SINGULARITIES AND NONCOMMUTATIVE GEOMETRY

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ABSTRACT. The general aim of this paper is to present and to give a contribution to the programme: prove an index theorem for singular varieties. Connes shown that cyclic homology is the good receptacle for the Chern character, one of the main ingredients in the definition of the topological index. On the other hand, for a compact manifold M, Connes expressed (periodic) cyclic cohomology of the algebra of smooth functions on M in terms of the classical de Rham cohomology of M. The precise aim of this paper is to prove an equivalent of the Connes result in the case of a singular variety defining and using a suitable cyclic homology and a suitable substitute to de Rham cohomology.

This is realized in the framework of intersection homology. The shadow forms give an explicit and geometric point of view for a de Rham theorem for stratified singular spaces. Moreover, they provide the good intuition for the construction of an algebra of 'controlled functions' and cyclic homology for singular varities.

INTRODUCTION

During the last decades, two important discoveries have provided fundamental progress in the areas of geometry and topology: Noncommutative Geometry and Intersection Homology Theory.

The purpose of noncommutative geometry, following Connes, is to extend the correspondence between geometric spaces and commutative algebras to the noncommutative case, in the framework of real analysis. Noncommutative geometry has been developed in particular for spaces such as the space of leaves of a foliation, which corresponds to a noncommutative algebra. Also, the theory provides new points of view for classical spaces. The example of the Gauss-Bonnet formula (see [14, Introduction]) is very instructive. Differential calculus and topology extend through new algebraic tools such as cyclic homology. While having its origins in physics, the theory has many applications in geometry and topology: the Atiyah-Singer index theorem, Atiyah-Hirzebruch topological K-theory, signature of manifolds, the Novikov conjecture, etc...

The Atiyah-Singer index theorem asserts an equality between the analytic index and the topological index of an elliptic differential operator on a smooth compact manifold. The general aim of this paper is to present and to give a contribution to the programme: prove an index theorem for singular varieties. To fulfil such a programme, it is necessary to define and to give sense to the different ingredients which appear in the formulæ of analytic and topological indices, in the singular case. Many authors already proposed generalizations of some of the ingredients and, for the moment, they appear as pieces of a puzzle to be fitted together in order to prove the result. It is impossible to give an exhaustive list: many authors introduced classes of (pseudo)differential operators on manifolds with edges, corners or manifolds with conical singularities, in particular: Brüning-Seeley [7], [8], [9], Lesch [21], [22], Melrose [25], Monthubert [26], Schrohe [27], Schülze, [28], [29], [30]. On another hand, the extension of the Chern character to singular varieties has been studied by Baum-Fulton-MacPherson [1], Schwartz [31], Kwieciński [20] and Yokura [33].

The Chern character is in fact one of the crucial ingredients in the definition of the topological index. As indicated by Connes, cyclic homology appears to be the good receptacle for the Chern character (in K-homology). For a compact manifold M, Connes expressed (periodic) cyclic cohomology of the

1991 *Mathematics Subject Classification.* 32S60, 58A35, 58J42, 14F40, 81R60, 16E40.

Key words and phrases. Noncommutative Geometry, Cyclic homology, Intersection homology, Singular varieties, de Rham complexes.

algebra of smooth functions on M in terms of the classical de Rham cohomology of M. The precise aim of this paper is to prove an equivalent of Connes' result in the case of a singular variety defining and using a suitable cyclic homology and a suitable substitute for de Rham cohomology.

This is realized in the framework of intersection homology. Intersection homology defined by Goresky and MacPherson, allows one to recover, for singular varieties, many of the properties which are no longer true using ordinary homology: Poincaré duality, the de Rham theorem, Hodge theory, Morse theory etc... In particular, Cheeger [10], [11] and Cheeger-Goresky-MacPherson [12], then many authors proved in particular cases an isomorphism between $\mathcal{L}^2$-cohomology of differential forms on the regular part of the variety and intersection homology. The shadow forms give an explicit and geometric point of view for such a 'singular' de Rham theorem. Moreover, they provide good intuition for the construction of an algebra of 'controlled functions' and cyclic homology for singular varieties, i.e. considering smooth functions defined on the regular part which have suitable behaviour in the neighbourhood of the singular strata.

In the first section of these notes, we recall briefly the ingredients used in the definition of analytic and topological indices. Section 2 is devoted to the definition of intersection homology, due to Mark Goresky and Robert MacPherson, then we introduce the shadow forms, common work with these two authors [2]. Controlled functions and controlled differential forms on a singular variety are defined using the idea of shadow forms. Restricting ourselves to the case of a cone on a manifold, we obtain a 'mixed complex' whose cyclic homology is related to intersection homology. This part comes from work with André Legrand ([3],[4],[5]). The author thanks IMPA, Rio de Janeiro for hospitality during the preparation of the manuscript.

1. The index theorem.

In this section, we recall the definitions of analytic and topological indices in the case of a smooth manifold, in particular the different ingredients used in their definitions. General references for this section are Hirzebruch [18] and Teleman [32].

Let M be a (compact) differentiable n-dimensional manifold with a Riemannian metric and let E and F be differentiable complex vector bundles over M. We denote by $\Gamma(E)$ and $\Gamma(F)$ the vector spaces of global differentiable sections. A differential operator D of order m is a linear operator $D:\Gamma(E)\to\Gamma(F)$ such that in local coordinates (frames) ε_i and η_j in E and F respectively, one has

$$D(f\varepsilon_i)=\sum_j D_i^j(f)\eta_j \quad \text{with} \quad D_i^j(f)(x)=\sum_{|\alpha|\leq m} A_{i\alpha}^j \frac{\partial^{|\alpha|}f}{\partial x^\alpha}(x)$$

for $f\in C^\infty(M)$ and $\alpha=(\alpha_1,\dots,\alpha_r)$ a multi-index of length $|\alpha|=\alpha_1+\cdots+\alpha_r$.

The completion $\Gamma_k(E)$ of $\Gamma(E)$ with metric

$$||\lambda||_k^2=\sum_{|\alpha|\leq k}\left\|\frac{\partial^{|\alpha|}f}{\partial x^\alpha}(x)\right\|^2$$

is a Hilbert space (the Sobolev space of order k of sections of the bundle E) and the differential operator D of order m induces a bounded operator

$$\Gamma_k(E)\to\Gamma_{k-m}(F)$$

Recall that a (bounded) operator between Hilbert spaces is called Fredholm if $\dim_{\mathbb{C}} KerD<\infty$, the image of D ImD is closed in $\Gamma(F)$ and $\dim_{\mathbb{C}} CokerD<\infty$.

