$ acting on E and $\widetilde{E}$ (where "id" means the identity in the respective finite-dimensional spaces that will be always clear by the context).
Set $r(\xi') := \mathrm{diag}\,(1, \langle \xi' \rangle^{\frac{1}{2}} \mathrm{id})$.

Definition 1.4. *An operator family $g(x', \xi') \in C^\infty(U \times \mathbb{R}^q, \mathcal{L}(E, \widetilde{E}))$ is said to be a Green symbol of order μ and type 0 if $b(x', \xi') := r^{-1}(\xi') g(x', \xi') r(\xi')$ satisfies the following relations:*

$$b(x', \xi') \in S^{\mu}_{\mathrm{cl}}(U \times \mathbb{R}^q; L^2(\mathbb{R}_+) \oplus \mathbb{C}^{j_-}, \mathcal{S}(\overline{\mathbb{R}}_+) \oplus \mathbb{C}^{j_+}),$$
$$b^*(x', \xi') \in S^{\mu}_{\mathrm{cl}}(U \times \mathbb{R}^q; L^2(\mathbb{R}_+) \oplus \mathbb{C}^{j_+}, \mathcal{S}(\overline{\mathbb{R}}_+) \oplus \mathbb{C}^{j_-}),$$

where $b^*(\cdot,\cdot)$ denotes the pointwise adjoint in the sense $(bu, v)_{L^2(\mathbb{R}_+)\oplus\mathbb{C}^{j_+}} = (u, b^*v)_{L^2(\mathbb{R}_+)\oplus\mathbb{C}^{j_-}}$ for all $u \in C_0^\infty(\mathbb{R}_+) \oplus \mathbb{C}^{j_-}$, $v \in C_0^\infty(\mathbb{R}_+) \oplus \mathbb{C}^{j_+}$.
Moreover, an operator family $g(x', \xi') : H^s(\mathbb{R}_+) \oplus \mathbb{C}^{j_-} \to \mathcal{S}(\overline{\mathbb{R}}_+) \oplus \mathbb{C}^{j_+}$, $s \geq d - \frac{1}{2}$ for some $d \in \mathbb{N}$ is called a Green symbol of order μ and type d if it has the form

$$g(x', \xi') = g_0(x', \xi') + \sum_{j=1}^{d} g_j(x', \xi')\, \mathrm{diag}(\partial^j_{x_n}, 0)$$

for Green symbols $g_j(x',\xi')$ of order $\mu-j$ and type 0 (here, x_n denotes the variable on the half-axis). Let $\mathcal{R}_G^{\mu,d}(U\times\mathbb{R}^q;j_-,j_+)$ denote the space of all Green symbols of order μ and type d.

Let us now introduce the homogeneous principal symbol of $g(x',\xi')$ of DN-homogeneity μ. First, the symbol $b(x',\xi')$ is classical of order μ and has a homogeneous principal symbol in the operator-valued sense, based on the group action $\{\kappa_\delta,\mathrm{id}\}_{\delta\in\mathbb{R}_+}$. For $g(x',\xi')$ itself that means

$$g(x',\xi')\in S^\mu_{\mathrm{cl}}(U\times\mathbb{R}^q;H^s(\mathbb{R}_+)\oplus\mathbb{C}^{j_-},\mathcal{S}(\overline{\mathbb{R}}_+)\oplus\mathbb{C}^{j_+})_{\boldsymbol{\kappa},\boldsymbol{\kappa}}$$

for $\{\boldsymbol{\kappa}_\delta\}_{\delta\in\mathbb{R}_+}:=\mathrm{diag}\,\{\kappa_\delta,\delta^{\frac{1}{2}}\}_{\delta\in\mathbb{R}_+}$, $s>d-\frac{1}{2}$. Then, the homogeneous principal symbol in that sense will also be called the principal boundary symbol of $g(x',\xi')$, written $\sigma_\partial(g)(x',\xi')$. Homogeneity then means

$$\sigma_\partial(g)(x',\delta\xi')=\delta^\mu\mathrm{diag}\,\{\kappa_\delta,\delta^{\frac{1}{2}}\}\sigma_\partial(g)(x',\xi')\mathrm{diag}\,\{\kappa_\delta,\delta^{\frac{1}{2}}\}^{-1} \tag{1.10}$$

for all $(x',\xi')\in U\times(\mathbb{R}^q\setminus\{0\})$, $\delta\in\mathbb{R}_+$.

1.4 Local amplitude functions for boundary value problems

We now return to the space $S^\mu_{\mathrm{cl}}(U\times\overline{\mathbb{R}}_+\times\mathbb{R}^{q+1})_{\mathrm{tr}}$ of classical symbols with the transmission property where $U\subseteq\mathbb{R}^p$ is an arbitrary open set of variables x', $x_n\in\overline{\mathbb{R}}_+$, and covariables $(\xi',\xi_n)\in\mathbb{R}^{q+1}$ of arbitrary dimension, where $\xi'\in\mathbb{R}^q$ (as noted in the beginning U may also be replaced by a set of the form $\overline{\mathbb{R}}_+\times U'$ or $\overline{\mathbb{R}}_+\ni x_n$ replaced by $\mathbb{R}$, etc.; such generalizations will be tacitly used). In particular, we have the space $S^\mu_{\mathrm{cl}}(U\times\mathbb{R}\times\mathbb{R}^{q+1})_{\mathrm{tr}}$ where elements $p(x',x_n,\xi',\xi_n)\in S^\mu_{\mathrm{cl}}(U\times\overline{\mathbb{R}}_+\times\mathbb{R}^{q+1})_{\mathrm{tr}}$ are defined by restrictions of corresponding symbols over $U\times\mathbb{R}\times\mathbb{R}^{q+1}$.
As noted in Section 1.2 for every $p(x',x_n,\xi',\xi_n)\in S^\mu_{\mathrm{cl}}(U\times\overline{\mathbb{R}}_+\times\mathbb{R}^{q+1})_{\mathrm{tr}}$ that is independent of x_n for large x_n we have an operator-valued symbol

$$\mathrm{op}^+(p)(x',\xi')\in S^\mu(U\times\mathbb{R}^q;H^s(\mathbb{R}_+),H^{s-\mu}(\mathbb{R}_+)) \tag{1.11}$$

for $s>-\frac{1}{2}$ (here, in $\mathrm{op}^+(p)=\mathrm{r}^+\mathrm{op}(p)\mathrm{e}^+$ we tacitly use any extension of p as a symbol on $U\times\mathbb{R}\times\mathbb{R}^{q+1}$, though $\mathrm{op}^+(p)$ does not depend on the specific extension).

Definition 1.5. *The space $\mathcal{R}^{\mu,d}(U\times\mathbb{R}^q;j_-,j_+)$, $(\mu,d)\in\mathbb{Z}\times\mathbb{N}$, is defined to be the set of all operator families*

$$a(x',\xi'):=\begin{pmatrix}\mathrm{op}^+(p)(x',\xi') & 0\\ 0 & 0\end{pmatrix}+g(x',\xi')$$

for arbitrary $p(x',x_n,\xi',\xi_n)\in S^\mu_{\mathrm{cl}}(U\times\overline{\mathbb{R}}_+\times\mathbb{R}^{q+1})_{\mathrm{tr}}$ (that is independent of x_n for large x_n) and arbitrary $g(x',\xi')\in\mathcal{R}_G^{\mu,d}(U\times\mathbb{R}^q;j_-,j_+)$.

Remark 1.6. *As a consequence of* (1.11) *and Definition* 1.4 *we see that for* $a(x',\xi') \in \mathcal{R}^{\mu,d}(U \times \mathbb{R}^q; j_-, j_+)$ *and* $b(x',\xi') = r^{-1}(\xi')a(x',\xi')r(\xi')$, *we have* $b(x',\xi') \in S^\mu(U\times\mathbb{R}^q; H^s(\mathbb{R}_+)\oplus\mathbb{C}^{j_-}, H^{s-\mu}(\mathbb{R}_+)\oplus\mathbb{C}^{j_+})$ *for every* $s > d-\frac{1}{2}$. *In addition, it can easily be verified that* $b(x',\xi') \in S^\mu(U\times\mathbb{R}^q; \mathcal{S}(\overline{\mathbb{R}}_+)\oplus\mathbb{C}^{j_-}, \mathcal{S}(\overline{\mathbb{R}}_+)\oplus\mathbb{C}^{j_+})$.

Definition 1.5 has a straightforward extension to the case of $l\times k$-block matrix valued upper left corners, in other words, where p is replaced by $(p_{ij})_{i=1,\dots,l;\, j=1,\dots,k}$, and $g(x',\xi')$ is operator-valued symbol between spaces $H^s(\mathbb{R}_+,\mathbb{C}^k) \oplus \mathbb{C}^{j_-}$ and $\mathcal{S}(\overline{\mathbb{R}}_+,\mathbb{C}^l) \oplus \mathbb{C}^{j_+}$. We then get $\mathcal{R}^{\mu,d}(U \times \mathbb{R}^q; \mathbf{w})$, $\mathbf{w} := (k,l;j_-,j_+)$, as the corresponding generalisation of the former space.
The numbers k,l and j_-,j_+ will play the role of fibre dimensions of corresponding complex vector bundles in the global calculus below. Then, for $\Omega \subseteq \mathbb{R}^q$ instead of U we have invariance of the corresponding spaces $\mathcal{R}^{\mu,d}(U \times \mathbb{R}^q; \mathbf{w})$ under substituting the transition maps $(\Omega\times\overline{\mathbb{R}}_+)\times\mathbb{C}^k \to (\Omega\times\overline{\mathbb{R}}_+)\times\mathbb{C}^k, \dots, \Omega\times\mathbb{C}^{j_\pm} \to \Omega\times\mathbb{C}^{j_\pm}$ of the respective bundles.
For the set U we either take $\Omega \times \Omega$ if we want to talk about "double symbols" or Ω for "left symbols". For simplicity, we look at the latter case. Every element $a(x',\xi') \in \mathcal{R}^{\mu,d}(\Omega \times \mathbb{R}^q; j_-, j_+)$ has a homogeneous principal interior symbol, namely

$$\sigma_\psi^\mu(a)(x,\xi) = p_{(\mu)}(x,\xi), \tag{1.12}$$

$(x,\xi) \in T^*(\Omega \times \overline{\mathbb{R}}_+) \setminus 0$ and a homogeneous principal boundary symbol, defined as

$$\sigma_\partial(a)(x',\xi') := \begin{pmatrix} \sigma_\partial(p)(x',\xi') & 0 \\ 0 & 0 \end{pmatrix} + \sigma_\partial(g)(x',\xi'), \tag{1.13}$$

$(x',\xi') \in T^*\Omega \setminus 0$. While (1.12) is a scalar function as usual, (1.13) is operator-valued and defines a family of continuous maps

$$\sigma_\partial(a)(x',\xi') : \begin{array}{c} H^s(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{j_-} \end{array} \to \begin{array}{c} H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{j_+} \end{array} \tag{1.14}$$

for every real $s > d - \frac{1}{2}$. For $\sigma_\partial(a)$ we have an analogous homogeneity as (1.10). In the discussion of ellipticity it will be sufficient to consider $\sigma_\partial(a)$ as a family of operators

$$\sigma_\partial(a)(x',\xi') : \begin{array}{c} \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{j_-} \end{array} \to \begin{array}{c} \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{j_+} \end{array}. \tag{1.15}$$

Proposition 1.7. $a \in \mathcal{R}^{\mu,d}(\Omega \times \mathbb{R}^q; (j_0, j_+))$, $b \in \mathcal{R}^{\nu,e}(\Omega \times \mathbb{R}^q; (j_-, j_0))$ *implies* $ab \in \mathcal{R}^{\mu+\nu,h}(\Omega \times \mathbb{R}^q; (j_-, j_+))$ *for* $h = \max(\nu + d, e)$, *and we have*

$$\sigma_\psi(ab) = \sigma_\psi(a)\sigma_\psi(b), \quad \sigma_\partial(ab) = \sigma_\partial(a)\sigma_\partial(b).$$

Moreover, if a or b is a Green symbol, the same is true for ab.

Proofs of this result may be found in [20], Chapter 2, or [31], Chapter 4.

1.5 Global calculus and ellipticity

We now pass to parameter-dependent boundary value problems on a (not necessarly compact) C^∞ manifold X with boundary ∂X. All manifolds in consideration are assumed to be locally compact and paracompact. Consider on X a collar neighbourhood V of ∂X diffeomorphic to $\partial X \times [0,1)$ and fix Riemannian metrics on X and ∂X such that the metric on X corresponds to the product metric of $\partial X \times [0,1)$ on V with dx_n on the x_n-interval $[0,1)$. Concerning charts on X for convenience we always assume transition maps near the boundary to be independent of x_n for small x_n.

Let $\mathrm{Vect}(\cdot)$ denote the set of all complex smooth vector bundles on the manifold in the brackets. All occurring complex bundles are assumed to be equipped with a Hermitian metric. Together with the Riemannian metrics on the respective base manifolds we then have local L^2-scalar products. E.g., for $E \in \mathrm{Vect}(X)$ there is a sesquilinear pairing

$$(u,v) := \int_X (u(x), v(x))_{E_x}\, dx$$

between $u, v \in C^\infty(X,E)$ where either u or v have compact support; here $(\cdot,\cdot)_{E_x}$ means the pairing between vectors in the fibres with respect to the Hermitian metric in the fibre. Similarly, we have local scalar products on ∂X between sections of bundles $J \in \mathrm{Vect}(\partial X)$. Corresponding local L^2-spaces are denoted by $L^2_{\mathrm{loc}}(X,E)$ and $L^2_{\mathrm{loc}}(\partial X, J)$, respectively.

If $\widetilde{X} = 2X$ is the double of X we also have a natural way to double up elements $E \in \mathrm{Vect}(X)$ to bundles $\widetilde{E} \in \mathrm{Vect}(\widetilde{X})$ such that $E = \widetilde{E}|_X$ (where X is identified with X_+) (i.e., $\widetilde{X} = X_- \bigcup_g X_+$ where $X_\pm$ are two copies of X and $\bigcup_g$ means glueing along ∂X). To introduce (local) Sobolev spaces of distributional sections on X in E we first consider the situation with $\widetilde{X}$ and $\widetilde{E}$. Then the notions $H^s_{\mathrm{comp}}(\widetilde{X},\widetilde{E})$ and $H^s_{\mathrm{loc}}(\widetilde{X},\widetilde{E})$ for $s \in \mathbb{R}$ are standard, where, in particular, for $s = 0$ we also write $L^2_{\mathrm{comp}}(\widetilde{X},\widetilde{E})$ and $L^2_{\mathrm{loc}}(\widetilde{X},\widetilde{E})$ with the above sesquilinear pairing $(\cdot,\cdot) : L^2_{\mathrm{comp}}(\widetilde{X},\widetilde{E}) \times L^2_{\mathrm{loc}}(\widetilde{X},\widetilde{E}) \longrightarrow \mathbb{C}$. Of course, such a construction makes sense on ∂X, i.e., we have the spaces $H^s_{\mathrm{comp}}(\partial X, J)$, $H^s_{\mathrm{loc}}(\partial X, J)$ and, in particular, the comp/loc-version of L^2 spaces. Concerning $X = X_+$ we set

$$H^s_{\mathrm{loc}}(X,E) = \{u|_{X_+} : u \in H^s_{\mathrm{loc}}(\widetilde{X},\widetilde{E})\}$$

and, similarly, for "comp", and we then have the comp/loc-versions of L^2 spaces on X. Our next objective is to introduce global smoothing operators

$$\mathcal{G} : \begin{array}{c} C_0^\infty(X,E) \\ \oplus \\ C_0^\infty(\partial X, J_-) \end{array} \longrightarrow \begin{array}{c} C^\infty(X,F) \\ \oplus \\ C^\infty(\partial X, J_+) \end{array} \tag{1.16}$$

for $E, F \in \mathrm{Vect}(X)$, $J_-, J_+ \in \mathrm{Vect}(\partial X)$. It will be advantageous to abbreviate the bundles by $\mathbf{v} = (E, F; J_-, J_+)$.

Let $\mathcal{B}^{-\infty,0}(X;\mathbf{v})$ denote the space of all operators (1.16) that extend to continuous operators

$$\mathcal{G} : \begin{array}{c} H^s_{\mathrm{comp}}(X,E) \\ \oplus \\ H^{s-\frac{1}{2}}_{\mathrm{comp}}(\partial X, J_-) \end{array} \longrightarrow \begin{array}{c} C^\infty(X,F) \\ \oplus \\ C^\infty(\partial X, J_+) \end{array}$$

for all real $s > -\frac{1}{2}$, where the formal adjoint $\mathcal{G}^*$, defined by

$$(\mathcal{G}u, v)_{L^2_{\mathrm{loc}}(X,F)\oplus L^2_{\mathrm{loc}}(\partial X,J_+)} = (u, \mathcal{G}^* v)_{L^2_{\mathrm{loc}}(X,E)\oplus L^2_{\mathrm{loc}}(\partial X,J_-)}$$

for all $u \in C_0^\infty(X,E) \oplus C_0^\infty(\partial X, J_-)$, $v \in C_0^\infty(X,F) \oplus C_0^\infty(\partial X, J_+)$, induces a continuous operator

$$\mathcal{G}^* : \begin{array}{c} H^s_{\mathrm{comp}}(X,F) \\ \oplus \\ H^{s-\frac{1}{2}}_{\mathrm{comp}}(\partial X, J_+) \end{array} \longrightarrow \begin{array}{c} C^\infty(X,E) \\ \oplus \\ C^\infty(\partial X, J_-) \end{array}$$

for all real $s > -\frac{1}{2}$.

Let T be a first order differential operator on X (with smooth coefficients up to ∂X) that acts as $C^\infty(X,E) \to C^\infty(X,E)$ where $Tu|_V = \frac{\partial}{\partial t} u|_V$ with t being the global normal direction in the collar neighbourhood $V \cong \partial X \times [0,1)$. Writing $\mathcal{G} = (G_{ij})_{i,j=1,2}$, the upper left corner G_{11} is also called a smoothing Green operator of type 0, G_{21} a smoothing trace operator of type 0, and G_{12} a smoothing potential operator. G_{22} is simply an element of $L^{-\infty}(\partial X; J_-, J_+)$ (i.e., smoothing on ∂X and acting between sections in J_- and J_+). Now $\mathcal{B}^{-\infty,d}(X;\mathbf{v})$ for any $d \in \mathbb{N}$ is defined to be the space of all operators of the form

$$\mathcal{G} = \mathcal{G}_0 + \sum_{j=1}^{d} \mathcal{G}_j \,\mathrm{diag}(T^j, 0) \tag{1.17}$$

with arbitrary $\mathcal{G}_j \in \mathcal{B}^{-\infty,0}(X;\mathbf{v})$, $j = 0, \ldots, d$. Note that the operators $\mathcal{G}_j$ in (1.17) are not unique for $d > 0$. We may pass to a unique representation of the form

$$\mathcal{G} = \mathcal{G}_0 + \sum_{j=0}^{d-1} \begin{pmatrix} K_j \gamma^j & 0 \\ B_j \gamma^j & 0 \end{pmatrix} \tag{1.18}$$

where $\gamma^j u := T^j u|_{\partial X}$ and K_j are smoothing potential operators, B_j smoothing operators on the boundary ∂X.

Notice that $\mathcal{B}^{-\infty,0}(X;\mathbf{v})$ is a Fréchet space in a natural way. Then, using the representation (1.18) for elements in $\mathcal{B}^{-\infty,d}(X;\mathbf{v})$, also that space becomes Fréchet. Now, if $\mathbb{R}^l \ni \lambda$ is a space of parameters, we set

$$\mathcal{B}^{-\infty,d}(X;\mathbf{v};\mathbb{R}^l) := \mathcal{S}(\mathbb{R}^l, \mathcal{B}^{-\infty,d}(X;\mathbf{v})),$$

that is again Fréchet, or, more generally,

$$\mathcal{B}^{-\infty,d}(X;\mathbf{v};U\times\mathbb{R}^l):=C^\infty(U,\mathcal{B}^{-\infty,d}(X;\mathbf{v};\mathbb{R}^l))$$

for any open set $U\subseteq\mathbb{R}^p$.

