Edge Problems on Configurations with Model Cones of Different Dimensions

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Abstract

Elliptic equations on configurations $W = W_1 \cup \ldots \cup W_N$ with edge Y and components W_j of different dimension can be treated in the frame of pseudo-differential analysis on manifolds with geometric singularities, here, edges. Starting from edge-degenerate operators on W_j, $j = 1, \ldots, N$, we construct an algebra with extra "transmission" conditions on Y that satisfy an analogue of the Shapiro-Lopatinskij condition. Ellipticity refers to a two-component symbolic hierarchy with an interior and an edge part; the latter one is operator-valued, operating on the union of different dimensional model cones. We construct parametrices within our calculus, where exchange of information between the various components is encoded in Green and Mellin operators that are smoothing on $W \setminus Y$. Moreover, we obtain regularity of solutions in weighted edge spaces with asymptotics.

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Introduction

This paper is aimed at studying elliptic operators on a configuration with edges, locally described by wedges with model cones of different dimensions. To be more precise, if X is a (say, compact) topological space and $X^\Delta := (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$ the cone with base X (where $\{0\} \times X$ in the quotient space corresponds to the tip v of the cone), the Cartesian product $X^\Delta \times \Omega$ with a C^∞ manifold Ω is a wedge with model cone X^Δ and edge Ω. In our case we assume $X = X_1 \cup \ldots \cup X_N$ to be a disjoint union of compact and closed C^∞ manifolds X_j of dimensions $n_j = \dim X_j$. Then X^Δ is a cone of the form $\cup_v X_j^\Delta := \cup X_j^\Delta / \sim$, where $\cup_v$ is the disjoint union combined with the quotient map that identifies the tips of the cones X_j^Δ with a single point v. Examples of edge configurations can easily be constructed in terms of transversal intersections of embedded C^∞ manifolds of different dimensions; the intersections are then the edges.

Configurations of that kind occur in a number of applications, for instance, in mechanics, heat diffusion and other models of applied sciences. Ellipticity in operator algebras with discrete asymptotics in the simpler case of cones $\cup_v X_j^\Delta$ with different-dimensional X_j has been investigated in [15]. Models with transmission effects in network-like situations have been studied by Ali Mehmeti for hyperbolic equations, see [1] and the references there. It is also interesting to consider operators on spaces with “higher” edges and corners, i.e., spaces composed of subspaces of different dimensions and (say, piecewise smooth) singular geometry, though such a calculus is not yet established; it would be of a similar complexity as a corresponding theory for manifolds with higher singularities in the sense of [17], [18].

In the present paper we concentrate on algebras of edge-degenerate operators on a stretched configuration with edges, parametrix constructions for elliptic edge problems and asymptotics of solutions. To formulate our calculus, we first deal with the transmission algebra $C^\mu(\boldsymbol{X}^\wedge, \boldsymbol{g}; \boldsymbol{v})$ on cones (see Section 1 below for notation): the edge-degenerate operators of our algebra will be defined by means of operator-valued symbols taking values in $C^\mu(\boldsymbol{X}^\wedge, \boldsymbol{g}; \boldsymbol{v})$, parametrised by $(y, \eta) \in T^*Y$, where Y is the edge. Parameter-dependent versions of the present calculi are also possible, and they would be necessary in analogous problems when the base spaces X_j themselves are configurations with edges and corners of different dimensions, see [18] for a special situation of that kind.

Anyway, we will not give here a detailed description of the parameter-dependent case.

Differential and pseudo-differential operators on manifolds with geometric singularities such as conical points or edges have a long history and are studied under different aspects by many authors before, cf. Kondratyev [7], Melrose, Mendoza [9], Mazzeo [8]. Concerning further references, cf. [5] or [14]. Here we focus on an approach for edge-degenerate operators, first established in [12] for "standard" manifolds with edges, that combines ideas from the analysis of boundary value problems in the sense of Boutet de Monvel [3] or Rempel and Schulze [11] with special Mellin quantisations in model cone direction, see [13], and quantisation in edge direction, see [14], [16], based on twisted homogeneity of operator-valued symbols, connected with strongly continuous groups of isomorphisms on weighted spaces on the model cones. What we obtain is an algebra of (2×2)-block matrix operators with trace and potential conditions along edges. The latter ones satisfy an analogue of the Shapiro-Lopatinskij condition in the elliptic case. Similarly to boundary value problems, cf. Atiyah and Bott [2], there is a topological obstruction for the existence of such conditions. For the edge case with different model cones it may happen (as for, say, the Cauchy-Riemann operator and transmission problems on a manifold with respect to an interface of codimension 1) that for operators from one side the obstruction may be non-vanishing though from both sides it vanishes. Transmission problems in general (for pseudo-differential operators with or without transmission property at the interface) are, in fact, special cases of our calculus. In that case the normal half lines in the two opposite directions are just the model cones of corresponding local wedges.

Ellipticity in our algebra is determined by a bijectivity condition for a symbolic hierarchy, consisting of interior and edge components, and we construct parametrices within the algebra. Regularity of solutions is controlled in weighted spaces and subspaces with discrete or continuous asymptotics.

A large variety of explicit examples of elliptic operators with "usual" edges may be found in [10], including operators where the above-mentioned topological obstruction does not vanish. Other examples may be constructed in terms of parameter-dependent ellipticity such that corresponding operators induce isomorphisms between the respective weighted Sobolev spaces, cf. also Dorschfeldt [4]. The approach can also be extended to the present algebra with different dimensional components.

1 Transmission algebras on cones

1.1 Mellin operators and cone Sobolev spaces

Let us first fix some notation around the Mellin transform and associated pseudo-differential operators. The Mellin transform $\mathcal{M}$ is given by the formula

$$\mathcal{M}u(z) = \int_0^\infty r^{z-1} u(r) dr, \tag{1.1.1}$$

in the simplest case for $u \in C_0^\infty(\mathbb{R}_+)$; then the covariable z varies in $\mathbb{C}$, and we have $\mathcal{M}u(z) \in \mathcal{A}(\mathbb{C})$ (here, $\mathcal{A}(U)$ for any open $U \subseteq \mathbb{C}$ denotes the space of all holomorphic functions in U). (1.1.1) will be then extended to various function and distribution spaces, also vector-valued ones; then z will vary on subsets of $\mathbb{C}$, for instance, lines

$$\Gamma_\beta := \{z \in \mathbb{C} \ : \ \operatorname{Re} z = \beta\} \tag{1.1.2}$$

for some real β. Define the weighted Mellin transform with weight $\gamma \in \mathbb{R}$ by $\mathcal{M}_\gamma u(z) := \mathcal{M}(r^{-\gamma}u)(z+\gamma)$. Then $\mathcal{M}_\gamma u = \mathcal{M}u|_{\Gamma_{\frac{1}{2}-\gamma}}$ for $u \in C_0^\infty(\mathbb{R}_+)$ is interpreted as a map $\mathcal{M} \ : \ C_0^\infty(\mathbb{R}_+) \to \mathcal{A}(\mathbb{C})$. Recall that $\mathcal{M}_\gamma$ extends to an isomorphism $\mathcal{M}_\gamma \ : r^\gamma L^2(\mathbb{R}_+) \to L^2(\Gamma_{\frac{1}{2}-\gamma})$, and the inverse is

$$\mathcal{M}_\gamma^{-1} g(r) = \frac{1}{2\pi i} \int_{\Gamma_{\frac{1}{2}-\gamma}} r^{-z} g(z) dz.$$

Here and in the sequel, function and distribution spaces, originally given on $\mathbb{R}$, will also be employed for $\Gamma_\beta \ni z$, where $\operatorname{Im} z$ plays the role of the real variable. In particular, we have symbol spaces $S^\mu(\mathbb{R}_+ \times \mathbb{R}_+ \times \Gamma_\beta)$ in the sense of the Hörmander classes, where we write $a(r, r', z)$ with $\tau := \operatorname{Im} z$ being the covariable. With symbols $a(r, r', z) \in S^\mu(\mathbb{R}_+ \times \mathbb{R}_+ \times \Gamma_{\frac{1}{2}-\gamma})$ we associate weighted Mellin pseudo-differential operators on the half-axis

$$\begin{aligned} \operatorname{op}_M^\gamma(a)u(r) \quad &:= \quad \mathcal{M}^{-1}_{\gamma, z \to r}\{\mathcal{M}_{\gamma, r' \to z} a(r, r', z) u(r')\} \\ &= \quad \int \int_0^\infty \left(\frac{r}{r'}\right)^{-\left(\frac{1}{2}-\gamma+i\tau\right)} a(r, r', z) u(r') \frac{dr'}{r'} đ\tau, \end{aligned} \tag{1.1.3}$$

where $z = \dfrac{1}{2} - \gamma + i\tau$. Note that we can also write

$$\operatorname{op}_M^\gamma(a) = r^\gamma \operatorname{op}_M(T^{-\gamma}a) r^{-\gamma}$$

where $(T^{-\gamma}a)(r, r', z) = a(r, r', z-\gamma)$ and $\operatorname{op}_M(.) = \operatorname{op}_M^0(.)$. Below we use such a notation also in the vector- and operator-valued case, where, for instance, $a(r, r', z) \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+, L^\mu(X; \Gamma_{\frac{1}{2}-\gamma}))$, and X is a closed compact C^∞ manifold. Here $L^\mu(X; \mathbb{R})$ denotes the space of all pseudo-differential operators of order $\mu \in \mathbb{R}$ on X that depend on a parameter $\tau \in \mathbb{R}$. Recall that such operators are locally described by amplitude functions $a(x, x', \xi, \tau)$ in the covariables $(\xi, \tau) \in \mathbb{R}^{n+1}$, $n = \dim X$, while $L^{-\infty}(X; \mathbb{R}) = \mathcal{S}(\mathbb{R}, L^{-\infty}(X))$ with the space $L^{-\infty}(X)$ of smoothing operators on X. In a similar sense there are parameter-dependent spaces $L^\mu(X; \mathbb{R}^l)$ for an l-dimensional parameter λ. All these spaces are equipped with natural Fréchet topologies.