Definition 1.1. *The analytic index of a Fredholm operator D is defined as*

$$i_a(D) = \dim_{\mathbb{C}} KerD - \dim_{\mathbb{C}} CokerD$$

Example 1.2. Consider the classical operators $*$ (Hodge), d and $\delta = - * d *$, then for $D = d + \delta$, we have $i_a(D) = 0$.

The symbol of the operator D corresponds to the highest order terms of the operator. More precisely, denote by T^*M the cotangent bundle of M and by π the projection $\pi : T^*M \to M$. Then the symbol of D, denoted by $\sigma_D : \pi^*E \to \pi^*F$, is the morphism of vector bundles defined by

$$\sigma_D(x,\xi) : [\pi^*E]_{(x,\xi)} \to [\pi^*F]_{(x,\xi)} \qquad \sigma_D(x,\xi)(\omega) = D(f^m\tilde{\omega})(x)$$

where $x \in M$, $\xi \in T^*_x(M)$, $\omega \in E_x$, f is a C^∞ function on M such that $f(x) = 0$ and $(df)_x = \xi$ and $\tilde{\omega}$ is a section of E such that $\tilde{\omega}(x) = \omega$.

Remark 1. σ_D is a morphism of vector bundles satisfying

$$\sigma_D(x,t\xi) = t^m\sigma_D(x,\xi)$$

Example 1.3. $\sigma_d(x,\xi)(\omega) = \xi \wedge \omega$, $\sigma_\delta(x,\xi)(\omega) = -i_\xi(\omega)$ and for the Laplace operator $\Delta = d\delta + \delta d$, $\sigma_\Delta(x,\xi)(\omega) = -\|\xi\|^2(\omega)$.

Definition 1.4. *The differential operator D is said to be elliptic if $\sigma_D(x,\xi)$ is an isomorphism for any $\xi \neq 0$.*

Denote by $(T^*M)_0$ the complement of the zero section. The first ingredient for the topological index is given by the following proposition.

Proposition 1.5. *If D is elliptic; it defines a class*

$$[\sigma_D] \in K^0(T^*M,(T^*M)_0) = K^0_c(T^*M)$$

(K_c denotes K-theory with compact supports).

The two other ingredients are the Chern character and the Todd class, which are defined by means of a formal factorisation of the Chern class (see [18], §10). Let E be a complex vector bundle on a manifold X, with Chern class $\sum_{j=0}^q c_j(E)$. In terms of the formal factorisation $\sum_{j=0}^q c_j(E)t^j = \prod_{i=1}^q(1+\gamma_i t)$, the Chern character is defined by $ch(E) = \sum_{i=1}^q e^{\gamma_i}$ and the Todd class by $\tau(E) = \prod_{i=1}^q \frac{\gamma_i}{1-e^{-\gamma_i}}$.

On one hand, the Chern character defines a map $ch : K(T^*M)\otimes\mathbb{Q} \to H^*(T^*M;\mathbb{Q})$. On the other hand, denoting by $T^*_{\mathbb{C}}M$ the complexified bundle of the cotangent bundle, the Todd class of M is defined by $\tau(M) = \tau(\pi^*T^*_{\mathbb{C}}M)$.

Definition 1.6. *Denote by $[T^*M] \in H_{2n}(T^*M)$ the fundamental class of the $2n$-dimensional manifold T^*M. The topological index of an elliptic operator D is defined as the evaluation*

$$i_t(D) = (-1)^n\left(ch[\sigma_D] \cup \tau(M)\right)[T^*M].$$

Theorem 1.7. (Index theorem) *Let D be an elliptic differential operator. Then the analytic and topological indices coincide : $i_a(D) = i_t(D)$.*

To prove an index theorem in the singular case, it is necessary to define in that context the different ingredients which appear in the formulae of analytic and topological indices. As mentioned in introduction, some of these ingredients have been defined by various authors in particular cases of singular varieties. The aim of the present paper is to provide a suitable general setup in which the formula can be established. This is made in the context of Hochschild homology, following Connes' ideas and using intersection homology.

2. INTERSECTION HOMOLOGY.

We consider a singular variety X which is a pseudovariety. In particular, the singular part has codimension at least 2, and we suppose that X is endowed with a Thom-Mather C^∞ stratification, i.e. a filtration of X by closed subsets

$$X = X_n \supset X_{n-1} \supset X_{n-2} \supset \cdots \supset X_1 \supset X_0 \supset X_{-1} = \emptyset$$

such that the connected components of $X_k - X_{k-1}$ are k-dimensional C^∞-manifolds, denoted by S_i^k and called the strata. For each of them, there exist: an open neighbourhood T_i of S_i^k in X, a continuous retraction π_i of T_i on S_i^k, and a continuous function $\rho_i : T_i \to [0,1[$, such that $S_i^k = \{x \in T_i | \rho_i(x) = 0\}$ and the (T_i, π_i, ρ_i) satisfy Mather's axioms (see [24]).

According to Thom's first isotopy lemma, these data imply the following local triviality condition: every point x in S_i^k admits an open neighbourhood $U_x \subset X$ such that there is a homeomorphism

$$\psi_x : U_x \to B^k \times c(L_x)$$

where B^k is the standard k-dimensional open ball and $c(L_x)$ is the (open) cone over the link L_x. (By definition $c(\emptyset) = \{point\}$). The link is assumed to be stratified and independent of the point $x \in S_i^k$. We stratify the cone $c(L_x)$ with the vertex as the 0-dimensional stratum V_0 and the (open) generators through a stratum of L_x forming a stratum of $c(L_x)$. There is a filtration

$$L_x = L_{n-k-1} \supset L_{n-k-2} \supset \cdots \supset L_0 \supset L_{-1} = \emptyset,$$

and ψ_x carries the stratification of U_x induced by that of X to the product stratification of $B^k \times c(L_x)$.

The parameter of the cone corresponds to the Mather distance function ρ_i. For a complete definition see for instance [16].

We now recall the definition of intersection homology due to M. Goresky and R. MacPherson [15]. Given a stratified singular variety, the idea of intersection homology is to consider only chains and cycles whose intersections with the strata are 'not too big'. The allowed chains and cycles meet the strata with a controlled and defect of transversality. This defect is given by a sequence of integers $\bar{p} = (p_0, p_1, \ldots, p_\alpha, \ldots, p_n)$, called the *perversity*, which satisfies:

$$p_0 = p_1 = p_2 = 0 \qquad \text{and} \qquad p_\alpha \le p_{\alpha+1} \le p_\alpha + 1 \quad \text{for } \alpha \ge 2.$$

Let $C_i(X)$ be any 'classical' chain complex on X with integer coefficients, and define the complex of *$\bar{p}$-allowable* chains:

$$IC_i^{\bar{p}}(X) = \{\xi \in C_i(X) \ : \ \dim(|\xi| \cap X_{n-\alpha}) \le i - \alpha + p_\alpha \\ \text{and } \dim(|\partial\xi| \cap X_{n-\alpha}) \le i - 1 - \alpha + p_\alpha\}.$$

The *intersection homology groups* (with $\mathbb{Z}$ coefficients) $IH_i^{\bar{p}}(X)$ are the homology groups of this complex.