Let us now pass to the definition of global pseudo-differential operators of the class $\mathcal{B}^{\mu,d}(X;\mathbf{v};U\times\mathbb{R}^l)$. For simplicity, we first omit U; the generalization of dependence on variables in U will be trivial.

First, we have the space $L^\mu_{\mathrm{cl}}(\operatorname{int}X;E,F;\mathbb{R}^l)$ of classical parameter-dependent pseudo-differential operators on $\operatorname{int}X$, acting between distributional sections of the respective bundles E and F.

Moreover, let us fix a locally finite open covering of V by coordinate neighbourhoods V_j, $j\in\mathbb{N}$, $V_j\cong V_j'\times[0,1)$, such that V_j', $j\in\mathbb{N}$ form a corresponding covering of ∂X, and consider charts $\chi_j:V_j\to\Omega_j\times[0,1)$ that induce charts $\chi_j':V_j'\to\Omega_j$ on the boundary. Let $\omega,\tilde\omega\in C^\infty(X)$ be functions supported in V that are equal to 1 in a neighbourhood of ∂X. Moreover, fix systems of functions $\{\phi_j\}_{j\in\mathbb{N}}$ and $\{\theta_j\}_{j\in\mathbb{N}}$, where $\operatorname{supp}\phi_j,\operatorname{supp}\theta_j\subset V_j$, $\sum\limits_{j\in\mathbb{N}}\phi_j=1$ in a neighbourhood of ∂X, and $\phi_j\theta_j=\phi_j$ for all j. Then, $\{\phi_j'\}_{j\in\mathbb{N}}$ for $\phi_j':=\phi_j|_{\partial X}$ is a partition of unity on ∂X subordinated to $\{V_j'\}_{j\in\mathbb{N}}$ and $\theta_j':=\theta_j|_{\partial X}$ satisfies $\phi_j'\theta_j'=\phi_j'$ for all j. Given any $\phi\in C^\infty(X)$ we set $\phi'=\phi|_{\partial X}$ and denote by $\mathcal{M}_\varphi$ the operator of multiplication by $\operatorname{diag}(\phi\,\mathrm{id}_E,\phi'\,\mathrm{id}_J)$ acting in the space $C^\infty(X,E)\oplus C^\infty(\partial X,J)$. For simplicity we will often omit id_E and id_J when the meaning is clear or corresponding identity maps refer to different bundles that are clear in the context.

Definition 1.8. *The space $\mathcal{B}^{\mu,d}(X;\mathbf{v};\mathbb{R}^l)$, $\mathbf{v}=(E,F;J_-,J_+)$, of parameter-dependent pseudo-differential boundary value problems of order $\mu\in\mathbb{Z}$ and type $d\in\mathbb{N}$ is defined to be the set of all operator families*

$$\mathcal{A}(\lambda)=\mathcal{M}_\omega\sum_{j\in\mathbb{N}}\mathcal{M}_{\phi_j}\mathcal{A}_j(\lambda)\mathcal{M}_{\theta_j}\mathcal{M}_{\tilde\omega}+\begin{pmatrix}(1-\omega)A_{\mathrm{int}}(\lambda)(1-\tilde\omega)&0\\0&0\end{pmatrix}+\mathcal{G}(\lambda)\tag{1.19}$$

such that

(i) *$\mathcal{A}_j(\lambda)$ is the operator push-forward under $\chi_j^{-1}:\Omega_j\times[0,1)\to V_j$ (that includes the chosen trivializations of the bundles in $\mathbf{v}=(E,F;J_-,J_+)$ on the respective coordinate neighbourhoods) of an operator $\mathrm{Op}(a_j)(\lambda)$ for any $a_j(x',\xi',\lambda)\in\mathcal{R}^{\mu,d}(\Omega_j\times\mathbb{R}^{n-1+l};\mathbf{w})$, where $\mathbf{w}=(m,k;j_-,j_+)$ is the tuple of fibres dimensions of the bundles in $\mathbf{v}$;*

(ii) *$A_{\mathrm{int}}(\lambda)\in L^\mu_{\mathrm{cl}}(\operatorname{int}X;E,F;\mathbb{R}^l)$;*

(iii) *$\mathcal{G}(\lambda)\in\mathcal{B}^{-\infty,d}(X;\mathbf{v};\mathbb{R}^l)$.*

For references below we formulate the following standard continuity result for operators in $\mathcal{B}^{\mu,d}(X;\mathbf{v};\mathbb{R}^l)$, see Boutet de Monvel [2], or the monographs [20], Chapter 2, [31], Chapter 4.

Theorem 1.9. *Every $\mathcal{A} \in \mathcal{B}^{\mu,d}(X;\mathbf{v};\mathbb{R}^l)$ induces a family of continuous operators*

$$\mathcal{A}(\lambda) : \begin{array}{c} H^s_{\mathrm{comp}}(X,E) \\ \oplus \\ H^{s-\frac{1}{2}}_{\mathrm{comp}}(\partial X, J_-) \end{array} \to \begin{array}{c} H^{s-\mu}_{\mathrm{loc}}(X,F) \\ \oplus \\ H^{s-\mu-\frac{1}{2}}_{\mathrm{loc}}(\partial X, J_+) \end{array}$$

for every $s > d - \frac{1}{2}$.

The local principal interior and boundary symbols of the amplitude functions $a_j(x',\xi',\lambda)$ in Definition 1.8 (i) as well as the principal symbols of $A_{\mathrm{int}}(\lambda)$ (where the parameter λ is treated as a part of the covariables (ξ,λ) and (ξ',λ), respectively) have an invariant meaning and give rise to corresponding global parameter-dependent principal symbols. We have the principal interior and boundary symbols $\sigma_\psi(\mathcal{A})(x,\xi,\lambda)$, $\quad \sigma_\partial(\mathcal{A})(x',\xi',\lambda)$, which are bundle morphisms

$$\sigma_\psi(\mathcal{A}) : \pi_X^* E \to \pi_X^* F \tag{1.20}$$

for $\pi_X : (T^*X \times \mathbb{R}^l) \setminus 0 \to X$, and

$$\sigma_\partial(\mathcal{A}) : \pi_{\partial X}^* \begin{pmatrix} E' \otimes H^s(\mathbb{R}_+) \\ \oplus \\ J_- \end{pmatrix} \to \pi_{\partial X}^* \begin{pmatrix} F' \otimes H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ J_+ \end{pmatrix} \tag{1.21}$$

for $\pi_{\partial X} : (T^*\partial X \times \mathbb{R}^l) \setminus 0 \to \partial X$. Similarly to (1.10) we have DN-homogeneity of $\sigma_\partial(\mathcal{A})$, namely

$$\sigma_\partial(\mathcal{A})(x',\delta\xi',\delta\lambda) = \delta^\mu \mathrm{diag}\,\{\kappa_\delta, \delta^{\frac{1}{2}}\}\sigma_\partial(\mathcal{A})(x',\xi',\lambda)\mathrm{diag}\,\{\kappa_\delta, \delta^{\frac{1}{2}}\}^{-1}$$

for all $(x',\xi',\lambda) \in (T^*\partial X \times \mathbb{R}^l) \setminus 0$.

Remark 1.10. *The spaces $\mathcal{B}^{\mu,d}(X;\mathbf{v};\mathbb{R}^l)$ are Fréchet in a natural way, cf.* [31], Section 4.3.2.

An operator family $\mathcal{A}(\lambda) \in \mathcal{B}^{\mu,d}(X;\mathbf{v};\mathbb{R}^l)$ for $\mathbf{v} = (E,F;J_-,J_+)$ is called parameter-dependent elliptic (with parameters $\lambda \in \mathbb{R}^l$), if both (1.20) and (1.21) are isomorphisms.
As is well-known, if $\mathcal{A}(\lambda) \in \mathcal{B}^{\mu,d}(X;\mathbf{v};\mathbb{R}^l)$ is parameter-dependent elliptic, there is a paramenter-dependent parametrix $\mathcal{P}(\lambda) \in \mathcal{B}^{-\mu,(d-\mu)^+}(X;\mathbf{v}^{-1};\mathbb{R}^l)$ of $\mathcal{A}(\lambda)$ (with proper support of kernels that make the following compositions possible) such that

$$\mathcal{I} - \mathcal{P}(\lambda)\mathcal{A}(\lambda) \in \mathcal{B}^{-\infty,\max(\mu,d)}(X;\mathbf{v}_\ell;\mathbb{R}^l)$$
$$\mathcal{I} - \mathcal{A}(\lambda)\mathcal{P}(\lambda) \in \mathcal{B}^{-\infty,(d-\mu)^+}(X;\mathbf{v}_r;\mathbb{R}^l),$$

with notation in the following meaning: $\alpha^+ = \max(\alpha, 0)$, $\mathbf{v}^{-1} = (F, E; J_+, J_-)$, $\mathbf{v}_\ell = (E, E; J_-, J_-)$, $\mathbf{v}_r = (F, F; J_+, J_+)$.

2 The parameter-dependent edge algebra

2.1 Holomorphic families of boundary value problems

In this section we assume X to be a compact C^∞ manifold with boundary ∂X. In Section 1.5 we have introduced the spaces $\mathcal{B}^{\mu,d}(X; \mathbf{v}; \mathbb{R}^l)$ of parameter-dependent boundary value problems of order $\mu \in \mathbb{Z}$ and type $d \in \mathbb{N}$, where $\mathbb{R}^l \ni \lambda$ are parameters, and $\mathbf{v} = (E, F; J_-, J_+)$ is a tuple of vector bundles. The space $\mathcal{B}^{\mu,d}(X; \mathbf{v}; \mathbb{R}^l)$ will be considered in its natural Fréchet topology. In the sequel we replace $\mathbb{R}^l$ by $\Gamma_\beta \times \mathbb{R}^l$. Here, $\Gamma_\beta := \{z \in \mathbb{C} : \Re z = \beta\}$ for any $\beta \in \mathbb{R}$, and $\Im z$ for $z \in \Gamma_\beta$ plays the role of parameter.

Definition 2.1. $\mathcal{M}^{\mu,d}_{\mathcal{O}}(X; \mathbf{v}; \mathbb{R}^l)$ *is defined to be the subspace of all the elements* $h(z) \in \mathcal{A}(\mathbb{C}, \mathcal{B}^{\mu,d}(X; \mathbf{v}; \mathbb{R}^l))$ *such that* $h(z)|_{\Gamma_\beta} \in \mathcal{B}^{\mu,d}(X; \mathbf{v}; \Gamma_\beta \times \mathbb{R}^l)$ *for every* $\beta \in \mathbb{R}$, *uniformly in* $c \le \beta \le c'$ *for arbitrary* $c \le c'$.

Notice that also $\mathcal{M}^{\mu,d}_{\mathcal{O}}(X; \mathbf{v}; \mathbb{R}^l)$ is a Fréchet space in a canonical way. If $l = 0$ we simply write $\mathcal{M}^{\mu,d}_{\mathcal{O}}(X; \mathbf{v})$ for the corresponding space.
Operators on a cone with base X and axial variable $r \in \mathbb{R}_+$ will be written as pseudo-differential operators with respect to the Mellin transform in r and with symbols taking values in $\mathcal{M}^{\mu,d}_{\mathcal{O}}(X; \mathbf{v}; \mathbb{R}^l)$.
We employ the Mellin transform in its classical form $Mu(z) = \int_0^\infty r^{z-1} u(r)\, dr$. This is well-defined for $u \in C_0^\infty(\mathbb{R}_+)$; then z varies over $\mathbb{C}$. Later on, M will be extended to several more general distribution spaces, also vector-valued ones. In this case we often restrict z to weight lines $\Gamma_\beta = \{z \in \mathbb{C} : \Re z = \beta\}$ for some real β. The Mellin transform will also be used in its weighted form with weight $\gamma \in \mathbb{R}$, and we write $M_\gamma u(z) = M(r^{-\gamma} u)(z + \gamma)$. Pseudo-differential operators with respect to M_γ are written as

$$\mathrm{op}_M^\gamma(h) u(r) = \int \int \Big(\frac{r}{r'}\Big)^{-(\frac{1}{2} - \gamma + i\rho)} h\big(r, r', \frac{1}{2} - \gamma + i\rho\big) u(r') \, \frac{dr'}{r'} \, đ\rho \tag{2.1}$$

for symbols $h(r, r', z) \in C^\infty(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+, \mathcal{M}^{\mu,d}_{\mathcal{O}}(X; \mathbf{v}))$, where z is restricted to $\Gamma_{\frac{1}{2}-\gamma}$.
Let us now define Sobolev spaces on a stretched cone $\mathbb{R}_+ \times N := N^\wedge$, first for a closed compact C^∞ manifold N. We use the fact that the space $L^\mu_{\mathrm{cl}}(N; \mathbb{R}^l)$ of parameter-dependent pseudo-differential operators of order μ on N contains elements $R^\mu(\lambda)$ that induce isomorphisms $R^\mu(\lambda) : H^s(N) \to H^{s-\mu}(N)$ for all $s \in \mathbb{R}$ and $\lambda \in \mathbb{R}^l$. Let us apply this for $l = 1$. Then $\mathcal{H}^{s,\gamma}(N^\wedge)$ for $s, \gamma \in \mathbb{R}$ is

defined as the completion of $C_0^\infty(N^\wedge)$ with respect to the norm

$$\Big\{\frac{1}{2\pi i}\int_{\Gamma_{\frac{n+1}{2}-\gamma}} \|R^s(\Im z)(Mu)(z)\|^2_{L^2(N)}\,dz\Big\}^{\frac{1}{2}},$$

for $n = \dim N$. Note that another choice of $R^s(\lambda)$ with the mentioned properties gives rise to an equivalent norm. The definition can be extended to the case of a compact C^∞ manifold N with boundary, see similar constructions below in a slightly modified situation.

Theorem 2.2. *Let* $\tilde{p}(r,\tilde{\rho},\lambda) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{B}^{\mu,d}(X;\mathbf{v};\mathbb{R}_{\tilde{\rho}}\times\mathbb{R}^l_\lambda))$ *and set* $p(r,\rho,\lambda) := \tilde{p}(r,r\rho,\lambda)$. *Then there exists an* $h(r,z,\lambda) \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}^{\mu;d}_{\mathcal{O}}(X;\mathbf{v};\mathbb{R}^l))$ *such that*

$$\mathrm{op}_r(p)(\lambda) - \mathrm{op}^\beta_M(h)(\lambda) \in \mathcal{B}^{-\infty,d}(X^\wedge;\mathbf{v};\mathbb{R}^l) \tag{2.2}$$

for every $\beta\in\mathbb{R}$, *where* $h(r,z,\lambda)$ *is uniquely determined* mod $C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}^{-\infty,d}_{\mathcal{O}}(X;\mathbf{v};\mathbb{R}^l))$.

A proof of this result may be found in [10], Section 4.6.7, see also the article [11].

Remark 2.3. *Operators in relation* (2.2) *are interpreted as mappings on the space* $C_0^\infty(X^\wedge,E)\oplus C_0^\infty((\partial X)^\wedge,J_-)$ *(*$\beta\in\mathbb{R}$ *is then arbitrary because of the holomorphy of* h *in* z *and the Cauchy theorem). Below we pass to extensions to weighted spaces and then specify the choice of* β.

2.2 Parameter-dependent operators on the model cone

We now turn to a calculus of families of boundary value problems on the infinite (stretched) cone $X^\wedge = \mathbb{R}_+\times X$ with boundary $(\partial X)^\wedge = \mathbb{R}_+\times\partial X$. There are specific smoothing operator families that we call Green symbols.

We will need another kind of weighted Sobolev spaces on infinite stretched cones. Let us give the definition first for the case $N^\wedge = \mathbb{R}_+\times N$ when N is a closed compact C^∞ manifold. Consider an arbitrary coordinate neighbourhood $U\subset N$, and let $\chi : U^\wedge\to\Gamma$ be a diffeomorphism to a conical set $\Gamma\subset\mathbb{R}^{n+1}_{\tilde{x}}\setminus\{0\}$ such that $\chi(\lambda r,x) = \lambda\chi(r,x)$ for all $\lambda\in\mathbb{R}_+$, $(r,x)\in U^\wedge$. Then $H^s_{\mathrm{cone}}(N^\wedge)$ denotes the subspace of all $u(r,x)\in H^s_{\mathrm{loc}}(\mathbb{R}\times N)|_{\mathbb{R}_+\times N}$ such that for every $\chi : U^\wedge\to\Gamma$ as described before we have $(\chi^{-1})^*(1-\omega)\phi u\in H^s(\mathbb{R}^{n+1})$ for arbitrary $\phi\in C_0^\infty(U)$ and any cut-off function $\omega(r)$. The space $H^s_{\mathrm{cone}}(N^\wedge)$ can be endowed with a scalar product such that we get a Banach space with the corresponding norm. We now form the space

$$\mathcal{K}^{s,\gamma}(N^\wedge) = \{\omega u + (1-\omega)v : u\in\mathcal{H}^{s,\gamma}(N^\wedge),\ v\in H^s_{\mathrm{cone}}(N^\wedge)\}.$$

In order to fix a norm in this space, connected with a scalar product, we define non-direct sums, for purposes below, for Fréchet spaces E_0, E_1 that are embedded in a Hausdorff topological vector space H. The space

$$E_0 + E_1 := \{e_0 + e_1 : e_0 \in E_0,\, e_1 \in E_1\} \tag{2.3}$$

is isomorphic to $E_0 \oplus E_1/\Delta$ for $\Delta := \{(e, -e) : e \in E_0 \bigcap E_1\}$; the latter space is closed in $E_0 \oplus E_1$, and we endow (2.3) with the corresponding quotient topology. Then (2.3) is again a Fréchet space, called the non-direct sum of E_0 and E_1. If E_0 and E_1 are Hilbert spaces, also $E_0 + E_1$ becomes a Hilbert space with the scalar product from $E_0 \oplus E_1$, restricted to the orthogonal complement of Δ.
If a Fréchet space E is a module over an algebra A, by $[a]E$ we denote the completion of $\{ae : e \in E\}$ in the space E. Then, in particular, we can write

$$\mathcal{K}^{s,\gamma}(N^\wedge) = [\omega]\mathcal{H}^{s,\gamma}(N^\wedge) + [1-\omega]H^s_{\mathrm{cone}}(N^\wedge)$$

as a non-direct sum.
Let X be a compact C^∞ manifold with boundary, and let $2X$ be the double, consisting of two copies $X_+ = X$ and X_-, glued together along ∂X. We then have the space $\mathcal{K}^{s,\gamma}((2X)^\wedge)$, and we set

$$\mathcal{K}^{s,\gamma}_0(X^\wedge_\pm) := \{u \in \mathcal{K}^{s,\gamma}((2X)^\wedge) : \operatorname{supp} u \subset X^\wedge_\pm\}$$

and

$$\mathcal{K}^{s,\gamma}(X^\wedge) := \{u|_{(\operatorname{int} X)^\wedge} : u \in \mathcal{K}^{s,\gamma}((2X)^\wedge)\}.$$

From the isomorphism $\mathcal{K}^{s,\gamma}(X^\wedge) \cong \mathcal{K}^{s,\gamma}((2X)^\wedge)/\mathcal{K}^{s,\gamma}_0(X^\wedge_-)$ we then get a Hilbert space structure on the space $\mathcal{K}^{s,\gamma}(X^\wedge)$ for every $s, \gamma \in \mathbb{R}$. In particular, we have $\mathcal{K}^{0,0}(X^\wedge) = r^{-\frac{n}{2}}L^2(\mathbb{R}_+ \times X)$ for $n = \dim X$, where L^2 refers to the measure $dr\,dx$, with dx being related to a Riemannian metric on X.
Similarly we can define spaces $\mathcal{K}^{s,\gamma}(X^\wedge, E)$ of distributional sections in vector bundles E on $X^\wedge$. Every E can be regarded as the pull-back of a corresponding bundle on X with respect to the canonical projection $\mathbb{R}_+ \times X \to X$, $(r, x) \to x$, and we assume every $E \in \operatorname{Vect}(X^\wedge)$ to be endowed with a Hermitian metric that is independent of r. In particular, this gives rise to a scalar product $(\cdot,\cdot)$ in $\mathcal{K}^{0,0}(X^\wedge, E)$, and $(\cdot,\cdot) : C^\infty_0(X^\wedge, E) \times C^\infty_0(X^\wedge, E) \to \mathbb{C}$ gives rise to non-degenerate sesquilinear pairings $(\cdot,\cdot) : \mathcal{K}^{s,\gamma}(X^\wedge, E) \times \mathcal{K}^{-s,-\gamma}_0(X^\wedge, E) \to \mathbb{C}$ for all $s, \gamma \in \mathbb{R}$. Analogous constructions hold for the case of a closed compact manifold as base (then, of course, without any zero in the corresponding sesquilinear pairings).