To define weighted Sobolev spaces on a stretched cone $X^\wedge := \mathbb{R}_+ \times X$ with base X, we employ the fact that for every $\mu \in \mathbb{R}$ there exists an element $R^\mu(\lambda) \in L^\mu(X; \mathbb{R}^l)$ that is parameter-dependent elliptic of order μ and induces isomorphisms $R^\mu(\lambda) \ : \ H^s(X) \to H^{s-\mu}(X)$ for all $\lambda \in \mathbb{R}^l, s \in \mathbb{R}$. Here, $H^s(X)$ are the standard Sobolev spaces of smoothness $s \in \mathbb{R}$ on X. We now choose such a family $R^\mu(\tau)$ with parameter $\tau \in \mathbb{R}$ and define $\mathcal{H}^{s,\gamma}(X^\wedge)$ as the completion of $C_0^\infty(X^\wedge)$ with respect to the norm

$$\left\{ \frac{1}{2\pi i} \int_{\Gamma_{\frac{n+1}{2}-\gamma}} \|R^\mu(\operatorname{Im} z)\mathcal{M}u(z)\|^2_{L^2(X)} dz \right\}^{\frac{1}{2}}, \tag{1.1.4}$$

for $n = \dim X$. The space $L^2(X)$ is equipped with a scalar product, defined in terms of a fixed Riemannian metric on X. Recall that when we choose another family $\tilde{R}^\mu(\tau)$ with analogous properties, we get an equivalent norm.

The spaces $\mathcal{H}^{s,\gamma}(X^\wedge)$ have the meaning of Sobolev spaces based on the Fuchs type derivative in $r \in \mathbb{R}_+$ and (local) usual derivatives on X. More precisely, for $s \in \mathbb{N}$ we have

$$\begin{aligned} \mathcal{H}^{s,\gamma}(X^\wedge) = \{&u(r,x) \in r^{-\frac{n}{2}+\gamma} L^2(\mathbb{R}_+ \times X) \;: \\ &(r\partial_r)^k D u(r,x) \in r^{-\frac{n}{2}+\gamma} L^2(\mathbb{R}_+ \times X) \text{ for all } k + \operatorname{ord} D \leq s\} \end{aligned} \tag{1.1.5}$$

Here D stands for arbitrary differential operators on X. It can be easily proved that (1.1.5) is an equivalent definition of $\mathcal{H}^{s,\gamma}(X^\wedge)$ for $s \in \mathbb{N}$, and the full scale could be defined by duality and interpolation. Notice that $\mathcal{H}^{0,0}(X^\wedge) = r^{-\frac{n}{2}} L^2(\mathbb{R}_+ \times X)$ (with L^2 being taken with measure $drdx$).

By a cut-off function on the half-axis we understand in this paper any real-valued $\omega(r) \in C_0^\infty(\overline{\mathbb{R}}_+)$ that equals 1 in a neighbourhood of $r = 0$.

In the considerations below we will also employ a modified scale of weighted Sobolev spaces, namely $\mathcal{K}^{s,\gamma}(X^\wedge)$, defined by

$$\mathcal{K}^{s,\gamma}(X^\wedge) := \{\omega u + (1-\omega)v \;:\; u \in \mathcal{H}^{s,\gamma}(X^\wedge), v \in H^s_{\text{cone}}(X^\wedge)\}$$

for any cut-off ω. Here, $H^s_{\text{cone}}(X^\wedge)$ for $X = S^n$ (the unit sphere in $\mathbb{R}^{n+1}$) is the subspace of all $v \in H^s_{\text{loc}}(X^\wedge)$ such that $(1-\omega)v \in H^s(\mathbb{R}^{n+1})$ where (r,x) are interpreted as polar coordinates in $\mathbb{R}^{n+1} \setminus \{0\} \cong (S^n)^\wedge$. For general X we first choose a chart $\chi_1 : G \to C$ on X for an open set $C \subset S^n$, and set $C^\wedge := \{\tilde{x} \in \mathbb{R}^{n+1} \setminus \{0\} \;:\; \tilde{x}/|\tilde{x}| \in C\}$, $H^s_{\text{cone}}(C^\wedge)_0 := \{u \in H^s(\mathbb{R}^{n+1}) \;:\; \operatorname{supp} u \subset C^\wedge\}$. Then, extending χ_1 to a diffeomorphism $\chi : G^\wedge \to C^\wedge$ by homogeneity (i.e., $\chi(s,x) := s\chi_1(x)$), we set $H^s_{\text{cone}}(G^\wedge)_0 := \{\chi^* u \;:\; u \in H^s_{\text{cone}}(C^\wedge)_0\}$. We then define $H^s_{\text{cone}}(X^\wedge)$ to be the space of all $\sum \varphi_j v_j$ where $v_j \in H^s_{\text{cone}}(G_j^\wedge)_0$ for an open covering $\{G_1, \ldots, G_N\}$ of X by coordinate neighbourhoods and $\{\varphi_1, \ldots, \varphi_N\}$ a subordinate partition of unity.

We will be interested in spaces on two (or finitely many) cones $X_1^\triangle \cup_v X_2^\triangle$ for bases X_1, X_2 of different dimensions, where $\cup_v$ means disjoint union combined with an identification of vertices. These will be the model cones of configurations with edges. In our transmission algebras on (say, local and stretched) wedges $(X_1^\wedge \cup X_2^\wedge) \times \Omega$ for an open set $\Omega \subseteq \mathbb{R}^q$ we consider operators that share information between $X_1^\wedge \times \Omega$ and $X_2^\wedge \times \Omega$, encoded below by a kind of smoothing Mellin and Green operators. For the non-smoothing part, because of "pseudo-locality", we can ignore for a while the interaction of operators and discuss edge-degenerate symbols for a single wedge $X^\wedge \times \Omega$ with a smooth compact cone base X.

Let $U \subseteq \mathbb{R}^n$ be an open set and let $\tilde{S}^\mu(\overline{\mathbb{R}}_+ \times U \times \Omega \times \mathbb{R}^{1+n+q})$ denote the space of all symbols $p(r,x,y,\rho,\xi,\eta)$ that have the form

$$p(r,x,y,\rho,\xi,\eta) = \tilde{p}(r,x,y,r\rho,\xi,r\eta),$$

where

$$\tilde{p}(r,x,y,\tilde{\rho},\xi,\tilde{\eta}) \in S^\mu(\overline{\mathbb{R}}_+ \times U \times \Omega \times \mathbb{R}^{1+n+q}) := S^\mu(\mathbb{R}_+ \times U \times \Omega \times \mathbb{R}^{1+n+q})|_{\overline{\mathbb{R}}_+ \times U \times \Omega}$$

(here, we use common notation, i.e., Hörmander's symbol spaces $S^\mu(V \times \mathbb{R}^N)$, $V \subseteq \mathbb{R}^m$ open).

Given an atlas $\chi_j : G_j \to U_j$, $j = 1, \ldots, N$ on X and a system of local symbols $p_j \in \tilde{S}^\mu(\overline{\mathbb{R}}_+ \times U_j \times \Omega \times \mathbb{R}^{1+n+q})$ we can pass to (r,y,ρ,η)-dependent families of pseudo-differential operators on X by setting

$$p(r,y,\rho,\eta) := \sum_{j=1}^{N} \varphi_j \{(\chi_j^{-1})_* \mathrm{op}_x(p_j)(r,y,\rho,\eta)\} \psi_j. \tag{1.1.6}$$

In formula (1.1.6), $\{\varphi_j\}_{j=1,\ldots,N}$ is a partition of unity subordinate to the covering $\{G_j\}_{j=1,\ldots,N}$, and $\{\psi_j\}_{j=1,\ldots,N}$ are functions in $C_0^\infty(G_j)$ such that $\varphi_j \psi_j = \varphi_j$ for all j, and $(\chi_j^{-1})_*$ is the operator push-forward under χ_j^{-1}.

Let us now introduce parameter-dependent families on X that are holomorphic in $z \in \mathbb{C}$.

Definition 1.1 *Let $M_{\mathcal{O}}^\mu(X; \mathbb{R}^q)$, $\mu \in \mathbb{R}$, denote the space of all operator families $h(z,\eta) \in \mathcal{A}(\mathbb{C}_z, L^\mu(X; \mathbb{R}^q_\eta))$ such that*

$$h(z,\eta)|_{\Gamma_\beta \times \mathbb{R}^q} \in L^\mu(X; \Gamma_\beta \times \mathbb{R}^q)$$

for every $\beta \in \mathbb{R}$, uniformly in compact β-intervals. For $q = 0$ we simply write $M_{\mathcal{O}}^\mu(X)$.

The space $M_{\mathcal{O}}^\mu(X; \mathbb{R}^q)$ is Fréchet in a canonical way, and we then have spaces of the kind $C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M_{\mathcal{O}}^\mu(X; \mathbb{R}^q))$. We now recall a Mellin quantisation result that will be essential in our operator algebra below.

Theorem 1.2 *For every $p(r,y,\rho,\eta)$ of the form* (1.1.6) *there is an $\tilde{h}(r,y,z,\eta) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M_{\mathcal{O}}^\mu(X; \mathbb{R}^q))$ such that $h(r,y,z,\eta) := \tilde{h}(r,y,z,r\eta)$ satisfies the following relation:*

$$\mathrm{op}_r(p)(y,\eta) - \mathrm{op}_M^\beta(h)(y,\eta) \in C^\infty(\Omega, L^{-\infty}(X^\wedge; \mathbb{R}^q)) \tag{1.1.7}$$

for every $\beta \in \mathbb{R}$ (where operators are interpreted in the sense $C_0^\infty(X^\wedge) \to C^\infty(X^\wedge)$), and $\tilde{h}$ is unique mod $C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M_{\mathcal{O}}^{-\infty}(X; \mathbb{R}^q))$.

