AN INDEX FORMULA ON MANIFOLDS WITH FIBERED CUSP ENDS

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Abstract. We consider a compact manifold X whose boundary is a locally trivial fiber bundle and an associated pseudodifferential algebra that models fibered cusps at infinity. Using trace-like functionals that generate the 0-dimensional Hochschild cohomology groups we express the index of a fully elliptic fibered cusp operator as the sum of a local contribution from the interior of X and a term that comes from the boundary. This answers the index problem formulated in [11]. We give a more precise answer in the case where the base of the boundary fiber bundle is S^1. In particular, for Dirac operators associated to a product metric of the form $g^X = \frac{dx^2}{x^4} + \frac{d\theta^2}{x^2} + g^F$ near the boundary $\{x = 0\}$ with twisting bundle T we obtain

$$\operatorname{index}(A) = \int_X \hat{A}(X)\operatorname{ch}(T) - \frac{\lim_a \eta(A_{|\partial X})}{2}$$

in terms of the integral of the Atiyah-Singer form in the interior of X, and the adiabatic limit of the η-invariant of the restriction of the operator to the boundary.

1. Introduction

Let X be a compact manifold whose boundary is the total space of a locally trivial fiber bundle $\varphi : \partial X \to Y$ of closed manifolds. Let $x : X \to \overline{\mathbb{R}}_+$ be a defining function for ∂X, i.e. $\partial X = \{x = 0\}$ and dx does not vanish at ∂X. One way to organize a (pseudo)differential analysis that reflects the geometry of X is to choose a Lie algebra $\mathcal{V}$ of vector fields on X, or more precisely a boundary fibration structure in the sense of Melrose [13]. The choice of the boundary fibration structure is by no means unique and different such structures on X require in fact completely different analytic tools – see, for instance, [6, 7, 9, 10, 11, 15]. In this paper we study the boundary fibration structure determined by the Lie algebra $\mathcal{V}_\Phi(X)$ of *fibered-cusp* or briefly Φ-*vector fields* where a smooth vector field V on X belongs to

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$\mathcal{V}_\Phi(X)$ provided V is tangent to the fibers of φ at the boundary and satisfies $Vx \in x^2\mathcal{C}^\infty(X)$. It is instructive to picture Φ-vector fields in local coordinates. Let $(x,y,z) : X \supseteq U \to \overline{\mathbb{R}}_+ \times \mathbb{R}^n_y \times \mathbb{R}^m_z$ be local product coordinates near the boundary such that y,z are coordinates in Y, respectively in the fiber F of φ. Then $V \in \mathcal{V}_\Phi(X)$ can be written as

$$V|_U(x,y,z) = a(x,y,z)x^2\partial_x + \sum_{j=1}^n b_j(x,y,z)x\partial_{y_j} + \sum_{k=1}^m c_k(x,y,z)\partial_z$$

with coefficients a, b_j, and c_k smooth down to $x = 0$. Taking these coefficients as local trivializations we see that the Lie algebra $\mathcal{V}_\Phi(X)$ can be realized as the space of smooth sections of a smooth vector bundle ${}^\Phi TX \to X$ that comes equipped with a natural homomorphism ${}^\Phi TX \to TX$.

The algebra of Φ-differential operators is by definition the enveloping algebra $\mathrm{Diff}^*_\Phi(X)$ of $\mathcal{V}_\Phi(X)$ over $\mathcal{C}^\infty(X)$. A corresponding Φ-pseudodifferential calculus $\Psi^*_\Phi(X)$ has been constructed by Mazzeo and Melrose in [11]. First, Φ-pseudodifferential operators act canonically on $\mathcal{C}^\infty(X)$, but since $\mathrm{Diff}^*_\Phi(X)$ as well as $\Psi^*_\Phi(X)$ are $\mathcal{C}^\infty(X)$-modules we can consider Φ-(pseudo)differential operators acting between sections of smooth vector bundles $\mathcal{E},\mathcal{F} \to X$ over X and we write $\mathrm{Diff}^*_\Phi(X;\mathcal{E},\mathcal{F})$ resp. $\Psi^*_\Phi(X;\mathcal{E},\mathcal{F})$ for the corresponding spaces. Important examples of Φ-differential operators are the Laplacian and the Dirac operators corresponding to *exact* Φ*-metrics*, for instance (1) (see [11]; these are certain complete metrics on the interior of X which induce Euclidean metrics on ${}^\Phi TX$).

As in the closed case, a Φ-pseudodifferential operator of order m_0 acts continuously as an operator of order m_0 on a scale of Φ-*Sobolev spaces* H^s_Φ, $s \in \mathbb{R}$. The Φ-pseudodifferential operators A that induce Fredholm operators $H^s_\Phi \to H^{s-p}_\Phi$ have been characterized by Mazzeo and Melrose [11] as being those operators with invertible principal symbol as well as invertible *normal operator* (see Section 3). Such operators are called fully elliptic. The index of a fully elliptic operator A is independent of the particular $s \in \mathbb{R}$. A preliminary index formula for fully elliptic operators has been obtained in [6] under the assumption that $\mathcal{E} = \mathcal{F}$ (this assumption is no restriction when a fully elliptic operator exists; an isomorphism $\mathcal{E} \to \mathcal{F}$ is given by the principal symbol of the operator applied to a non-vanishing Φ-vector field, which exists whenever $\partial X \neq \emptyset$).

Let us briefly recall the index formula. We need several trace-like functionals on the Φ-calculus whose definition has been adapted from a similar context in [16]. Let $Q \in \Psi^1_\Phi(X)$ be a positive self-adjoint fully elliptic operator and $Q^{-\lambda} \in \Psi^{-\lambda}_\Phi(X)$ the family of complex powers constructed as in [5]. Then for $A \in \Psi^{m_0}_\Phi(X)$, the operator $x^zAQ^{-\lambda}$ is of trace class for $\mathrm{Re}(z) > n+1$ and $\mathrm{Re}(\lambda) > m_0 + \dim(X)$, and the map $(\lambda, z) \mapsto z\lambda \operatorname{Tr} x^zAQ^{-\lambda}$ admits a meromorphic extension to $\mathbb{C}^2$ which is analytic near $(z,\lambda) = (0,0)$; thus, we can define

$$\begin{aligned} z\lambda \operatorname{Tr} x^z A Q^{-\lambda} \quad =: \quad & \operatorname{Tr}_{\partial,\sigma}(A) + \lambda \widehat{\operatorname{Tr}}_\partial(A) + z\widehat{\operatorname{Tr}}_\sigma(A) \\ & + \lambda^2 W(\lambda,z) + \lambda z W'(\lambda,z) + z^2 W''(\lambda,z)\,, \end{aligned}$$

where the error terms W, W', W'' are holomorphic near $0 \in \mathbb{C}^2$. The functionals obtained in this way have been studied in [6].

Theorem 1 ([6]). *Let $A \in \Psi_\Phi^{m_0}(M,\mathcal{E})$ be a fully elliptic Φ-operator. Then*

$$\operatorname{index} A = \widehat{\operatorname{Tr}}_\sigma(A[B,\log Q]) - \widehat{\operatorname{Tr}}_\partial([A,\log x]B)$$

where $B \in \Psi_\Phi^{-p}(X)$ is any inverse of A up to trace class remainders, and

$$\begin{aligned} [B,\log Q] \quad &:= \quad \frac{d}{d\lambda}(Q^{-\lambda} B Q^{\lambda})_{|\lambda=0} \in \Psi_\Phi^{-p}(X,\mathcal{E}); \\ [\log x, A] \quad &:= \quad \frac{d}{dz}(x^z A x^{-z})_{|z=0} \in x\Psi_\Phi^{p-1}(X,\mathcal{E}). \end{aligned}$$

Formally this is the same simplification of the computation of Melrose and Nistor [16] as in [7, 8]. From [6] we know already that the first contribution to the index is local, i.e. does not change if we modify A by an operator of sufficiently negative order, whereas the second contribution is global but depends only on the behavior of the operators at the boundary, i.e. it does not change if we modify A by an operator that vanishes to sufficiently high order at the boundary. We identify in Proposition 12 the local term in Theorem 1 with the regularized Atiyah-Singer integral for the index, defined in terms of heat kernel asymptotics.