Note that if ξ and $X_{n-\alpha}$ are transverse, then their intersection dimension is precisely $i - \alpha$, so the perversity corresponds to an allowed defect of transversality.

Intersection homology does not satisfy all properties of classical homology. For instance, intersection homology is not a homotopy invariant. The main property satisfied by intersection homology is Poincaré duality. If $\bar{p}$, $\bar{q}$ and $\bar{r}$ are perversities such that $\bar{p} + \bar{q} \le \bar{r}$, there is an unique pairing

$$IH_i^{\bar{p}}(X) \times IH_j^{\bar{q}}(X) \to IH_{i+j-n}^{\bar{r}}(X)$$

realized by intersection of cycles.

Theorem 2.1. (Poincaré duality [15]) *Let X be a compact stratified pseudovariety, let $\bar{p}$ and $\bar{q}$ be complementary perversities (i.e. $p_\alpha + q_\alpha = t_\alpha = \alpha - 2$ for $\alpha \geq 2$), then the augmented pairing*

$$IH_i^{\bar{p}}(X;\mathbb{Q}) \times IH_{n-i}^{\bar{q}}(X;\mathbb{Q}) \to IH_0^{\bar{t}}(X;\mathbb{Q}) \xrightarrow{\varepsilon} \mathbb{Q}$$

is nondegenerate.

Intersection homology satisfies the relative exact sequence and the Mayer-Vietoris exact sequence but the Künneth formula does not hold in general. It holds, for example, if one of the varieties involved in the product is smooth (Goresky).

In section 4, we will define complexes of sheaves on the pseudovariety X and will prove that their hypercohomology is equal to the intersection homology of X. This is a consequence of the Theorem 2.2: if a complex of sheaves on X satisfies the so-called *perverse sheaves* axioms (see [16]), then the hypercohomology of (X with value in) this perverse sheaf is the intersection homology of X. In fact, the main axioms of perverse sheaves originate from the following local property: Let x be a point in an α-codimensional stratum, and $U_x \cong B^{n-\alpha} \times c(L_x)$ a distinguished neighbourhood of x. Let $\bar{p}$ be any perversity, then:

$$IH_i^{\bar{p}}(U_x) \cong \begin{cases} IH_i^{\bar{p}}(L_x) & i < \alpha - p_\alpha - 1 \\ 0 & i \geq \alpha - p_\alpha - 1. \end{cases}$$

Let us give a sketch of the easy and very instructive proof of this local computation: The Künneth formula (true in this case) implies

$$IH_i^{\bar{p}}(U_x) \cong IH_i^{\bar{p}}(B^{n-\alpha} \times c(L_x)) \cong IH_i^{\bar{p}}(c(L_x))$$

where $c(L_x)$ is an α-dimensional pseudovariety V, with the stratification induced from that of L_x.

(i) If $i < \alpha - p_\alpha - 1$ and ξ is an allowed i-cycle in $c(L_x)$, then

$$\dim(|\xi| \cap V_0) \leq i - \alpha + p_\alpha < -1$$

so ξ does not contain the vertex of the cone. We can deform ξ along the generatrices of the cone to an allowed i-cycle of the base L_x. We obtain in this way a cycle in $IC_i^{\bar{p}}(L_x)$ homologous to ξ by an allowed homology.

(ii) If $i \geq \alpha - p_\alpha - 1$ and ξ is an allowed i-cycle in $c(L_x)$, then the cone on ξ with vertex $\{x\}$, denoted by $c(\xi)$, is an allowed $i+1$-chain. We have, in particular

$$0 = \dim(|c(\xi)| \cap V_0) \leq (i+1) - \alpha + p_\alpha.$$

The chain $\xi = \partial(c(\xi))$ is an allowed boundary in $IC_i^{\bar{p}}(c(L_x))$ and is homologous to 0. □

Let us now denote, in general $U_j = X - X_{n-j} \quad (2 \leq j \leq n+1)$ and by i_j the natural inclusion $i_j : U_j \hookrightarrow U_{j+1}$. For every closed subset F of X, let i be the inclusion $i : X - F \hookrightarrow X$. If $\mathcal{F}^\bullet$ is a complex of sheaves on X, the composition :

$$\mathcal{F}^\bullet \longrightarrow i_* i^* \mathcal{F}^\bullet \xrightarrow{\mu} Ri_* i^* \mathcal{F}^\bullet$$

is called the *attaching map*. It is a quasi-isomorphism at all $x \in X - F$. If $\mathcal{F}^\bullet$ is a complex of soft sheaves, μ is a quasi-isomorphism for all $x \in X$.

Theorem 2.2. [16] *Let $\mathcal{F}^\bullet$ be a complex of fine (or soft) sheaves on X satisfying the following axioms:*

1. *$\mathcal{F}^\bullet$ is a bounded complex, $\mathcal{F}^k = 0$ for $k < 0$, and $\mathcal{F}^\bullet_{U_2} = \mathbb{C}$,*
2. *for all $x \in X_{n-j} - X_{n-j-1}$, $H^k(\mathcal{F}^\bullet_x) = 0$, for $k > q_j$, $(j = 2, \ldots, n)$,*
3. *the attaching map $\mathcal{F}^k|_{U_{j+1}} \to Ri_{j*}\mathcal{F}^k|_{U_j}$ is a quasi-isomorphism for $k \leq q_j$,*

then the hypercohomology groups (with compact support) $\mathbb{H}^k_c(X;\mathcal{F}^\bullet)$ *are isomorphic to* $Hom(IH^{\bar{p}}_k(X);\mathbb{R})$, *where* $\bar{p}$ *is the complementary perversity.*

3. THE DE RHAM THEOREM.

The Whitney forms, defined by Whitney in the smooth case in order to prove the de Rham theorem, are the main tool used in the definition of shadow forms, providing a de Rham theorem in the singular case.

3.1. **The smooth case: Whitney forms.**

In the case of a compact manifold M, the integration map:

$$\Omega^r(M) \to C^r(M) \qquad \omega \mapsto \left(c \to \int_c \omega \right)$$

from differential forms to singular cochains (with respect to smooth chains c in $C_r(M)$) gives rise to the de Rham isomorphism in cohomology. The proof given by Whitney [34] uses the following Whitney map, providing a homotopy inverse to the integration map.