Remark 1.3 *If we define* $p_0(r,y,\rho,\eta)$ *in terms of symbols* $p_{j,0}(r,y,\rho,\eta) := \tilde{p}_j(0,y,r\rho,r\eta)$ *and set* $h_0(r,y,z,\eta) := \tilde{h}(0,y,z,r\eta)$*, relation* (1.1.7) *implies*

$$\mathrm{op}_r(p_0)(y,\eta) - \mathrm{op}_M^{\beta}(h_0)(y,\eta) \in C^\infty(\Omega, L^{-\infty}(X^\wedge;\mathbb{R}^q)).$$

The local edge-degenerate symbols $p_j(r,x,y,\varrho,\xi,\eta)$ give rise to homogeneous principal symbols in $(\varrho,\xi,\eta)\neq 0$ of order μ, denoted by $\sigma_\psi(\mathrm{op}_r(p))(r,x,y,\varrho,\xi,\eta)$. As usual, these are invariantly defined functions on $T^*(\mathbb{R}_+ \times X \times \Omega) \setminus 0$.

1.2 Asymptotics and Green operators

Our next objective is to study particular operator-valued symbols, acting in weighted Sobolev spaces on $X_1^\wedge \cup X_2^\wedge$ and mapping to spaces with asymptotics. Since symbols depend on variables and covariables (y,η), asymptotics also will be variable; in fact they will depend on y (not on η). For this reason we need a sufficiently flexible concept of asymptotics, not only discrete but also so-called continuous ones. To introduce basic notions, we first look at a single cone $X^\wedge$.

Let us define subspaces $\mathcal{K}_P^{s,\gamma}(X^\wedge)$ of $\mathcal{K}^{s,\gamma}(X^\wedge)$ with asymptotics of type P, associated with weight data $\boldsymbol{g} = (\gamma,\Theta)$ where $\Theta = (\vartheta,0]$, $-\infty \leq \vartheta < 0$, is a weight interval. By a discrete asymptotic type P we understand a sequence

$$P = \{(p_j, m_j, L_j)\}_{j=0,\ldots,N} \tag{1.2.1}$$

for an $N = N(P)$ that is finite for finite Θ, where

$$\pi_{\mathbb{C}} P := \{p_j\}_{j=0,\ldots,N} \subset \left\{ z \in \mathbb{C} \ : \ \frac{n+1}{2} - \gamma + \vartheta < \mathrm{Re}\, z < \frac{n+1}{2} - \gamma \right\},$$

$n = \dim X$, and (for the case $N(P) = \infty$) $\mathrm{Re}\, p_j \to -\infty$ as $j \to \infty$, further $m_j \in \mathbb{N}$, while $L_j \subset C^\infty(X)$ is a subspace of finite dimension. Set

$$\mathcal{K}_\Theta^{s,\gamma}(X^\wedge) = \varprojlim_{\varepsilon>0} \mathcal{K}^{s,\gamma-\vartheta-\varepsilon}(X^\wedge)$$

endowed with the Fréchet topology of the projective limit. If Θ is finite, we denote by $\mathcal{E}_P(X^\wedge)$ the vector space spanned by all functions $\omega(r)c_{jk}(x)r^{-p_j}\log^k r$ for all $p_j \in \pi_{\mathbb{C}} P$, $0 \leq k \leq m_j$, $c_{jk} \in L_j$, $j = 0,\ldots,N$, where ω is a fixed cut-off function. Observe that then $\mathcal{E}_P(X^\wedge) \subset \mathcal{K}^{\infty,\gamma}(X^\wedge)$, and we have $\mathcal{E}_P(X^\wedge) \cap \mathcal{K}_\Theta^{s,\gamma}(X^\wedge) = \{0\}$. We then define $\mathcal{K}_P^{s,\gamma}(X^\wedge) := \mathcal{K}_\Theta^{s,\gamma}(X^\wedge) + \mathcal{E}_P(X^\wedge)$ in the Fréchet topology of the direct sum. To define $\mathcal{K}_P^{s,\gamma}(X^\wedge)$ for $\vartheta = -\infty$ we choose an arbitrary sequence $\{\vartheta_k\}_{k\in\mathbb{N}}$, such that $\vartheta_{k+1} < \vartheta_k < 0$ and $\lim\limits_{k\to\infty} \vartheta_k = -\infty$, set

$$P_k = \left\{ (p,m,L) \in P \ : \ \frac{n+1}{2} - \gamma + \vartheta_k < \mathrm{Re}\, p < \frac{n+1}{2} - \gamma \right\}$$

and define

$$\mathcal{K}_{P_k}^{s,\gamma}(X^\wedge) := \mathcal{K}_{\Theta_k}^{s,\gamma}(X^\wedge) + \mathcal{E}_{P_k}(X^\wedge) \tag{1.2.2}$$

where $\Theta_k := (\vartheta_k, 0]$. This is a direct sum for every k, and the corresponding Fréchet topology in (1.2.2) is independent of the choice of ω. We have $\mathcal{K}^{s,\gamma}_{P_{k+1}}(X^\wedge) \hookrightarrow \mathcal{K}^{s,\gamma}_{P_k}(X^\wedge)$ for all k, and we then set

$$\mathcal{K}^{s,\gamma}_P(X^\wedge) = \varprojlim_{k\in\mathbb{N}} \mathcal{K}^{s,\gamma}_{P_k}(X^\wedge)$$

in the (Fréchet) topology of the projective limit. The elements of $\mathcal{E}_{P_k}(X^\wedge)$ are called singular functions of the discrete asymptotics for the cone.

To pass to continuous asymptotics we first reinterpret the discrete ones in the following form. Let us first assume that Θ is finite; then $K := \pi_{\mathbb{C}}P$ is a finite set. Choose a (say, smooth) curve $\mathcal{C}$ in the strip $\frac{n+1}{2} - \gamma + \vartheta < \mathrm{Re}\, z < \frac{n+1}{2} - \gamma$ surrounding the set K counter-clockwise. Fix a cut-off function, and set $f(z) := \mathcal{M}_{\gamma-\frac{n}{2}}(\omega u)(z)$ for $u \in \mathcal{K}^{s,\gamma}_P(X^\wedge)$, which is a meromorphic ($C^\infty(X)$-valued) function with poles at p_j of multiplicities m_j+1 and Laurent coefficients $k!(-1)^k c_{jk}(x)$ at $(z-p_j)^{-(k+1)}$, where $c_{jk} \in L_j$, $0 \le k \le m_j$, $j = 0, \ldots, N(P)$. Then we have

$$u(r) - \frac{1}{2\pi i}\int_{\mathcal{C}} f(z) r^{-z} dz \in \mathcal{K}^{s,\gamma}_{\Theta}(X^\wedge). \tag{1.2.3}$$

This relation has the following more general background. Let $f(z)$ be an arbitrary meromorphic $C^\infty(X)$-valued function with the indicated poles, multiplicities and Laurent coefficients. Then with f we can associate an analytic functional ζ_f in $\mathbb{C}$ carried by K, namely

$$\langle \zeta_f, h\rangle := \frac{1}{2\pi i}\int_{\mathcal{C}} f(z) h(z) dz, \quad h \in \mathcal{A}(\mathbb{C}). \tag{1.2.4}$$

In the present notation we just have

$$\langle \zeta_f, h\rangle = \sum_j \sum_{k=0}^{m_j} (-1)^k\, c_{jk}(x)\, \frac{d^k}{dz^k} h(z)\Big|_{z=p_j}. \tag{1.2.5}$$

In other words, if we denote for a moment by $\mathcal{F}(K)$ the set of all meromorphic functions f associated with P in the described way, we have

$$\mathcal{E}_P(X^\wedge) = \{\langle \zeta_f, r^{-z}\rangle\, \omega(r) \;:\; f \in \mathcal{F}(K)\}.$$

Thus, we can produce all singular functions in terms of a certain set of $C^\infty(X)$-valued analytic functionals carried by $K = \pi_{\mathbb{C}}P$. The idea of continuous asymptotics is now to replace K by an arbitrary compact set in $\mathbb{C}$ and to admit arbitrary $\zeta \in \mathcal{A}'(K, C^\infty(X))$. Here, $\mathcal{A}'(K)$ denotes the space of all (scalar) analytic functionals carried by K in its (nuclear) Fréchet topology, and $\mathcal{A}'(K, E)$ ($= \mathcal{A}'(K)\hat{\otimes}_\pi E$) for any Fréchet space E is the corresponding vector-valued variant.

To define carrier sets of continuous asymptotics we define $\mathcal{V}$ to be the system of all closed subsets $V \subset \mathbb{C}$ such that $V \cap \{z \in \mathbb{C} \;:\; c \le \mathrm{Re}\, z \le c'\}$ is compact for every $c \leq c'$ and $z_0, z_1 \in V$, $\operatorname{Re} z_0 = \operatorname{Re} z_1$ implies $(1-\lambda)z_0 + \lambda z_1 \in V$ for all $0 \leq \lambda \leq 1$. Given a $V \in \mathcal{V}$ with $V \subset \left\{z \in \mathbb{C} \ : \ \operatorname{Re} z < \frac{n+1}{2} - \gamma\right\}$ and a finite weight interval $\Theta = (\vartheta, 0]$, we consider the compact set $K := V \cap \left\{z \in \mathbb{C} \ : \ \operatorname{Re} z \geq \frac{n+1}{2} - \gamma + R\right\}$ for any $R < \vartheta$. Observe that we then have $u_\zeta(r,x) := \langle \zeta, r^{-z} \rangle \, \omega(r) \in \mathcal{K}^{\infty,\gamma}(X^\wedge)$ for every $\zeta \in \mathcal{A}'(K, C^\infty(X))$. Moreover, $K \subset \left\{z \in \mathbb{C} \ : \ \operatorname{Re} z \leq \frac{n+1}{2} - \gamma + \vartheta\right\}$ implies $u_\zeta(r,x) \in \mathcal{K}^{\infty,\gamma}_\Theta(X^\wedge)$. Let us form the space

$$\mathcal{E}_K(X^\wedge) := \{\langle \zeta, r^{-z} \rangle \, \omega \ : \ \zeta \in \mathcal{A}'(K, C^\infty(X))\} \subset \mathcal{K}^{\infty,\gamma}(X^\wedge),$$

and write $u \sim v$ for $u, v \in \mathcal{E}_K(X^\wedge)$ if and only if $u - v \in \mathcal{K}^{\infty,\gamma}_\Theta(X^\wedge)$. The quotient space $\mathcal{E}_K(X^\wedge)/\sim$ is called a continuous asymptotic type, associated with the weight data $\boldsymbol{g} = (\gamma, \Theta)$ (clearly, this does not depend on the choice of R). Let $\operatorname{As}(X, \boldsymbol{g})$ denote the set of all such continuous asymptotic types P. We then define

$$\mathcal{K}^{s,\gamma}_P(X^\wedge) := \mathcal{K}^{s,\gamma}_\Theta(X^\wedge) + \mathcal{E}_K(X^\wedge),$$

endowed with the Fréchet topology of the non-direct sum.