Our main interest lies in the case where the base Y of the fiber bundle $\varphi : \partial X \to Y$ is the circle S^1, for first-order differential operators A modeled on Dirac operators. Choose a connection in the boundary fiber bundle, that is a rule for lifting the horizontal vector field ∂_θ. Choose a smooth metric on the interior of X that with respect to a product decomposition $\partial X \times [0,1] \subset X$ close to the boundary looks like

$$g^X = \frac{dx^2}{x^4} + \frac{d\theta^2}{x^2} + g^F \tag{1}$$

where θ is the variable in the circle and g^F is a family of metrics on the fibers. Such a metric is called a *product Φ-metric*; it induces an Euclidean metric on the vector bundle ${}^\Phi TX$. Assume moreover that $\mathcal{E}_{|\partial X} = E^+ \oplus E^-$ and fix Hermitian metrics and connections ∇ in $E^\pm$.

Theorem 2. *Let $\mathcal{E}, \mathcal{F} \to X$ be Hermitian vector bundles and let*

$$A : C^\infty(X,\mathcal{E}) \to C^\infty(X,\mathcal{F})$$

be a first-order differential operator which in a product decomposition $\partial X \times [0,1] \subset X$, $\mathcal{E}_{|\partial X} = E^+ \oplus E^-$ looks like

$$A = \sigma\left((x^2\partial_x - \frac{x}{2})I_2 + \begin{bmatrix} -ix\tilde{\nabla}_{\partial_\theta} & D^* \\ D & ix\tilde{\nabla}_{\partial_\theta} \end{bmatrix} \right) \tag{2}$$

where D is a family of invertible operators on the fibers of φ, D^* is the formal adjoint of D, σ is an isometry $\mathcal{E}_{|\partial X} \to \mathcal{F}_{|\partial X}$, and $\tilde{\nabla}_{\partial_\theta} := \nabla_{\partial_\theta} + \frac{1}{4}\operatorname{Tr}(L_{\partial_\theta} g^F)$. Then A is Fredholm as an unbounded operator on $L^2(X, \mathcal{E}, \mathcal{F}, g^X)$ and its index is given by

$$\operatorname{index}(A) = \overline{AS}(A) - \frac{\lim_a \eta(\delta_x)}{2}.$$

Here $\overline{AS}(A)$ is the integral on X of the pointwise supertrace of the heat kernel of A, which is a local expression in the full symbol of A and in the metric, while $\lim_a \eta(\delta_x)$ is the adiabatic limit (the limit as x tends to 0) of the eta invariant of the "boundary operator"

$$\delta_x := \begin{bmatrix} -ix\tilde{\nabla}_{\partial_\theta} & D^* \\ D & ix\tilde{\nabla}_{\partial_\theta} \end{bmatrix} : C^\infty(\partial X, \mathcal{E}) \to C^\infty(\partial X, \mathcal{E}).$$

Note that we do not need to actually construct the heat kernel of A in order to define the local index density. This density is slightly less than L^1, see Proposition 17. It is instructive to compare this result with the classical Atiyah-Patodi-Singer index formula [1].

Corollary 3. *Let (X, g^X) be a compact spin manifold with boundary and $T \to X$ a Hermitian vector bundle with constant metric and connection near ∂X. Then the twisted Dirac operator A on (X, g^X) is of the form (2). Moreover, A is Fredholm on $L^2(X, \mathcal{E}, \mathcal{F}, g^X)$ if and only if the family D of Dirac operators on the fibers is invertible, and*

$$\operatorname{index}(A) = \int_X \hat{A}(X)\operatorname{ch}(T) - \frac{\lim_a \eta(\delta_x)}{2}.$$

This follows immediately from the local index theorem (see [2]) and Theorem 2. One checks directly in this case that the curvature of g^X is a smooth 2-form on X with values in $\operatorname{End}({}^\Phi TX)$ so $\hat{A}(X)\operatorname{ch}(T)$ is a smooth form on X, thus in L^1 (see Proposition 13).

Corollary 4. *Let A be as in Theorem 2. Then the integral of the index density is an integer if and only if the determinant bundle of the boundary family D has trivial holonomy.*

This is a trivial application, since the adiabatic limit of the eta invariant equals the logarithm of the holonomy (see Section 2 for the definitions). In particular, under the assumptions of Corollary 3 the Atiyah-Singer volume form defines an integral cohomology class.

A related index formula has been obtained by Nye and Singer [19] for the spin Dirac operator on $X = S^1 \times \mathbb{R}^3$ where the boundary fiber bundle is the projection $S^1 \times S^2 \to S^2$. In the general case Vaillant [20] gave a formula for the index of the Dirac operator of a d-metric on manifolds with fibered cusps. Vaillant's formula contains the adiabatic limit of the eta invariant in the form computed by Bismut and Cheeger [3]. It seems therefore likely that Theorem 2 continues to hold for boundary fiber bundles with higher dimensional base, as conjectured in [19].

The paper is structured as follows: in Section 2 we recall results about the eta invariant of self-adjoint operators. Section 3 is devoted to an introduction to fibered cusp pseudodifferential operators, with a focus on traces. The index theorem 1 is reviewed in Section 4. Finally the proof of Theorem 2 occupies Section 5. A curious feature of the proof is the appearance and then cancellation of the integral over S^1 of the determinant of D^*D in the boundary term of the index formula.

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2. Review of eta invariants and adiabatic limits

The eta function of an elliptic self-adjoint differential operator δ was defined by Atiyah, Patodi and Singer [1] as

$$\eta(\delta, s) := \operatorname{Tr}\left((\delta^2)^{-\frac{s+1}{2}}\delta\right).$$

(assuming that δ is invertible).

Consider the family of operators on $\mathcal{C}^\infty(\partial X, \mathcal{E})$ defined for $0 \leq x < \infty$ by

$$\delta_x := \begin{bmatrix} -ix\tilde{\nabla}_{\partial_\theta} & D^* \\ D & ix\tilde{\nabla}_{\partial_\theta} \end{bmatrix}$$

(see (2)), where D is a family of elliptic invertible first-order differential operators on the fibers of ∂X, acting from E^+ to E^-, and $\mathcal{E} = E^+ \oplus E^-$.

Lemma 5. *The operator δ_x is symmetric on $L^2(\partial X, \mathcal{E}, \frac{d\theta^2}{x^2} + g^F)$.*

Proof. We must show that $\tilde{\nabla}_{\partial_\theta}$ is skew-symmetric. Let $s_1, s_2 \in \mathcal{C}^\infty(\partial X, \mathcal{E})$. Then

$$\begin{aligned}
&(\tilde{\nabla}_{\partial_\theta} s_1, s_2) + (s_1, \tilde{\nabla}_{\partial_\theta} s_2) \\
&= \int_{\partial X} \left((\tilde{\nabla}_{\partial_\theta} s_1, s_2) + (s_1, \tilde{\nabla}_{\partial_\theta} s_2)\right) dg^F d\theta \\
&= \int_{\partial X} (\partial_\theta (s_1, s_2) + \frac{(s_1, s_2)}{2} \operatorname{Tr}(L_{\partial_\theta} g)) dg^F d\theta \\
&= \int_{\partial X} L_{\partial_\theta}((s_1, s_2) dg^F) d\theta \\
&= \int_{S^1} \partial_\theta \left(\int_{\partial X / S^1} (s_1, s_2) dg^F\right) d\theta = 0.
\end{aligned}$$

$\square$

We will see in Section 5 that δ_x is also invertible for small enough $x > 0$.

The eta invariant of δ_x is by definition the regularized value of $\eta(\delta_x, s)$ at $s = 0$. In fact, the eta function is regular at $s = 0$ (see [1]). By the adiabatic limit, denoted $\lim_a \eta(\delta_x)$, we mean the limit $\lim_{x\to 0} \eta(\delta_x)$. Intuitively it corresponds to separating the fibers of $\partial X \to S^1$ in the limit since the Riemannian distance between distinct fibers tends to infinity.

Recall the definition of the determinant line bundle of the family D with the Bismut-Freed connection. Since D is invertible, $\det(D)$ is defined as the trivial bundle $\mathbb{C} \times S^1 \to S^1$ with the connection $d + \omega^{BF}(0)$, where $\omega^{BF}(0)$ is the finite value at $s = 0$ of the meromorphic family of 1-forms

$$\omega^{BF}(s) := \operatorname{Tr}\left((D^*D)^{-\frac{s}{2}} D^{-1} \tilde{\nabla}_{\partial_\theta}(D)\right).$$

Clearly, the holonomy of the Bismut-Freed connection is

$$\operatorname{hol}(\det(D), \omega^{BF}(0)) = e^{-\int_{S^1} \omega^{BF}(0)}.$$

We define the logarithm of the holonomy of $\det(D)$ as

$$\log \operatorname{hol}(\det(D)) = -\int_{S^1} \omega^{BF}(0) \in i\mathbb{R}. \tag{3}$$

Determinant bundles and eta invariants are linked by the global anomaly formula of Witten [21], initially proved in [4] for Dirac operators. The general result that we need is taken from [17, 18].