Let K be a triangulation of M and σ an r-simplex of K, with vertices v^i, $1 \leq i \leq r+1$. Then barycentric coordinates on σ are functions x_i such that for every point x in σ, $x = \sum_{i=1}^{r+1} x_i v^i$ with $\sum_{i=1}^{r+1} x_i = 1$ and $0 \leq x_i \leq 1$.

In terms of such coordinates, the Whitney form associated to σ is defined by

$$W(\sigma) = r! \sum_{j=1}^{r+1} (-1)^{j+1} x_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{r+1}.$$

In fact, the Whitney construction uses smoothing of the barycentric coordinate functions as these are piecewise linear. The Whitney map associates to the elementary r-cochain whose value is 1 on the r-simplex σ and 0 on other simplices, the Whitney form $W(\sigma)$.

Composed with Poincaré duality, the de Rham isomorphism gives isomorphisms

$$H^{n-k}_{\mathrm{dR}}(M) \cong H^{n-k}(M) \cong H_k(M) \qquad \text{(we put } r = n-k\text{)}.$$

3.2. **The singular case: shadow forms.**

In the remainder of this section, X will be a n-pseudomanifold triangulated by a simplicial complex K linearly embedded in a euclidean space.

The idea of shadow forms is to extend the Whitney map to the singular case, providing an isomorphism between the (de Rham) cohomology of a suitable complex of differential forms defined on the regular part of X and the intersection homology of X (there is no longer a Poincaré isomorphism as X is singular).

Denote by Σ the singular part of X. The construction of shadow forms associates explicitly a differential $(n-k)$-form $\omega(\xi)$ on $X - \Sigma$ to any k-simplex ξ of the barycentric subdivision K' of K, transverse to the $(n-1)$-skeleton of K. There is a natural stratification of X corresponding to the filtration of K by skeleta of successive dimensions. The shadow forms satisfy:

i) if ξ is transverse to a stratum of K, then $\omega(\xi)$ is defined and regular on this stratum,

ii) if ξ meets an α-codimensional stratum S with a failure of transversality p_α, then the shadow form $\omega(\xi)$ has a pole of maximum order p_α on the stratum S.

We will recall briefly the construction of [2] on the standard n-simplex and we will make precise the behaviour of shadow forms in the neighbourhood of singular strata.

Let $\Delta = \Delta^n$ be the standard n-simplex. The shadow forms are defined for k-simplices ξ of the barycentric subdivision Δ' which do not lie in the boundary of Δ. Such a barycentric subdivision can be realized for each point p in the interior of Δ, requiring that p is the barycenter of Δ and that for each pair $F' < F$

of faces of Δ, the barycenters of F, F' and of the face opposite to F' in F are collinear. The corresponding barycentric subdivision of Δ will be denoted by $\Delta'(p)$. Every k-simplex ξ of the (abstract) barycentric subdivision Δ' admits a geometrical realisation $\xi(p)$ in $\Delta'(p)$. For c a singular $(n-k)$-chain with support in the interior of Δ, the shadow $S_\xi(c)$ cast by c with respect to ξ is the set of all points p such that $\xi(p)$ intersects c.

Theorem 3.1. [2] *The current defined by $c \mapsto$ volume $S_\xi(c)$ is realizable by a differential form $\omega(\xi)$ (the shadow form), i.e. $\omega(\xi)$ satisfies*

$$\int_c \omega(\xi) = \textit{volume } S_\xi(c).$$

We give an explicit formula for $\omega(\xi)$, allowing one to determine explicitly the order of its poles. The Whitney form associated to a face of Δ with vertices $v^{i_1}, \dots, v^{i_{r+1}}$ is the differential form defined on Δ by

$$W_\Delta(x_{i_1}, \dots, x_{i_{r+1}}) = r! \sum_{j=1}^{r+1} (-1)^{j+1} x_{i_j} dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_j}} \wedge \cdots \wedge dx_{i_{r+1}}.$$

We will denote it also by $W_\Delta(J)$, J being the set of indices $\{i_1, \dots, i_{r+1}\}$.

For example, in Δ of any dimension, $W_\Delta(x_1, x_2) = x_1 dx_2 - x_2 dx_1$.

Let ξ be a k-simplex of Δ', not lying in the boundary of Δ, the vertices of ξ are barycenters of incident faces of Δ:

$$F_1 < F_2 < \cdots < F_{k+1} = \Delta.$$

We denote the sets of indices of vertices as follows. J_1 is the set of indices of vertices in F_1 and, in general, J_ℓ is the set of indices of vertices in F_ℓ, which are not in $F_{\ell-1}$. The set of indices $J_1, \dots, J_{k+1}$ defines a partition of $\{1, \dots, n+1\}$. We write ξ as

$$\xi = (\{x_i\}_{i \in J_1})(\{x_i\}_{i \in J_2}) \cdots (\{x_i\}_{i \in J_{k+1}}) \tag{3.1}$$

and we denote $s_j = \mathrm{card}(J_j)$ and $\lambda_j = \sum_{i \in J_j} x_i$. Also, we define the $(k+1)$-tuples $s = (s_1, \dots, s_{k+1})$ and $\lambda = (\lambda_1, \dots, \lambda_{k+1})$.

Define the function $\phi_s(u_1, \dots, u_k)$ by

$$\phi_s(u_1, \dots, u_k) = (u_1)^{s_1 - 1} \cdots (u_k)^{s_k - 1} (1 - u_1 - \cdots - u_k)^{s_{k+1} - 1}$$

and the domains

$$\begin{aligned} R &= \{(u_1, \dots, u_k) \in \mathbb{R}^k : u_1 \geq 0, \dots, u_k \geq 0, \sum_{i=1}^k u_i \leq 1\}, \\ R(\lambda) &= \{(u_1, \dots, u_k) \in R : 0 \leq \frac{u_1}{\lambda_1} \leq \cdots \frac{u_k}{\lambda_k} \leq \frac{1 - u_1 - \cdots - u_{k-1}}{1 - \lambda_1 - \cdots - \lambda_{k-1}}. \end{aligned}$$

The following function is defined in terms of Dirichlet type integrals (in fact a function of the λ_j's):

$$\Psi(x_1, \dots, x_{n+1}) = \frac{\int_{R(\lambda)} \phi_s(u_1, \dots, u_k) du_k \cdots du_1}{\int_R \phi_s(u_1, \dots, u_k) du_k \cdots du_1}.$$

We can now give the explicit expression of the shadow forms:

Theorem 3.2. [2] *The shadow form $\omega(\xi)$ associated to the k-simplex (3.1) is given by*

$$\omega(\xi) = \varepsilon(s) \frac{W_\Delta(J_1)}{(\lambda_1)^{s_1}} \wedge \cdots \wedge \frac{W_\Delta(J_{k+1})}{(\lambda_{k+1})^{s_{k+1}}} \; \Psi(x_1, \dots, x_{n+1})$$

where $\varepsilon(s) = (-1)^{s_2 + 2s_3 + \cdots k s_{k+1}}$.