Let us now extend the definition to the infinite weight strip $\Theta = (-\infty, 0]$ and arbitrary $V \in \mathcal{V}$, $V \subset \left\{z \in \mathbb{C} \ : \ \operatorname{Re} z < \frac{n+1}{2} - \gamma\right\}$. In this case we choose a sequence $\{\vartheta_k\}_{k\in\mathbb{N}}$, where $\vartheta_{k+1} < \vartheta_k$ for all k, $\vartheta_k \to -\infty$ for $k \to \infty$, and form corresponding sets $K_k := V \cap \left\{z \in \mathbb{C} \ : \ \operatorname{Re} z \geq \frac{n+1}{2} - \gamma + R_k\right\}$ for arbitrary $R_k < \vartheta_k$. For the associated continuous asymptotic types $P_k \in \operatorname{As}(X, \boldsymbol{g}_k)$, $\boldsymbol{g}_k := (\gamma, \Theta_k)$, we then have continuous embeddings $\mathcal{K}^{s,\gamma}_{P_{k+1}}(X^\wedge) \hookrightarrow \mathcal{K}^{s,\gamma}_{P_k}(X^\wedge)$ for all k, and we set

$$\mathcal{K}^{s,\gamma}_P(X^\wedge) := \varprojlim_{k\in\mathbb{N}} \mathcal{K}^{s,\gamma}_{P_k}(X^\wedge) \tag{1.2.6}$$

in the (Fréchet) topology of the projective limit. This defines the symbol P, called a continuous asymptotic type associated with $\boldsymbol{g} = (\gamma, \Theta)$ in the infinite case, and the set of all such P is again denoted $\operatorname{As}(X, \boldsymbol{g})$.

If P is an asymptotic type connected with weight data (γ, Θ) (discrete or continuous), we set

$$\mathcal{S}^\gamma_P(X^\wedge) = \{\omega u + (1-\omega)v \ : \ u \in \mathcal{K}^{\infty,\gamma}_P(X^\wedge), v \in \mathcal{S}(\overline{\mathbb{R}}_+, C^\infty(X))\} \tag{1.2.7}$$

for any cut-off function ω. Clearly, the space (1.2.7) is independent of the specific ω. It is a Fréchet space in a natural way.

Remark 1.4 *The spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$ are equipped with a strongly continuous group of isomorphisms $\{\kappa^n_\lambda\}_{\lambda\in\mathbb{R}_+}$, defined by $(\kappa^n_\lambda u)(r,x) = \lambda^{\frac{n+1}{2}} u(\lambda r, x)$ where $n = \dim X$. In addition, the spaces $\mathcal{K}^{s,\gamma}_P(X^\wedge)$ as well as $\mathcal{S}^\gamma_P(X^\wedge)$ (both for discrete and continuous asymptotic types P) can be written as projective limits of Hilbert spaces $\{H_j\}_{j\in\mathbb{N}}$ with continuous embeddings $H_{j+1} \hookrightarrow H_j$ for all j and*

$H_0 = \mathcal{K}^{s,\gamma}(X^\wedge)$, where $\{\kappa_\lambda^n\}_{\lambda\in\mathbb{R}_+}$ restricts to a strongly continuous group of isomorphisms on every H_j.

Definition 1.5 *Let X and Y be closed compact C^∞ manifolds, $n = \dim X$, $m = \dim Y$, and choose reals $\gamma, \delta \in \mathbb{R}$ and a weight interval $\Theta = (\vartheta, 0]$, $-\infty \le \vartheta < 0$. Then $C_G(X^\wedge, Y^\wedge, (\gamma, \delta, \Theta))$ is defined to be the space of all continuous maps*

$$G \,:\, \mathcal{K}^{s+\frac{n}{2},\gamma+\frac{n}{2}}(X^\wedge) \to \mathcal{K}^{\infty,\delta+\frac{m}{2}}(Y^\wedge),$$

$s \in \mathbb{R}$, that induce continuous operators

$$G \,:\, \mathcal{K}^{s+\frac{n}{2},\gamma+\frac{n}{2}}(X^\wedge) \to \mathcal{S}_P^{\delta+\frac{m}{2}}(Y^\wedge)$$

and

$$G^* \,:\, \mathcal{K}^{s-\frac{m}{2},-\delta+\frac{m}{2}}(Y^\wedge) \to \mathcal{S}_Q^{-\gamma+\frac{n}{2}}(X^\wedge)$$

for all $s \in \mathbb{R}$, with asymptotic types $P \in \mathrm{As}\Big(Y, \Big(\delta+\dfrac{m}{2}, \Theta\Big)\Big)$ and $Q \in \mathrm{As}\Big(X, \Big(-\gamma + \dfrac{n}{2}, \Theta\Big)\Big)$. Here, G^* is the formal adjoint of G in the sense

$$(Au, v)_{\mathcal{K}^{0,\frac{m}{2}}(Y^\wedge)} = (u, A^* v)_{\mathcal{K}^{0,\frac{n}{2}}(X^\wedge)}$$

for all $u \in C_0^\infty(X^\wedge)$, $v \in C_0^\infty(Y^\wedge)$ via the scalar products of $\mathcal{K}^{0,\frac{n}{2}}(X^\wedge)$ and $\mathcal{K}^{0,\frac{m}{2}}(Y^\wedge)$, respectively. The elements of $C_G(X^\wedge, Y^\wedge, (\gamma, \delta, \Theta))$ are called Green operators of the transmission cone algebra with continuous asymptotics.

Remark 1.6 *An analogous definition makes sense for discrete asymptotic types P, Q. Moreover, there is a straightforward extension of Definition 1.5 to operators*

$$\mathcal{K}^{s+\frac{n}{2},\gamma+\frac{n}{2}}(X^\wedge, E) \to \mathcal{K}^{\infty,\delta+\frac{m}{2}}(Y^\wedge, F)$$

acting between distributional sections of vector bundles E on $X^\wedge$ and F on $Y^\wedge$, endowed with suitable Hermitian metrics (with an obvious generalisation of asymptotic types), cf. Section 2.1 below.

1.3 Mellin operators with asymptotics

We now turn to a specific class of pseudo-differential operators on $X^\wedge = \mathbb{R}_+ \times X \ni (r, x)$ for a compact, closed C^∞ manifold X, based on the Mellin transform in $r \in \mathbb{R}_+$, with operator-valued symbols that reflect asymptotics. A sequence

$$R := \{(p_j, m_j, L_j)\}_{j\in\mathbb{Z}} \tag{1.3.1}$$

is called a discrete asymptotic type for Mellin symbols, if for $\pi_{\mathbb{C}} R := \{p_j\}_{j\in\mathbb{Z}} \subset \mathbb{C}$ the set $\pi_{\mathbb{C}} R \cap \{z : c \le \mathrm{Re}\, z \le c'\}$ is finite for every $c \le c'$, moreover, $m_j \in \mathbb{N}$, and $L_j \subset L^{-\infty}(X)$ are finite-dimensional subspaces of operators of finite rank. We also admit finite sequences (1.3.1), where a triple (p, m, L) may be ignored as soon as $L = \{0\}$.

$M_R^{-\infty}(X)$ for R given as (1.3.1) is defined to be the set of all $f \in \mathcal{A}(\mathbb{C} \setminus \pi_{\mathbb{C}} R, L^{-\infty}(X))$ such that f is meromorphic with poles at p_j of multiplicity $m_j + 1$ and Laurent coefficients at $(z - p_j)^{-(k+1)}$ in L_j for $0 \leq k \leq m_j$, where

$$\chi(z)\, f(z)|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_\beta, L^{-\infty}(X))$$

for every real β, uniformly in compact β-intervals; here χ denotes any $\pi_{\mathbb{C}} R$-excision function, that is $\chi \in C^\infty(\mathbb{C})$ and $\chi(z) = 0$ in a neighbourhood of $\pi_{\mathbb{C}} R$ and $\chi(z) = 1$ for $\operatorname{dist}(z, \pi_{\mathbb{C}} R) > \varepsilon$ for some $\varepsilon > 0$.

Let us now pass to an analogue of continuous asymptotics. First, fix $c < c'$, and let $M_{\mathcal{O}}^{-\infty}(X)_{(c,c')}$ defined to be the set of all $h \in \mathcal{A}(\{c < \operatorname{Re} z < c'\}, L^{-\infty}(X))$ such that $h|_{\Gamma_\beta} \in L^{-\infty}(X; \Gamma_\beta)$ for every real β, uniformly in compact β-intervals of (c, c'). Now, let $V \in \mathcal{V}$ and set $V_{(c,c')} := V \cap \{c \leq \operatorname{Re} z \leq c'\}$, which is a compact set. There is then a map

$$\mathcal{A}'(V_{(c,c')}, L^{-\infty}(X)) \to \mathcal{A}(\mathbb{C} \setminus V_{(c,c')}, L^{-\infty}(X)),$$