Remark 2.4. *On the spaces $\mathcal{K}^{s,\gamma}(X^\wedge, E)$ and $\mathcal{K}^{s,\gamma}((\partial X)^\wedge, J)$ we define the group actions $\{\kappa^{(n)}_\lambda\}_{\lambda\in\mathbb{R}_+}$ and $\{\kappa^{(n-1)}_\lambda\}_{\lambda\in\mathbb{R}_+}$, respectively, where $n = \dim X$, and $\kappa^{(k)}_\lambda f(r, x) := \lambda^{\frac{k+1}{2}} f(\lambda r, x)$ for $\lambda \in \mathbb{R}_+$.*

It will also be convenient to employ Sobolev spaces on $X^\wedge \ni (r,x)$ with arbitrary weights for $r \to 0$ and $r \to \infty$. Let us set $\mathcal{K}^{s,\gamma;\beta}(X^\wedge, E) := \langle r\rangle^{-\beta}\mathcal{K}^{s,\gamma}(X^\wedge, E)$, $E \in \mathrm{Vect}\,(X^\wedge)$, and, similarly, $\mathcal{K}^{s,\gamma;\beta}((\partial X)^\wedge, J) := \langle r\rangle^{-\beta}\mathcal{K}^{s,\gamma}((\partial X)^\wedge, J)$, $J \in \mathrm{Vect}\,((\partial X)^\wedge)$, for $s,\beta,\gamma \in \mathbb{R}$. Then $\{\kappa_\lambda^{(n)}\}_{\lambda\in\mathbb{R}_+}$ and $\{\kappa_\lambda^{(n-1)}\}_{\lambda\in\mathbb{R}_+}$ are strongly continuous groups of isomorphism on the respective spaces on $X^\wedge$ and $(\partial X)^\wedge$, respectively, for all $s,\beta,\gamma \in \mathbb{R}$. Let us set

$$\mathcal{S}_{\mathcal{O}}(X^\wedge, E) := \varprojlim \mathcal{K}^{s,\gamma;\beta}(X^\wedge, E), \quad \mathcal{S}_{\mathcal{O}}(X^\wedge, J) := \varprojlim \mathcal{K}^{s,\gamma;\beta}((\partial X)^\wedge, J)$$

where the projective limits are taken over all $s,\gamma,\beta \in \mathbb{R}$. (Clearly, it suffices to take projective limits over all integers, and we then get countable semi-norms systems in the respective Fréchet spaces). In the following definition we set $\mathbf{v} = (E,F;J_-,J_+)$ for $E,F \in \mathrm{Vect}\,(X^\wedge)$, $J_-,J_+ \in \mathrm{Vect}\,((\partial X)^\wedge)$. It will be convenient to set

$$\mathcal{K}^{s,\gamma;\beta}(X^\wedge;\mathbf{m}) := \mathcal{K}^{s,\gamma;\beta}(X^\wedge, E) \oplus \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2};\beta}((\partial X)^\wedge, J_-)$$

for $\mathbf{m} := (E,J_-)$ and, similarly, for $\mathbf{n} := (F,J_+)$, with the group action given by $\mathrm{diag}(\{\kappa_\lambda^{(n)}\},\{\kappa_\lambda^{(n-1)}\})_{\lambda\in\mathbb{R}_+}$. Then, on spaces of the kind $\mathcal{K}^{s,\gamma;\beta}(X^\wedge;\mathbf{m}) \oplus \mathbb{C}^l$ for some $l \in \mathbb{N}$ we take the group action $\mathrm{diag}(\{\kappa_\lambda^{(n)}\},\{\kappa_\lambda^{(n-1)}\},\mathrm{id})_{\lambda\in\mathbb{R}_+}$, where id is the identity on $\mathbb{C}^l$.

Let $r(\eta) := \mathrm{diag}(\mathrm{id}, \langle\eta\rangle^{\frac{1}{2}}\mathrm{id}, \langle\eta\rangle^{\frac{n+1}{2}})$ where id means identity maps on spaces of distributional sections of vector bundles on $X^\wedge$ and $(\partial X)^\wedge$, respectively, while $\langle\eta\rangle^{\frac{n+1}{2}}$ in the third component is composed with the identity in $\mathbb{C}^l$ for the corresponding dimension l. We shall use the same notation $r(\eta)$ for different bundles and dimensions that will be clear by the context.

Let T denote any first order differential operator on X with smooth coefficients that equals $\frac{d}{dt}$ in a collar neighbourhood of the boundary (with respect to the chosen splitting of variables $x = (x',t)$).

Definition 2.5. *The space $\mathcal{R}_G^{\nu,0}(U \times \mathbb{R}^q;\mathbf{w})_{\mathcal{O}}$ for $\mathbf{w} := (\mathbf{v};l_-,l_+)$ is defined to be the set of all operator families $g(y,\eta) \in \bigcap C^\infty(U \times \mathbb{R}^q, \mathcal{L}(\mathcal{K}^{s,\gamma;\beta}(X^\wedge;\mathbf{m}),$ $\mathcal{K}^{s',\gamma';\beta'}(X^\wedge;\mathbf{n}))$, $U \subseteq \mathbb{R}^p$ open, where the intersection is taken over all indices $s,\gamma,\beta,s',\gamma',\beta' \in \mathbb{R}$, such that $b(y,\eta) := r^{-1}(\eta)g(y,\eta)r(\eta)$ are operator-valued symbols in the following sense:*

$$b(y,\eta) \in \bigcap S_{\mathrm{cl}}^\nu(U \times \mathbb{R}^q; \mathcal{K}^{s,\gamma;\beta}(X^\wedge;\mathbf{m}) \oplus \mathbb{C}^{l_-}, \mathcal{S}_{\mathcal{O}}(X^\wedge;\mathbf{n}) \oplus \mathbb{C}^{l_+}), \tag{2.4}$$

$$b^*(y,\eta) \in \bigcap S_{\mathrm{cl}}^\nu(U \times \mathbb{R}^q; \mathcal{K}^{s',\gamma';\beta'}(X^\wedge;\mathbf{n}) \oplus \mathbb{C}^{l_+}, \mathcal{S}_{\mathcal{O}}(X^\wedge;\mathbf{m}) \oplus \mathbb{C}^{l_-}) \tag{2.5}$$

where the intersections are taken over all $s,s' > -\frac{1}{2}$ and $\gamma,\beta,\gamma',\beta' \in \mathbb{R}$. More generally, $\mathcal{R}_G^{\nu,d}(U \times \mathbb{R}^q;\mathbf{w})_{\mathcal{O}}$ for $d \in \mathbb{N}$ is defined to be the set of all

$$g(y,\eta) = g_0(y,\eta) + \sum_{j=1}^{d} g_j(y,\eta)\,\mathrm{diag}(T^j,0,0) \tag{2.6}$$

for arbitrary $g_j(y,\eta) \in \mathcal{R}_G^{\nu-j,0}(U \times \mathbb{R}^q; \mathbf{w})_{\mathcal{O}}$. *The elements of* $\mathcal{R}_G^{\nu,d}(U \times \mathbb{R}^q; \mathbf{w})_{\mathcal{O}}$ *are called flat Green symbols (of the edge calculus) of order* ν *and type* d.

Remark 2.6. *Notice that, similarly to formula* (1.18), *there is a unique representation of Green symbols* $g(y,\eta)$ *of order* ν *and type* d *as*

$$g(y,\eta) = g_0(y,\eta) + \sum_{j=0}^{d-1} \begin{pmatrix} k_j(y,\eta)\gamma^j & 0 & 0 \\ b_j(y,\eta)\gamma^j & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.7}$$

where $k_j(y,\eta)$ *and* $b_j(y,\eta)$ *are Green symbols of order* $\nu - j$ *and type* 0 *of the form of* 12- *and* 22-*entries in a* 3×3-*block matrix belonging to* $\mathcal{R}_G^{\nu-j,0}$. *The space* $\mathcal{S}_{\mathcal{O}}(X^\wedge, \mathbf{n}) \oplus \mathbb{C}^{l_+}$ *is a countable projective limit of Hilbert spaces and condition* (2.4) *reduces to a corresponding expression for each space in that limit. In addition, it suffices to require the conditions for countably many* s *and* β. *Thus, from* (2.4) *we get a countable semi-norm system in the space* $\mathcal{R}_G^{\nu,0}(U \times \mathbb{R}^q; \mathbf{w})_{\mathcal{O}}$. *Together with an analogous semi-norm system from the condition* (2.5) *the space* $\mathcal{R}_G^{\nu,0}(U \times \mathbb{R}^q; \mathbf{w})_{\mathcal{O}}$ *becomes Fréchet. Moreover, the unique decomposition* (2.7) *allows us to represent* $\mathcal{R}_G^{\nu,d}(U \times \mathbb{R}^q; \mathbf{w})_{\mathcal{O}}$ *as a subspace of a finite Cartesian product of spaces of the type* $\mathcal{R}_G^{\nu-j,0}(U \times \mathbb{R}^q; \mathbf{w})_{\mathcal{O}}$, $0 \le j \le d-1$. *This gives us a Fréchet topology also in* $\mathcal{R}_G^{\nu,d}(U \times \mathbb{R}^q; \mathbf{w})_{\mathcal{O}}$ *for every* $d \in \mathbb{N}$.

Remark 2.7. *Given* $g \in \mathcal{R}_G^{\nu,d}(U \times \mathbb{R}^q; \mathbf{w})_{\mathcal{O}}$, *the corresponding operator-valued symbol* $b(y,\eta)$, *cf. Definition* 2.5, *is classical, and so we can consider its homogeneous principal symbol* $b_{(\nu)}(y,\eta)$, *which satisfies*

$$b_{(\nu)}(y,\delta\eta) = \delta^\nu \mathrm{diag}\,(\kappa_\delta^{(n)}, \kappa_\delta^{(n-1)}, \mathrm{id}) b_{(\nu)}(y,\eta)\, \mathrm{diag}\,(\kappa_\delta^{(n)}, \kappa_\delta^{(n-1)}, \mathrm{id})^{-1},$$

for all $\delta \in \mathbb{R}_+$, $(y,\eta) \in U \times (\mathbb{R}^q \setminus \{0\})$, *where* $\kappa_\delta^{(k)}$ *are as in Remark* 2.4. *Then we have the corresponding DN-homogeneous principal symbol* $g_{(\nu)}(y,\eta)$ *of* $g(y,\eta)$, $g_{(\nu)}(y,\eta) = r(\eta) b_{(\nu)}(y,\eta) r^{-1}(\eta)$, *satisfying*

$$g_{(\nu)}(y,\delta\eta) = \delta^\nu \mathrm{diag}\,(\kappa_\delta^{(n)}, \delta^{\frac{1}{2}}\kappa_\delta^{(n-1)}, \delta^{\frac{n+1}{2}}) g_{(\nu)}(y,\eta)\, \mathrm{diag}\,(\kappa_\delta^{(n)}, \delta^{\frac{1}{2}}\kappa_\delta^{(n-1)}, \delta^{\frac{n+1}{2}})^{-1},$$

for all $\delta \in \mathbb{R}_+$, $(y,\eta) \in U \times (\mathbb{R}^q \setminus \{0\})$; *the notation "DN" stands for Douglis Nirenberg.*

Remark 2.8. *Let* $\mathcal{S}_0(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times X \times X) := \{u(r,r',x,x') \in \mathcal{S}(\mathbb{R} \times \mathbb{R}, C^\infty(X \times X)) : \operatorname{supp} u \subset \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times X \times X\}$ *endowed with its canonical Fréchet topology. Then, if* $S_{\mathrm{cl}}^\nu(U \times \mathbb{R}^q)$ *is the space of standard classical scalar symbols in* $(y,\eta) \in U \times \mathbb{R}^q$ *of order* ν, *we can form the space*

$$\mathcal{S}_0(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times X \times X, S_{\mathrm{cl}}^\nu(U \times \mathbb{R}^q)) := \mathcal{S}_0(\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ \times X \times X) \hat{\otimes}_\pi S_{\mathrm{cl}}^\nu(U \times \mathbb{R}^q). \tag{2.8}$$

If $\eta \to [\eta]$ is any strictly positive function in $C^\infty(\mathbb{R}^q)$ such that $[\eta] = |\eta|$ for $|\eta| > c$ for some $c > 0$, for any $g(r, r', x, x'; y, \eta)$ from the space (2.8) *we can form $g_\eta(r, r', x, x'; y, \eta) := g(r[\eta], r'[\eta], x, x'; y, \eta)$. Then*

$$u(r,x) \to \int_X \int_0^\infty g_\eta(r, r', x, x'; y, \eta) u(r', x')\, dr'\, dx'$$

for $\nu = \mu + \frac{n+1}{2}$ represents an element in the upper left corner of $\mathcal{R}_G^{\mu,0}(U \times \mathbb{R}^q)_\mathcal{O}$ (here, for trivial bundles of fibre dimension 1). It can be shown that this is even an equivalent characterisation. A similar observation is true for the other entries with the spaces $C^\infty(X \times (\partial X))$, $C^\infty((\partial X) \times X)$ or $C^\infty((\partial X) \times (\partial X))$ in place of $C^\infty(X \times X)$ in the above-mentioned description.

Proposition 2.9. *Let $g_j(y, \eta) \in \mathcal{R}_G^{\nu-j,d}(U \times \mathbb{R}^q; \mathbf{w})_\mathcal{O}$, $j \in \mathbb{N}$, be an arbitrary sequence. Then there is a $g(y, \eta) \in \mathcal{R}_G^{\nu,d}(U \times \mathbb{R}^q; \mathbf{w})_\mathcal{O}$ such that $g - \sum_{j=0}^N g_j \in \mathcal{R}_G^{\nu-(N+1),d}(U \times \mathbb{R}^q; \mathbf{w})_\mathcal{O}$, and $g(y, \eta)$ is uniquely determined* mod $\mathcal{R}_G^{-\infty,d}(U \times \mathbb{R}^q; \mathbf{w})_\mathcal{O}$.

The proof follows from a corresponding general result on asymptotic summation of operator-valued symbols, here specified to the case of Definition 2.5
We now pass to edge symbols for boundary value problems that contain the typical non-smoothing contributions from the interior of our configuration with edges. Starting point are edge-degenerate families of boundary value problems

$$p(r, y, \rho, \eta) := \tilde{p}(r, y, \tilde{\rho}, \tilde{\eta})|_{\tilde{\rho}=r\rho, \tilde{\eta}=r\eta} \tag{2.9}$$

where $\tilde{p}(r, y, \tilde{\rho}, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times U, \mathcal{B}^{\nu,d}(X; \mathbf{v}; \mathbb{R}_{\tilde{\rho}} \times \mathbb{R}^q_{\tilde{\eta}}))$. We apply an analogue of Theorem 2.2 for the case of y-dependent operator families, i.e., there is an $\tilde{h}(r, y, z, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times U, \mathcal{M}_\mathcal{O}^{\nu,d}(X; \mathbf{v}; \mathbb{R}^q_{\tilde{\eta}}))$ such that for

$$h(r, y, z, \eta) := \tilde{h}(r, y, z, \tilde{\eta})|_{\tilde{\eta}=r\eta} \tag{2.10}$$

we have

$$\mathrm{op}_r(p)(y, \eta) = \mathrm{op}_M^\beta(h)(y, \eta) \mod C^\infty(U, \mathcal{B}^{-\infty,d}(X^\wedge; \mathbf{v}; \mathbb{R}^q)) \tag{2.11}$$

for every $\beta \in \mathbb{R}$.

Remark 2.10. *To control principal edge symbols below we set $p_0(r, y, \rho, \eta) := \tilde{p}(0, y, r\rho, r\eta)$, $h_0(r, y, z, \eta) := \tilde{h}(0, y, z, r\eta)$. Then* (2.11) *implies*

$$\mathrm{op}_r(p_0)(y, \eta) = \mathrm{op}_M^\beta(h_0)(y, \eta) \mod C^\infty(U, \mathcal{B}^{-\infty,d}(X^\wedge; \mathbf{v}; \mathbb{R}^q))$$

for all β.

In the following we fix arbitrary cut-off functions $\sigma(r)$, $\tilde{\sigma}(r)$ and $\omega(r)$, $\tilde{\omega}(r)$, set $\chi(r) := 1 - \omega(r)$, and choose any $\tilde{\chi}(r) \in C^\infty(\overline{\mathbb{R}}_+)$ that vanishes near $r = 0$ and equals 1 on $\operatorname{supp} \tilde{\omega}$.

Proposition 2.11. *Let $p(r, y, \rho, \eta)$ and $h(r, y, z, \eta)$ be operator families as before and set*

$$\begin{aligned} a_M(y,\eta) &:= r^{-\nu}\omega(r[\eta])\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(y,\eta)\tilde{\omega}(r[\eta]), \\ a_\psi(y,\eta) &:= r^{-\nu}\chi(r[\eta])\mathrm{op}_r(p)(y,\eta)\tilde{\chi}(r[\eta]), \end{aligned}$$

$\gamma \in \mathbb{R}$. Then we have

$$a_M(y,\eta) + a_\psi(y,\eta) = r^{-\nu}\mathrm{op}_r(p)(y,\eta) \mod C^\infty(U, \mathcal{B}^{-\infty,d}(X^\wedge; \mathbf{v}; \mathbb{R}^q)). \quad (2.12)$$

Moreover, setting $b(y,\eta) := \mathrm{diag}(\mathrm{id}, \langle\eta\rangle^{-\frac{1}{2}}\mathrm{id})a(y,\eta)\mathrm{diag}(\mathrm{id}, \langle\eta\rangle^{\frac{1}{2}}\mathrm{id})$ for

$$a(y,\eta) := \sigma(r)\{a_M(y,\eta) + a_\psi(y,\eta)\}\tilde{\sigma}(r) \quad (2.13)$$

we have

$$b(y,\eta) \in S^\mu(U \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge; \mathbf{m}), \mathcal{K}^{s-\nu,\gamma-\nu}(X^\wedge; \mathbf{n})) \quad (2.14)$$

for every $s > d - \frac{1}{2}$.

Relation (2.12) easily follows from Theorem 2.2. A proof of (2.14) is given in [10], Chapter 4.

The operator function

$$\sigma_M(a)(y,z) := \tilde{h}(0, y, z, 0), \quad (2.15)$$

$y \in U$, $z \in \mathbb{C}$, takes values in $\mathcal{B}^{\mu,d}(X; \mathbf{v})$; as such it represents a family of continuous operators

$$\sigma_M(a)(y,z) : \begin{array}{c} H^s(X,E) \\ \oplus \\ H^{s-\frac{1}{2}}(\partial X, J_-) \end{array} \rightarrow \begin{array}{c} H^{s-\mu}(X,F) \\ \oplus \\ H^{s-\mu-\frac{1}{2}}(\partial X, J_+) \end{array} \quad (2.16)$$

for $s > d - \frac{1}{2}$, cf. Theorem 1.9.