For example, in Δ^2:

$$\xi_1 = (x_1, x_2)(x_3) \qquad \omega(\xi_1) = -W_\Delta(x_1, x_2),$$
$$\xi_2 = (x_1)(x_2, x_3) \qquad \omega(\xi_2) = W_\Delta(x_2, x_3)\frac{x_1(2-x_1)}{(1-x_1)^2},$$
$$\xi_3 = (x_1)(x_2)(x_3) \qquad \omega(\xi_3) = \frac{x_1 x_2}{1-x_1}.$$

Remark that the simplices ξ_1 and ξ_3 are transverse to all faces of Δ, and the shadow forms $\omega(\xi_1)$ and $\omega(\xi_3)$ are smooth. The simplex ξ_2 is not transverse to the vertex $\{v^1\}$ and the shadow from $\omega(\xi_2)$ has a pole of order 1 in this point.

The explicit formula for a shadow form provides its behaviour in the neighbourhood of the singular strata, with respect to the position of ξ relative to these strata. The position of ξ relative to the faces of Δ is given by its profile. The profile p_ξ is a function graph defined as follows:

1) We mark successive points ($\bullet$) on the real axis, 0, then $s_{k+1}, s_{k+1}+s_k, \cdots, s_{k+1}+s_k+\cdots+s_2$,
2) The starting point of the graph is the origin of the real axis,
3) Each time we reach a marked point ($\bullet$) of the real axis, the graph goes on the right for one unit, then grows along a parallel to the first diagonal until reaching the next marked point ($\bullet$).

The profile gives the bound of the intersection dimension of the k-simplex ξ and faces of Δ. For a face F with codimension α, we have:

$$\dim(\xi \cap F) \le (k - \alpha + p_\xi(\alpha)). \tag{3.2}$$

We remark that, if ξ and F are transverse and intersect, the dimension of $\xi \cap F$ is $k-\alpha$ and there are faces which realize the equality.

As ξ is transverse to the $(n-1)$-skeleton of Δ, the profile satisfies all conditions to be a perversity. In fact, the profile is the smallest perversity for which ξ is an allowable chain. On a face F of codimension α such that (3.2) is an equality, the shadow form $\omega(\xi)$ admits a pole of order $p_\xi(\alpha)$ and that is the maximum order possible for poles of $\omega(\xi)$.

Theorem 3.3. ([2] Corollary 9.3) *Let X be a triangulated pseudovariety, let $1 \le q \le \infty$ and write $\bar{p}_q$ for the largest perversity such that $\bar{p}(\alpha) < \alpha/q$ for all α. The shadow forms corresponding to simplices ξ such that $p_\xi \le \bar{p}_q$ generate a complex whose cohomology is the intersection homology $IH_*^{\bar{p}_q}(X)$.*

Thus shadow forms provide a good intuition for what can substitute for the de Rham complex in regard to Hochschild homology.

4. Controlled differential forms

We return to the situation of a singular pseudovariety endowed with a Thom-Mather stratification. Intersection differential forms will be 'controlled' differential forms with poles of maximum fixed order along every stratum of the singular part (the order depends only on the stratum). The complex generated by these forms is an intersection complex whose perversity is related to the order of poles. The construction (Brasselet-Legrand [4] [5]) is mainly inspired by the properties of shadow forms.

4.1. **The case of the cone.** Remarking that the cone $c(*)$ over a point is $[0,1[$, we define the algebra

$$A = \{a \in C^\infty(]0,1[) \ : \ \forall\, k \ \sup_{r\in]0,1[} |(r\partial_r)^k a(r)| < +\infty\}$$

and, for $\beta \in \mathbb{R}$, the module of differential forms

$$B^*_\beta = r^{-\beta}\left(A \oplus A\frac{dr}{r}\right) \subset \Omega^*(]0,1[)\,.$$

Here, we have to be cautious: in order to show that controlled geometry is naturally associated to an algebra of differential functions, we need topological versions of the standard algebraic constructions of homological algebra. In particular, for the natural Fréchet topology, the set of C^∞ functions with compact support in $]0,1[$ is not dense in A. This is true for the locally convex topology and that is the reason why we use locally convex spaces, injective and projective tensor products (see [17]). This is carefully done in [4] where the reader will find suitable justifications.

For each graded A-module $\mathcal{D}^*$ of locally convex spaces we define the complex

$$\left(B^*_\beta \;\widehat{\otimes}_A\; \mathcal{D}^*\right)^k = r^{-\beta}\left(\mathcal{D}^k \oplus \frac{dr}{r} \wedge \mathcal{D}^{k-1}\right)$$

with the adapted locally convex topology and injective tensor product.

Let L be a smooth $(n-1)$-dimensional compact Riemannian manifold. We can now define an intersection complex of differential forms on the cone $c(L) = L\times[0,1]/L\times\{0\}$ on L.

Definition 4.1. [5] *The algebra of intersection functions on the cone cL on a smooth manifold L is the 'algebraic cone' of $C^\infty L$:*

$$IC^\infty(cL) = rA\,\widehat{\otimes} C^\infty L + A.$$

In the de Rham subalgebra of $\Omega^*(]0,1[\times L)$ generated by elements $f \in IC^\infty(cL)$ and dg where $g \in IC^\infty(cL)$, we consider the subalgebra $\widetilde{\Omega}^*(cL)$ of differential forms which do not contain terms with element dr. The associated de Rham complex of forms with poles of order β is defined by

$$I\Omega^*_\beta(cL) = B^*_\beta \;\widehat{\otimes}_A\; \widetilde{\Omega}^*(cL).$$

Proposition 4.2. [5] (Poincaré lemma for $I\Omega^*_\beta(cL)$). *Let $\beta > 0$, $\beta \notin \mathbb{N}$, then*

$$H^k(I\Omega^*_\beta(cL)) = \begin{cases} H^k(\Omega^* L) & \text{if } k \leq [\beta]-1 \\ 0 & \text{if } k > [\beta]-1. \end{cases}$$

There are (at least) two ways to prove such a result, the first one geometric, inspired by classical proofs of the Poincaré lemma (see [10]); the second algebraic, using spectral sequences (cf [5]). We give a sketch of the first. Decompose ω as $\omega = \eta + dr\wedge\phi$, and define, for $r_0 > 0$, a homotopy operator

$$\mathcal{K}_{r_0}(\omega) = \int_{r_0}^{r} \phi\, dr.$$

It is easy to see that

$$(\mathcal{K}_{r_0} d + d\mathcal{K}_{r_0})(\omega) = \omega - \omega(r_0)$$

If $k \leq \beta - 1$, then $\mathcal{K}_{r_0}(\omega) \in I\Omega^{k-1}_\beta(cL)$ and $d\mathcal{K}_{r_0}(\omega) \in I\Omega^k_\beta(cL)$.