$\zeta \mapsto f_\zeta$, by setting

$$f_\zeta(z) := \mathcal{M}_{\gamma, r \to z}(\langle \zeta_w, \omega(r) r^{-w} \rangle)$$

with the weighted Mellin transform $\mathcal{M}_\gamma$ for any $\gamma < \dfrac{1}{2} - c'$. The space $\mathcal{F}_{V_{(c,c')}}$ of all functions f_ζ that belong to $\mathcal{A}(\mathbb{C} \setminus V_{(c,c')}, L^{-\infty}(X))$ is isomorphic to $\mathcal{A}'(V_{(c,c')}, L^{-\infty}(X))$, and therefore has a canonical Fréchet topology. We then define $M_V^{-\infty}(X)_{(c,c')}$ as the space of all elements $h(z) + f_\zeta(z)|_{\{c < \operatorname{Re} z < c'\}}$ for $h \in M_{\mathcal{O}}^{-\infty}(X)_{(c,c')}$, $\zeta \in \mathcal{A}'(V_{(c,c')}, L^{-\infty}(X))$, endowed with the topology of the non-direct sum of $M_{\mathcal{O}}^{-\infty}(X)_{(c,c')} + \mathcal{F}_{V_{(c,c')}}$ taken in the space $\mathcal{A}(\{c < \operatorname{Re} z < c'\} \setminus V_{(c,c')}, L^{-\infty}(X))$ (clearly, the space $\mathcal{F}_{V_{(c,c')}}$ depends on the choice of ω but the non-direct sum is independent of the specific cut-off function). Notice that for any $\tilde{c} \leq c$, $c' \leq \tilde{c}'$ we have a continuous embedding $M_V^{-\infty}(X)_{(\tilde{c},\tilde{c}')} \hookrightarrow M_V^{-\infty}(X)_{(c,c')}$. We then define the space

$$M_V^{-\infty}(X) := \varprojlim_{N \in \mathbb{N}} M_V^{-\infty}(X)_{(-N,N)}$$

as a projective limit. The elements of $M_V^{-\infty}(X)$ are interpreted as smoothing Mellin symbols with continuous asymptotics of type V. Setting again $n = \dim X$, if $\omega(r)$ and $\tilde{\omega}(r)$ are cut-off functions, with every $f \in M_V^{-\infty}(X)$ we can associate a continuous operator

$$\omega \operatorname{op}_M^\gamma(f)\, \tilde{\omega} \;:\; \mathcal{K}^{s,\gamma+\frac{n}{2}}(X^\wedge) \to \mathcal{K}^{\infty,\gamma+\frac{n}{2}}(X^\wedge), \tag{1.3.2}$$

provided $V \cap \Gamma_{\frac{1}{2}-\gamma} = \emptyset$. More precisely, (1.3.2) induces continuous operators

$$\omega \operatorname{op}_M^\gamma(f)\, \tilde{\omega} \;:\; \mathcal{K}_P^{s,\gamma+\frac{n}{2}}(X^\wedge) \to \mathcal{K}_Q^{\infty,\gamma+\frac{n}{2}}(X^\wedge)$$

for every $P \in \operatorname{As}\left(X, \left(\gamma + \dfrac{n}{2}, \Theta\right)\right)$ and some resulting $Q \in \operatorname{As}\left(X, \left(\gamma + \dfrac{n}{2}, \Theta\right)\right)$, that depends on P and f, not on $s \in \mathbb{R}$. A similar result is true for discrete asymptotic types R for Mellin symbols instead of V and discrete asymptotic types $\tilde{P}, \tilde{Q}$ in place of P, Q, cf. the notation in Section 1.2. Recall that some Riemannian metric on X is kept fixed. There is then an identification between $L^{-\infty}(X)$ and the space of integral operators on X with kernels in $C^\infty(X \times X)$. Let us now generalise the construction of spaces of smoothing Mellin symbols to the case $Y \times X$. To simplify notation, we identify the space of all operators $\cap_{s\in\mathbb{R}} \mathcal{L}(H^s(X), H^\infty(Y))$ with the space $C^\infty(Y \times X)$ via integral kernels. Let $\mathbf{As}^\bullet(X,Y)$ denote the set of all sequences (1.3.1) with $(p_j, m_j) \in \mathbb{C} \times \mathbb{N}$ as before, whileL_j is a finite-dimensional subspace of $C^\infty(Y \times X)$.

Let $M_R^{-\infty}(X,Y)$ for $R \in \mathbf{As}^\bullet(X,Y)$ denote the set of all $f(z) \in \mathcal{A}(\mathbb{C} \setminus \pi_{\mathbb{C}} R, C^\infty(Y \times X))$ such that $f(z)$ is meromorphic with poles at the points p_j of multiplicities $m_j + 1$ and Laurent coefficients at $(z - p_j)^{k+1}$, $0 \le k \le m_j$, in L_j, where $\chi(z) f(z) \in \mathcal{S}(\Gamma_\beta, C^\infty(Y \times X))$ for every $\beta \in \mathbb{R}$, uniformly in compact β-intervals, for every $\pi_{\mathbb{C}} R$-excision function χ.

In a similar manner, for every $V \in \mathcal{V}$ we can define a Fréchet space $M_V^{-\infty}(X,Y)$ by replacing $L^{-\infty}(X)$ in the above construction by $C^\infty(Y \times X)$. The extension of this definition that allows us to introduce the space $M_V^{-\infty}(X,Y;E,F)$ with closed compact manifolds X, Y and vector bundles $E \in \mathrm{Vect}(X)$, $F \in \mathrm{Vect}(Y)$, is immediate, cf. Remark 1.6.

Definition 1.7 *Let X,Y be smooth closed, compact manifolds and $E \in \mathrm{Vect}(X)$, $F \in \mathrm{Vect}(Y)$ vector bundles, further (γ, δ, Θ), $\gamma, \delta \in \mathbb{R}$, $\Theta = (-(k+1), 0]$, $k \in \mathbb{N}$ weight data. Then, the space $C_{M+G}(X^\wedge, Y^\wedge; (\gamma, \delta, \Theta); E, F)$ is defined to be the set of all the operators $A = M + G$ where G is a Green operator in $C_G(X^\wedge, Y^\wedge, (\gamma, \delta, \Theta); E, F)$ (as introduced in Definition 1.5 and Remark 1.6) while*

$$M = r^{\delta-\gamma} \omega \sum_{j=0}^{k} r^j \{\mathrm{op}_M^{\beta}(f_j) + \mathrm{op}_M^{\tilde{\beta}}(\tilde{f}_j)\}\, \tilde{\omega} \;:\; \mathcal{K}^{s,\gamma+\frac{n}{2}}(X^\wedge, E) \to \mathcal{K}^{\infty,\delta+\frac{m}{2}}(Y^\wedge, F) \tag{1.3.3}$$

with $f_j \in M_{W_j}^{-\infty}(X,Y;E,F)$, $\tilde{f}_j \in M_{\widetilde{W}_j}^{-\infty}(X,Y;E,F)$ and reals $\beta = \beta(j), \tilde{\beta} = \tilde{\beta}(j)$ such that $W_j \cap \Gamma_{\frac{1}{2}-\beta} = \emptyset$, $\widetilde{W}_j \cap \Gamma_{\frac{1}{2}-\tilde{\beta}} = \emptyset$, $j + \beta \ge \gamma \ge \beta$, $j + \tilde{\beta} \ge \gamma \ge \tilde{\beta}$, $j = 0, \ldots, k$. The space $C_{M+G}(X^\wedge, Y^\wedge; (\gamma, \delta, \Theta); E, F)$ for the infinite weight interval $\Theta = (-\infty, 0]$ is defined by taking intersections over the corresponding spaces for $\Theta_k = (-k, 0]$, $k \in \mathbb{N}$.

Note that the terms involving $\tilde{f}_j$, $j = 1, \ldots, k$ in (1.3.3) can be suppressed in the case of discrete asymptotics. Moreover, in the case $\Theta = (-(k+1), 0]$, a term like those in the sum (1.3.3) with $j \ge k+1$ is in fact a Green operator. Let us set

$$\sigma_{\mathrm{M}}(M)(z) := f_0(z) + \tilde{f}_0(z) \tag{1.3.4}$$

regarded as a z-dependent family of continuous operators

$$\sigma_{\mathrm{M}}(M) \;:\; H^s(X, E) \to H^s(Y, F),$$

$s \in \mathbb{R}$, called the principal conormal symbol of the operator M. More generally, similarly to the standard cone theory, we can introduce lower order conormal symbols

$$\sigma_{\mathrm{M}}^{(j)}(M)(z) := f_j(z) + \tilde{f}_j(z), \qquad j = 0, \ldots, k,$$

$\sigma_{\mathrm{M}}^{(0)} =: \sigma_{\mathrm{M}}$. It can easily be verified that the functions f_j, $\tilde{f}_j$ are well-defined by the action on special argument functions of the form $r^{-w}\omega(r)$, cf. [14], Section 1.3.1. Then $\sigma_{\mathrm{M}}^{(j)}(M) = \sigma_{\mathrm{M}}^{(j)}(\tilde{M})$, $j = 0, \ldots, k$ for two elements $M, \tilde{M} \in C_{M+G}(X^\wedge, Y^\wedge, (\gamma, \delta, \Theta); E, F)$ entails $M = \tilde{M} \bmod C_G(X^\wedge, Y^\wedge, (\gamma, \delta, \Theta); E, F)$.

Observe that M is compact as an operator $\mathcal{K}^{s,\gamma+\frac{n}{2}}(X^\wedge) \to \mathcal{K}^{s,\delta+\frac{m}{2}}(Y^\wedge)$ if and only if $\sigma_{\mathrm{M}}(M)$ vanishes.