Remark 2.12. *We shall employ below operator families $a(y, \eta, \lambda)$ that have an analogous structure as (2.13) where $\eta \in \mathbb{R}^q$ is replaced by $(\eta, \lambda) \in \mathbb{R}^{q+l}$. Then $a(y, \eta, \lambda_0)$ for any fixed $\lambda_0 \in \mathbb{R}^l$ is an operator-valued symbol in the sense of Proposition 2.11. Setting $U = \Omega$ for open $\Omega \subseteq \mathbb{R}^q_y$ and applying $\mathrm{Op}_y(\cdot)$ to $a(y, \eta, \lambda)$ gives us a λ-dependent family of continuous operators*

$$\begin{aligned} \mathrm{Op}(a)(\lambda) : \mathcal{W}^s_{\mathrm{comp}}(\Omega, \mathcal{K}^{s,\gamma}(X^\wedge, E)) \oplus \mathcal{W}^{s-\frac{1}{2}}_{\mathrm{comp}}(\Omega, \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}((\partial X)^\wedge, J_-)) \\ \rightarrow \mathcal{W}^{s-\nu}_{\mathrm{loc}}(\Omega, \mathcal{K}^{s-\nu,\gamma-\nu}(X^\wedge, F)) \oplus \mathcal{W}^{s-\nu-\frac{1}{2}}_{\mathrm{loc}}(\Omega, \mathcal{K}^{s-\nu-\frac{1}{2},\gamma-\nu-\frac{1}{2}}((\partial X)^\wedge, J_+)) \end{aligned} \quad (2.17)$$

for all $s > d - \frac{1}{2}$.

There is a useful alternative representation of operator families (2.13) (modulo flat Green symbols) that we will employ below.

Theorem 2.13. *Given $a(y,\eta)$ in the form* (2.13) *there exists a Green symbol $g(y,\eta) \in \mathcal{R}^{\nu,d}(U \times \mathbb{R}^q; \mathbf{v})_{\mathcal{O}}$ for $\mathbf{v} = (E, F; J_-, J_+)$ such that*

$$a(y,\eta) = \sigma_0(r)\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(y,\eta)\tilde{\sigma}_0(r) + g(y,\eta)$$

for every choice of cut-off functions σ_0, $\tilde{\sigma}_0$ such that $\sigma_0 \equiv 1$ on $\operatorname{supp}\sigma$, $\tilde{\sigma}_0 \equiv 1$ on $\operatorname{supp}\tilde{\sigma}$.

Such a result for closed compact X has been proved in Gil, Seiler and Schulze [7]. The case of boundary value problems as we need it here was treated in [11].

2.3 The edge algebra

We now introduce a parameter-dependent algebra of edge-boundary value problems on a (stretched) manifold $\mathbb{W}$ with edge Y, cf. the notation in Section 1.1. For convenience we make some additional assumptions on the nature of transition maps (1.1) between local wedges. From now on we will assume that there is an $\epsilon > 0$ such that $(\tilde{r}, \tilde{x}, \tilde{y})$ do not depend on r for $0 \leq r < \epsilon$. This will be assumed for $\mathbb{W}$ itself as well as for the double $\widetilde{\mathbb{W}}$. In addition we assume that $\partial\widetilde{\mathbb{W}}$ has the form $\widetilde{X} \times Y$ (i.e., is a trivial $\widetilde{X}$-bundle on Y) and choose a neighbourhood $\widetilde{\mathbb{M}}$ of $\partial\widetilde{\mathbb{W}}$ in $\mathbb{W}$ of the form $\widetilde{\mathbb{M}} = [0,\epsilon) \times \widetilde{X} \times Y$ with a global splitting of variables (r, x, y). Clearly, in local descriptions the choice of ϵ is unessential, and, in fact, we often talk about $\overline{\mathbb{R}}_+ \times \widetilde{X} \times Y$ (that is diffeomorphic to $[0,\epsilon) \times \widetilde{X} \times Y$). Similarly, we proceed for $\mathbb{W}$ and then write $\mathbb{M} = \widetilde{\mathbb{M}} \bigcap \mathbb{W}$ which corresponds to $[0,\epsilon) \times X \times Y$ (or $\overline{\mathbb{R}}_+ \times X \times Y$).

For convenience we study our edge algebra for the case that W (and then also Y and $\mathbb{W}$) are compact; the non-compact case can be considered as well, but this does not concern our main application.

On $\widetilde{\mathbb{W}}$ we fix a Riemannian metric that induces the product metric on $\widetilde{\mathbb{M}} = [0,\epsilon) \times \widetilde{X} \times Y$ for some Riemannian metrics on $\widetilde{X}$ and Y, respectively. We then have corresponding metrics on $\mathbb{W}$. Note that $\mathbb{W}$ has corners; smooth objects on $\mathbb{W}$ are defined as restrictions of corresponding smooth ones on the double $\widetilde{\mathbb{W}}$. In particular, if we talk about $\mathrm{Vect}(\mathbb{W})$, the set of smooth complex vector bundles on $\mathbb{W}$, we mean $\{\widetilde{E}|_{\mathbb{W}} : \widetilde{E} \in \mathrm{Vect}(\widetilde{W})\}$. In a similar sense we define Hermitian metrics in bundles $E \in \mathrm{Vect}(\mathbb{W})$ as restrictions of Hermitian metrics in $\widetilde{E}$. From now on we assume that $\mathbb{W}$ is compact. Fix elements $E, F \in \mathrm{Vect}(\mathbb{W})$, $J_-, J_+ \in \mathrm{Vect}(\mathbb{V})$ and $L_-, L_+ \in \mathrm{Vect}(Y)$, and set $\mathbf{v} = (E, F; J_-, J_+)$, $\mathbf{w} = (E, F; J_-, J_+; L_-, L_+)$.

Definition 2.14. *Define $\mathcal{Y}^{-\infty,0}(\mathbb{W}; \mathbf{w}; \mathbb{R}^l)_{\mathcal{O}}$ to be the space of all operator families*

$$\mathcal{C}(\lambda) \in \bigcap_{\substack{s>-\frac{1}{2} \\ \gamma\in\mathbb{R}}} \mathcal{S}(\mathbb{R}^l, \mathcal{L}(\mathcal{E}^{s,\gamma}, \mathcal{F}))$$

such that

$$\mathcal{C}^*(\lambda) \in \bigcap_{\substack{s>-\frac{1}{2} \\ \gamma\in\mathbb{R}}} \mathcal{S}(\mathbb{R}^l, \mathcal{L}(\mathcal{F}^{s,\gamma}, \mathcal{E}))$$

for

$$\mathcal{E}^{s,\gamma} := \mathcal{W}^{s,\gamma}(\mathbb{W}, E) \oplus \mathcal{W}^{s,\gamma}(\mathbb{V}, J_-) \oplus H^s(Y, L_-),$$
$$\mathcal{F} := \mathcal{W}^{\infty,\infty}(\mathbb{W}, F) \oplus \mathcal{W}^{\infty,\infty}(\mathbb{V}, J_+) \oplus H^\infty(Y, L_+),$$

and

$$\mathcal{F}^{s,\gamma} := \mathcal{W}^{s,\gamma}(\mathbb{W}, F) \oplus \mathcal{W}^{s,\gamma}(\mathbb{V}, J_+) \oplus H^s(Y, L_+),$$
$$\mathcal{E} := \mathcal{W}^{\infty,\infty}(\mathbb{W}, E) \oplus \mathcal{W}^{\infty,\infty}(\mathbb{V}, J_-) \oplus H^\infty(Y, L_-),$$

where "$*$" denotes the pointwise formal adjoint with respect to the $\mathcal{W}^{0,0}(\mathbb{W}, \cdot) \oplus \mathcal{W}^{0,0}(\mathbb{V}, \cdot) \oplus H^0(Y, \cdot)$*-scalar product. Moreover we define* $\mathcal{Y}^{-\infty,d}(\mathbb{W}; \mathbf{w}; \mathbb{R}^l)_\mathcal{O}$*, for* $d \in \mathbb{N}$*, as the space of all*

$$\mathcal{C}(\lambda) = \mathcal{C}_0(\lambda) + \sum_{j=1}^{d} \mathcal{C}_j(\lambda)\, \mathrm{diag}(D^j, 0, 0)$$

for arbitrary $\mathcal{C}_j(\lambda) \in \mathcal{Y}^{-\infty,0}(\mathbb{W}; \mathbf{w}; \mathbb{R}^l)_\mathcal{O}$ *where* D^j *are differential operators of order* j *that differentiate transversally to the boundary of* $\mathbb{V}$*.*

Remark 2.15. *The space* $\mathcal{Y}^{-\infty,d}(\mathbb{W}; \mathbf{w}; \mathbb{R}^l)_\mathcal{O}$ *is Fréchet in a natural way. An adequate semi-norm system immediately follows from the definition.*

We now fix a chart $\chi : V \to \Omega$ on Y, where $\Omega \subseteq \mathbb{R}^q$ is open and consider a $g_\Omega(y, \eta, \lambda) \in \mathcal{R}_G^{\nu,d}(\Omega \times \mathbb{R}^{q+l}_{\eta,\lambda}; \mathbf{w}_\Omega)_\mathcal{O}$ where $\mathbf{w}_\Omega$ is a tuple of bundles as required in Definition 2.5, obtained from the bundles $(E, F; J_-, J_+; L_-, L_+)$ of Definition 2.14. We express the involved bundles in local form and write E_Ω, F_Ω for the bundles on $X^\wedge \times \Omega$ induced by E, F near $r = 0$, further $J_{-,\Omega}$, $J_{+,\Omega}$ for the bundles on $(\partial X)^\wedge \times \Omega$ induced by J_-, J_+, and $\Omega \times \mathbb{C}^{l_-}$, $\Omega \times \mathbb{C}^{l_+}$ for the corresponding trivial bundles on Ω induced by L_-, L_+ (as is customary we also write $l_\pm$ in place of the trivial bundles $\Omega \times \mathbb{C}^{l_\pm}$). Without of loss of generality we assume that E_Ω, F_Ω ($J_{\pm,\Omega}$) are pull-backs of corresponding bundles on $X^\wedge$ ($(\partial X)^\wedge$) under the canonical projection $X^\wedge \times \Omega \to X^\wedge$ ($(\partial X)^\wedge \times \Omega \to (\partial X)^\wedge$), and we denote those bundles on $X^\wedge$ and $(\partial X)^\wedge$ by E_Ω, F_Ω and $J_{\pm,\Omega}$, respectively. In that sense we set $\mathbf{w}_\Omega := (E_\Omega, F_\Omega; J_{-,\Omega}, J_{+,\Omega}; l_-, l_+)$.
If $\tilde{\chi} : V \to \widetilde{\Omega}$ is another chart there is a transition diffeomorphism $\Omega \to \widetilde{\Omega}$ and associated transition isomorphisms $E_\Omega \underset{\cong}{\to} E_{\widetilde{\Omega}}$, $F_\Omega \to F_{\widetilde{\Omega}}$, $J_{\pm,\Omega} \to J_{\pm,\widetilde{\Omega}}$, as well as $\Omega \times \mathbb{C}^{l_\pm} \to \widetilde{\Omega} \times \mathbb{C}^{l_\pm}$ and it is a simple lemma, left to the reader, that this induces a symbol push-forward $g_\Omega(y, \eta, \lambda) \to g_{\widetilde{\Omega}}(y, \eta, \lambda)$ that is canonical modulo

Green symbols of order $-\infty$ and type d. Now if we fix our chart $\chi : V \to \Omega$ and choose arbitrary functions $\phi, \theta \in C_0^\infty(V)$ we can lift

$\phi_0 \mathrm{Op}_y(g_\Omega)(\lambda)\theta_0$:

$$\begin{array}{ccc} \mathcal{W}^s_{\mathrm{loc}}(\Omega, \mathcal{K}^{s,\gamma}(X^\wedge, E_\Omega)) & & \mathcal{W}^{s-\nu}_{\mathrm{comp}}(\Omega, \mathcal{K}^{\infty,\infty}(X^\wedge, F_\Omega)) \\ \oplus & & \oplus \\ \mathcal{W}^{s-\frac{1}{2}}_{\mathrm{loc}}(\Omega, \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}((\partial X)^\wedge, J_{-,\Omega})) & \longrightarrow & \mathcal{W}^{s-\nu-\frac{1}{2}}_{\mathrm{comp}}(\Omega, \mathcal{K}^{\infty,\infty}((\partial X)^\wedge, J_{+,\Omega})), \\ \oplus & & \oplus \\ H^{s-\frac{n+1}{2}}_{\mathrm{loc}}(\Omega, \mathbb{C}^{l_-}) & & H^\infty_{\mathrm{comp}}(\Omega, \mathbb{C}^{l_+}) \end{array} \tag{2.18}$$

$s > d - \frac{1}{2}$ (where $\phi_0, \theta_0 \in C_0^\infty(\Omega)$ are defined by $\phi = \chi^*\phi_0$, $\theta = \chi^*\theta_0$) from Ω to V by operator push-forward under χ^{-1} which also takes into account the bundle transition isomorphisms. We employ the notation $(\boldsymbol{\chi}^{-1})_*$ for the pull-back that includes the bundle structure in this sense. In other words we get an operator family from (2.18) that we denote by

$$\phi(\boldsymbol{\chi}^{-1})_* \mathrm{Op}(g_\Omega)\theta. \tag{2.19}$$

Because of the nature of mappings it is natural to localize the operators to a neighbourhood of $r = 0$ by forming

$$\mathrm{diag}(\omega, \omega, 1)\phi(\boldsymbol{\chi}^{-1})_* \mathrm{Op}(g_\Omega)\theta \, \mathrm{diag}(\tilde{\omega}, \tilde{\omega}, 1)$$

for some cut-off functions $\omega(r)$, $\tilde{\omega}(r)$, with obvious meaning of notation.
Let us now fix a system of charts $\chi_j : V_j \to \Omega_j$, $j = 1, \dots, N$, where $\{V_j\}_{j=1,\dots,N}$ is an open covering of Y, and choose a subordinate partition of unity $\{\phi_j\}_{j=1,\dots,N}$ and another system $\{\theta_j\}_{j=1,\dots,N}$ of functions $\theta_j \in C_0^\infty(V_j)$ such that $\phi_j\theta_j = \phi_j$ for all j. Given a system of Green symbols

$$g_j(y, \eta, \lambda) \in \mathcal{R}^{\nu,d}_G(\Omega_j \times \mathbb{R}^{q+l}; \mathbf{w}_{\Omega_j})_{\mathcal{O}}, \;\; j = 1, \dots, N \tag{2.20}$$

we then form an operator family

$$\mathcal{G}(\lambda) := \mathrm{diag}(\omega, \omega, 1) \sum_{j=1}^N \phi_j(\boldsymbol{\chi}_j^{-1})_* \mathrm{Op}(g_j)(\lambda)\theta_j \, \mathrm{diag}(\tilde{\omega}, \tilde{\omega}, 1). \tag{2.21}$$

Notice that the specific choice of the cut-off functions ω, $\tilde{\omega}$ only affects $\mathcal{G}(\lambda)$ by an element in $\mathcal{Y}^{-\infty,d}(\mathbb{W}; \mathbf{w}; \mathbb{R}^l)_{\mathcal{O}}$. Let $\mathcal{Y}^{\nu,d}_G(\mathbb{W}; \mathbf{w}; \mathbb{R}^l)_{\mathcal{O}}$ denote the space of all $\mathcal{G}(\lambda) + \mathcal{C}(\lambda)$ for arbitrary operator families of the form (2.21) and $\mathcal{C}(\lambda) \in \mathcal{Y}^{-\infty,d}(\mathbb{W}; \mathbf{w}; \mathbb{R}^l)_{\mathcal{O}}$.

Operators of the kind (2.17) can easily be lifted to $\mathbb{W}$ by an analogous procedure as before in the case of Green operators. In other words, we start from "symbols"

$$p_j(r,y,\rho,\eta,\lambda)=\tilde{p}_j(r,y,\tilde{\rho},\tilde{\eta},\tilde{\lambda})|_{\tilde{\rho}=r\rho,\tilde{\eta}=r\eta,\tilde{\lambda}=r\lambda},$$

where $\tilde{p}_j(r,y,\tilde{\rho},\tilde{\eta},\tilde{\lambda})\in C^\infty(\overline{\mathbb{R}}_+\times\Omega_j,\mathcal{B}^{\nu,d}(X;\mathbf{v}_\Omega;\mathbb{R}_{\tilde{\rho}}\times\mathbb{R}^{q+l}_{\tilde{\eta},\tilde{\lambda}}))$, form associated Mellin symbols

$$h_j(r,y,z,\eta,\lambda)=\tilde{h}_j(r,y,z,\tilde{\eta},\tilde{\lambda})|_{\tilde{\eta}=r\eta,\tilde{\lambda}=r\lambda}$$

and pass to operator-valued symbols

$$a_j(y,\eta,\lambda)=\sigma(r)\{a_{j,M}(y,\eta,\lambda)+a_{j,\psi}(y,\eta,\lambda)\}\tilde{\sigma}(r)$$

by the same scheme as in Proposition 2.11. We form local operators $\mathrm{Op}_y(a_j)(\lambda)$ and corresponding global operators

$$M(\lambda):=\sum_{j=1}^N\phi_j(\chi_j^{-1})_*\mathrm{Op}_y(a_j)(\lambda)\theta_j. \tag{2.22}$$

Definition 2.16. *$\mathcal{Y}^{\nu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^l)_{\mathcal{O}}$ for $\nu\in\mathbb{Z}$, $d\in\mathbb{N}$ and $\mathbf{w}=(E,F;J_-,J_+;L_-,L_+)$ denotes the space of all operator families*

$$\mathcal{A}(\lambda):=\mathcal{M}(\lambda)+\mathcal{P}(\lambda)+\mathcal{G}(\lambda) \tag{2.23}$$

for arbitrary $\mathcal{G}(\lambda)\in\mathcal{Y}_G^{\nu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^l)_{\mathcal{O}}$, further

$$\mathcal{P}(\lambda):=\begin{pmatrix}(1-\sigma)P(\lambda)(1-\tilde{\tilde{\sigma}}) & 0\\ 0 & 0\end{pmatrix} \tag{2.24}$$

for arbitrary $P(\lambda)\in\mathcal{B}^{\nu,d}(\mathbb{W}_{\mathrm{reg}};\mathbf{v};\mathbb{R}^l)$, $\mathbf{v}:=(E,F;J_-,J_+)$, where $\sigma(r)$, $\tilde{\tilde{\sigma}}(r)$ are cut-off functions satisfying $\sigma\tilde{\tilde{\sigma}}=\tilde{\tilde{\sigma}}$, and finally

$$\mathcal{M}(\lambda):=\begin{pmatrix}M(\lambda) & 0\\ 0 & 0\end{pmatrix} \tag{2.25}$$

where $M(\lambda)$ has the form (2.22) where the involved cut-off functions σ, $\tilde{\sigma}$ are assumed to satisfy $\sigma\tilde{\sigma}=\sigma$.

Let us set

$$\mathcal{Y}^{-\infty,d}_{M+G}(\mathbb{W};\mathbf{w};\mathbb{R}^l)_{\mathcal{O}}:=\bigcap_{\nu\in\mathbb{Z}}\mathcal{Y}^{\nu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^l)_{\mathcal{O}}.$$

In Definition 2.16 the dimension $l\in\mathbb{N}$ is arbitrary. For the case $l=0$ we simply omit $\mathbb{R}^l$ in the notation, i.e., we get the operator spaces $\mathcal{Y}^{\nu,d}(\mathbb{W};\mathbf{w})_{\mathcal{O}}$.
For abbreviation we set $\mathbf{m}:=(E,J_-)$ for vector bundles $E\in\mathrm{Vect}(\mathbb{W})$, $J_-\in\mathrm{Vect}(\mathbb{V})$, and $\mathcal{W}^{s,\gamma}(\mathbb{W};\mathbf{m}):=\mathcal{W}^{s,\gamma}(\mathbb{W},E)\oplus\mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{V},J_-)$, $s\in\mathbb{R}$.