If $k > \beta - 1$, then, for r_0 going to 0, the integral $\mathcal{K}_0(\omega)$ converges and $\mathcal{K}_{r_0} d(\omega)$ converges to $\mathcal{K}_0 d(\omega)$ in $I\Omega^*_\beta(cL)$. As $\omega(r_0)$ converges to 0 (the coefficients are bounded by $Cr_0^{k-\beta}$), we obtain $(\mathcal{K}_0 d + d\mathcal{K}_0)(\omega) = \omega$ and the Proposition.

4.2. **A model of stratified space.** The elementary example of stratified variety satisfying the Thom-Mather conditions which is our model for the atlas charts of singular varieties is the *ℓ-th iterated cone* $V = M_0 \times c(M_1 \times c(M_2 \times \cdots \times c(M_\ell)\cdots))$ on a family of smooth manifolds $(M_i)_{0\le i\le \ell}$.

The regular stratum is

$$V_{\text{reg}} = M_0 \times]0,1[\times M_1 \times]0,1[\times M_2 \times \cdots \times]0,1[\times M_\ell.$$

The singular strata are $S^{k_i} = M_0 \times]0,1[\times M_1 \times]0,1[\times M_2 \times \cdots \times]0,1[\times M_i$ where $0 \le i < \ell$ and the corresponding link is the iterated cone

$$N_i = M_{i+1} \times c(M_{i+2} \times \cdots \times c(M_\ell)\cdots) = M_{i+1} \times c(N_{i+1})$$

for $0 \le i < \ell - 1$ and $N_{\ell-1} = M_\ell$.

We now extend the definition of the algebra $IC^\infty(cL)$ of intersection functions on a cone cL (Definition 4.1) to an iterated cone $V = M_0 \times c(M_1) \times \ldots \times c(M_\ell) \ldots$.

Definition 4.3. [5] *The intersection function algebra on the iterated cone V, denoted by $IC^\infty(V)$, is the locally convex algebra of differentiable functions defined iteratively by*

$$IC^\infty(N_{\ell-1}) = C^\infty(M_\ell), \quad \text{and}$$
$$IC^\infty(N_{i-1}) = C^\infty(M_i)\, \widehat{\otimes}\, c(IC^\infty(N_i)).$$

To define the de Rham complex of differential forms with polar coefficients, we use the following definition of the cone over a positively graded A-module of locally convex spaces C^*:

$$(cC)^k = r^{k+1} A\, \widehat{\otimes} C^k + r^k A\, \widehat{\otimes} dC^{k-1}.$$

Definition 4.4. [5] *For any sequence $\bar\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_\ell)$ of real numbers, the intersection de Rham differential module $I\Omega^*_{\bar\gamma}(V)$ of forms on $IC^\infty(V)$, with poles of order less than $\bar\gamma$, is defined iteratively by:*

$$\Omega^*_{\emptyset}(N_{\ell-1}) = \Omega^*(M_\ell) \quad \text{and}$$
$$I\Omega^*_{(\gamma_{i+1},\ldots,\gamma_\ell)}(N_{i-1}) = \Omega^*(M_i)\, \widehat{\otimes} \left(B^*_{\gamma_{i+1}}\, \widehat{\otimes}_{A_{i+1}}\, c[I\Omega^*_{(\gamma_{i+2},\ldots,\gamma_\ell)}(N_i)]\right).$$

4.3. **The case of stratified spaces.** In order to generalize the previous constructions to stratified varieties, we will use atlases whose charts are iterated cones. Let x be in a stratum $S_{n-\alpha_1}$ of the stratified space X and $\psi_x : U_x \overset{\cong}{\to} B^{n-\alpha_1} \times cL_x$ a distinguished open neighbourhood of x. The link L_x is itself a singular variety covered by distinguished open sets of the same type. By iteration, we obtain a system of *iterated cone* charts of the type:

$$W_{\bar\alpha} = \mathbf{B}^{n-\alpha_1} \times c\left(\mathbf{B}^{\alpha_1-1-\alpha_2} \times c\left(\mathbf{B}^{\alpha_2-1-\alpha_3} \times \cdots \times c(L^{\alpha_\ell - 1})\cdots\right)\right)$$

where $\bar\alpha = \{n+1 = \alpha_0 > \alpha_1 > \alpha_2 > \cdots > \alpha_\ell > \alpha_{\ell+1} = 0\}$ denotes a decreasing sequence of integers, $\mathbf{B}^{\alpha_t - 1 - \alpha_{t+1}}$ is an open ball in $\mathbb{R}^{\alpha_t - 1 - \alpha_{t+1}}$ (possibly a point) and $L^{\alpha_\ell - 1}$ is a finite disjoint union of $(\alpha_\ell - 1)$-dimensional balls. Via the homeomorphism ψ_x, this chart corresponds to the following chain of elements of the filtration of X:

$$\emptyset = X_{n-\alpha_0} \subset X_{n-\alpha_1} \subset X_{n-\alpha_2} \subset \cdots \subset X_{n-\alpha_\ell} \subset X = X_n = X_{n-\alpha_{\ell+1}}\,.$$

We will denote in the same way the iterated cone and its image in X. We obtain in this way a covering of U_x by charts which are iterated cones and correspond to different sequences $\bar\alpha$. Each chart $W_{\bar\alpha}$ is identified with a model iterated cone $V = M_0 \times c(M_1 \times \cdots \times c(M_\ell)\cdots)$.

Let us fix a sequence of real numbers $\bar\beta = (\beta_2, \beta_3, \ldots, \beta_n)$, each β_α being associated with the stratum $S_{n-\alpha}$. Given an iterated cone chart $W_{\bar\alpha}$, hence a sequence $\bar\alpha$ as above, we associate the sequence $\bar\beta_{\bar\alpha} = (\beta_{\alpha_1}, \beta_{\alpha_2}, \ldots, \beta_{\alpha_\ell})$, i.e. such that $(\bar\beta_{\bar\alpha})_i = \beta_{\alpha_i}$.