Remark 1.8 *The choice of the weights β, $\tilde{\beta}$ (under the mentioned conditions) as well as of cut-off functions ω, $\tilde{\omega}$ is arbitrary. If $\tilde{M}$ is an expression of analogous structure as M in* (1.3.3) *with the same Mellin symbols but other weights or cut-off functions, then we have $M - \tilde{M} \in C_G(X^\wedge, Y^\wedge, (\gamma, \delta, (-\infty, 0]); E, F)$. In view of this, except for $j = 0$ (where necessarily $\beta(0) = \tilde{\beta}(0) = \gamma$), we can choose them in a "normalised way", setting $\beta(j) = \gamma - \frac{1}{3}$, $\tilde{\beta}(j) = \gamma - \frac{2}{3}$ for $j = 1, \ldots, k$.*

Proposition 1.9 *Let $\omega, \tilde{\omega}$ be cut-off functions, and let $f(z) \in M_V^{-\infty}(X, Y)$, $V \in \mathcal{V}$, where $V \cap \Gamma_{\frac{1}{2}-\gamma} = \emptyset$ for some $\gamma \in \mathbb{R}$. Then*

$$\omega \operatorname{op}_M^\gamma(f)\, \tilde{\omega} \ : \ \mathcal{K}^{s,\gamma+\frac{n}{2}}(X^\wedge) \to \mathcal{K}^{\infty,\gamma+\frac{m}{2}}(Y^\wedge) \tag{1.3.5}$$

is a continuous operator for all $s \in \mathbb{R}$, and (1.3.5) *induces continuous operators*

$$\omega \operatorname{op}_M^\gamma(f)\, \tilde{\omega} \ : \ \mathcal{K}_P^{s,\gamma+\frac{n}{2}}(X^\wedge) \to \mathcal{K}_Q^{\infty,\gamma+\frac{m}{2}}(Y^\wedge)$$

for every $P \in \mathrm{As}\Big(X, \Big(\gamma + \dfrac{n}{2}, \Theta\Big)\Big)$ with some $Q \in \mathrm{As}\Big(Y, \Big(\gamma + \dfrac{m}{2}, \Theta\Big)\Big)$, $s \in \mathbb{R}$. In addition, the formal adjoint of (1.3.5) *in the sense $(Au, v)_{\mathcal{K}^{0,\frac{m}{2}}(Y^\wedge)} = (u, A^* v)_{\mathcal{K}^{0,\frac{n}{2}}(X^\wedge)}$ for all $u \in C_0^\infty(X^\wedge)$, $v \in C_0^\infty(Y^\wedge)$ has the form*

$$\tilde{\omega} \operatorname{op}_M^{-\gamma}(f^{(*)})\, \omega$$

where $f^{(*)}(z) := f^*(1 - \overline{z})$ with subscript * denoting the pointwise formal adjoint in the sense $(f\,\varphi, \psi)_{L^2(Y)} = (\varphi, f^*\psi)_{L^2(X)}$ for all $\varphi \in C^\infty(X)$, $\psi \in C^\infty(Y)$.

Proof. The continuity follows adapting the proof of the analogous result for the standard cone algebra to the present situation. To prove the formula for the adjoint, note that

$$\begin{aligned}
(\omega \operatorname{op}_M^\gamma(f)\, \tilde{\omega})^* &= (\omega\, r^\gamma \operatorname{op}_M(T^{-\gamma} f) r^{-\gamma}\, \tilde{\omega})^* = \tilde{\omega}\, r^{-\gamma} (\operatorname{op}_M(T^{-\gamma} f))^* r^\gamma\, \omega \\
&= \tilde{\omega}\, r^{-\gamma} \operatorname{op}_M((T^{-\gamma} f)^*) r^\gamma\, \omega = \tilde{\omega}\, r^{-\gamma} \operatorname{op}_M((f(z-\gamma))^*) r^\gamma\, \omega.
\end{aligned}$$

Now, taking into account that, setting for a moment $g(z) := f(z - \gamma)$,

$$g^{(*)}(z) = g^*(1 - \overline{z}) = f^*(1 - \overline{z} - \gamma) = f^*(1 - \overline{(z + \gamma)}) = T^\gamma f^{(*)}(z),$$

we can conclude

$$(\omega\,\mathrm{op}_M^\gamma(f)\,\tilde{\omega})^* = \tilde{\omega}\,r^{-\gamma}\mathrm{op}_M(T^\gamma f^{(*)})r^\gamma\,\omega = \tilde{\omega}\,\mathrm{op}_M^{-\gamma}(f^{(*)})\,\omega$$

as claimed. $\square$

1.4 Transmission algebras

Let us now introduce algebras of block matrix operators on $X_1^\wedge \cup X_2^\wedge$, that we call transmission algebras. Let us fix weights $\gamma, \delta \in \mathbb{R}$, a weight interval $\Theta = (-(k+1), 0]$, $k \in \mathbb{N}$, vector bundles $E_j, F_j \in \mathrm{Vect}\,(X_j^\wedge)$, $j = 1, 2$. Set $\boldsymbol{v} := (E_1, F_1; E_2, F_2)$ and $\boldsymbol{g} = (\gamma, \delta; \Theta)$. We will also use the following abbreviations: $\boldsymbol{X}^\wedge := (X_1^\wedge, X_2^\wedge)$, $\boldsymbol{n} := (n_1, n_2)$, $\boldsymbol{E} := (E_1, E_2)$, $\boldsymbol{F} := (F_1, F_2)$. Similar notation will be used for pairs of asymptotic types, $\boldsymbol{P} := (P_1, P_2)$, $\boldsymbol{Q} := (Q_1, Q_2)$, and for direct sums of spaces,

$$C_0^\infty(\boldsymbol{X}^\wedge, \boldsymbol{E}) := \begin{array}{c} C_0^\infty(X_1^\wedge, E_1) \\ \oplus \\ C_0^\infty(X_2^\wedge, E_2) \end{array}, H^{s+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^\wedge, \boldsymbol{E}) := \begin{array}{c} H^{s+\frac{n_1}{2}}(X_1^\wedge, E_1) \\ \oplus \\ H^{s+\frac{n_2}{2}}(X_2^\wedge, E_2) \end{array},$$

$$\mathcal{K}^{s+\frac{\boldsymbol{n}}{2},\gamma+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^\wedge, \boldsymbol{E}) := \begin{array}{c} \mathcal{K}^{s+\frac{n_1}{2},\gamma+\frac{n_1}{2}}(X_1^\wedge, E_1) \\ \oplus \\ \mathcal{K}^{s+\frac{n_2}{2},\gamma+\frac{n_2}{2}}(X_2^\wedge, E_2) \end{array}, \mathcal{S}_{\boldsymbol{P}}^{\gamma+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^\wedge, \boldsymbol{E}) := \begin{array}{c} \mathcal{S}_{P_1}^{\gamma+\frac{n_1}{2}}(X_1^\wedge, E_1) \\ \oplus \\ \mathcal{S}_{P_2}^{\gamma+\frac{n_2}{2}}(X_2^\wedge, E_2) \end{array},$$

as well as for $\mathcal{K}_{\boldsymbol{P}}^{s+\frac{\boldsymbol{n}}{2},\gamma+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^\wedge, \boldsymbol{E})$. We will use the subscript "$(\boldsymbol{P})$" when a formula or result holds in the cases with and without asymptotics.

Finally, we fix arbitrary cut-off functions ω, $\tilde{\omega}$, $\tilde{\tilde{\omega}}$ (their specific choice is unimportant, but, for convenience, we assume $\omega\,\tilde{\omega} = \omega$, $\omega\,\tilde{\tilde{\omega}} = \tilde{\tilde{\omega}}$).

Definition 1.10 *The space $C^\mu(\boldsymbol{X}^\wedge, \boldsymbol{g}; \boldsymbol{v})$ is defined to be the set of all operators of the form*

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} + M + G, \tag{1.4.1}$$

where the ingredients are as follows:

(i) *$A_j = r^{\delta-\gamma}\omega\,\mathrm{op}_M^\gamma(h_j)\,\tilde{\omega} + (1-\omega)A_{j\psi}(1-\tilde{\tilde{\omega}})$, for arbitrary $h_j(r,z) \in C^\infty(\overline{\mathbb{R}}_+, M_{\mathcal{O}}^\mu(X_j; E_j, F_j))$ and $A_{j\psi} \in L^{\mu;0}(X_j^\wedge; E_j, F_j)$, $j = 1, 2$;*

(ii) *$M + G = (M_{ij} + G_{ij})_{i,j=1,2}$ is a block matrix of operators belonging to $C_{M+G}(X_j^\wedge, X_i^\wedge, (\gamma, \delta, \Theta); E_j, F_i)$.*

We write $C_{M+G}(\boldsymbol{X}, \boldsymbol{g}; \boldsymbol{v})$ or $C_G(\boldsymbol{X}, \boldsymbol{g}; \boldsymbol{v})$ for the subspaces of operators (1.4.1) where A_1 and A_2 vanish or, respectively, A_1, A_2 and M vanish. Setting for a moment $\boldsymbol{g}_k := (\gamma, \delta, (-(k+1), 0])$, we can pass to a space $C^\mu(\boldsymbol{X}^\wedge, \boldsymbol{g}; \boldsymbol{v})$ for $\boldsymbol{g} = (\gamma, \delta, (-\infty, 0])$ by

$$C^\mu(\boldsymbol{X}^\wedge, \boldsymbol{g}; \boldsymbol{v}) = \bigcap_{k\in\mathbb{N}} C^\mu(\boldsymbol{X}^\wedge, \boldsymbol{g}_k; \boldsymbol{v}).$$

All essential elements of our calculus for a finite weight interval $(-(k+1), 0]$ remain true also for $\Theta = (-\infty, 0]$; for this reason we mainly discuss the finite case.

Theorem 1.11 *An operator $A \in C^{\mu}(\boldsymbol{X}^{\wedge}, \boldsymbol{g}; \boldsymbol{v})$ induces continuous operators*

$$A \, : \, \mathcal{K}_{(\boldsymbol{P})}^{s+\frac{\boldsymbol{n}}{2}, \gamma+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^{\wedge}, \boldsymbol{E}) \rightarrow \mathcal{K}_{(\boldsymbol{Q})}^{s-\mu+\frac{\boldsymbol{n}}{2}, \delta-\mu+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^{\wedge}, \boldsymbol{F})$$

for every $s \in \mathbb{R}$ and every pair of asymptotic types $\boldsymbol{P}$ with some resulting $\boldsymbol{Q}$, dependent on $\boldsymbol{P}$ and on A.

The components of asymptotic types $\boldsymbol{P} := (P_1, P_2)$ and $\boldsymbol{Q} := (Q_1, Q_2)$ are assumed to be associated to weight data corresponding to the weights in the spaces and the chosen weight interval $\Theta = (-(k+1), 0]$, and we admit discrete as well as continuous asymptotic types.