Theorem 6. *The adiabatic limit of the eta invariant of δ_x satisfies*

$$\lim_a \eta(\delta_x) = -\frac{1}{i\pi} \log \operatorname{hol}(\det(D)).$$

3. The fibered cusp calculus of pseudo-differential operators

In this section we introduce the basic facts about the fibered cusp calculus from [6] that are used in the next sections. For a thorough treatment of the fibered cusp calculus we refer the reader to [11, 20]. We continue to use the notations from [6].

Blowing-up a submanifold N of a smooth manifold M means replacing N by the set of its real normal directions inside M, i.e. the sphere bundle of its normal bundle; this procedure is equally defined for manifolds with corners. The result of the blow-up is a new manifold with corners of codimension possibly higher by 1 than those of the initial manifold.

3.1. The construction of Φ-operators.

Let

$$X^2_\Phi := [X \times X; \partial X \times \partial X, (\partial X)^2_\varphi \times \{0\}]$$

be the fibered-cusp double space, obtained by an iterated real blow-up from X^2 as follows: first blow up the corner $\partial X \times \partial X$ (at this stage we get the celebrated b-double space X^2_b). The new boundary hyperface introduced by this blow-up is diffeomorphic to $\partial X \times \partial X \times [-1,1]$ under the following map: the class (modulo $\mathbb{R}_+^*$) of the non-zero normal vector (V_1, V_2) at (y_1, y_2) maps to $\frac{V_1(x)+V_2(x)}{V_1(x)-V_2(x)}$. Thus $\partial X \times \partial X \times \{0\}$ is a well-defined submanifold of X_b^2 provided we fixed the boundary-defining function x. The second stage of the construction involves blowing-up the fiber diagonal

$$(\partial X)^2_\varphi := \partial X \times_\varphi \partial X = \{(p,q) \in \partial X \times \partial X; \varphi(p) = \varphi(q)\}$$

of the boundary fiber bundle which by the discussion above is also a submanifold of X_b^2. The space X_Φ^2 comes equipped with a canonical smooth structure as a manifold with corners of codimension at most 2, and with a smooth blow-down map

$$\beta : X_\Phi^2 \to X^2$$

which extends the identical diffeomorphism $(X_\Phi^2)^\circ = (X^2)^\circ$ of the interiors. The last face introduced by blow-up is called the Φ-front face, denoted ff_Φ. The lifted diagonal Δ_Φ is by definition the closure in X_Φ^2 of the preimage under β of the interior of the diagonal in X^2.

The motivation of the construction is the fact [11, Corollary 1] that the space of Φ-differential operators described in the Introduction is canonically isomorphic to the space of distributions on X_Φ^2 supported on Δ_Φ, conormal to Δ_Φ and extendable across ff_Φ, with values in the bundle $\mathcal{F} \boxtimes \mathcal{E}^* \otimes \Omega'$. Here Ω' is the pull-back through the projection on the right factor of the Φ-density bundle $\Omega({}^\Phi TX)$. Note that

$$\mathcal{C}^\infty(X, \Omega({}^\Phi TX)) = x^{-\dim(Y)-2}\mathcal{C}^\infty(X, \Omega(TX)).$$

This singularity of order $\dim(Y)+2$ will play a great role in the rest of the paper. In fact, the main reason for assuming $Y = S^1$ in Theorem 2 is making this order of singularity relatively small. One defines then [11] $\Psi_\Phi^{m_0}(X)$ as the space of linear operators $A : \dot{C}^\infty(X, \mathcal{E}) \to \dot{C}^\infty(X, \mathcal{F})$ such that the lift κ_A of the Schwartz kernel k_A to X_Φ^2 is a classical conormal distribution of order m_0 on X_Φ^2 with values in $\mathcal{E}^* \boxtimes \mathcal{F} \otimes \Omega'$, vanishing rapidly to all boundary faces other than ff_Φ and extendable across ff_Φ. These operators extend to bounded operators between appropriate Φ-Sobolev spaces. The fibered cusp calculus is closed under composition [11, Theorem 2].

If φ is the identity map, fibered-cusp operators are nothing else than *scattering* operators [12]. In that case the identifier Φ for double spaces, tangent bundles etc. will be replaced by *sc*.

3.2. **The normal operator.** There exist two symbol maps on $\Psi_\Phi(X)$, both multiplicative under composition of operators. One is the usual conormal principal symbol (living on ${}^\Phi T^*X$). To describe the second symbol $\mathcal{N}$, called the *normal operator*, first assume that $Y = S^1$. In this case the interior of the Φ-front face ff_Φ is the total space of a trivial 2-dimensional real vector bundle over $(\partial X)^2_\varphi$. The two real directions correspond to the normal direction to

∂X in X, and to the normal direction to $(\partial X)^2_\varphi$ inside $\partial X \times \partial X$. Then $\mathcal{N}$ is obtained by "freezing coefficients" at ff_Φ and then Fourier–transforming in the two real directions:

$$\mathcal{N}(A) := \widehat{\kappa_A|_{\mathrm{ff}_\Phi}}.$$

Note that in the general case there are $\dim Y + 1$ suspending variables. This new symbol map $\mathcal{N}$ surjects onto the algebra $\Psi^{\mathbb{Z}}_{\mathrm{sus}({}^\Phi N^*Y)-\varphi}(\partial X)$ of families of classical pseudo-differential operators on $\{Z_p \times \mathbb{R}^2\}_{p\in S^1}$ invariant with respect to translations in $\mathbb{R}^2$ (2-*suspended operators* in the terminology of [14]) where Z_p is the fiber over $p \in S^1$ of the boundary fiber bundle $\varphi : \partial X \to S^1$. An operator A in $\Psi_\Phi(X)$ is called *elliptic* if its principal conormal symbol is pointwise invertible, and *fully elliptic* if, in addition, the corresponding normal operator $\mathcal{N}(A)$ consists of a pointwise invertible family of pseudodifferential operators.

3.3. **The formal boundary symbol.** The principal symbol and the normal operator are invariantly defined. As for standard pseudodifferential operators there exists a more refined notion of *formal symbol* map, associating to an operator the Laurent series of its full symbol at the sphere at infinity inside the radial compactification $\overline{{}^\Phi T^*X}$ (however, this symbol depends on choices except for its first term, the principal symbol). Similarly, we associate to an operator its *formal boundary symbol*

$$q : \Psi_\Phi(X) \to \Psi^{\mathbb{Z}}_{\mathrm{sus}({}^\Phi N^*Y)-\varphi}(\partial X)[[x]]\,. \tag{4}$$

To construct q, first choose a product decomposition $\partial X \times [0,\epsilon) \hookrightarrow X$ of X near ∂X so that $x(y,t) = t$. Let X_ϵ be the image of this map, and $Y_\epsilon := Y \times [0,\epsilon)$. Thus X_ϵ fibers over Y_ϵ via $\varphi \times Id$ with fiber type F and $(X_\epsilon)^2_\varphi$ fibers over $(Y_\epsilon)^2_{sc}$ with fiber type $F \times F$.