Definition 4.5. *Let X be a stratified singular variety endowed with an atlas $(W_{\bar\alpha})$ of iterated cones. We denote by $I\Omega^k_{\bar\beta}$ the sheaf on X whose sections over every open set U of X are C^∞-differential forms ω on $U \cap X_{\mathrm{reg}}$ such that, for each iterated cone $W_{\bar\alpha}$, $\omega \in I\Omega^*_{\bar\beta_{\bar\alpha}}(W_{\bar\alpha} \cap U)$.*

The sheaf $I\Omega^k_{\bar\beta}$ satisfies the following Poincaré lemma [4].

Proposition 4.6. *Let cX be the cone on a stratified $(n-1)$-variety X. Let $\bar\beta$ be a sequence of positive numbers as previously defined. Assume that no β_α is an integer. Then, there is an isomorphism*

$$H^k(I\Omega^*_{\bar\beta_{\bar\alpha}}(cX) = \begin{cases} H^k(I\Omega^*_{\bar\beta_{\bar\alpha}}(X) & \text{if } k \leq [\beta_n]-1 \\ 0 & \text{if } k > [\beta_n]-1. \end{cases}$$

As a consequence, we obtain the following de Rham theorem (see [3] and [5]):

Theorem 4.7. *Let X be a stratified singular variety, endowed with an atlas of iterated cones. Let $\bar\beta$ be a sequence of positive real numbers, satisfying the perversity condition:*

$$[\beta_\alpha] \leq [\beta_{\alpha+1}] \leq [\beta_\alpha]+1 \ ,$$

for all $2 \leq \alpha \leq n$ and such that no β_α is an integer. Then,

1. *$I\Omega^*_{\bar\beta}$ is a perverse sheaf for the perversity $\bar q$ defined by $q_\alpha = [\beta_\alpha]-1$,*
2. *there is an isomorphism*

$$H^k(I\Omega^*_{\bar\beta}(X)) \cong Hom(IH^{\bar p}_k(X,\mathbb{R});\mathbb{R}),$$

where $\bar p$ is the complementary perversity of $\bar q$.

5. Hochschild and cyclic homology.

In this section, we recall the definitions of Hochschild and cyclic homology: the basic references are [14] and [23]. Let Λ be a ring and $\mathcal{A}$ be an Λ-algebra with unit, so $\Lambda \subset \mathcal{A}$. For any $\mathcal{A}$-bimodule N, the Hochschild complex $(C_*(\mathcal{A},N),b)$ with coefficients in N is defined by $C_k(\mathcal{A},N) = N \otimes_\Lambda \mathcal{A}^{(\otimes^k_\Lambda)}$, with boundary map

$$(5.3) \quad b(n\otimes a_1 \otimes \cdots \otimes a_k) = na_1 \otimes a_2 \otimes \cdots \otimes a_k + \sum_{j=1}^{k-1} (-1)^j n \otimes a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_k + (-1)^k a_k n \otimes a_1 \otimes \cdots \otimes a_{k-1}.$$

Its homology, called Hochschild homology, is denoted by $HH_*(\mathcal{A},N)$.

When $N = \mathcal{A}$, the Hochschild complex is denoted by $C_*(\mathcal{A})$ and Hochschild homology by $HH_*(\mathcal{A})$. In that case, $C_*(\mathcal{A})$ is a cyclic module, i.e. it admits a cyclic action

$$\tau(a_0 \otimes \cdots \otimes a_k) = (-1)^k a_k \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{k-1} \ .$$

We have $\tau^{k+1} = id$, so, for each k, τ defines an action of $\mathbb{Z}/(k+1)\mathbb{Z}$ on $C_k(\mathcal{A})$.

If Λ is a field of characteristic 0, the Connes cyclic homology of $\mathcal{A}$, denoted by $HC_*(\mathcal{A})$, is the homology of the complex $(C_*(\mathcal{A})/(1-\tau),b)$ where b is induced by the Hochschild boundary. The relation between Hochschild and cyclic homology is given by the Connes exact sequence

$$\cdots \to HH_k(\mathcal{A}) \xrightarrow{I} HC_k(\mathcal{A}) \xrightarrow{S} HC_{k-2}(\mathcal{A}) \xrightarrow{B} HH_{k-1}(\mathcal{A}) \to \cdots$$

Using the so called *periodicity operator* S, we define the periodic cyclic homology (odd or even)

$$PHC_{\mathrm{o/e}}(\mathcal{A}) = \lim_{k} \left[HC_k(\mathcal{A}) \xrightarrow{S} HC_{k-2}(\mathcal{A}) \right].$$

The importance of the above definitions is shown by the following result. Let M be a compact C^∞ manifold, $\mathcal{A} = C^\infty(M)$ the Fréchet algebra of differentiable functions on M, and $\Omega^*(M)$ the associated de Rham algebra. Replace everywhere $\otimes$ by the projective tensor product $\widehat{\otimes}$ (Grothendieck [17]). Then the map $\pi : C_k(C^\infty(M)) \to \Omega^k(M)$ defined by $\pi(f_0 \otimes \cdots \otimes f_k) = f_0 df_1 \wedge \cdots \wedge df_k$ induces the following isomorphisms [13]:

$$HH_*(C^\infty(M)) \cong \Omega^*(M); \quad PHC_{\mathrm{o/e}}(C^\infty(M)) \cong \oplus_{\mathrm{o/e}} H^*_{dR}(M). \tag{5.4}$$

From this result, Connes recovers the de Rham complex of the manifold X without using the commutativity of the algebra $\mathcal{A}$, which is needed in the construction of the exterior algebra.

If Λ is a ring, as it will be the case in the singular framework, the definition of cyclic homology needs a more general setting, the notion of mixed complex that we briefly describe now (see [19]).

A *mixed complex* is a triple (E_*, b, B), where E_* is a graded module and b, resp. B, is a differential on E_* of degree -1, resp. $+1$, such that $bB + Bb = 0$. It defines a bicomplex $E_*[u]$ with differentials $b(xu^k) = (bx)u^k$, $B(xu^k) = (Bx)u^{k-1}$ where degree$(u) = 2$. We define the *Hochschild homology* of (E_*, b, B) as $HH_*(E_*) = H_*(E_*, b)$ and the *cyclic homology* as $HC_*(E_*) = H_*(E_*[u], b + B)$. There is again a Connes exact sequence and we can define the periodic cyclic homology $PHC(E_*)$.