Theorem 2.17. *Every $\mathcal{A}(\lambda) \in \mathcal{Y}^{\nu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^l)_{\mathcal{O}}$, $\mathbf{w} = (E, F; J_-, J_+; L_-, L_+)$ for $\mathbb{W}$ compact induces a family of continuous operators*

$$\mathcal{A}(\lambda) : \begin{array}{c} \mathcal{W}^{s,\gamma}(\mathbb{W};\mathbf{m}) \\ \oplus \\ H^{s-\frac{n+1}{2}}(Y, L_-) \end{array} \longrightarrow \begin{array}{c} \mathcal{W}^{s-\nu,\gamma-\nu}(\mathbb{W};\mathbf{n}) \\ \oplus \\ H^{s-\frac{n+1}{2}}(Y, L_+) \end{array} \tag{2.26}$$

for every $s > d - \frac{1}{2}$ and $\gamma \in \mathbb{R}$, where $\mathbf{m} := (E, J_-)$, $\mathbf{n} := (F, J_+)$.

Remark 2.18. *A similar result holds for non-compact $\mathbb{W}$; in that case we have continuity in spaces with subscript* "comp" *for the domain and* "loc" *for the image of operators. The proof is easy as well.*

Remark 2.19. *The spaces $\mathcal{Y}^{\nu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^l)_{\mathcal{O}}$ can be endowed with natural Fréchet topologies for every l. Then it is possible to differentiate $\mathcal{A}(\lambda) \in \mathcal{Y}^{\nu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^l)_{\mathcal{O}}$ with respect to λ or to consider holomorphy of families $\mathcal{A}(\lambda, w)$ in $w \in \mathbb{C}$ with values in $\mathcal{Y}^{\nu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^q)_{\mathcal{O}}$, cf. Section* 3 *below. However, for convenience, we characterize C^∞ (or holomorphy) of a function $\mathcal{A}(\lambda, w)$ directly by requiring the amplitude functions involved in* (2.23) *as well as the smoothing operators to be smooth (holomorphic) in λ (w). For the smoothing families, cf. Remark* 2.15*; concerning amplitude functions, it is evident how to proceed, see, for instance, Definition* 2.5.

Remark 2.20. *We have*

$$D_\lambda^\alpha \mathcal{Y}^{\nu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^l)_{\mathcal{O}} \subset \mathcal{Y}^{\nu-|\alpha|,d}(\mathbb{W};\mathbf{w};\mathbb{R}^l)_{\mathcal{O}}$$

for every $\alpha \in \mathbb{N}^l$.

Let us now introduce the (parameter-dependent) principal symbolic structure of operator families $\mathcal{A}(\lambda) \in \mathcal{Y}^{\nu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^l)_{\mathcal{O}}$. First, $\mathcal{A} := \mathcal{A}(\lambda)$ can be viewed as a 3×3-block matrix $\mathcal{A} = (A_{ij}(\lambda))_{i,j=1,2,3}$. Then the 2×2-block matrix upper left corner $(A_{ij}(\lambda))_{i,j=1,2}$ belongs to Boutet de Monvel's calculus (with parameters) on the smooth (non-compact) manifold $\mathbb{W}_{\mathrm{reg}}$ with boundary. As such there are the parameter-dependent principal symbol of order ν

$$\sigma_\psi(\mathcal{A}) := \sigma_\psi(A_{11}(\lambda)) : \pi^*_{\mathbb{W}_{\mathrm{reg}}} E \to \pi^*_{\mathbb{W}_{\mathrm{reg}}} F, \tag{2.27}$$

$\pi_{\mathbb{W}_{\mathrm{reg}}} : T^*\mathbb{W}_{\mathrm{reg}} \times \mathbb{R}^l \setminus 0 \to \mathbb{W}_{\mathrm{reg}}$, and the parameter-dependent principal (in the sense of DN-homogeneity, cf. Section 1.5) boundary symbol

$$\sigma_\partial(\mathcal{A}) := \sigma_\partial((A_{ij}(\lambda))_{i,j=1,2}),$$

$$\sigma_\partial(\mathcal{A}) : \pi^*_{\mathbb{V}_{\mathrm{reg}}} \begin{pmatrix} E' \otimes H^s(\mathbb{R}_+) \\ \oplus \\ J_- \end{pmatrix} \to \pi^*_{\mathbb{V}_{\mathrm{reg}}} \begin{pmatrix} F' \otimes H^{s-\nu}(\mathbb{R}_+) \\ \oplus \\ J_+ \end{pmatrix}, \tag{2.28}$$

$\pi_{\mathbb{V}_{\mathrm{reg}}} : T^*\mathbb{V}_{\mathrm{reg}} \times \mathbb{R}^l \setminus 0 \to \mathbb{V}_{\mathrm{reg}}$, where $E' := E|_{\mathbb{V}}$, $F' := F|_{\mathbb{V}}$. Because of the assumed edge-degenerate nature of our operator family, near $r = 0$ we can write

$$\begin{aligned}\sigma_\psi(\mathcal{A}) &= \sigma_\psi(\mathcal{A})(r, x, y, \rho, \xi, \eta, \lambda) \\ &= r^{-\nu}\tilde{\sigma}_\psi(\mathcal{A})(r, x, y, \tilde{\rho}, \xi, \tilde{\eta}, \tilde{\lambda})|_{\tilde{\rho}=r\rho, \tilde{\eta}=r\eta, \tilde{\lambda}=r\lambda},\end{aligned}$$

where $\tilde{\sigma}_\psi(\mathcal{A})(r, x, y, \tilde{\rho}, \xi, \tilde{\eta}, \tilde{\lambda})$, $(\tilde{\rho}, \xi, \tilde{\eta}, \tilde{\lambda}) \neq 0$, is smooth up to $r = 0$. Moreover,

$$\begin{aligned}\sigma_\partial(\mathcal{A}) &= \sigma_\partial(\mathcal{A})(r, x', y, \rho, \xi', \eta, \lambda) \\ &= r^{-\nu}\tilde{\sigma}_\partial(\mathcal{A})(r, x', y, \tilde{\rho}, \xi', \tilde{\eta}, \tilde{\lambda})|_{\tilde{\rho}=r\rho, \tilde{\eta}=r\eta, \tilde{\lambda}=r\lambda},\end{aligned}$$

where $\tilde{\sigma}_\partial(\mathcal{A})(r, x', y, \tilde{\rho}, \xi', \tilde{\eta}, \tilde{\lambda})$, $(\tilde{\rho}, \xi', \tilde{\eta}, \tilde{\lambda}) \neq 0$, is smooth up to $r = 0$. Finally, we have the parameter-dependent principal (in the sense of DN-homogeneity) edge symbol

$$\sigma_\wedge(\mathcal{A}) := \sigma_\wedge((A_{ij}(\lambda))_{i,j=1,2,3}),$$

$$\sigma_\wedge(\mathcal{A}) : \pi_Y^* \begin{pmatrix} \mathcal{K}^{s,\gamma}(X^\wedge; \mathbf{m}^\wedge) \\ \oplus \\ L_- \end{pmatrix} \to \pi_Y^* \begin{pmatrix} \mathcal{K}^{s-\nu,\gamma-\nu}(X^\wedge; \mathbf{n}^\wedge) \\ \oplus \\ L_+ \end{pmatrix}, \tag{2.29}$$

$\pi_Y : T^*Y \times \mathbb{R}^l \setminus 0 \to Y$, where $\mathbf{m}^\wedge := (E^\wedge, J_-^\wedge)$, $\mathbf{n}^\wedge := (F^\wedge, J_+^\wedge)$ and moreover $\mathcal{K}^{s,\gamma}(X^\wedge; \mathbf{m}^\wedge) := \mathcal{K}^{s,\gamma}(X^\wedge, E^\wedge) \oplus \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}((\partial X)^\wedge, J_-^\wedge)$. The definition of $\sigma_\wedge(\mathcal{A})$ is as follows: because of an easy coordinate invariance it suffices to define $\sigma_\wedge(\mathcal{A})$ in a neighbourhood of a point on the edge in local coordinates $y \in \Omega$, $\Omega \subseteq \mathbb{R}^q$ open, with the covariable η (and the parameter λ that is also treated as a component of the full covariables). By linearity we may consider the summands in (2.23) separately; the only non-vanishing contributions come from $\mathcal{M}(\lambda)$ and $\mathcal{G}(\lambda)$. The operators $\mathcal{G}(\lambda)$ are locally pseudo-differential operators with symbols in $\mathcal{R}_G^{\nu,d}(\Omega \times \mathbb{R}^{q+l}; \mathbf{w}_\Omega)_\mathcal{O}$. Those symbols are classical and DN-homogeneous and have a principal term, namely $\sigma_\wedge(\mathcal{G}(\lambda))$. Concerning $\mathcal{M}(\lambda)$ it suffices to look at the upper left corner $M(\lambda)$, cf. (2.25) and (2.22). First we have

$$\sigma_\wedge(M(\lambda)) = \sum_{j=1}^{N} \sigma_\wedge\big(\phi_j(\boldsymbol{\chi}^{-1})_*\mathrm{Op}(a_j)(\lambda)\theta_j\big).$$

In local term we may omit the operator push-forward. Since ϕ_j and θ_j are only y-dependent factors, it suffices to define $\sigma_\wedge(\mathrm{Op}(a_j))$, where we now omit j. For

$$a(y, \eta, \lambda) = \sigma(r)\{a_M(y, \eta, \lambda) + a_\psi(y, \eta, \lambda)\}\tilde{\sigma}(r)$$

we set (in the notation of Remark 2.10, Proposition 2.11, and with (η, λ) in place of η)

$$\begin{aligned}\sigma_\wedge(\mathrm{Op}(a))(y, \eta, \lambda) = r^{-\nu}\{&\omega(r|\eta, \lambda|)\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_0)(y, \eta, \lambda)\tilde{\omega}(r|\eta|) \\ &+ \chi(r|\eta, \lambda|)\mathrm{op}_r(p_0)(y, \eta, \lambda)\tilde{\chi}(r|\eta, \lambda|)\}.\end{aligned}$$

The homogeneity of $\sigma_\partial(\mathcal{A})$ and $\sigma_\wedge(\mathcal{A})$ in DN-order is connected with shifts of smoothness and weights by $\frac{1}{2}$ in the spaces on the boundary. We have, when $\kappa_\delta : H^s(\mathbb{R}_+) \to H^s(\mathbb{R}_+)$ means $(\kappa_\delta u)(x_n) = \delta^{\frac{1}{2}} u(\delta x_n)$, $\delta \in \mathbb{R}_+$,

$$\begin{aligned}\sigma_\partial(\mathcal{A})(r,&x', y, \delta\rho, \delta\xi', \delta\eta, \delta\lambda) \\ &= \delta^\nu \begin{pmatrix} \kappa_\delta & 0 \\ 0 & \delta^{\frac{1}{2}} \end{pmatrix} \sigma_\partial(\mathcal{A})(r, x', y, \rho, \xi', \eta, \lambda) \begin{pmatrix} \kappa_\delta & 0 \\ 0 & \delta^{\frac{1}{2}} \end{pmatrix}^{-1} \end{aligned} \tag{2.30}$$

for all $\delta \in \mathbb{R}_+$. The homogeneity of $\sigma_\wedge(\mathcal{A})$ in DN-orders is based on group actions $\kappa_\delta^{(n)} : \mathcal{K}^{s,\gamma}(X^\wedge, E^\wedge) \to \mathcal{K}^{s,\gamma}(X^\wedge, E^\wedge)$, $\kappa_\delta^{(n)} u(r,x) = \delta^{\frac{n+1}{2}} u(\delta r, x)$, $n = \dim X$, and $\kappa_\delta^{(n-1)} : \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}((\partial X)^\wedge, J) \to \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}((\partial X)^\wedge, J)$, $\kappa_\delta^{(n-1)} v(r,x') = \delta^{\frac{n}{2}} v(\delta r, x')$ and then

$$\begin{aligned}&\sigma_\wedge(\mathcal{A})(y, \delta\eta, \delta\lambda) = \\ &= \delta^\nu \begin{pmatrix} \kappa_\delta^{(n)} & 0 & 0 \\ 0 & \delta^{\frac{1}{2}}\kappa_\delta^{(n-1)} & 0 \\ 0 & 0 & \delta^{\frac{n+1}{2}} \end{pmatrix} \sigma_\wedge(\mathcal{A})(y, \eta, \lambda) \begin{pmatrix} \kappa_\delta^{(n)} & 0 & 0 \\ 0 & \delta^{\frac{1}{2}}\kappa_\delta^{(n-1)} & 0 \\ 0 & 0 & \delta^{\frac{n+1}{2}} \end{pmatrix}^{-1}\end{aligned} \tag{2.31}$$

for all $\delta \in \mathbb{R}_+$. The triple

$$\sigma(\mathcal{A}) := (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A}), \sigma_\wedge(\mathcal{A})) \tag{2.32}$$

is called the principal symbol of the operator family $\mathcal{A} \in \mathcal{Y}^{\nu,d}(\mathbb{W}; \mathbf{w}; \mathbb{R}^l)_\mathcal{O}$. For convenience, the order ν of $\mathcal{A}$ is not explicitly indicated in (2.32); clearly, the homogeneity of $\sigma_\psi(\mathcal{A})$ in the standard meaning as well as the twisted homogeneity in relations (2.30) and (2.31) are always assumed to be connected with the order of $\mathcal{A}$ in the described way.
The 2×2-upper left corners of $\sigma_\wedge(\mathcal{A})(y, \eta, \lambda)$ are families of operators in the cone algebra of boundary value problems on $X^\wedge$. As such they have a principal (and also lower order) conormal symbolic structure. Let us set

$$\sigma_M \sigma_\wedge(\mathcal{A})(y, z) = \sum_{j=1}^N \phi_j \tilde{h}_j(0, y, z, 0, 0)\theta_j,$$

cf. also formula (2.15). This is just the principal conormal symbol of the respective cone operator $\sigma_\wedge(\mathcal{A})(y, \eta, \lambda)$ which is independent of η, λ. It is regarded as a subordinate symbolic level because it is uniquely determined by $\sigma_\wedge(\mathcal{A})$. Since $\sigma_\wedge(\mathcal{A})$ is completely determined by $\mathcal{A}$ itself, we may also write

$$\sigma_M(\mathcal{A})(y, z) := \sigma_M \sigma_\wedge(\mathcal{A})(y, z), \tag{2.33}$$

interpreted as a family of elements of $\mathcal{B}^{\nu,d}(X, \mathbf{v})$, see also formula (2.16).

Remark 2.21. *$\mathcal{A} \in \mathcal{Y}^{\nu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^l)_{\mathcal{O}}$, $\sigma(\mathcal{A}) = 0$ imply $\mathcal{A} \in \mathcal{Y}^{\nu-1,d}(\mathbb{W};\mathbf{w};\mathbb{R}^l)_{\mathcal{O}}$; in that case the operators (2.26) are compact for every $s > d - \frac{1}{2}$.*

Theorem 2.22. *If $\mathcal{A} \in \mathcal{Y}^{\nu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^l)_{\mathcal{O}}$ and $\mathcal{B} \in \mathcal{Y}^{\mu,e}(\mathbb{W};\mathbf{v};\mathbb{R}^l)_{\mathcal{O}}$, for $\mathbf{w} = (E_0, F; J_0, J_+; L_0, L_+)$ and $\mathbf{v} = (E, E_0; J_-, J_0; L_-, L_0)$, then $\mathcal{AB} \in \mathcal{Y}^{\mu+\nu,h}(\mathbb{W};\mathbf{w}\circ\mathbf{v};\mathbb{R}^l)_{\mathcal{O}}$ for $\mathbf{w}\circ\mathbf{v} = (E, F; J_-, J_+; L_-, L_+)$, $h = \max(\mu + d, e)$, and we have $\sigma(\mathcal{AB}) = \sigma(\mathcal{A})\sigma(\mathcal{B})$ in the sense of componentwise composition.*

Proof. The proof is close to a corresponding result in a larger operator algebra of boundary value problems without parameters $\lambda \in \mathbb{R}^l$, cf. [10], Chapter 4. The new element here is that we control dependence on parameters that are involved as additional covariables in edge-degenerate form and that the corresponding subalgebra with holomorphic Mellin symbols and flat Green operators is preserved under composition. What concerns parameters there is no essential extra difficulty, because, modulo smoothing operator families that are Schwartz functions in $\lambda \in \mathbb{R}^l$ with values in smoothing edge-boundary value problems, the composition may be discussed for localised operators. Then, similarly to standard pseudo-differential operators, the composition is reduced to Leibniz-multiplied amplitude functions, that are here operator-valued, cf. relation (2.14). In this consideration the parameter is simply to be treated as an extra covariable. What concerns holomorphy (in $z \in \mathbb{C}$) of Mellin symbols and flatness (in cone axis direction) of Green amplitude functions, we can pass to the alternative description of amplitude functions as in Theorem 2.13. The composition then preserves, indeed, the holomorphy and flatness, as it may be found in [7] for the case of a closed cone base and in analogously in [10] for the case of boundary value problems. $\square$

2.4 Ellipticity

Definition 2.23. *An operator $\mathcal{A} \in \mathcal{Y}^{\mu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^l)_{\mathcal{O}}$ is said to be (parameter-dependent) elliptic of order μ and with respect to a weight $\gamma \in \mathbb{R}$, if $\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A}), \sigma_\wedge(\mathcal{A}))$ is elliptic in the following sense:*

(i)
$$\sigma_\psi(\mathcal{A}) : \pi^*_{\mathbb{W}_{\mathrm{reg}}} E \longrightarrow \pi^*_{\mathbb{W}_{\mathrm{reg}}} F$$
is an isomorphism, and also $\tilde{\sigma}_\psi(\mathcal{A})$ is bijective for $(\tilde{\rho}, \xi, \tilde{\eta}, \tilde{\lambda}) \neq 0$ up to $r = 0$;

(ii)
$$\sigma_\partial(\mathcal{A}) : \pi^*_{\mathbb{V}_{\mathrm{reg}}} \begin{pmatrix} E' \otimes H^s(\mathbb{R}_+) \\ \oplus \\ J_- \end{pmatrix} \to \pi^*_{\mathbb{V}_{\mathrm{reg}}} \begin{pmatrix} F' \otimes H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ J_+ \end{pmatrix}$$
is an isomorphism for an $s = s_0 > \max(\mu, d) - \frac{1}{2}$, and also $\tilde{\sigma}_\partial(\mathcal{A})$ is bijective for $(\tilde{\rho}, \xi', \tilde{\eta}, \tilde{\lambda}) \neq 0$ up to $r = 0$;

(iii)

$$\sigma_\wedge(\mathcal{A}): \pi_Y^* \begin{pmatrix} \mathcal{K}^{s,\gamma}(X^\wedge;\mathbf{m}^\wedge) \\ \oplus \\ L_- \end{pmatrix} \to \pi_Y^* \begin{pmatrix} \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge;\mathbf{n}^\wedge) \\ \oplus \\ L_+ \end{pmatrix}$$

is an isomorphism for an $s = s_0 > \max(\mu, d) - \frac{1}{2}$.

Note that the bijectivities in (ii) and (iii) for an arbitrary $s = s_0$ imply the same for all $s > \max(\mu, d) - \frac{1}{2}$.

Theorem 2.24. *Let* $\mathcal{A} \in \mathcal{Y}^{\mu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^l)_{\mathcal{O}}$ *be elliptic. Then*

$$\mathcal{A}(\lambda): \begin{array}{c} \mathcal{W}^{s,\gamma}(\mathbb{W};\mathbf{m}) \\ \oplus \\ H^{s-\frac{n+1}{2}}(Y,L_-) \end{array} \to \begin{array}{c} \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W};\mathbf{n}) \\ \oplus \\ H^{s-\mu-\frac{n+1}{2}}(Y,L_+) \end{array} \tag{2.34}$$

is a family of Fredholm operators for all $s > \max(\mu, d) - \frac{1}{2}$. *For* $l \geq 1$ *we have* $\operatorname{ind}\mathcal{A}(\lambda) = 0$ *for every* $\lambda \in \mathbb{R}^l$ *and there is a* $C > 0$ *such that* (2.34) *are isomorphisms for all* $|\lambda| \geq C$.