Lift through β the diagonal embedding $(\partial X)^2_\varphi \times [0,\epsilon) \hookrightarrow X^2$ to an embedding

$$(\partial X)^2_\varphi \times [0,\epsilon) \hookrightarrow X^2_\Phi. \tag{5}$$

Note that $(\partial X)^2_\varphi \times \{0\}$ maps identically to itself as the zero section in ff_Φ. Moreover, the image of (5) is exactly the preimage of Δ_{sc} under the fibration

$$(X_\epsilon)^2_\Phi \to (Y_\epsilon)^2_{sc}. \tag{6}$$

Thus the normal bundle to $(\partial X)^2_\varphi \times [0,\epsilon)$ inside X^2_Φ is the pull-back via (6) of the normal bundle to Δ_{sc}, which is canonically isomorphic to ${}^{sc}TY_\epsilon$. Consequently we use the notation $N((\partial X)^2_\varphi \times [0,\epsilon)) = {}^{sc}TX_\epsilon$. The total space of ${}^{sc}TX_\epsilon|_{x=0}$ coincides with the interior of the front face. Construct a collar neighborhood map $\mu : {}^{sc}TX_\epsilon \hookrightarrow X^2_\Phi$ which on ff_Φ is the identity. Take a Φ-operator A and pull back its lifted Schwartz kernel κ_A from X^2_Φ to the total space of ${}^{sc}TX_\epsilon$ using μ. Note that $\mu^*(\Omega') = \Omega_{fiber,R} \otimes \pi^*(\Omega({}^{sc}TY_\epsilon))$ is the tensor product of the density bundle in the second factor of $(\partial X)^2_\varphi$ and the pull-back of the scattering density bundle from Y_ϵ. At the zero section in ${}^{sc}TX_\epsilon$ we can identify the scattering density factor with an Euclidean density on the fibers of ${}^{sc}TX_\epsilon$ (this density will alow us to take Fourier transforms in the fibers). Identify $\mathcal{E}, \mathcal{F}$ and ${}^{sc}TX_\epsilon$ with their pull-backs from the zero-section of the front face, take the Taylor series of $\mu^*(\kappa_A)$ at $x = 0$ (we can differentiate with respect to x a distribution conormal to $\Delta_{\partial X} \times [0, \epsilon)_x$) and then Fourier transform the coefficients in the fibers of ${}^{sc}TX_\epsilon$. This defines the map q.

3.4. **Product on formal boundary symbols.** The formal boundary map q depends on choices of connections and trivializations except for its leading term which is just the normal operator. We denote by $*$ the product induced by q on $\Psi^m_{\mathrm{sus}({}^\Phi N^*Y)-\varphi}(\partial X)[[x]]$. Recall that in the standard pseudodifferential case we can choose the formal symbol map so that the induced product on formal series of homogeneous symbols (the so-called star product) takes the form

$$a(y,\xi) * b(y,\xi) = a(y,\xi)b(y,\xi) + \frac{1}{i}\sum \partial_{\xi_i} a(y,\xi)\nabla_{\partial_{y_i}} b(y,\xi) + \dots$$

Similarly in the context of Theorem 2 we can choose the formal boundary symbol map q with the properties:

$$q(x\tilde{\nabla}_{\partial_\theta}) = i\tau \tag{7}$$

$$q(x^2\partial_x) = i\xi \tag{8}$$

$$q(P) = P$$

where τ is the suspending variable cotangent to the base S^1 of the boundary, ξ is the suspending variable conormal to ∂X in X and P is a fiberwise differential operator which is constant in the fixed product decomposition near the boundary.

Lemma 7. *For $U, V \in \Psi^{\mathbb{Z}}_{\mathrm{sus}({}^\Phi N^*Y)-\varphi}(\partial X)[[x]]$, the product induced by q takes the form*

$$U * V = UV + \frac{x}{i}\frac{\partial U}{\partial \xi}\left(x\frac{\partial V}{\partial x} + \tau\frac{\partial V}{\partial \tau}\right) + \frac{x}{i}\frac{\partial U}{\partial \tau}\tilde{\nabla}_{\partial_\theta}(V) + O(x^2). \tag{9}$$

where the product in the right-hand side is the standard product of power series with coefficients in the algebra $\Psi^{\mathbb{Z}}_{\mathrm{sus}({}^\Phi N^*Y)-\varphi}(\partial X)$.

Proof. It is enough to prove the formula for Φ-differential operators since the product is given by bi-differential operators with polynomial coefficients (see e.g. [9, Proposition 3.11] for details of this argument in a similar context). But $\mathrm{Diff}^*_\Phi(X, \mathcal{E})$ is generated near ∂X by $ix\tilde{\nabla}_{\partial_\theta}$, $ix^2\partial_x$ and by differential operators along the fibers. The lemma follows easily from (7) and (8) since it is valid on the generators. $\square$

3.5. **Traces densities of Φ-operators.** Of main interest for us are traces of Φ-operators. We study them using the more refined notion of trace density. It is a standard fact that on a closed manifold M, any operator $A \in \Psi^\lambda(M, \mathcal{E})$ with $\mathrm{Re}(\lambda) < -\dim(M)$ is of trace class, and

$$\mathrm{Tr}(A) = \int_\Delta \mathrm{tr}(k_A|_\Delta)$$

(the Schwartz kernel k_A is continuous on $M \times M$ and its restriction to the diagonal is a smooth 1-density with values in $\mathrm{End}(\mathcal{E})$). The same remains true for Φ-operators modulo an integrability issue at the boundary. By pulling-back via the blow-down map we write

$$\mathrm{Tr}(A) = \int_{\Delta_\Phi} \mathrm{tr}(\kappa_A|_{\Delta_\Phi}).$$

We identify Δ_Φ with X via the right projection. The restriction to Δ_Φ of the density bundle Ω' is precisely $\Omega({}^\Phi TX) = x^{-\dim(Y)-2}\Omega(X)$. It is then clear that A is of trace class if and only if $A \in x^z\Psi^\lambda_\Phi(X, \mathcal{E})$ with $\mathrm{Re}(z) > \dim(Y) + 1, \mathrm{Re}(\lambda) < -\dim(X)$. The density $\mathrm{tr}(\kappa_A|_{\Delta_\Phi})$, viewed as a density on X, is called the *trace density* of A.

Recall that for the definition of the formal boundary symbol we have chosen a product decomposition of X near ∂X, as well as a local isomorphism of $\mathcal{E}$ with its pull-back from ∂X. We can thus expand $\kappa_A|_{\Delta_\Phi}$ in powers of x near $x = 0$. Let ${}^\Phi N\partial X$ be the restriction of the vector bundle ${}^{sc}TX_\epsilon$ to $\partial X \times \{0\} \subset \mathrm{ff}_\Phi$ and ${}^\Phi N^*\partial X$ its dual.

Proposition 8. *Let* $A \in x^z\Psi^\lambda_\Phi(X, \mathcal{E})$ *with* $\mathrm{Re}(\lambda) < -\dim(X)$*. Then*

$$\kappa_A|_{\Delta_\Phi} \sim_{x\to 0} \frac{1}{(2\pi)^{\dim(Y)+1}} \int_{{}^\Phi N^*\partial X/\partial X} q(A) \frac{\omega_{sc}^{\dim(Y)+1}}{(\dim(Y)+1)!}.$$

PROOF. This is precisely the Fourier inversion formula in each fiber of ${}^\Phi N\partial X$, where ω_{sc} is the canonical (singular) symplectic form on ${}^{sc}T^*Y_\epsilon$ pulled back to ${}^\Phi N^*\partial X$. □

In the case $Y = S^1$ this takes a somewhat simpler form.

Corollary 9. *Let* $A \in x^z\Psi^\lambda_\Phi(X, \mathcal{E})$ *with* $\mathrm{Re}(\lambda) < -\dim(X)$ *and assume* $Y = S^1$*. Then*

$$\kappa_A|_{\Delta_\Phi} \sim_{x\to 0} \left(\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} q(A)(\tau, \xi) d\tau d\xi\right) \frac{d\theta dx}{x^3}.$$

PROOF. Notice that ${}^\Phi N\partial X$ is a trivial $\mathbb{R}^2$ bundle in this case, while $\omega_{sc} = \frac{d\xi\wedge dx}{x^2} + \frac{d\tau\wedge d\theta}{x}$. □

If $\mathbb{C} \ni \lambda \mapsto A(\lambda) \in \Psi^\lambda_\Phi(X, \mathcal{E})$ is an entire family of Φ-operators then both $\kappa_A|_{\Delta_\Phi}$ and $q(A)|{}^\Phi N^*\partial X$ extend meromorphically to $\mathbb{C}$ with at most simple poles. Moreover, Corollary 9 remains valid for the meromorphic extensions for all values of λ (at a pole, the expansions are valid for the residues and for the regularized parts separately).

We close this section with a description in terms of the map q of the trace functional $\widehat{\mathrm{Tr}}_\partial$ defined in the Introduction (compare with [6, Proposition 7.6]).