Proof. The only terms that have to be explicitly considered are those from M, in particular the mixed terms M_{21} and M_{12}, that have been treated in Proposition 1.9. The other ingredients of A belong to the standard cone algebras on $X_j^{\wedge}$, $j = 1, 2$, except for G_{21} and G_{12}, where the mapping properties follow from Definition 1.5. $\square$

We now pass to the symbolic structure of operators $A \in C^{\mu}(\boldsymbol{X}^{\wedge}, \boldsymbol{g}; \boldsymbol{v})$. First, for the operators A_j in (1.4.1) we have homogeneous principal symbols of order μ that are (up to weight factors) Fuchs-degenerate near $r = 0$, namely

$$\sigma_{\psi}(A_j) \, : \, \pi_j^* E_j \rightarrow \pi_j^* F_j, \tag{1.4.2}$$

where $\pi_j \, : \, T^* X_j^{\wedge} \setminus 0 \rightarrow X_j^{\wedge}$ denotes the canonical projections. Locally, near $r = 0$ we can write

$$\sigma_{\psi}(A_j)(r, x_j, \varrho, \xi_j) = r^{\delta-\gamma} \tilde{\sigma}_{\psi}(A_j)(r, x_j, \tilde{\varrho}, \xi_j)|_{\tilde{\varrho}=r\varrho} \tag{1.4.3}$$

for bundle homomorphisms $\tilde{\sigma}_{\psi}(A_j)$ that are smooth in r up to $r = 0$. To illustrate this structure in more detail, let us look at the scalar case, i.e., trivial bundles of fiber dimension 1. Then, as is known from Mellin pseudo-differential operators of the prescribed form, in local coordinates $x_j \in \mathbb{R}^{n_j}$ on X_j the operators A_j have amplitude functions

$$p_j(r, x_j, \varrho, \xi_j) = r^{\delta-\gamma} \tilde{p}_j(r, x_j, \tilde{\varrho}, \xi_j)|_{\tilde{\varrho}=r\varrho}$$

where $\tilde{p}_j(r, x_j, \tilde{\varrho}, \xi_j) \in S_{\mathrm{cl}}^{\mu}(\overline{\mathbb{R}}_+ \times \Sigma \times \mathbb{R}_{\tilde{\varrho}, \xi_j}^{n_j+1})$, $\Sigma \subseteq \mathbb{R}^{n_j}$ open. In other words, locally A_j equals $\mathrm{op}_{r,x_j}(p_j) \bmod L^{-\infty}(\mathbb{R}_+ \times \Sigma)$. Then, $\tilde{\sigma}_{\psi}(A_j)$ just corresponds to the homogeneous principal part of $\tilde{p}_j$ of order μ.

As operators on cones $X_j^{\wedge}$ with exit to infinity there are exit symbols of A_j locally on $\mathbb{R}_+ \times \Sigma$, $\Sigma \subset \mathbb{R}^{n_j}$ open, that are invariantly defined modulo Schwartz functions in variables and covariables. The finite system of such local exit symbols (corresponding to a finite covering of X_j by coordinate neighbourhoods) will be denoted by $\sigma_{\mathrm{e}}(A_j)$, and the pair $\sigma_{\mathrm{e}}(A) = (\sigma_{\mathrm{e}}(A_1), \sigma_{\mathrm{e}}(A_2))$ is called the exit symbol of the operator A.

Finally, the principal conormal symbol $\sigma_{\mathrm{M}}(A)$ is defined to be the family of maps

$$\sigma_{\mathrm{M}}(A)(z) = \begin{pmatrix} \sigma_{\mathrm{M}}(A_1)(z) & 0 \\ 0 & \sigma_{\mathrm{M}}(A_2)(z) \end{pmatrix} + \sigma_{\mathrm{M}}(M)(z) \tag{1.4.4}$$

where, according to the common cone calculus,

$$\sigma_{\mathrm{M}}(A_j) = h_j(0,z), \qquad j = 1,2,$$

cf. Definition 1.10(i), and

$$\sigma_{\mathrm{M}}(M)(z) = (\sigma_{\mathrm{M}}(M_{jk})(z))_{j,k=1,2},$$

cf. formula (1.4.4). Note that $\sigma_{\mathrm{M}}(A)$ gives rise to a z-dependent family of maps $\sigma_{\mathrm{M}}(A) \ : \ H^s(\boldsymbol{X},\boldsymbol{E}) \to H^{s-\mu}(\boldsymbol{X},\boldsymbol{F})$, where we have used the same letters for the restrictions of the bundles to the cone bases X_1, X_2. Let us set

$$\sigma(A) = (\sigma_\psi(A), \sigma_{\mathrm{M}}(A), \sigma_{\mathrm{e}}(A)), \tag{1.4.5}$$

called the symbol of A.

Let us now pass to the composition of operators

$$A \in C^\mu(\boldsymbol{X}^\wedge, \boldsymbol{g}; \boldsymbol{v}), B \in C^\nu(\boldsymbol{X}^\wedge, \boldsymbol{h}; \boldsymbol{w})$$

for $\boldsymbol{g} = (\gamma,\delta,\Theta)$, $\boldsymbol{v} = (F_1, G_1; F_2, G_2)$ and $\boldsymbol{h} = (\beta,\gamma,\Theta)$, $\boldsymbol{w} = (E_1, F_1; E_2, F_2)$. Set $\boldsymbol{g} \circ \boldsymbol{h} = (\beta,\delta,\Theta)$, $\boldsymbol{v} \circ \boldsymbol{w} = (E_1, G_1; E_2, G_2)$.

Theorem 1.12 *We have $AB \in C^{\mu+\nu}(\boldsymbol{X}^\wedge, \boldsymbol{g}\circ\boldsymbol{h}; \boldsymbol{v}\circ\boldsymbol{w})$ and $\sigma(AB) = \sigma(A)\,\sigma(B)$ with componentwise composition, where*

$$\begin{aligned} \sigma_\psi(AB) &= \sigma_\psi(A)\,\sigma_\psi(B), \\ \sigma_{\mathrm{e}}(AB) &= \sigma_{\mathrm{e}}(A) \# \sigma_{\mathrm{e}}(B), \\ \sigma_{\mathrm{M}}(AB)(z) &= (T^{\beta-\gamma}\sigma_{\mathrm{M}}(A)(z))\,\sigma_{\mathrm{M}}(B)(z), \end{aligned}$$

and # denotes the Leibniz product of local representatives. If A or B belongs to the class with subscript $M+G$ or G, then the same is true of the composition.

Proof. Let us write

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} + M + G \text{ and } B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} + M' + G',$$

where the various terms are described in Definition 1.10. The product is of the type

$$\begin{pmatrix} A_1B_1 & 0 \\ 0 & A_2B_2 \end{pmatrix} + C,$$

and, due to the algebra property in the case of a single cone, we only have to show that $C \in C_{M+G}(\boldsymbol{X}^\wedge, \boldsymbol{g}\circ\boldsymbol{h}; \boldsymbol{v}\circ\boldsymbol{w})$. Most of the terms appearing in the diagonal components of C turn out to be of the correct type, since they come from compositions of operators belonging to the cone algebras on $X_1^\wedge$ and $X_2^\wedge$. Moreover, all the compositions with Green operators coming from G and G' are Green operators, as one can easily verify by their mapping properties, cf. Definition 1.5 and Theorem 1.11. In addition to this, the terms of the block (21) are of the same nature of those of the block (12), by exchanging the role of $X_1^\wedge$ and $X_2^\wedge$. So the only terms that we have to examine explicitly are $Q_{11} = M_{12}M'_{21}$, $Q_{22} = M_{21}M'_{12}$, $Q^1_{12} = A_1M'_{12}$, $Q^2_{12} = M_{12}B_2$, $Q^3_{12} = M_{11}M'_{12}$ and $Q^4_{12} = M_{12}M'_{22}$. For simplicity, we consider, from now on, trivial bundles.

Let us start by focusing on Q_{11}. Of course, it is enough to take into account only one of the terms arising from this composition. Adapting Lemmas 2.3.69, 2.3.70 and 2.3.72 of [16] to the present situation, we can, modulo Green remainders, modify the expressions of M and M' by commuting r powers with Mellin operators having meromorphic smoothing symbol, changing the cut-off functions and shifting the weight lines, provided the carriers of the asymptotics do not meet the weight lines themselves, cf. Remark 1.8.

Similarly, cf. Lemma 2.3.73 of [16], we have

$$\omega_1 \mathrm{op}_M^\gamma(f)(1-\omega)\mathrm{op}_M^\gamma(f')\omega_2 \in C_G(X_1^\wedge, X_1^\wedge, (\gamma, \gamma, (-\infty, 0]))$$

for arbitrary cut-off functions $\omega, \omega_1, \omega_2$ and smoothing meromorphic symbols $f' \in M^{-\infty}_{V'}(X_1, X_2)$, $f \in M^{-\infty}_{V}(X_2, X_1)$, when the carriers V and V' do not intersect the weight line $\Gamma_{\frac{1}{2}-\gamma}$. Then, modulo Green remainders, it suffices to consider terms of the kind

$$r^{\delta-\gamma}\omega r^{j+\kappa}\mathrm{op}_M(T^{-\kappa}f)\omega r^{-\kappa+j'+\kappa'+\gamma-\beta}\mathrm{op}_M(T^{-\kappa'}f')r^{-\kappa'}\omega$$

with $j+\kappa \geq \gamma \geq \kappa$, $j'+\kappa' \geq \beta \geq \kappa'$, and we can follow the same argument of the last part of the proof of Theorem 2.3.84 of [16], obtaining smoothing Mellin operators on X_1 of the type

$$r^{\delta-\beta}\omega\, r^{j+j'}\mathrm{op}_M^{\tilde{\kappa}}((T^{\beta-\gamma-j'}f)f')\,\omega \tag{1.4.6}$$

with $j+j'+\tilde{\kappa} \geq \beta \geq \tilde{\kappa}$.

The required property for Q^3_{12} follows in the same way, since the involved terms are essentially of the same kind of those in Q_{11} (the only difference being the fact that M_{11} takes values in operators with kernel in $C^\infty(X_1 \times X_1)$ instead of $C^\infty(X_1 \times X_2)$).