Proof. The ellipticity of $\mathcal{A}$ here is a special case of a generalisation of ellipticity for the "standard" edge algebra of boundary value problems from [10], Chapter 4. In fact, the standard edge algebra admits a parameter-dependent version, as explained in the proof of Theorem 2.22, and then $\mathcal{Y}^{\mu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^l)_{\mathcal{O}}$ is simply a subalgebra. Ellipticity then entails the existence of a parametrix within the larger edge algebra (of course, not, in the subclass $\mathcal{Y}^{-\mu,(d-\mu)^+}(\mathbb{W};\mathbf{w};\mathbb{R}^l)_{\mathcal{O}}$), and we now use such a result in the analogous case with parameters which gives us a corresponding parameter-dependent parametrix $\mathcal{P}(\lambda)$ of $\mathcal{A}(\lambda)$. In this framework we then obtain that $\mathcal{P}(\lambda)\mathcal{A}(\lambda) - \mathcal{I}$ is a Schwartz function of $\lambda \in \mathbb{R}^l$ in the space of smoothing operators in the larger algebra; the same is true of $\mathcal{A}(\lambda)\mathcal{P}(\lambda) - \mathcal{I}$. This yields that $\mathcal{A}(\lambda)$ becomes invertible for sufficiently large $|\lambda|$. At the same time we see that $\mathcal{A}(\lambda)$ is a family of Fredholm operators, since the remainders are compact for every λ. $\square$

3 Kernel cut-off and meromorphic symbols

3.1 Kernel cut-off for Green operators

In the following we will take $\mathbb{R}^{l+1}$ as the space of parameters (λ, τ) for $\lambda \in \mathbb{R}^l$, $\tau \in \mathbb{R}$. We want to define a class of operator families $\mathcal{Y}^{\mu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^l \times \mathbb{C})_{\mathcal{O}} \ni \mathcal{A}(\lambda, w)$ with holomorphic dependence on $w \in \mathbb{C}$ and $\mathcal{A}(\lambda, \tau + i\beta) \in \mathcal{Y}^{\mu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^l_\lambda \times \mathbb{R}_\tau)_{\mathcal{O}}$ for each $\beta \in \mathbb{R}$, with some uniformity condition with respect to β.
In this section we consider families $\mathcal{G}(\lambda, \tau) \in \mathcal{Y}_G^{\mu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^{l+1})_{\mathcal{O}}$ with parameters $(\lambda, \tau) \in \mathbb{R}^{l+1}$ and construct maps $\mathcal{Y}_G^{\mu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^{l+1})_{\mathcal{O}} \to \mathcal{Y}_G^{\mu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^l \times \mathbb{C})_{\mathcal{O}}$ by

a so-called kernel cut-off procedure. Let us illustrate the idea first for the simpler situation of operator-valued symbols.
Let E and $\widetilde{E}$ be Hilbert spaces, endowed with strongly continuous group of isomorphisms $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ and $\{\tilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$, respectively. We then have the symbol spaces $S^\mu_{\mathrm{cl}}(U\times\mathbb{R}^{m+1}_{\xi,\tau};E,\widetilde{E})$, $U\subseteq\mathbb{R}^p$ open, cf. Section 1.1. Let $S^\mu_{\mathrm{cl}}(U\times\mathbb{R}^m\times\mathbb{C};E,\widetilde{E})$ denote the subspace of all $h(x,\xi,w)\in\mathcal{A}(\mathbb{C}_w,S^\mu_{\mathrm{cl}}(U\times\mathbb{R}^m;E,\widetilde{E}))$ such that

$$h(x,\xi,\tau+i\beta)\in S^\mu_{\mathrm{cl}}(U\times\mathbb{R}^m\times\mathbb{R};E,\widetilde{E})$$

for every $\beta\in\mathbb{R}$, uniformly in compact β-intervals. The space $S^\mu_{\mathrm{cl}}(U\times\mathbb{R}^m\times\mathbb{C};E,\widetilde{E})$ is Fréchet in a canonical way. An adequate semi-norm system is as follows. Let first $(p_j)_{j\in\mathbb{N}}$ and $(q_k)_{k\in\mathbb{N}}$ be countable semi-norm systems in the spaces $S^\mu_{\mathrm{cl}}(U\times\mathbb{R}^m;E,\widetilde{E})$ and $S^\mu_{\mathrm{cl}}(U\times\mathbb{R}^m\times\mathbb{R};E,\widetilde{E})$, respectively. Then, for $h(x,\xi,w)\in S^\mu_{\mathrm{cl}}(U\times\mathbb{R}^m\times\mathbb{C};E,\widetilde{E})$ we form

$$\alpha_{j,K}(h):=\sup_{w\in K}p_j(h(x,\xi,w)) \tag{3.1}$$

for every $K\subset\subset\mathbb{C}$ and

$$\delta_{k,[c,c']}(h):=\sup_{c\le\beta\le c'}q_k(h(x,\xi,\tau+i\beta)) \tag{3.2}$$

for every compact interval $[c,c']$. Here, p_j is the semi-norm linked to the variables (x,ξ) while q_k is linked to (x,ξ,τ). Then, if we take, for instance a countable system of $K\subset\subset\mathbb{C}$ and compact intervals $[c,c']$ exausting $\mathbb{C}$ and $\mathbb{R}$, respectively, (3.1), (3.2) form altogether a countable system for our Fréchet topology. Given an element $a(x,\xi,\tau)\in S^\mu_{\mathrm{cl}}(U\times\mathbb{R}^{m+1}_{\xi,\tau};E,\widetilde{E})$ we set

$$k(a)(x,\xi,\vartheta)=\int e^{i\vartheta\tau}a(x,\xi,\tau)\,đ\tau. \tag{3.3}$$

This integral defines an element in $\mathcal{S}'(\mathbb{R},S^\mu_{\mathrm{cl}}(\mathbb{R}^m;E,\widetilde{E}))$ for every fixed $x\in U$, and it can easily be verified that for every $\psi(\vartheta)\in C^\infty_0(\mathbb{R})$ such that $\psi(\vartheta)\equiv1$ in a neighbourhood of $\vartheta=0$ (such elements will be called cut-off functions) we have

$$(1-\psi(\vartheta))k(a)(x,\xi,\vartheta)\in\mathcal{S}(\mathbb{R},S^\mu_{\mathrm{cl}}(\mathbb{R}^m;E,\widetilde{E})) \tag{3.4}$$

for every x. Let $\varphi(\vartheta)\in C^\infty_0(\mathbb{R})$ and form

$$H(\varphi)a(x,\xi,w):=\int e^{-i\vartheta w}\varphi(\vartheta)k(a)(x,\xi,\vartheta)\,d\vartheta \tag{3.5}$$

which is convergent for every $w\in\mathbb{C}$. In particular, we can define $H(\varphi)a(x,\xi,\tau)$ by inserting $\Im w=0$ in (3.5).

Theorem 3.1. *For every $a(x,\xi,\tau) \in S^{\mu}_{\mathrm{cl}}(U \times \mathbb{R}^{m+1}; E, \widetilde{E})$ the map $\varphi(\vartheta) \to H(\varphi)a(x,\xi,\tau)$ induces a coutinuous operator*

$$C_0^{\infty}(\mathbb{R}) \longrightarrow S^{\mu}_{\mathrm{cl}}(U \times \mathbb{R}^{m+1}; E, \widetilde{E}).$$

Moreover, for fixed $\varphi \in C_0^{\infty}(\mathbb{R})$ the map $a(x,\xi,\tau) \to H(\varphi)a(x,\xi,w)$ defines a coutinuous operator

$$H(\varphi) : S^{\mu}_{\mathrm{cl}}(U \times \mathbb{R}^{m+1}; E, \widetilde{E}) \longrightarrow S^{\mu}_{\mathrm{cl}}(U \times \mathbb{R}^{m} \times \mathbb{C}; E, \widetilde{E}).$$

A proof of this result in a slightly different form may be found in Dorschfeldt [4], Section 1.5.2.
The operator

$$H(\psi) : S^{\mu}_{\mathrm{cl}}(U \times \mathbb{R}^{m+1}; E, \widetilde{E}) \longrightarrow S^{\mu}_{\mathrm{cl}}(U \times \mathbb{R}^{m} \times \mathbb{C}; E, \widetilde{E})$$

for a cut-off function $\psi(\vartheta) \in C_0^{\infty}(\mathbb{R})$ (i.e., when $\psi(\vartheta) \equiv 1$ near $\vartheta = 0$) will be called a kernel cut-off operator. For any such operator we then have

$$a(x,\xi,\tau) - H(\psi)a(x,\xi,\tau) \in S^{-\infty}(U \times \mathbb{R}^{m+1}; E, \widetilde{E}), \tag{3.6}$$

cf. relation (3.4).

Proposition 3.2. *Let us set $\psi_r(\vartheta) = \psi(r\vartheta)$ for a cut-off function ψ. Then we have*

$$H(\psi_r)a(x,\xi,\tau) \to a(x,\xi,\tau)$$

as $r \to 0$ in the topology of $S^{\mu}_{\mathrm{cl}}(U \times \mathbb{R}^{m+1}; E, \widetilde{E})$.

Proof. Observe first that

$$H(\psi_r)a(x,\xi,\tau) - a(x,\xi,\tau) = \mathcal{F}_{\vartheta \to \tau}[(\psi_r(\vartheta) - 1)k(a)(x,\xi,\vartheta)], \tag{3.7}$$

and consider $f_r(x,\xi,\vartheta) = (\psi_r(\vartheta) - 1)k(a)(x,\xi,\vartheta)$. Since $\psi_r(\vartheta) - 1 = 0$ in a neighbourhood of $(\xi,\vartheta) = (0,0)$, we have that $f_r(x,\xi,\vartheta) \in C^{\infty}(U, \mathcal{S}(\mathbb{R}^{m+1}, \mathcal{L}(E,\widetilde{E})))$. We want to prove that

$$f_r(x,\xi,\vartheta) \longrightarrow 0 \text{ in } C^{\infty}(U, \mathcal{S}(\mathbb{R}^{m+1}, \mathcal{L}(E,\widetilde{E}))) \text{ as } r \to 0. \tag{3.8}$$

Let us write $\ell(\cdot) := \|\tilde{\kappa}^{-1}_{\langle \xi,\vartheta\rangle}\{\cdot\}\kappa_{\langle \xi,\vartheta\rangle}\|_{\mathcal{L}(E,\widetilde{E})}$ and fix $N \in \mathbb{N}_0$; observe that there exists a constant $r_0 > 0$ such that, for every $r < r_0$, $(\partial^h_{\vartheta}(\psi_r(\vartheta) - 1))(1 - \psi(\vartheta)) = \partial^h_{\vartheta}(\psi_r(\vartheta) - 1)$ for all $h \geq 0$. Then, putting $1 - \psi(\vartheta) := \varphi(\vartheta)$, for $m + |\alpha| + k \leq N$ and $r < r_0$ we have:

$$\sup_{(\xi,\vartheta) \in \mathbb{R}^{m+1}} \ell\big(\partial^{\alpha}_{\xi}\partial^{k}_{\vartheta} f_r(x,\xi,\vartheta)\big)\langle \xi,\vartheta\rangle^m$$

$$\leq C \sup_{(\xi,\vartheta) \in \mathbb{R}^{m+1}} \sum_{k_1+k_2=k} \ell\Big([\partial^{k_1}_{\vartheta}(\psi_r(\vartheta) - 1)]\varphi(\vartheta)\partial^{k_2}_{\vartheta} \int e^{i\vartheta\tau}\partial^{\alpha}_{\xi} a(x,\xi,\tau)\, đ\tau\Big)\langle \xi,\vartheta\rangle^m$$

$$\leq C \sup_{(\xi,\vartheta) \in \mathbb{R}^{m+1}} \sum_{k_1+k_2=k} |\partial^{k_1}_{\vartheta}(\psi_r(\vartheta) - 1)|\, \ell\Big(\varphi(\vartheta)\partial^{k_2}_{\vartheta} \int e^{i\vartheta\tau}\partial^{\alpha}_{\xi} a(x,\xi,\tau)\, đ\tau\Big)\langle \xi,\vartheta\rangle^m;$$

because of the excision function φ, we have that $\varphi(\vartheta)\partial_\vartheta^{k_2}\int e^{i\vartheta\tau}\partial_\xi^\alpha a(x,\xi,\tau)\,đ\tau \in C^\infty(U,\mathcal{S}(\mathbb{R}^{m+1},\mathcal{L}(E,\widetilde{E})))$, and so the following estimate holds:

$$\ell\Big(\varphi(\vartheta)\partial_\vartheta^{k_2}\int e^{i\vartheta\tau}\partial_\xi^\alpha a(x,\xi,\tau)\,đ\tau\Big) \le C\langle\xi,\vartheta\rangle^{-m-1};$$

moreover,

$$|\partial_\vartheta^{k_1}(\psi_r(\vartheta)-1)| \text{ is } \begin{cases} \le C & \text{for } \vartheta\in\mathbb{R} \\ = 0 & \text{for } \vartheta \le \frac{c}{r} \end{cases}.$$

Then for $r<r_0$ we obtain

$$\sup_{(\xi,\vartheta)\in\mathbb{R}^{m+1}} \ell\big(\partial_\xi^\alpha\partial_\vartheta^k f_r(x,\xi,\vartheta)\big)\langle\xi,\vartheta\rangle^m \le C \sup_{\substack{(\xi,\vartheta)\in\mathbb{R}^{m+1}\\ \vartheta\ge\frac{c}{r}}} \langle\xi,\vartheta\rangle^{-1} \le C'r \underset{r\to 0}{\longrightarrow} 0.$$

So (3.8) holds; by (3.7) and the fact that the Fourier transform is an isomorphism in the Schwartz classes, we have that $H(\psi_r)a(x,\xi,\tau)\to a(x,\xi,\tau)$ in $C^\infty(U,\mathcal{S}(\mathbb{R}^{m+1},\mathcal{L}(E,\widetilde{E})))$ as $r\to 0$; we then get the convergence in $S_{\mathrm{cl}}^\mu(U\times\mathbb{R}^{m+1};E,\widetilde{E})$ by the inclusion $C^\infty(U,\mathcal{S}(\mathbb{R}^{m+1},\mathcal{L}(E,\widetilde{E})))\hookrightarrow S_{\mathrm{cl}}^\mu(U\times\mathbb{R}^{m+1};E,\widetilde{E})$. $\square$

Corollary 3.3. $H(\psi_r)a(x,\xi,\tau)\in C^\infty(\overline{\mathbb{R}}_+,S_{\mathrm{cl}}^\mu(U\times\mathbb{R}^{m+1};E,\widetilde{E}))$.

Proposition 3.4. *Let $\varphi\in C_0^\infty(\mathbb{R}_+)$, let $\beta\in\mathbb{R}$. Then for every $a(x,\xi,\tau)\in S_{\mathrm{cl}}^\mu(U\times\mathbb{R}^{m+1};E,\widetilde{E})$ we have an asymptotic expansion*

$$H(\varphi)a(x,\xi,\tau+i\beta)\sim\sum_{k=0}^\infty c_k(\varphi,\beta)D_\tau^k a(x,\xi,\tau)$$

in the space $S_{\mathrm{cl}}^\mu(U\times\mathbb{R}^{m+1};E,\widetilde{E})$, with constants $c_k(\varphi,\beta)$ that are independent of the symbol a. In particular, for any cut-off function $\psi(\vartheta)$ in place of $\varphi(\vartheta)$ we have $c_0(\psi,\beta)=1$.

Corollary 3.5. *Let $a(x,\xi,\tau)\in S_{\mathrm{cl}}^\mu(U\times\mathbb{R}^{m+1};E,\widetilde{E})$ and let $\psi(\vartheta)$ be a cut-off function. Then for every $\alpha,\beta\in\mathbb{R}$ we have*

$$H(\psi)a(x,\xi,\tau+i\alpha)-H(\psi)a(x,\xi,\tau+i\beta)\in S_{\mathrm{cl}}^{\mu-1}(U\times\mathbb{R}^{m+1};E,\widetilde{E}).$$

More precisely, we have the following result.

Proposition 3.6. *For every $h(x,\xi,w)\in S_{\mathrm{cl}}^\mu(U\times\mathbb{R}^m\times\mathbb{C};E,\widetilde{E})$ we have the following properties:*

(i)
$$h(x,\xi,\tau+i\beta)-h(x,\xi,\tau+i\alpha)\in S^{\mu-1}_{\mathrm{cl}}(U\times\mathbb{R}^{m+1}_{\xi,\tau};E,\widetilde{E}) \tag{3.9}$$
for every $\alpha,\beta\in\mathbb{R}$, and there is an asymptotic expansion
$$h(x,\xi,\tau+i\beta)\sim\sum_{k=0}^{\infty}c_k(\alpha,\beta)D^k_\tau h(x,\xi,\tau+i\alpha) \tag{3.10}$$
in $S^{\mu}_{\mathrm{cl}}(U\times\mathbb{R}^{m+1};E,\widetilde{E})$, with constants $c_k(\alpha,\beta)$, depending on α,β,k (not on h);

(ii) *$h(x,\xi,\tau+i\alpha)\in S^{\mu-1}_{\mathrm{cl}}(U\times\mathbb{R}^{m+1}_{\xi,\tau};E,\widetilde{E})$ for any fixed $\alpha\in\mathbb{R}$ implies that $h(x,\xi,w)\in S^{\mu-1}_{\mathrm{cl}}(U\times\mathbb{R}^m\times\mathbb{C};E,\widetilde{E})$; in particular, if $h(x,\xi,\tau+i\beta)\in S^{-\infty}(U\times\mathbb{R}^{m+1};E,\widetilde{E})$ then $h(x,\xi,w)\in S^{-\infty}(U\times\mathbb{R}^m\times\mathbb{C};E,\widetilde{E})$.*

(iii) *$\{h(x,\xi,\tau+i\beta)\in S^{\mu}_{\mathrm{cl}}(U\times\mathbb{R}^{m+1};E,\widetilde{E}):h(x,\xi,w)\in S^{\mu}_{\mathrm{cl}}(U\times\mathbb{R}^m\times\mathbb{C};E,\widetilde{E})\}+S^{-\infty}(U\times\mathbb{R}^{m+1};E,\widetilde{E})=S^{\mu}_{\mathrm{cl}}(U\times\mathbb{R}^{m+1};E,\widetilde{E})$.*

Proof. (i) By the Taylor formula we get
$$h(x,\xi,\tau+i\beta)=\sum_{k=0}^{N-1}\frac{(\beta-\alpha)^k}{k!}D^k_\tau h(x,\xi,\tau+i\alpha)+\frac{(\beta-\alpha)^N}{N!}D^N_\tau h(x,\xi,\tau+i\rho) \tag{3.11}$$
for a suitable ρ between α and β. Now, since ρ belongs to a compact interval and $h(x,\xi,w)\in S^{\mu}_{\mathrm{cl}}(U\times\mathbb{R}^m\times\mathbb{C};E,\widetilde{E})$, we have $\frac{(\beta-\alpha)^N}{N!}D^N_\tau h(x,\xi,\tau+i\rho)\in S^{\mu-N}_{\mathrm{cl}}(U\times\mathbb{R}^{m+1}_{\xi,\tau};E,\widetilde{E})$; then (3.11) gives us the asymptotic expansion (3.10) with constants $c_k(\alpha,\beta)=\frac{(\beta-\alpha)^k}{k!}$. In particular, for $N=1$ we have (3.9).