Lemma 10. *Let $A \in \Psi_\Phi^{m_0}(X)$. Then $\widehat{\mathrm{Tr}}_\partial(A)$ is explictly given by*

$$\widehat{\mathrm{Tr}}_\partial(A) = \frac{1}{(2\pi)^2}\left(\int_{S^1\times\mathbb{R}^2} \mathrm{Tr}(q(AQ^{-\lambda})_{[-2]})d\theta d\tau d\xi\right)_{\lambda=0}$$

where Tr *in the right-hand side denotes the trace of operators on the fibers of $\varphi : \partial X \to S^1$, $(\cdot)_{[k]}$ is the coefficient of x^{-k} in a power expansion in x, and $(\cdot)_{\lambda=0}$ stands for the regularized value at $\lambda = 0$.*

PROOF. We can assume that $\kappa_{AQ^{-\lambda}}$ is supported in $(X_\epsilon)^2_\Phi$. Then

$$\begin{aligned}\widehat{\mathrm{Tr}}_\partial(A) &= \mathrm{Res}_{z=0}\,\mathrm{Tr}(x^z AQ^{-\lambda})_{|\lambda=0}\\ &= \mathrm{Res}_{z=0}\int_{\Delta_\Phi}\mathrm{tr}(\kappa_{x^z AQ^{-\lambda}})_{|\lambda=0}\\ &= \mathrm{Res}_{z=0}\int_0^\epsilon\int_{\partial X} x^z\,\mathrm{tr}(\kappa_{AQ^{-\lambda}})_{|\lambda=0}\end{aligned}$$

Obviously,

$$\mathrm{Res}_{z=0}\left(\int_0^\epsilon x^{z-k}dx\right)$$

equals 1 if $k = 1$ and 0 otherwise, so

$$\begin{aligned}\widehat{\mathrm{Tr}}_\partial(A) &= \left(\int_{\partial X}\mathrm{tr}(\kappa_{AQ^{-\lambda}})_{[1]}\right)_{|\lambda=0}\\ &= \left(\int_{S^1}\int_{\partial X/S^1}\left(\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\mathrm{tr}\,q(AQ^{-\lambda})_{[-2]}d\tau d\xi\right)d\theta\right)_{|\lambda=0}\end{aligned}$$

(we used Corollary 9 and the remark following it in the last equality). Now

$$\int_{\partial X/S^1}\mathrm{tr}\,q(AQ^{-\lambda})_{[-2]} = \mathrm{Tr}(q(AQ^{-\lambda})_{[-2]})$$

so the result follows by Fubini's theorem. □

4. THE ABSTRACT INDEX FORMULA

The results of this section hold for general boundary fiber bundles, i.e. not necessarily with base S^1.

Fibered-cusp operators have two types of principal symbols. Accordingly, elliptic regularity has a new aspect in fibered-cusp theory concerning regularity at the boundary. The following lemma and its proof are quite standard; we include them for future reference.

Lemma 11. *Let $A \in \Psi_\Phi(X, \mathcal{E}, \mathcal{F})$ be fully elliptic. Then the L^2 solutions of $A\psi = 0$ belong to $x^\infty C^\infty(M, \mathcal{E})$.*

Proof. Since A is fully elliptic there exists a parametrix B of A inverting A up to $R \in x^\infty \Psi_\Phi^{-\infty}(X, \mathcal{E})$. Let ψ be a distributional solution of the pseudo-differential equation $A\psi = 0$. It follows

$$0 = BA\psi = (I + R)\psi = \psi + R\psi$$

so $\psi = -R\psi$. But $R \in x^\infty \Psi_\Phi^{-\infty}(X, \mathcal{E})$ implies $R\psi \in x^\infty C^\infty(M, \mathcal{E})$. □

Since R is compact on L^2_Φ (the compact operators in $\Psi_\Phi^{\mathbb{Z}}(X)$ are precisely those in $x\Psi_\Phi^{-1}(X)$) it follows that $\ker A$ is finite dimensional and moreover the orthogonal projection $P_{\ker A}$ belongs to the ideal $x^\infty \Psi_\Phi^{-\infty}(X, \mathcal{E})$. We can define therefore invertible Φ-operators

$$\begin{aligned} Q_1 &:= (AA^* + P_{\ker A^*})^{1/2}; \\ Q_2 &:= (A^*A + P_{\ker A})^{1/2}. \end{aligned}$$

Note that $q(Q_1) = q(AA^*)^{1/2}$. Let

$$B := A^* Q_1^{-1}$$

be a parametrix of A.

4.1. **The index formula.** Let us reprove the index formula from [6]. Assume for simplicity that A is of order 1. For technical reasons we would like to work with operators acting from $\mathcal{E}$ to itself. Recall that $\partial X \neq \emptyset$ implies the existence of a non-vanishing vector field on X (the obstruction to the existence of such a vector field lives in $H^{\dim(X)}(X)$ and this space is 0 when $\partial X \neq \emptyset$). There exist non-canonical isomorphisms between TX, ${}^\Phi TX$ and ${}^\Phi T^*X$, and thus a non-vanishing section in ${}^\Phi T^*X$. The principal conormal symbol of A evaluated on this section gives an isomorphism u between $\mathcal{E}$ and $\mathcal{F}$. Finally, $v := u^*(uu^*)^{-1/2}$ is an isometry $\mathcal{F} \to \mathcal{E}$. Thus $U := vA \in \Psi_\Phi^1(X, \mathcal{E})$ has the property

$$\begin{aligned} R_1 &:= (UU^* + P_{\ker U^*})^{1/2} = vQ_1v^* \\ R_2 &:= (U^*U + P_{\ker U})^{1/2} = Q_2. \end{aligned}$$

Set $V := Bv^*$. Note the commutations $UR_2^{-\lambda} = R_1^{-\lambda}U$, $VR_1^{-\lambda} = R_2^{-\lambda}V$. The index formula is obtained as follows:

$$\begin{aligned}
\operatorname{index}(A) &= \operatorname{index}(U)\\
&= \operatorname{Tr}(UV - VU)\\
&= \operatorname{Tr}(x^z(UV - VU)R_2^{-\lambda})_{\lambda=0,z=0}\\
&\overset{(10.1)}{=} \operatorname{Tr}(x^zUVR_2^{-\lambda} - Ux^zVR_1^{-\lambda})_{\lambda=0,z=0}\\
&= \operatorname{Tr}(x^zUV(R_2^{-\lambda} - R_1^{-\lambda}) + [x^z, U]VR_1^{-\lambda})_{\lambda=0,z=0}\\
&\overset{(10.2)}{=} (\operatorname{Tr}(x^z(R_2^{-\lambda} - R_1^{-\lambda}))_{\lambda=0,z=0}\\
&\quad + \operatorname{Tr}([x^z, A]BQ_1^{-\lambda})_{\lambda=0,z=0}.
\end{aligned} \tag{10}$$

For (10.1) we have used the fact that $\operatorname{Tr}[C, D] = 0$ for $C \in x^c\Psi_\Phi^a(X)$, $D \in x^d\Psi_\Phi^b(X)$ with $a, b, c, d \in \mathbb{C}$, $\operatorname{Re}(a + b) < -\dim(X)$, $\operatorname{Re}(c + d) > 1 + \dim(Y)$. Thus (10.1) is true for large real parts of λ, z, hence for any $\lambda, z \in \mathbb{C}$ by unique continuation. (10.2) holds at $\lambda = 0, z = 0$ because $UV - 1 \in x^\infty\Psi_\Phi^{-\infty}(X)$ while v, v^* cancel in the trace.

4.2. **The interior term.** We claim that the first term in (10) is the regularized integral on X of a local expression in the full symbol expansion of U. Indeed, for $j = 1, 2$ let $r_j(\lambda) \in C^\infty(X, \operatorname{End}(\mathcal{E}) \otimes {}^\Phi\Omega(X))$ be the meromorphic extension of the lifted Schwartz kernel $\kappa_{R_j^{-\lambda}}$ restricted to Δ_Φ. As in the case of closed manifolds it is easy to see that $r_j(\lambda)$ is regular at $\lambda = 0$. The first term in (10) is $\int_X x^z \operatorname{tr}(r_2(0) - r_1(0))|_{z=0}$. By Corollary 9 and the remark after it, the density $\operatorname{tr}(r_2(0))$ has a Laurent expansion at $x = 0$ starting with x^{-3} so the previous integral is absolutely integrable and holomorphic in z for $\operatorname{Re}(z) > 2$ and extends to $\mathbb{C}$ with possible simple poles at $z = 2 - \mathbb{N}$. Now

$$\begin{aligned}
R_2^{-\lambda} - R_1^{-\lambda} &= U[R_2^{-\lambda}, V] + O(\lambda)x^\infty\Psi_\Phi^{-\infty}(X)\\
&= \lambda U[V, \log R_2]R_2^{-\lambda} + O(\lambda)x^\infty\Psi_\Phi^{-\infty}(X) + O(\lambda^2)
\end{aligned}$$

where $O(\lambda)$ denotes an analytic multiple of λ near $\lambda = 0$. Clearly then

$$(\operatorname{Tr}(x^z(R_2^{-\lambda} - R_1^{-\lambda}))_{\lambda=0,z=0} = \widehat{\operatorname{Tr}}_\sigma(U[V, \log R_2]). \tag{11}$$

On the more refined level of trace densities, by [6, Proposition 7.4],

$$r_2(0) - r_1(0) = \frac{1}{(2\pi)^N}\int_{{}^\Phi S^*X/X} \sigma_{[-N]}(U[V, \log R_2])\imath_{\mathcal{R}}\omega_\Phi^N$$

is given in terms of the component of homogeneity $-\dim(X)$ of the formal symbol of $U[V, \log R_2]$, so clearly depends only on the jets of the full symbol of U.