The Hochschild complex $C_*(\mathcal{A})$ admits a mixed complex structure where b is the Hochschild boundary and B is the operator defined by

$$\begin{aligned} B(a_0 \otimes \cdots \otimes a_k) = \sum_{j=0}^{k-1} (-1)^{kj} 1 \otimes a_j \otimes \cdots \otimes a_k \otimes a_0 \otimes \cdots \otimes a_{j-1} \\ - (-1)^{k(j-1)} a_{j-1} \otimes 1 \otimes a_j \otimes \cdots \otimes a_k \otimes a_0 \otimes \cdots \otimes a_{j-2}. \end{aligned} \tag{5.5}$$

Let us remark that, when Λ is a field of characteristic 0, we recover the previous definition of cyclic homology of $\mathcal{A}$ in the following way. Replacing the quotient $C_k(\mathcal{A})/(1-\tau)$ by a $(\mathbb{Z}/(k+1)\mathbb{Z})$-free resolution of $C_k(\mathcal{A})$, we obtain a bicomplex which is quasi-isomorphic to the bicomplex associated to the mixed complex structure of $C_*(\mathcal{A})$. Then the two definitions of cyclic homology agree.

6. The singular case.

The aim of this section is to prove a version of Connes' result (5.4). This is performed in the case of a cone on a smooth manifold. The general result requires more sophisticated techniques (see [5]).

In the case of the cone, we use a slight generalization of the previous definition, taking $\Lambda = A$, $\mathcal{A} = IC^\infty(cL)$ and we define a mixed complex structure on the Hochschild complex $C^A_*(IC^\infty(cL), B^*_\beta)$ of $IC^\infty(cL)$ with coefficients in the differential module B^*_β (see [4]).

More precisely, consider the A-Hochschild complex of the Fréchet algebra $IC^\infty(cL)$,

$$C^A_k(IC^\infty(cL)) = IC^\infty(cL) \widehat{\otimes}_A \cdots \widehat{\otimes}_A IC^\infty(cL) \qquad (k+1)\text{-terms}$$

and denote by b_A its differential. We define the *Hochschild intersection complex* by

$$IC^\beta_k(cL) = C^A_k(IC^\infty(cL), B^*_\beta) = B^*_\beta \,\widehat{\otimes}_A C^A_{k-*}(IC^\infty(cL)).$$

The elements of degree k are sums of terms

$$r^{-\beta} f_0 \otimes f_1 \otimes \cdots \otimes f_k + r^{-\beta} \frac{dr}{r} g_0 \otimes g_1 \otimes \cdots \otimes g_{k-1}$$

such that $f_i, g_j \in IC^\infty(cL)$. The differential b is given by

$$\begin{array}{ccccc} IC_k^\beta(cL) & = & r^{-\beta}\, C_k^A(IC^\infty(cL)) & \oplus\, r^{-\beta}A\dfrac{dr}{r}\widehat{\otimes}_A\, C_{k-1}^A(IC^\infty(cL)) \\ \uparrow b & & b_A^{k+1}\uparrow\downarrow B_A^k \qquad \searrow d_r & b_A^k \uparrow\downarrow B_A^{k-1} \\ IC_{k+1}^\beta(cL) & = & r^{-\beta}\, C_{k+1}^A(IC^\infty(cL)) & \oplus\, r^{-\beta}A\dfrac{dr}{r}\widehat{\otimes}_A\, C_k^A(IC^\infty(cL)) \end{array}$$

where the operators b_A and B_A are defined in (5.3) and (5.5), with cyclic action defined by

$$\tau(r^{-\beta}g_0\otimes\cdots\otimes g_k) = (-1)^k r^{-\beta} g_k\otimes g_0\otimes\cdots\otimes g_{k-1}$$

and d_r is the following derivation

$$d_r(r^{-\beta}f_0\otimes f_1\otimes\cdots\otimes f_k) = \\ (-1)^k r^{-\beta}\frac{dr}{r}\wedge\left[(-\beta)f_0\otimes\cdots\otimes f_k + \sum_{i=0}^k f_0\otimes\cdots\otimes\partial_r f_i\otimes\cdots\otimes f_k\right].$$

So $(b_A^*\oplus b_A^{*-1})$ has degree -1 and $(B_A^*\oplus B_A^{*-1}\ +\ d_r)$ has degree $+1$.

Lemma 6.1. *The following triple is a mixed complex:*

$$(IC_*^\beta(cL),\ b = b_A^*\oplus b_A^{*-1},\ B = B_A^*\oplus B_A^{*-1}\ +\ d_r)$$

Denote by $\pi : IC_*^\beta(cL)\to I\Omega_\beta^*(cL)$ the morphism defined by

$$\pi(r^{-\beta}f_0\otimes f_1\otimes\cdots\otimes f_k + r^{-\beta}\frac{dr}{r}g_0\otimes g_1\otimes\cdots\otimes g_{k-1}) \\ = r^{-\beta}f_0 d_L f_1\wedge\cdots\wedge d_L f_k + r^{-\beta}\frac{dr}{r}\wedge g_0 d_L g_1\wedge\cdots\wedge d_L g_{k-1}$$

and consider the kernel (K_*, b, B) of π equipped with the induced mixed complex structure.

Denote by C_* the Hochschild complex of $C^\infty(L)$ and by $s : C_{k-1}\to C_k$ the homotopy operator $s(f_1,\dots,f_k) = (1, f_1,\dots,f_k)$.

Lemma 6.2. [4] *The Hochschild homology of K_* satisfies the following exact sequence*

$$0\to\frac{r^{k+1}A\,\widehat{\otimes}\,bC_{k+1}/bsC_k}{r^{k+2}A\,\widehat{\otimes}\,bC_{k+1}/bsC_k}\to HH_k(K_*)\to\frac{r^k A\,\widehat{\otimes}\,bC_{k+1}\cap sC_{k-1}}{r^{k+1}A\,\widehat{\otimes}\,bC_{k+1}\cap sC_{k-1}}\to 0$$

There is also a similar formula for

$$PHC_{\mathrm{o/e}}(K_*) = \lim_k\left[HC_k(K_*)\xrightarrow{S} HC_{k-2}(K_*)\right].$$

Theorem 6.3. [4] *The Hochschild homology and the cyclic homology of the mixed complex $(IC_*^\beta(cL), b, B)$ enter in the following exact sequences*

$$0\to HH_k(K_*)\to HH_k(IC_*^\beta(cL))\xrightarrow{\pi} I\Omega_\beta^k(cL)\to 0$$

$$0\to PHC_{\mathrm{o/e}}(K_*)\to PHC_{\mathrm{o/e}}(IC_*^\beta(cL))\to\bigoplus_{\mathrm{o/e}} IH_{\bar q}^*(cL)\to 0$$

where the perversity $\bar q$ satisfies $q_n = [\beta]$.

The result can be generalized to the case of a stratified variety, using the results of [5]. This will appear in a forthcoming publication.

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