(ii) By (3.9) we have that
$$h(x,\xi,\tau+i\beta)=h(x,\xi,\tau+i\alpha)\ \mathrm{mod}\ S^{\mu-1}_{\mathrm{cl}}(U\times\mathbb{R}^{m+1};E,\widetilde{E}),$$
and so $h(x,\xi,\tau+i\beta)\in S^{\mu-1}_{\mathrm{cl}}(U\times\mathbb{R}^{m+1};E,\widetilde{E})$ for every β. Moreover, the Taylor formula gives us, for a fixed γ,
$$h(x,\xi,\tau+i\beta)=h(x,\xi,\tau+i\gamma)+(\beta-\gamma)D_\tau h(x,\xi,\tau+i\rho) \tag{3.12}$$
for ρ between β and γ. Now $h(x,\xi,\tau+i\gamma)\in S^{\mu-1}_{\mathrm{cl}}(U\times\mathbb{R}^{m+1};E,\widetilde{E})$, and moreover $(\beta-\gamma)D_\tau h(x,\xi,w)\in S^{\mu-1}_{\mathrm{cl}}(U\times\mathbb{R}^m\times\mathbb{C};E,\widetilde{E})$, since $h\in S^{\mu}_{\mathrm{cl}}(U\times\mathbb{R}^m\times\mathbb{C};E,\widetilde{E})$. Note in particular that $(\beta-\gamma)D_\tau h(x,\xi,\tau+i\rho)$ satisfies uniform $S^{\mu-1}$ estimates for ρ in compact intervals; then by (3.12) in which we keep γ fixed, we get the uniform $S^{\mu-1}$ estimates of $h(x,\xi,\tau+i\beta)$ for β in compact intervals.

(iii) Let us consider $a(x,\xi,\tau) \in S^{\mu}_{\mathrm{cl}}(U \times \mathbb{R}^{m+1}_{\xi,\tau}; E, \widetilde{E})$, and form $H(\varphi)a(x,\xi,w) \in S^{\mu}_{\mathrm{cl}}(U \times \mathbb{R}^{m} \times \mathbb{C}; E, \widetilde{E})$ for a cut-off function $\varphi(\vartheta)$; we have to prove that

$$a(x,\xi,\tau) - H(\varphi)a(x,\xi,\tau) \in S^{-\infty}(U \times \mathbb{R}^{m+1}; E, \widetilde{E}).$$

By (3.5) we have

$$a(x,\xi,\tau) - H(\varphi)a(x,\xi,\tau) = \int e^{-i\vartheta\tau}(1-\varphi(\vartheta))k(a)(x,\xi,\vartheta)\,d\vartheta;$$

since $1-\varphi(\vartheta) = 0$ in a neighbourhood of $(\xi,\vartheta) = (0,0)$, we have that $(1-\varphi(\vartheta))k(a)(x,\xi,\vartheta) \in C^{\infty}(U, \mathcal{S}(\mathbb{R}^{m+1}, \mathcal{L}(E,\widetilde{E})))$, and so by the isomorphism of the Fourier transform in the Schwartz classes we get $a(x,\xi,\tau) - H(\varphi)a(x,\xi,\tau) \in C^{\infty}(U, \mathcal{S}(\mathbb{R}^{m+1}, \mathcal{L}(E,\widetilde{E}))) = S^{-\infty}(U \times \mathbb{R}^{m+1}; E, \widetilde{E})$. $\square$

The above scheme of constructing kernel cut-offs for operator-valued symbols can be specialised to Green symbols in the sense of Definition 2.5, here applied for covariables $(\xi,\tau) \in \mathbb{R}^{m+1}$ instead of η.
Applying Remark 2.6 the procedure can be completely reduced to the situation before of operator-valued symbols between Hilbert spaces $E, \widetilde{E}$. The slight modification with DN-orders does not change anything, because we could consider separately the spaces of symbols consisting of the entries of different orders. So we can directly formulate the corresponding results.
First we have the spaces $\mathcal{R}^{\mu,d}_{G}(U \times \mathbb{R}^{m} \times \mathbb{C}; \mathbf{w})_{\mathcal{O}}$ of Green symbols $g(y,\xi,w)$ belonging to $\mathcal{A}(\mathbb{C}, \mathcal{R}^{\mu,d}_{G}(U \times \mathbb{R}^{m}; \mathbf{w})_{\mathcal{O}})$ such that

$$g(y,\xi,\tau+i\beta) \in \mathcal{R}^{\mu,d}_{G}(U \times \mathbb{R}^{m+1}; \mathbf{w})_{\mathcal{O}}$$

for every $\beta \in \mathbb{R}$, uniformly in compact β-intervals.

Remark 3.7. *By replacing $S^{\mu}_{\mathrm{cl}}(U \times \mathbb{R}^{m+1}; E, \widetilde{E})$, $S^{\mu}_{\mathrm{cl}}(U \times \mathbb{R}^{m} \times \mathbb{C}; E, \widetilde{E})$ etc. by $\mathcal{R}^{\mu,d}_{G}(U \times \mathbb{R}^{m+1}; \mathbf{w})_{\mathcal{O}}$, $\mathcal{R}^{\mu,d}_{G}(U \times \mathbb{R}^{m} \times \mathbb{C}; \mathbf{w})_{\mathcal{O}}$, etc., the results of Theorem* 3.1, *Proposition* 3.2, 3.4 *and Corollary* 3.5 *remain true in analogous form.*

We now pass to the kernel cut-off for families of Green operators $\mathcal{G}(\lambda,\tau) \in \mathcal{Y}^{\mu,d}_{G}(\mathbb{W}; \mathbf{w}; \mathbb{R}^{l+1})_{\mathcal{O}}$ themselves. For convenience, we assume $\mathbb{W}$ to be compact, though the constructions can also be performed for non-compact $\mathbb{W}$. To this end we first introduce a natural Fréchet space structure in $\mathcal{Y}^{\mu,d}_{G}(\mathbb{W}; \mathbf{w}; \mathbb{R}^{l+1})_{\mathcal{O}}$. The space of smoothing operator families $\mathcal{Y}^{-\infty,d}(\mathbb{W}; \mathbf{w}; \mathbb{R}^{l+1})_{\mathcal{O}}$ is Fréchet in an obvious way. Moreover, we have a representation as a non-direct sum

$$\mathcal{Y}^{\mu,d}_{G}(\mathbb{W}; \mathbf{w}; \mathbb{R}^{l+1})_{\mathcal{O}} = \sum_{j=1}^{N} \mathcal{Y}_j + \mathcal{Y}^{-\infty,d}(\mathbb{W}; \mathbf{w}; \mathbb{R}^{l+1})_{\mathcal{O}} \tag{3.13}$$

where $\mathcal{Y}_j$ denotes the subspace of all elements of $\mathcal{Y}_G^{\mu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^{l+1})_{\mathcal{O}}$ that are of the form

$$\operatorname{diag}(\omega,\omega,1)\phi_j\mathcal{G}(\lambda,\tau)\theta_j\operatorname{diag}(\tilde{\omega},\tilde{\omega},1),$$

for arbitrary $\mathcal{G}(\lambda,\tau)\in\mathcal{Y}_G^{\mu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^{l+1})_{\mathcal{O}}$, cf. formula (2.21). Let us choose functions $\tilde{\theta}_j\in C_0^\infty(V_j)$ that equal 1 on $\operatorname{supp}\theta_j$; then we have $\phi_j\tilde{\theta}_j=\phi_j$, $\theta_j\tilde{\theta}_j=\theta_j$ for all j. The space $\mathcal{Y}_j$ is contained in the space $\widetilde{\mathcal{Y}}_j$ of all elements of

$$\operatorname{diag}(\omega,\omega,1)\,\mathcal{Y}_G^{\mu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^{l+1})_{\mathcal{O}}\operatorname{diag}(\tilde{\omega},\tilde{\omega},1)$$

the (operator-valued) distributional kernel of which is supported in $V_j\times V_j$. Let us pass to local representations of operators of $\widetilde{\mathcal{Y}}_j$ in $\Omega_j\times\Omega_j\ni(y,y')$ via push-forward under $\chi:V_j\to\Omega_j$. Then, similarly to the general pseudo-differential calculus with operator-valued symbols, cf. the remarks at the end of Section 1.2, every element $\mathcal{G}(\lambda,\tau)\in\widetilde{\mathcal{Y}}_j$, can be written as a sum $\mathcal{G}(\lambda,\tau)=\mathcal{G}_0(\lambda,\tau)+\mathcal{G}_1(\lambda,\tau)$, where $\mathcal{G}_0(\lambda,\tau)$ is properly supported in the (y,y')-variables, and $\mathcal{G}_1(\lambda,\tau)$ is smoothing in the sense that when we multiply $\mathcal{G}_1(\lambda,\tau)$ from both sides with $\tilde{\theta}_j$ we get (by extension by zero outside our neighbourhood) an element in $\mathcal{Y}^{-\infty,d}(\mathbb{W};\mathbf{w};\mathbb{R}^{l+1})_{\mathcal{O}}$. Without loss of generality we may assume that the kernel of $\mathcal{G}_0(\lambda,\tau)$ is supported for all (λ,τ) in a fixed proper compact set $K\subset\Omega_j\times\Omega_j$ (that contains $\operatorname{diag}\Omega_j\times\Omega_j$ in its interior). Denote by $\mathcal{Y}_{j,K}$ the subspace of all elements in $\widetilde{\mathcal{Y}}_j$ with support K in that sense. There is then a one-to-one map

$$\sigma:\mathcal{Y}_{j,K}\longrightarrow\mathcal{R}_G^{\mu,d}(\Omega_j\times\mathbb{R}^{q+l};\mathbf{w}_{\Omega_j})_{\mathcal{O},K}\tag{3.14}$$

to a closed subspace of $\mathcal{R}_G^{\mu,d}(\Omega_j\times\mathbb{R}^{q+l};\mathbf{w}_{\Omega_j})_{\mathcal{O}}$ such that $\mathcal{G}(\lambda,\tau)=\operatorname{Op}_y(g)(\lambda,\tau)$ for $g=\sigma(\mathcal{G})$, cf. the constructions at the end of Section 1.2. From the bijection (3.14) we get a Fréchet topology in the space $\mathcal{Y}_{j,K}$ for every j. Because of the properties concerning supports of distributional kernels we also have

$$\mathcal{Y}_G^{\mu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^{l+1})_{\mathcal{O}}=\sum_{j=1}^N\mathcal{Y}_{j,K}+\mathcal{Y}^{-\infty,d}(\mathbb{W};\mathbf{w};\mathbb{R}^{l+1})_{\mathcal{O}}\tag{3.15}$$

instead of (3.13). We then finally endow the space $\mathcal{Y}_G^{\mu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^{l+1})_{\mathcal{O}}$ with the Fréchet topology of the non-direct sum.
Let $\mathcal{Y}_G^{\mu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^l\times\mathbb{C})_{\mathcal{O}}$ denote the space of all $\mathcal{G}(\lambda,z)\in\mathcal{A}(\mathbb{C},\mathcal{Y}_G^{\mu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^l_\lambda)_{\mathcal{O}})$ such that $\mathcal{G}(\lambda,\tau+i\beta)\in\mathcal{Y}_G^{\mu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^l_\lambda\times\mathbb{R}_\tau)_{\mathcal{O}}$ for every $\beta\in\mathbb{R}$, uniformly in compact β-intervals. From the definition we immediately get a Fréchet space structure in the space $\mathcal{Y}_G^{\mu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^l\times\mathbb{C})_{\mathcal{O}}$.
Every $\mathcal{G}(\lambda,\tau)\in\mathcal{Y}_G^{\mu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^l_\lambda\times\mathbb{R}_\tau)_{\mathcal{O}}$ has the form

$$\mathcal{G}(\lambda,\tau)=\operatorname{diag}(\omega,\omega,1)\sum_{j=1}^N\phi_j(\boldsymbol{\chi}_j^{-1})_*\operatorname{Op}(g_j)(\lambda,\tau)\theta_j\operatorname{diag}(\tilde{\omega},\tilde{\omega},1)+\mathcal{C}(\lambda,\tau),$$

cf. formulas (2.21), (3.13), where $g_j(y, \eta, \lambda, \tau)$ are local Green symbols, cf. (2.20), and $\mathcal{C}(\lambda, \tau) \in \mathcal{Y}_G^{-\infty,d}(\mathbb{W}; \mathbf{w}; \mathbb{R}^l_\lambda \times \mathbb{R}_\tau)_\mathcal{O}$. Similarly as before for amplitude functions we can apply kernel cut-off to operator families. In particular, setting $k(\mathcal{G})(\lambda, \vartheta) = \int e^{i\vartheta\tau}\mathcal{G}(\lambda, \tau)\, d\tau$, $\mathcal{G}(\lambda, \tau) \in \mathcal{Y}_G^{\mu,d}(\mathbb{W}; \mathbf{w}; \mathbb{R}^{l+1}_{\lambda,\tau})_\mathcal{O}$, we can pass to

$$H(\varphi)\mathcal{G}(\lambda, w) = \int e^{-i\vartheta w}\varphi(\vartheta)k(\mathcal{G})(\lambda, \vartheta)\, d\vartheta,$$

$\varphi \in C_0^\infty(\mathbb{R})$, $w \in \mathbb{C}$. A consequence of the above general results is the following theorem.

Theorem 3.8. *The map $H(\varphi)$, $\varphi \in C_0^\infty(\mathbb{R})$, induces a continuous operator*

$$H(\varphi) : \mathcal{Y}_G^{\mu,d}(\mathbb{W}; \mathbf{w}; \mathbb{R}^{l+1}_{\lambda,\tau})_\mathcal{O} \longrightarrow \mathcal{Y}_G^{\mu,d}(\mathbb{W}; \mathbf{w}; \mathbb{R}^l_\lambda \times \mathbb{C}_w)_\mathcal{O} \tag{3.16}$$

for every $\mu \in \mathbb{Z}$, $d \in \mathbb{N}$. In particular, if ψ is a cut-off function, it follows that $\mathcal{G}(\lambda, \tau) - H(\psi)\mathcal{G}(\lambda, \tau) \in \mathcal{Y}^{-\infty,d}(\mathbb{W}; \mathbf{w}; \mathbb{R}^{l+1}_{\lambda,\tau})_\mathcal{O}$.
Moreover, given any $\mathcal{H}(\lambda, w) \in \mathcal{Y}_G^{\mu,d}(\mathbb{W}; \mathbf{w}; \mathbb{R}^l \times \mathbb{C})$, the (parameter-dependent) homogeneous principal edge symbol $\sigma_\wedge(\mathcal{H}_\beta)(y, \eta, \lambda, \tau)$ of $\mathcal{H}_\beta(\lambda, \tau) := \mathcal{H}(\lambda, \tau + i\beta) \in \mathcal{Y}_G^{\mu,d}(\mathbb{W}; \mathbf{w}; \mathbb{R}^{l+1})$ is independent of $\beta \in \mathbb{R}$.

Proof. Let us set

$$(H(\varphi)\mathcal{G})(\lambda, w) = \operatorname{diag}(\omega, \omega, 1) \sum_{j=1}^{N} \phi_j(\boldsymbol{\chi}_j^{-1})_* \operatorname{Op}(H(\varphi)(g_j))(\lambda, w)\theta_j \operatorname{diag}(\tilde{\omega}, \tilde{\omega}, 1),$$

with $(H(\varphi)g_j)(y, \eta, \lambda, w)$ begin interpreted as in Remark 3.7. The conclusions then follow from the preceding results on Green symbols, or, more generally, operator-valued symbols, cf. in particular Theorem 3.1 and (3.6). □

Remark 3.9. *Propositions* 3.2, 3.4 *and Corollary* 3.5 *have immediate analogoues for the case of families of Green operators.*

3.2 Kernel cut-off for the edge calculus

The next step in the program to construct holomorphic functions of $w \in \mathbb{C}$ with values in $\mathcal{Y}^{\mu,d}(\mathbb{W}; \mathbf{w}; \mathbb{R}^l)_\mathcal{O}$ is to focus on the summand in the middle of the right of (2.23), again with (λ, τ) instead of λ. Concerning bundle data we use an analogous notation as in Definition 2.16. Because of (2.24) it suffices to consider $P(\lambda, \tau) \in \mathcal{B}^{\mu,d}(\mathbb{W}_{\mathrm{reg}}; \mathbf{v}; \mathbb{R}^{l+1})$. Let

$$\mathcal{B}^{-\infty,d}(\mathbb{W}_{\mathrm{reg}}; \mathbf{v}; \mathbb{R}^l \times \mathbb{C}) \tag{3.17}$$

denote the space of all elements $\mathcal{G}(\lambda, w) \in \mathcal{A}(\mathbb{C}, \mathcal{B}^{-\infty,d}(\mathbb{W}_{\mathrm{reg}}; \mathbf{v}; \mathbb{R}^l))$ such that $\mathcal{G}(\lambda, \tau + i\beta) \in \mathcal{B}^{-\infty,d}(\mathbb{W}_{\mathrm{reg}}; \mathbf{v}; \mathbb{R}^{l+1}_{\lambda,\tau})$ for every $\beta \in \mathbb{R}$, uniformly in compact β-intervals. Moreover, $\mathcal{B}^{\mu,d}(\mathbb{W}_{\mathrm{reg}}; \mathbf{v}; \mathbb{R}^l \times \mathbb{C})$ is defined as the set of all operator families

$$\mathcal{P}(\lambda, w) = \mathcal{A}_0(\lambda, w) + \mathcal{A}_1(\lambda, w) + \mathcal{G}(\lambda, w)$$

such that $\mathcal{G}(\lambda, w) \in \mathcal{B}^{-\infty,d}(\mathbb{W}_{\mathrm{reg}}; \mathbf{v}; \mathbb{R}^l \times \mathbb{C})$, while $\mathcal{A}_0(\lambda, w)$ and $\mathcal{A}_1(\lambda, w)$ are of the following structure: using similar notation as in Definition 1.8, here with $\mathbb{W}_{\mathrm{reg}}$ instead of X, we set:

$$\mathcal{A}_1(\lambda, w) = \begin{pmatrix} (1-\omega)(H(\psi)A_{\mathrm{int}})(\lambda, w)(1-\tilde{\omega}) & 0 \\ 0 & 0 \end{pmatrix}$$

for a cut-off function ψ and an operator $A_{\mathrm{int}}(\lambda, \tau) \in L^{\mu}_{\mathrm{cl}}(\mathbb{W}_{\mathrm{reg}}; E, F; \mathbb{R}^{l+1}_{\lambda,\tau})$. As in the constructions of the preceding section the kernel cut-off operator is applied to the covariable τ in local parameter-dependent amplitude functions of $A_{\mathrm{int}}(\lambda, \tau)$. Moreover, we define

$$\mathcal{A}_0(\lambda, w) = \mathcal{M}_{\omega} \sum_{j \in \mathbb{N}} \mathcal{M}_{\phi_j}(H(\psi)\mathcal{A}_j)(\lambda, w)\mathcal{M}_{\theta_j}\mathcal{M}_{\tilde{\omega}},$$

again in analogy to a corresponding expression in Definition 1.8, where now the kernel cut-off operator is applied to corresponding local amplitude functions $a_j(\tilde{x}', \tilde{\xi}', \lambda, \tau)$ of $\mathcal{A}_j$ with respect to τ, where $a_j(\tilde{x}', \tilde{\xi}', \lambda, \tau) \in \mathcal{R}^{\mu,d}(U_{j,\tilde{x}'} \times \mathbb{R}^{n-1+q+(l+1)}_{\tilde{\xi}',\lambda,\tau}; (m, k; j_-, j_+))$ with local coordinates $\tilde{x}'$ for $\partial(\mathbb{W}_{\mathrm{reg}})$ and corresponding covariables $\tilde{\xi}'$; m and k are the fibre dimensions of E and F, respectively.
Note that the definition of (3.17) is correct in the sense that a change of the cut-off function ψ only affects the result by an operator family in $\mathcal{B}^{-\infty,d}(\mathbb{W}_{\mathrm{reg}}; \mathbf{v}; \mathbb{R}^l \times \mathbb{C})$; also the choice of local charts, partition of unity, etc., are not essential. Using the general structures on kernel cut-off from the beginning of Section 3.1 we get the following result:

Theorem 3.10. *For every $P_0(\lambda, \tau) \in \mathcal{B}^{\mu,d}(\mathbb{W}_{\mathrm{reg}}; \mathbf{v}; \mathbb{R}^{l+1}_{\lambda,\tau})$ there exists a $P(\lambda, w) \in \mathcal{B}^{\mu,d}(\mathbb{W}_{\mathrm{reg}}; \mathbf{v}; \mathbb{R}^l \times \mathbb{C})$ such that*

$$P_0(\lambda, \tau) - P(\lambda, \tau) \in \mathcal{B}^{-\infty,d}(\mathbb{W}_{\mathrm{reg}}; \mathbf{v}; \mathbb{R}^{l+1}).$$

Moreover, for every $P(\lambda, w) \in \mathcal{B}^{\mu,d}(\mathbb{W}_{\mathrm{reg}}; \mathbf{v}; \mathbb{R}^l \times \mathbb{C})$ we have

$$P(\lambda, \tau + i\beta) - P(\lambda, \tau + i\alpha) \in \mathcal{B}^{\mu-1,d}(\mathbb{W}_{\mathrm{reg}}; \mathbf{v}; \mathbb{R}^{l+1})$$

for arbitrary $\alpha, \beta \in \mathbb{R}$. In particular, it follows that (parameter-dependent) principal interior and boundary symbols of $P(\lambda, w)$ (with parameters $(\lambda, \Re w)$) are independent of $\Im w$.