4.3. **The boundary term.** Similarly for the second term from (10) we have

$$[x^z, A]B = zx^z[\log x, A]B + O(z^2)$$

and $\mathrm{Tr}(O(z^2)) = O(z)$ so

$$\mathrm{Tr}([x^z, A]BQ_1^{-\lambda})_{\lambda=0,z=0} = -\widehat{\mathrm{Tr}}_\partial([A, \log x]B). \tag{12}$$

By Lemma 10 this last quantity is concentrated at the boundary.

So far, combining (10), (11) and (12) we have proved the general index Theorem 1. Note that in (11) we can assume $v = 1$ (and so $U = A$, $V = B$, $Q = R_2$) since in Theorem 1 we suppose $\mathcal{E} = \mathcal{F}$.

4.4. **Relationship with heat kernel expansions.** Here is a more familiar interpretation of the local term (11). It is worth stressing that we do not prove a heat kernel expansion for A^*A. Rather we use the existence of heat kernel expansions for pseudo-differential operators on closed manifolds as well as the locality of the two quantities we want to relate.

Proposition 12. *The local quantity* $\mathrm{tr}(r_2(0) - r_1(0))$ *equals the index density, defined as the universal expression in the jets of the full symbol of* U *which gives the pointwise supertrace of the constant term in the heat kernel expansion.*

PROOF. Fix a point p in the interior of X and modify the operator U far from p so that it extends to an elliptic operator on the double of X. Denote the extensions to $2X$ by the same letters as before. Then for $j = 1, 2$ use the Mellin transformation formula

$$\Gamma(\lambda)R_j^{-2\lambda} = \int_0^\infty t^{\lambda-1}e^{-tR_j^2}dt$$

to identify the value at $\lambda = 0$ of the analytic extension of the Schwartz kernel of $R_j^{-\lambda}$ on the diagonal with the coefficient of t^0 in the asymptotic expansion as $t \searrow 0$ of the Schwartz kernel of $e^{-tR_j^2}$ on the diagonal. Remember that $R_2 = Q_2$, and observe that Q_1^2 and R_1^2 are conjugate via v so the pointwise trace of their Schwartz kernels is the same. $\square$

Such a formula for the local term is not surprising in index theory. Our point is getting it without having to construct heat kernels for Φ-operators. In this respect the approach via complex powers, which are already objects in the calculus, presents a great advantage.

Although the local term (11) is smooth on X up to the boundary as a Φ-density, its integral might in principle diverge since as an usual density on X it has a singularity of order 3 at $x = 0$. Thus, we cannot set directly $z = 0$ in the above evaluations. To prove Theorem 2 we must identify the boundary term with the adiabatic limit of the eta invariant and show that the index density is integrable in a restricted sense. We will do this in Section 5.

However, we can prove directly that when A is a twisted Dirac operator corresponding to the metric g^X, and with twisting bundle T with metric and connection constant in x in a neighborhood of the boundary, the local term in the index formula equals the integral of the Atiyah-Singer density $\hat{A}(X, g^X)\operatorname{ch}(T)$ *without regularization.* Indeed,

Proposition 13. *The Riemannian curvature R of (X, g^X) induces a smooth 2-form on X with values in $\operatorname{End}({}^\Phi TX)$ down to $x = 0$. Thus $\hat{A}(X, g^X)$ and $\operatorname{ch}(T)$ are smooth forms on X.*

Proof. Let $e_1, \ldots, e_m$ be a local orthonormal frame in the fibers of ϕ. It is straightforward to compute $R(g^X)$ using the local orthonormal Φ-vector fields $x^2\partial_x$, $x\partial_\theta$ (lifted to ∂X using the connection involved in the definition of g^X), $e_1, \ldots, e_m$. We observe that while for instance $R(\partial_x, e_i)\partial_\theta$ diverges like $1/x$ as $x \to 0$, the induced action of R on ${}^\Phi TX$ is smooth down to $x = 0$. To conclude that $\hat{A}(X, g^X)$ is smooth it is enough to prove that $\operatorname{tr}(R^k)$ is smooth down to $x = 0$ for all $k \in \mathbb{N}$. Of course it does not matter for the trace if we view the 2-form R as acting on TX or on ${}^\Phi TX$, so the conclusion follows. $\operatorname{ch}(T)$ is obviously smooth in x (it is in fact constant in x in a neighborhood of $x = 0$). $\square$

5. The index of first-order Φ-differential operators

For the rest of the paper we assume that $Y = S^1$ and that A satisfies the hypothesis of Theorem 2. Note that $x^2\partial_x - x/2$ is skew-symmetric with respect to the metric g^X. It follows directly from the properties of the quantization q that

$$q(\delta_x^2) = \begin{bmatrix} \tau^2 + D^*D & -ix\tilde{\nabla}_{\partial_\theta}(D^*) \\ ix\tilde{\nabla}_{\partial_\theta}(D) & \tau^2 + DD^* \end{bmatrix}$$

$$q(A) = \sigma \begin{bmatrix} i\xi - \frac{x}{2} + \tau & D^* \\ D & i\xi - \frac{x}{2} - \tau \end{bmatrix}$$

$$q(A^*) = \begin{bmatrix} -i\xi + \frac{x}{2} + \tau & D^* \\ D & -i\xi + \frac{x}{2} - \tau \end{bmatrix} \sigma^* \tag{13}$$

$$\mathcal{N}(AA^*) = \sigma \left(\begin{bmatrix} \xi^2 + \tau^2 + D^*D & 0 \\ 0 & \xi^2 + \tau^2 + DD^* \end{bmatrix} \right) \sigma^* \tag{14}$$

$$q(AA^*) = \sigma \left(\begin{bmatrix} \xi^2 + \tau^2 + D^*D & 0 \\ 0 & \xi^2 + \tau^2 + DD^* \end{bmatrix} \right.$$

$$\left. + x \begin{bmatrix} i\xi + \tau & -i\tilde{\nabla}_{\partial_\theta}(D^*) \\ i\tilde{\nabla}_{\partial_\theta}(D) & i\xi - \tau \end{bmatrix} + \frac{x^2}{4} \right) \sigma^*. \tag{15}$$

Set $\Delta_+ := D^*D$, $\Delta_- := DD^*$, $\Delta := \begin{bmatrix} \Delta_+ & 0 \\ 0 & \Delta_- \end{bmatrix}$.