We consider now the parameter-dependent edge algebra $\mathcal{Y}^{\mu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^{l+1})_{\mathcal{O}}$ on a compact (stretched) manifold $\mathbb{W}$, with parameters $(\lambda,\tau)\in\mathbb{R}^{l+1}$, cf. Definition 2.16. We want to apply the kernel cut-off procedure to the parameter τ. According to the representation (2.23) there remains the summand $\mathcal{M}(\lambda,\tau)$, since Green elements have been studied in the preceding section and $\mathcal{P}(\lambda,\tau)$ has been treated before. From (2.25) and relation (2.22) we see that the kernel cut-off can be reduced to a corresponding operation on amplitude functions $a_j(y,\eta,\lambda,\tau)$ with respect to τ. Let us omit subscript "j" for fixed j. By virtue of Theorem 2.13, and since Green elements are already treated, we may look at

$$a(y,\eta,\lambda,\tau)=\sigma(r)\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(y,\eta,\lambda,\tau)\tilde{\sigma}(r)$$

for an operator-valued Mellin symbol $h(r,y,z,\eta,\lambda,\tau)$. The variable $y\in\Omega$ is completely unessential for the following computation; therefore, for simplicity, we assume that h is independent of y. Because of (2.10) we have $h(r,z,\eta,\lambda,\tau)=\tilde{h}(r,z,r\eta,r\lambda,r\tau)$ for some $\tilde{h}(r,z,\tilde{\eta},\tilde{\lambda},\tilde{\tau})\in C^\infty(\overline{\mathbb{R}}_+,\mathcal{M}_{\mathcal{O}}^{\mu,d}(X;\mathbf{v};\mathbb{R}^{q+l+1}_{\tilde{\eta},\tilde{\lambda},\tilde{\tau}}))$.
Let $\psi(\vartheta)$ be a cut-off function as in the preceding section. We then have

$$H(\psi)a(\eta,\lambda,w)=\sigma(r)\mathrm{op}_M^{\gamma-\frac{n}{2}}(H(\psi)h)(\eta,\lambda,w)\tilde{\sigma}(r);$$

so we calculate $H(\psi)h$. Let us set

$$k_\tau(h)(r,z,\eta,\lambda,\vartheta)=\int e^{i\tau\vartheta}\tilde{h}(r,z,r\eta,r\lambda,r\tau)\,d\tau$$

and

$$k_{\tilde{\tau}}(\tilde{h})(r,z,\tilde{\eta},\tilde{\lambda},\tilde{\vartheta})=\int e^{i\tilde{\tau}\tilde{\vartheta}}\tilde{h}(r,z,\tilde{\eta},\tilde{\lambda},\tilde{\tau})\,d\tilde{\tau}.$$

We then obtain

$$k_\tau(h)(r,z,\eta,\lambda,\vartheta)=r^{-1}\int e^{i\tilde{\tau}\frac{\vartheta}{r}}\tilde{h}(r,z,\tilde{\eta},\tilde{\lambda},\tilde{\tau})\,d\tilde{\tau}=r^{-1}k_{\tilde{\tau}}(\tilde{h})\big(r,z,r\eta,r\lambda,\frac{\vartheta}{r}\big).$$

Then, it follows that

$$\begin{aligned}H(\psi)h(r,z,\eta,\lambda,w)&=\int e^{-i\vartheta w}\psi(\vartheta)k_\tau(h)(r,z,\eta,\lambda,\vartheta)\,d\vartheta\\&=r^{-1}\int e^{-i\vartheta w}\psi(\vartheta)k_{\tilde{\tau}}(\tilde{h})\big(r,z,r\eta,r\lambda,\frac{\vartheta}{r}\big)\,d\vartheta\\&=\int e^{-i\tilde{\vartheta}rw}\psi(r\tilde{\vartheta})k_{\tilde{\tau}}(\tilde{h})(r,z,r\eta,r\lambda,\tilde{\vartheta})\,d\tilde{\vartheta}\\&=\big(H(\psi_r)\tilde{h}\big)(r,z,r\eta,r\lambda,rw),\end{aligned}\tag{3.18}$$

where $\psi_r(\tilde{\vartheta}):=\psi(r\tilde{\vartheta})$, and $w\in\mathbb{C}$. Now we write $\mathcal{M}_{\mathcal{O}}^{-\infty,d}(X;\mathbf{v};\mathbb{R}^{q+l}_{\eta,\lambda}\times\mathbb{C}_w)$ for the set of all $m\in\mathcal{A}(\mathbb{C}_w,\mathcal{M}_{\mathcal{O}}^{-\infty,d}(X;\mathbf{v};\mathbb{R}^{q+l}_{\eta,\lambda}))$ such that $m(z,\eta,\lambda,\tau+i\beta)\in\mathcal{M}_{\mathcal{O}}^{-\infty,d}(X;\mathbf{v};\mathbb{R}^{q+l+1}_{\eta,\lambda,\tau})$, uniformly in compact β intervals.

Remark 3.11. *Let ψ, $\tilde{\psi}$ be two excision functions. Then $H(\psi)h(r,z,\eta,\lambda,w) - H(\tilde{\psi})h(r,z,\eta,\lambda,w)$ has the form $f(r,z,\eta,\lambda,w) = \tilde{f}(r,z,r\eta,r\lambda,rw)$ for an element $\tilde{f}(r,z,\tilde{\eta},\tilde{\lambda},\tilde{w})$ in the space $C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_{\mathcal{O}}^{-\infty,d}(X;\mathbf{v};\mathbb{R}^{q+l}_{\tilde{\eta},\tilde{\lambda}} \times \mathbb{C}_{\tilde{w}}))$.*

Recall that if we have a chart $\chi : V \to \Omega$ for a coordinate neighbourhood V on Y we denote by $(\boldsymbol{\chi}^{-1})_*$ the pull-back of operators from Ω to V which takes into account chosen trivialisations of vector bundles on Y restricted to V. Then a typical ingredient of our edge calculus are operators of the form

$$M_0(\lambda,\tau) := \phi(\boldsymbol{\chi}^{-1})_* \operatorname{Op}_y(a)(\lambda,\tau)\theta \tag{3.19}$$

for $\phi, \theta \in C_0^\infty(V)$ and an amplitude function $a(y,\eta,\lambda,\tau)$ as in (2.22) (here, with (λ,τ) in place of λ). According to Theorem 2.13 we may take

$$a(y,\eta,\lambda,\tau) = \sigma(r) \operatorname{op}_M^{\gamma-\frac{n}{2}}(h)(y,\eta,\lambda,\tau)\tilde{\sigma}(r)$$

modulo flat Green symbols (the latter ones are treated in the preceding section), $h(r,y,z,\eta,\lambda,\tau) = \tilde{h}(r,y,z,r\eta,r\lambda,r\tau)$ for an element $\tilde{h}(r,y,z,\tilde{\eta},\tilde{\lambda},\tilde{\tau}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \mathcal{M}_{\mathcal{O}}^{\mu,d}(X;\mathbf{v};\mathbb{R}^{q+l+1}_{\tilde{\eta},\tilde{\lambda},\tilde{\tau}}))$. Applying the kernel cut-off operator $H(\psi)$ to (3.19) with respect to the variable τ we get

$$M(\lambda,w) := (H(\psi)M_0)(\lambda,w) = \phi(\boldsymbol{\chi}^{-1})_* \operatorname{Op}_y(H(\psi)a)(\lambda,w)\theta \tag{3.20}$$

for

$$(H(\psi)a)(y,\eta,\lambda,w) = \sigma(r) \operatorname{op}_M^{\gamma-\frac{n}{2}}(H(\psi)h)(y,\eta,\lambda,w)\tilde{\sigma}(r).$$

Here, according to (3.18), $(H(\psi)h)(r,y,z,\eta,\lambda,w) = (H(\psi_r)\tilde{h})(r,y,z,r\eta,r\lambda,rw)$.

Definition 3.12. *Let $\mu \in \mathbb{Z}$, $d \in \mathbb{N}$ and $\mathbf{w} = (E,F;J_-,J_+;L_-,L_+)$. We write $\mathcal{Y}^{\mu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^l \times \mathbb{C})_{\mathcal{O}}$ for the space of all operator families*

$$\mathcal{M}(\lambda,w) + \mathcal{P}(\lambda,w) + \mathcal{G}(\lambda,w) + \mathcal{N}(\lambda,w),$$

where

$$\mathcal{M}(\lambda,w) = \begin{pmatrix} M(\lambda,w) & 0 \\ 0 & 0 \end{pmatrix}$$

with $M(\lambda,w) = \sum_{j=1}^N \phi_j(\boldsymbol{\chi}_j^{-1})_* \operatorname{Op}_y(H(\psi)a_j)(\lambda,w)\theta_j$, cf. the description on operators (3.20) before,

$$\mathcal{P}(\lambda,w) = \begin{pmatrix} (1-\sigma)P(\lambda,w)(1-\tilde{\tilde{\sigma}}) & 0 \\ 0 & 0 \end{pmatrix},$$

where $P(\lambda,w) \in \mathcal{B}^{\mu,d}(\mathbb{W}_{\mathrm{reg}};\mathbf{v};\mathbb{R}^l \times \mathbb{C})$; $\mathcal{G}(\lambda,w) \in \mathcal{Y}_G^{\mu,d}(\mathbb{W};\mathbf{w},\mathbb{R}^l \times \mathbb{C})_{\mathcal{O}}$, and finally

$$\mathcal{N}(\lambda,w) = \begin{pmatrix} N(\lambda,w) & 0 \\ 0 & 0 \end{pmatrix},$$

where $N(\lambda,w)$ has the form $N(\lambda,w) = \sum_{j=1}^N \phi_j(\boldsymbol{\chi}_j^{-1})_* \operatorname{Op}_y(m_j)(\lambda,w)\theta_j$, $m_j \in C^\infty(\overline{\mathbb{R}}_+, \mathcal{M}_{\mathcal{O}}^{-\infty,d}(X;\mathbf{v};\mathbb{R}^{q+l}_{\eta,\lambda} \times \mathbb{C}_w))$.

Observe that, by virtue of Remark 3.11, Definition 3.12 is correct, i.e., independent of the specific choice of the cut-off function ψ.

Theorem 3.13. *For every $\mathcal{A}_0(\lambda,\tau) \in \mathcal{Y}^{\mu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^{l+1})_{\mathcal{O}}$ there exists $\mathcal{A}(\lambda,w) \in \mathcal{Y}^{\mu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^l \times \mathbb{C})_{\mathcal{O}}$ such that*

$$\mathcal{A}_0(\lambda,\tau) - \mathcal{A}(\lambda,\tau) \in \mathcal{Y}^{-\infty,d}(\mathbb{W};\mathbf{w};\mathbb{R}^{l+1})_{\mathcal{O}}. \tag{3.21}$$

Moreover, for every $\mathcal{A}(\lambda,w) \in \mathcal{Y}^{\mu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^l \times \mathbb{C})_{\mathcal{O}}$ we have $\mathcal{A}(\lambda,\tau+i\beta) - \mathcal{A}(\lambda,\tau+i\alpha) \in \mathcal{Y}^{\mu-1,d}(\mathbb{W};\mathbf{w};\mathbb{R}^{l+1})_{\mathcal{O}}$ for arbitrary $\alpha,\beta \in \mathbb{R}$. In particular, it follows that for $\mathcal{A}_\beta(\lambda,\tau) := \mathcal{A}(\lambda,\tau+i\beta)$, $\beta \in \mathbb{R}$, the (parameter-dependent) homogeneous principal interior, boundary and edge symbols $\sigma_\psi(\mathcal{A}_\beta)$, $\sigma_\partial(\mathcal{A}_\beta)$ and $\sigma_\wedge(\mathcal{A}_\beta)$ are independent of β. The same is true for subordinate conormal symbols $\sigma_M(\mathcal{A}_\beta)(y,z)$.

Proof. The proof consists of applying kernel cut-off constructions with a cut-off function ψ to the summands in formula (2.23) separately, here, of course in the version $\mathcal{M}_0(\lambda,\tau)$, $\mathcal{P}_0(\lambda,\tau)$ and $\mathcal{G}_0(\lambda,\tau)$, cf. Theorem 3.8 for $\mathcal{G}_0(\lambda,\tau)$, Theorem 3.10 for $\mathcal{P}_0(\lambda,\tau)$, and formula (3.20) for $\mathcal{M}_0(\lambda,\tau)$. $\square$

Clearly, we also have (in the notation of Theorem 3.13)

$$\sigma_\psi(\mathcal{A}_0(\lambda,\tau)) = \sigma_\psi(\mathcal{A}(\lambda,w)), \quad \sigma_\partial(\mathcal{A}_0(\lambda,\tau)) = \sigma_\partial(\mathcal{A}(\lambda,w)),$$
$$\sigma_\wedge(\mathcal{A}_0(\lambda,\tau)) = \sigma_\wedge(\mathcal{A}(\lambda,w)), \quad \sigma_M(\mathcal{A}_0(\lambda,\tau)) = \sigma_M(\mathcal{A}(\lambda,w))$$

for $w = \tau + i\beta$ with arbitrary $\beta \in \mathbb{R}$.

Theorem 3.14. *$\mathcal{A} \in \mathcal{Y}^{\nu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^l \times \mathbb{C})_{\mathcal{O}}$, $\mathcal{B} \in \mathcal{Y}^{\mu,e}(\mathbb{W};\mathbf{v};\mathbb{R}^l \times \mathbb{C})_{\mathcal{O}}$ for $\mathbf{w}$ and $\mathbf{v}$ as in Theorem 2.22 implies $\mathcal{AB} \in \mathcal{Y}^{\nu+\mu,h}(\mathbb{W};\mathbf{w}\circ\mathbf{v};\mathbb{R}^l \times \mathbb{C})_{\mathcal{O}}$ for $h = \max(\mu+d,e)$, and we have $\sigma(\mathcal{AB}) = \sigma(\mathcal{A})\sigma(\mathcal{B})$ in the sense of componentwise composition.*

Proof. Let us write $\mathcal{A}(\lambda,w) = \mathcal{A}(\lambda,\tau+i\beta)$ and $\mathcal{B}(\lambda,w) = \mathcal{B}(\lambda,\tau+i\beta)$. Then, for fixed β we get $\mathcal{A}(\lambda,\tau+i\beta)\mathcal{B}(\lambda,\tau+i\beta) \in \mathcal{Y}^{\mu+\nu,h}(\mathbb{W};\mathbf{w}\circ\mathbf{v};\mathbb{R}^{l+1}_{\lambda,\tau})_{\mathcal{O}}$ which is a consequence of Theorem 2.22. An inspection of the proof of Theorem 2.22 shows also the holomorphy of the composition in the parameter $w \in \mathbb{C}$. $\square$

3.3 Ellipticity of holomorphic families

We now turn to ellipticity of families of edge boundary value problems on $\mathbb{W}$.

Definition 3.15. *An element $\mathcal{A}(\lambda,w) \in \mathcal{Y}^{\mu,d}(\mathbb{W};\mathbf{w};\mathbb{R}^l \times \mathbb{C})_{\mathcal{O}}$ is called elliptic, if there is a $\beta \in \mathbb{R}$ such that $\mathcal{A}(\lambda,\tau+i\beta)$ is elliptic in the sense of Definition 2.23.*

Because of Theorem 3.13 the definition is correct, i.e., independent of the choice of β. Moreover, if $\mathcal{A}(\lambda,\tau+i\beta)$ is elliptic, so is $\mathcal{A}(\lambda,\tau+i\alpha)$ for every $\alpha \in \mathbb{R}$.

Theorem 3.16. *Let $\mathcal{A}(\lambda, w) \in \mathcal{Y}^{\mu,d}(\mathbb{W}; \mathbf{w}; \mathbb{R}^l \times \mathbb{C})_{\mathcal{O}}$ be elliptic. Then*

$$\mathcal{A}(\lambda, w) : \begin{array}{c} \mathcal{W}^{s,\gamma}(\mathbb{W}; \mathbf{m}) \\ \oplus \\ H^{s-\frac{n+1}{2}}(Y, L_-) \end{array} \longrightarrow \begin{array}{c} \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W}; \mathbf{n}) \\ \oplus \\ H^{s-\mu-\frac{n+1}{2}}(Y, L_+) \end{array} \tag{3.22}$$

(*with the same notation as in Theorem* 2.24) *is a family of Fredholm operators for every $s > \max(\mu, d) - \frac{1}{2}$, and for every $c, c' \in \mathbb{R}$, $c < c'$, there exists a $C > 0$ such that the operators* (3.22) *are isomorphisms for all $(\lambda, w) \in \mathbb{R}^l \times \mathbb{C}$, $c < \Im w < c'$ and $|\Re w| \geq C$.*

Proof. Let us write $\mathcal{A}(\lambda, w) = \mathcal{A}(\lambda, \tau + i\beta)$. Then, for every fixed $\beta \in \mathbb{R}$ we have an elliptic element in $\mathcal{Y}^{\mu,d}(\mathbb{W}; \mathbf{w}; \mathbb{R}^{l+1}_{\lambda,\tau})_{\mathcal{O}}$, and hence we can apply Theorem 2.24. This gives us the Fredholm property of (3.22) and isomorphisms for $|\lambda, \tau| \geq C$ for some $C > 0$. It is then sufficient to require $|\tau| = |\Re w| \geq C$. The dependence of our operators on β is C^∞ with respect to the operator norm. Therefore, we find a continuous choice of the constant $C = C(\beta)$. This allows us to find a C for all β varying in a compact interval $[c, c']$ which completes the proof. $\square$

Holomorphic operator families (3.22) for the case $l = 0$ play the role of conormal symbols of a next higher calculus of corner boundary value problems and for the evaluation of iterated edge/corner asymptotics. In this connection the following observation is essential:

Proposition 3.17. *Let $\mathcal{A}(w) \in \mathcal{Y}^{\mu,d}(\mathbb{W}; \mathbf{w}; \mathbb{C})_{\mathcal{O}}$ be elliptic. Then there exists a countable set $D \subset \mathbb{C}$ with finite intersection $D \bigcap \{w : c \leq \Im w \leq c'\}$ for every $c < c'$, such that $\mathcal{A}(w)$ is bijective for all $w \in \mathbb{C} \setminus D$ (and all $s > \max(\mu, d) - \frac{1}{2}$).*

This is a consequence of a corresponding general result on holomorphic Fredholm families between Hilbert spaces; it employs the fact that there is at least one $w \in \mathbb{C}$ where $\mathcal{A}(w)$ is an isomorphism, cf., e.g., [31], Section 1.2.4. Theorem 3.16 tells us that the latter condition holds in our case, namely for $|\Re w|$ sufficiently large. The non-bijectivity points $w \in D$ of $\mathcal{A}(w)$ give rise to poles of $\mathcal{A}^{-1}(w)$. Its meromorphic structure fits to a general functional analytic framework as is studied in [8]. In the present case the inverse $\mathcal{A}^{-1}(w)$ has interesting specific properties, since, similarly to the boundaryless case, cf. [16], the poles contribute to iterated edge/corner asymptotics of solutions to corresponding elliptic corner boundary value problems. In a forthcoming joint paper with T. Krainer we investigate these phenomena in more detail, using an extension of our holomorphic operator spaces by smoothing meromorphic elements from the article [3].

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