Lemma 14. *The operator A defined by* (2) *is a fully elliptic Φ-operator if and only if the family D is invertible.*

Proof. It is clear from (14) that $\mathcal{N}(AA^*) = \sigma(\xi^2+\tau^2+\Delta)\sigma^*$ is elliptic and non-negative as a 2-suspended operator; moreover it is invertible for each value of the parameters $\tau, \xi \in \mathbb{R}$ (thus invertible as a suspended operator, see [14]) if and only if D is invertible. $\square$

Lemma 15. [17, 18] *The differential operator $\delta_x \in \mathrm{Diff}^1(\partial X, \mathcal{E})$ is invertible for $0 < x < \epsilon$ for some $\epsilon > 0$.*

Proof. (sketch) We can view δ_x as an adiabatic family of operators (i.e. an adiabatic differential operator in the sense of [18]). The adiabatic normal operator of this family is invertible as in Lemma 14, so exactly like in Lemma 11 there exists an inverse $\mu_x \in \Psi_a^{-1}(\partial X, \mathcal{E})$ modulo $x^\infty \Psi_a^{-\infty}(\partial X, \mathcal{E})$:

$$\delta_x \mu_x = I + r_x$$

Now the residual adiabatic ideal $x^\infty \Psi_a^{-\infty}(\partial X, \mathcal{E})$ equals the space of rapidly vanishing families of smoothing operators on ∂X as $x \to 0$. Thus $r_x \to 0$ inside bounded operators as $x \to 0$. The conclusion follows for ϵ chosen small enough so that $\|r_x\| < 1, \forall x < \epsilon$. $\square$

Proposition 16. *The boundary term $\widehat{\mathrm{Tr}}_\partial([A, \log x]B)$ from the index formula* (10) *equals $-\frac{i}{2\pi}\log(\mathrm{hol}(\det D))$.*

Proof. From (2) we know $q([A, \log x]) = x\sigma$. We claim that we can assume $\sigma = 1$. Let $S, T \in \mathcal{C}^\infty(X, \mathrm{End}(\mathcal{E})) \subset \Psi_\Phi^0(M, \mathcal{E})$ be such that $q(S) = \sigma, q(T) = \sigma^{-1}$. Since $q(ST) = 1$ it follows

$$\begin{aligned}
\mathrm{Tr}(x^z[A, \log x]BQ_1^{-\lambda}) &= \mathrm{Tr}(x^z ST[A, \log x]BQ_1^{-\lambda}) + O(z^0) \\
&= \mathrm{Tr}(x^z T[A, \log x]BQ_1^{-\lambda}S) + O(z^0).
\end{aligned}$$

Observe that $q(T[A, \log x])$ and $q(BQ_1^{-\lambda}S)$ do not contain σ anymore; on the other hand the term regular in z at $z = 0$ does not affect the residue, which proves our claim. From Lemma 10,

$$\begin{aligned}
&\widehat{\mathrm{Tr}}_\partial([A, \log x]B) \\
&\quad = \frac{1}{(2\pi)^2}\int_{S^1\times\mathbb{R}^2} \mathrm{Tr}\left(q([A, \log x]BQ_1^{-\lambda})_{[-2]}\right) d\tau d\xi d\theta_{|\lambda=0} \\
&\quad = \frac{1}{(2\pi)^2}\int_{S^1\times\mathbb{R}^2} \mathrm{Tr}\left(q(A^*(AA^*)^{-\frac{\lambda}{2}-1})_{[-1]}\right) d\tau d\xi d\theta_{|\lambda=0} \qquad (16)
\end{aligned}$$

where we use the formulas (13), (15) for $q(A^*), q(AA^*)$ with $\sigma = 1$. There are three types of terms occurring in (16) as explained below and we write accordingly

$$\widehat{\mathrm{Tr}}_\partial([A, \log x]B) = (I(\lambda) + II(\lambda) + III(\lambda))_{|\lambda=0}.$$

5.1. **The terms of type I.** First there are those terms where $q(A^*)$ and $\mathcal{N}((AA^*)^{-\lambda/2-1})$ are composed according to the product rule (9). Since $\frac{\partial q(A^*)}{\partial \xi}$ and $\frac{\partial q(A^*)}{\partial \tau}$ are constants, the corresponding integrands are exact forms so the terms containing them vanish. The non-vanishing terms of type I come from $\frac{x}{2}\mathcal{N}((AA^*)^{-\lambda/2-1})$ and $\frac{x\tau}{i}\frac{\partial \mathcal{N}(A^*)}{\partial \xi}\frac{\partial \mathcal{N}(AA^*)}{\partial \tau}$. Using (14) and polar coordinates in the (τ,ξ) plane we get

$$\begin{aligned} I_1(\lambda) &= \frac{1}{(2\pi)^2}\int_{S^1\times\mathbb{R}^2}\frac{1}{2}\operatorname{Tr}\left(\mathcal{N}(AA^*)^{-\lambda/2-1}\right)d\theta d\tau d\xi \\ &= \frac{1}{4\pi\lambda}\int_{S^1}\operatorname{Tr}\left(\Delta^{-\frac{\lambda}{2}}\right)d\theta. \end{aligned} \tag{17}$$

Thus at $\lambda=0$ we get the average of the logarithm of the determinant of the family D. Although in the end this term will cancel away, it is worth recalling the definition of the determinant of Δ, not to confuse with the determinant line bundle with connection $(\det D, d+\omega^{BF})$ defined in Section 2. The zeta function of any positive pseudo-differential operator is regular at $\lambda=0$; the logarithm of the determinant of Δ is defined as the derivative $\zeta'(\Delta,0)$. This derivative clearly equals the finite part at $\lambda=0$ of $-\frac{1}{\lambda}\operatorname{Tr}(\Delta^{-\frac{\lambda}{2}})$.

Similarly we get

$$\begin{aligned} I_2(\lambda) &= \frac{1}{(2\pi)^2}\int_{S^1\times\mathbb{R}^2}\operatorname{Tr}\left(\frac{\tau}{i}\frac{\partial q(A^*)}{\partial\xi}\frac{\partial\mathcal{N}(AA^*)^{-\frac{\lambda}{2}-1}}{\partial\tau}\right)d\theta d\tau d\xi \\ &\overset{(18.1)}{=} \frac{1}{(2\pi)^2}\int_{S^1\times\mathbb{R}^2}\operatorname{Tr}(\tau^2+\xi^2+\Delta)^{-\frac{\lambda}{2}-1}d\theta d\tau d\xi \\ &\overset{(18.2)}{=} \frac{1}{2\pi\lambda}\int_{S^1}\operatorname{Tr}\left(\Delta^{-\frac{\lambda}{2}}\right)d\theta. \end{aligned} \tag{18}$$

(in (18.1) we used the formulas (13) and (14) with $\sigma=1$ while in (18.1) we used polar coordinates in the (τ,ξ) plane).

5.2. **The terms of type II.** The second type of terms in (16) come from the coefficient of x in $(\xi^2+\tau^2+\Delta)_{[-1]}^{-\lambda/2-1}$, where the power is taken with respect to the product (9). This diagonal matrix is not explicitly computable, however the trace of $\mathcal{N}(A^*)$ times it *is*, because of two facts:

- The diagonal of $\mathcal{N}(A^*)$ is made of central elements modulo x.
- The partial derivatives of $\xi^2+\tau^2+\Delta$ with respect to ξ and τ, are central elements in $\Psi_\Phi^{\mathbb{Z}}(X)/x^\infty\Psi_\Phi^{\mathbb{Z}}(X)$ modulo x.

Thus we can compute $\mathrm{Tr}\left(\mathcal{N}(A^*)(\xi^2+\tau^2+\Delta)^{-\lambda/2-1}_{[-1]}\right)$ as if all the operators involved commute:

$$\begin{aligned} II(\lambda) &= \frac{1}{(2\pi)^2}\int_{S^1\times\mathbb{R}^2} \mathrm{Tr}\left(\mathcal{N}(A^*)(\xi^2+\tau^2+\Delta)^{-\lambda/2-1}_{[-1]}\right) d\tau d\theta d\xi \\ &= \frac{1}{(2\pi)^2}\int_{S^1\times\mathbb{R}^2} \mathrm{Tr}\left(\left(-i\xi+\tau\begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix}\right)\frac{\left(\frac{\lambda}{2}+1\right)\left(\frac{\lambda}{2}+2\right)}{2}\right. \\ &\qquad \left.\left(\frac{4}{i}\tau^2\xi+\frac{2}{i}\tau\tilde{\nabla}_{\partial_\theta}(\Delta)\right)(\xi^2+\tau^2+\Delta)^{-\frac{\lambda}{2}-3}\right) d\tau d\xi d\theta. \end{aligned}$$

We first eliminate the terms which are odd in ξ or τ and thus vanish after integration. The term containing $\tau^2\begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix}\tilde{\nabla}_{\partial_\theta}(\Delta)$ is also seen to vanish because the traces on E^+ and E^- cancel each other. We are left with the term containing $-4\xi^2\tau^2$.

$$\begin{aligned} II(\lambda) &= -\frac{4}{(2\pi)^2}\frac{\left(\frac{\lambda}{2}+1\right)\left(\frac{\lambda}{2}+2\right)}{2} \\ &\qquad \int_{S^1\times\mathbb{R}^2} \xi^2\tau^2\,\mathrm{Tr}(\xi^2+\tau^2+\Delta)^{-\frac{\lambda}{2}-3} d\tau d\xi d\theta \\ &= -\frac{1}{4\pi\lambda}\int_{S^1}\mathrm{Tr}(D^*D)^{-\frac{\lambda}{2}}d\theta. \end{aligned} \tag{19}$$

(we integrated by parts in τ and ξ and then used polar coordinates in the plane (τ,ξ).)

5.3. **The terms of type III.** These are the terms coming from the coefficient of x in $q(AA^*)$, i.e. the matrix $x\begin{bmatrix} i\xi+\tau & -i\tilde{\nabla}_{\partial_\theta}(D^*) \\ i\tilde{\nabla}_{\partial_\theta}(D) & i\xi-\tau\end{bmatrix}$. Again, it is impossible to compute these terms before taking the trace, however the other two factors involved commute (modulo x) so the trace behaves as if all operators involved commuted. We get the following contribution to (16):

$$\begin{aligned} III(\lambda) &= \left(-\frac{\lambda}{2}-1\right)\frac{1}{(2\pi)^2}\int_{S^1\times\mathbb{R}^2}\mathrm{Tr}\left(\begin{bmatrix} -i\xi+\tau & D^* \\ D & -i\xi-\tau\end{bmatrix}\right. \\ &\qquad \begin{bmatrix} \xi^2+\tau^2+D^*D & 0 \\ 0 & \xi^2+\tau^2+DD^*\end{bmatrix}^{-\frac{\lambda}{2}-2} \\ &\qquad \left.\begin{bmatrix} i\xi+\tau & -i\tilde{\nabla}_{\partial_\theta}(D^*) \\ i\tilde{\nabla}_{\partial_\theta}(D) & i\xi-\tau\end{bmatrix}\right) d\tau d\xi d\theta. \end{aligned}$$

The middle matrix is diagonal. Let us look first at the terms coming from the diagonal entries in the first and third matrix. They give

$$\begin{aligned} III_1(\lambda) &= \frac{(-\frac{\lambda}{2}-1)}{(2\pi)^2}\int_{S^1\times\mathbb{R}^2}(\xi^2+\tau^2)\operatorname{Tr}(\xi^2+\tau^2+\Delta)^{-\frac{\lambda}{2}-2}d\tau d\xi d\theta \\ &= -\frac{1}{(2\pi)^2}\int_{S^1\times\mathbb{R}^2}\operatorname{Tr}(\xi^2+\tau^2+\Delta)^{-\frac{\lambda}{2}-1}d\tau d\xi d\theta \\ &= -\frac{1}{2\pi\lambda}\int_{S^1}\operatorname{Tr}(\Delta^{-\frac{\lambda}{2}})d\theta. \qquad (20) \end{aligned}$$

Finally let us compute the contribution coming from anti-diagonal entries:

$$\begin{aligned} III_2(\lambda) &= -\frac{i(\lambda+2)}{(2\pi)^2}\int_{S^1\times\mathbb{R}^2}\Big[\operatorname{Tr}\Big(D^*(\xi^2+\tau^2+DD^*)^{-\frac{\lambda}{2}-2}\tilde{\nabla}_{\partial_\theta}(D)\Big) \\ &\quad -\operatorname{Tr}\Big(D(\xi^2+\tau^2+D^*D)^{-\frac{\lambda}{2}-2}\tilde{\nabla}_{\partial_\theta}(D^*)\Big)\Big]d\tau d\xi d\theta \\ &= -\frac{i}{4\pi}\int_{S^1}\Big(\operatorname{Tr}\Big(D^*(DD^*)^{-\frac{\lambda}{2}-1}\tilde{\nabla}_{\partial_\theta}(D)\Big) \\ &\quad -\operatorname{Tr}\Big(D(D^*D)^{-\frac{\lambda}{2}-1}\tilde{\nabla}_{\partial_\theta}(D^*)\Big)\Big)d\theta \\ &= -\frac{i}{2\pi}\int_{S^1}\omega^{BF}(\lambda)d\theta. \qquad (21) \end{aligned}$$

The terms (17), (18), (19) and (20) cancel so Proposition 16 follows from (21) specialized at $\lambda=0$ and from (3). □

By Theorem 6 the quantity computed in Proposition 16 equals half the adiabatic limit (the limit as x tends to 0) of the eta invariant of the family δ_x. To complete the proof of the index Theorem 2 we must show the integrability of the index density.

Proposition 17. *Under the assumptions of Theorem 2, the local index density* $\operatorname{tr}(r_1(0)-r_2(0))$ *is a smooth multiple of* $1/x$. *Moreover,*

$$\lim_{\varepsilon\to 0}\int_{X\cap\{x\geq\varepsilon\}}\operatorname{tr}(r_1(0)-r_2(0))$$

exists and gives the $\overline{AS}$ *term in the index formula without regularization with* x^z.

Proof. By Corollary 9 we know that

$$\operatorname{tr}(r_j(0))\sim_{x\to 0}\frac{1}{(2\pi)^2}\left(\int_{\mathbb{R}^2}\operatorname{tr}q(R_j^{-\lambda})d\xi d\tau\right)\frac{d\theta dx}{x^3} \qquad (22)$$

has a Laurent expansion at $x=0$ with a possible singularity of order 3. Thus we first want to show that the coefficients of x^{-3}, x^{-2} in $\operatorname{tr}(r_1(0))-\operatorname{tr}(r_2(0))$ vanish. We caution the reader that the products in this proof are with respect to the rule (9). Since tr is invariant under conjugation by linear isomorphisms, we can replace the operator U near $x = 0$ with

$$P := (x^2\partial_x - \frac{x}{2})I_2 + \delta_x.$$

We have

$$\begin{aligned} q(PP^*) &= \xi^2 + ix\xi + \frac{x^2}{2} + q(\delta_x^2) + x\tau \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\ q(P^*P) &= \xi^2 + ix\xi + \frac{x^2}{2} + q(\delta_x^2) - x\tau \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

From this we see that $q(PP^*)^{-\lambda} = q(P^*P)^{-\lambda}$ modulo x, and that $(q(PP^*)^{-\lambda} - q(P^*P)^{-\lambda})_{[-1]}$ is odd in τ hence vanishes after integration. Therefore (22) proves the first part of the theorem.

Let us now examine the integral on ∂X of the coefficient of x^{-1} in the index density. By (22) this is

$$\int_{\mathbb{R}^2\times S^1} \mathrm{Tr}(q(PP^*)^{-\lambda} - q(P^*P)^{-\lambda})_{[-2]} d\tau d\xi d\theta.$$

We cannot give an explicit formula for $(q(PP^*)^{-\lambda} - q(P^*P)^{-\lambda})_{[-2]}$, but we can do it for its trace. We proceed as in the proof of Proposition 16 to eliminate the terms odd in τ or ξ. We are left with

$$\int_{\mathbb{R}^2\times S^1} \frac{-\frac{\lambda}{2}\left(-\frac{\lambda}{2}-1\right)}{2} \mathrm{Tr}\left(\frac{1}{i}x^2 \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tilde{\nabla}_{\partial_\theta}(\Delta)(\xi^2+\tau^2+\Delta)^{-\frac{\lambda}{2}-2}\right) d\xi d\tau d\theta$$

(coming from the composition of $x\tau \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ with $(\xi^2+\tau^2+\Delta)^{-\frac{\lambda}{2}-1}$) and

$$\int_{\mathbb{R}^2\times S^1} \frac{\left(-\frac{\lambda}{2}\right)\left(-\frac{\lambda}{2}-1\right)\left(-\frac{\lambda}{2}-2\right)}{2} \mathrm{Tr}\left(\frac{1}{i}x^2 2\tau^2 \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tilde{\nabla}_{\partial_\theta}(\Delta)(\xi^2+\tau^2+\Delta)^{-\frac{\lambda}{2}-3}\right) d\xi d\tau d\theta$$

(coming from $(\xi^2+\tau^2+\Delta)^{-\frac{\lambda}{2}-1}_{[-1]}$). Integration by parts with respect to τ in the second term gives the negative of the first term. Note that this canceling occurs before the integration in θ. Both terms are actually 0 after integration in all variables. □

Note that we proved slightly more than we claimed, namely that the fiber-wise integral of the index density is smooth as a density in x, θ. It seems reasonable to ask if the index density itself is smooth down to $x = 0$ (as in the case of Dirac operators), however we were unable to prove or to disprove this fact.

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