Now, note that $Q^1_{12} = r^{\delta-\gamma}\omega\mathrm{op}_M^\gamma(h_1)\tilde{\omega}M'_{12} + (1-\omega)A_{1\psi}(1-\tilde{\tilde{\omega}})M'_{12}$. The first term can again be treated as above, if h does not depend on r: indeed, h_1 is holomorphic and its pointwise composition with the symbols f_j, $\tilde{f}_j$ appearing in M'_{12} gives an operator with kernel in $C^\infty(X_1 \times X_1)$. In the general case, the result is obtained via a Taylor expansion argument, since remainders with high enough r power are Green operators, cf. the proof of Proposition 2.3.69 and Theorem 2.4.15 in [16]. Choosing a suitable cut-off function ω', the second term can be written as $H = (1-\omega)A_{1\psi}(1-\tilde{\tilde{\omega}})\omega' M'_{12}$. Since $(1-\tilde{\tilde{\omega}})\omega' \in C_0^\infty(\mathbb{R}_+)$, it turns out, by the mapping properties of the involved factors, that H is again a Green operator.

Finally, Q_{22}, Q_{12}^2 and Q_{12}^4 can be treated as Q_{11}, Q_{12}^1 and Q_{12}^3, respectively.

The symbolic rules for the ψ- and e-components of $\sigma(AB)$ are immediate, both following from the usual composition rules for the diagonal non-smoothing terms of A and B. The formula for $\sigma_M(AB)$ is a consequence of the similar one for the standard cone algebra and of (1.4.6) with $j = j' = 0$. $\square$

For $A \in C^\mu(\boldsymbol{X}^\wedge, \boldsymbol{g}; \boldsymbol{v})$, $\boldsymbol{g} = (\gamma, \delta, (-(k+1), 0])$, $\boldsymbol{v} = (E_1, F_1; E_2, F_2)$ we can define the formal adjoint by

$$(Au, v)_{\mathcal{K}^{0,\frac{n}{2}}(\boldsymbol{X}^\wedge, \boldsymbol{F})} = (u, A^* v)_{\mathcal{K}^{0,\frac{n}{2}}(\boldsymbol{X}^\wedge, \boldsymbol{E})}$$

for $u \in C_0^\infty(\boldsymbol{X}^\wedge, \boldsymbol{E})$, $v \in C_0^\infty(\boldsymbol{X}^\wedge, \boldsymbol{F})$, cf. Definition 1.5. We omit the proof of the next theorem, which follows by Definition 1.5, Proposition 1.9 and the similar result for the standard cone algebra.

Theorem 1.13 *$A \in C^\mu(\boldsymbol{X}^\wedge, \boldsymbol{g}; \boldsymbol{v})$ implies $A^* \in C^\mu(\boldsymbol{X}^\wedge, \boldsymbol{g}^*; \boldsymbol{v}^*)$ for $\boldsymbol{g}^* = (-\delta, -\gamma, (-(k+1), 0])$, $\boldsymbol{v}^* = (F_1, E_1; F_2, E_2)$ and we have $\sigma(A^*) = \sigma(A)^*$, where * refers to each component in the symbolic triple. More precisely, $\sigma_\psi(A)^*$ is the adjoint symbol from the standard pseudo-differential calculus, $\sigma_{\mathrm{M}}(A^*)(z) = T^{\gamma-\delta}\sigma_{\mathrm{M}}(A)^*$, while $\sigma_{\mathrm{e}}(A)^*$ is again the standard rule from the exit calculus of pseudo-differential operators.*

Definition 1.14 *An operator $A \in C^\mu(\boldsymbol{X}^\wedge, \boldsymbol{g}; \boldsymbol{v})$ (in the notation of Definition* 1.10*) is said to be elliptic, if it is elliptic with respect to the three components of $\sigma(A)$, that is*

(i) *the interior symbols* (1.4.2) *are bundle isomorphisms, $j = 1, 2$, where also $\tilde{\sigma}_\psi(A_j)$ from* (1.4.3) *are isomorphisms up to $r = 0$;*

(ii) *the conormal symbol*

$$\sigma_{\mathrm{M}}(A)(z) \;:\; H^{s+\frac{n}{2}}(\boldsymbol{X}, \boldsymbol{E}) \to H^{s-\mu+\frac{n}{2}}(\boldsymbol{X}, \boldsymbol{F})$$

is a family of isomorphisms for all $z \in \Gamma_{\frac{1}{2}-\gamma}$;

(iii) *the exit symbol $\sigma_{\mathrm{e}}(A)$ is elliptic.*

Given an operator $A \in C^\mu(\boldsymbol{X}^\wedge, \boldsymbol{g}; \boldsymbol{v})$, a $P \in C^{-\mu}(\boldsymbol{X}^\wedge, \boldsymbol{g}^{-1}; \boldsymbol{v}^{-1})$ for $\boldsymbol{g}^{-1} = (\delta, \gamma, \Theta)$, $\boldsymbol{v}^{-1} = (F_1, E_1; F_2, E_2)$ is called a parametrix of A if

$$C_l := 1 - PA \text{ and } C_r = 1 - AP \tag{1.4.7}$$

belong to $C_G(\boldsymbol{X}^\wedge, \boldsymbol{g}_l; \boldsymbol{v}_l)$ and $C_G(\boldsymbol{X}^\wedge, \boldsymbol{g}_r; \boldsymbol{v}_r)$, respectively, where

$$\begin{aligned} \boldsymbol{g}_l &= (\gamma, \gamma, \Theta), & \boldsymbol{v}_l &= (E_1, E_1; E_2, E_2), \\ \boldsymbol{g}_r &= (\delta, \delta, \Theta), & \boldsymbol{v}_r &= (F_1, F_1; F_2, F_2). \end{aligned}$$

Theorem 1.15 *Let $A \in C^\mu(\boldsymbol{X}^\wedge, \boldsymbol{g}; \boldsymbol{v})$ be elliptic. Then, A admits a parametrix $P \in C^{-\mu}(\boldsymbol{X}^\wedge, \boldsymbol{g}^{-1}; \boldsymbol{v}^{-1})$.*

Proof. We only sketch the proof (cf., e.g., [16], Section 2.4.3). From conditions (i) and (iii) of Definition 1.14, it is possible to build two pseudo-differential operators $P_j \in L^{-\mu,0}(X_j^\wedge, F_j, E_j)$, $j = 1, 2$, which are parametrices, respectively, of A_j on $X_j^\wedge$, $j = 1, 2$, in the sense of the calculus on manifolds with exits. Moreover, owing to the hypothesis on $\tilde{\sigma}_\psi(A_j)$, "close to the tip" the P_j are of the form $P_j = r^\mu \mathrm{op}(p_j)$ with $p_j(r, \varrho) = \tilde{p}_j(r, r\varrho)$ and $\tilde{p}_j \in C^\infty(\overline{\mathbb{R}}_+, L^{-\mu}(X_j, \mathbb{R}_\varrho))$. Through the so-called Mellin quantisation of p_j, cf. Theorem 1.2, Remark 1.3 and [6], it is possible to find two holomorphic symbols $h_j \in M_{\mathcal{O}}^{-\mu}(\overline{\mathbb{R}}_+ \times X_j)$ such that

$$P_j' = \omega r^{\gamma-\delta} \mathrm{op}_M^\delta(h_j)\tilde{\omega} + (1-\omega)P_j(1-\tilde{\tilde{\omega}})$$

differs from P_j by an element in $L^{-\infty,-\infty}(\mathrm{int}X_j^\wedge)$. Setting $P' = \mathrm{diag}(P_1', P_2')$, by Theorem 1.12 and the definition of the P_j' we get

$$1 - P'A \in C^0(\boldsymbol{X}^\wedge, \boldsymbol{g}_l, \boldsymbol{v}_l) \cap L^{-\infty,-\infty}(\mathrm{int}\boldsymbol{X}^\wedge).$$

This means that the matrix $1 - P'A$ belongs to $C_{M+G}(\boldsymbol{X}, \boldsymbol{g}_l, \boldsymbol{v}_l)$. To get a parametrix modulo Green operators we use now hypothesis (ii) of Definition 1.14 and define

$$\begin{aligned} f &= T^{\gamma-\delta}((1-\sigma_{\mathrm{M}}(P'A))\sigma_{\mathrm{M}}(A)^{-1}), \\ P'' &= P' + \omega\, r^{\gamma-\delta}\mathrm{op}_M^\delta(f)\,\tilde{\omega} \in C^{-\mu}(\boldsymbol{X}^\wedge, \boldsymbol{g}_l, \boldsymbol{v}_l). \end{aligned}$$

By the formula for the Mellin symbol of the composition, we get $\sigma_{\mathrm{M}}(1-P''A) = 0$, which implies that $(1-P''A)^{k+1}$ is a Green operator (recall the observations before Remark 1.8). Then, $P_l = \sum_{j=0}^{k}(1-P''A)^j P'' \in C^{-\mu}(\boldsymbol{X}^\wedge, \boldsymbol{g}_l, \boldsymbol{v}_l)$ is a left parametrix of A modulo Green operators. A right parametrix P_r can be built in a similar way and $P_l - P_r$ is clearly a Green operator, so that either P_l or P_r can be chosen as the parametrix P. $\square$

The following three theorems are consequence of Theorem 1.15, by arguments similar to those valid for the standard cone algebra.

Theorem 1.16 *For an operator $A \in C^\mu(\boldsymbol{X}^\wedge, \boldsymbol{g}; \boldsymbol{v})$ the following conditions are equivalent:*

(i) *A is elliptic;*

(ii) *the operator*

$$A \,:\, \mathcal{K}^{s,\gamma+\frac{n}{2}}(\boldsymbol{X}^\wedge, \boldsymbol{E}) \to \mathcal{K}^{s-\mu,\gamma-\mu+\frac{n}{2}}(\boldsymbol{X}^\wedge, \boldsymbol{F}) \tag{1.4.8}$$

is Fredholm for certain $s \in \mathbb{R}$.

Theorem 1.17 *Let $A \in C^\mu(\boldsymbol{X}^\wedge, \boldsymbol{g}; \boldsymbol{v})$ be an element that induces a Fredholm operator* (1.4.8) *for certain fixed $s \in \mathbb{R}$. Then A is a Fredholm operator* (1.4.8) *for arbitrary $s \in \mathbb{R}$. The parametrix P of A can be chosen in such a way that C_l is a projection to $V := \ker A$ and C_r a projection to a complement W of* $\mathrm{im}\, A$

for every fixed $s \in \mathbb{R}$*. Moreover, there are asymptotic types* $\boldsymbol{P}$ *and* $\boldsymbol{Q}$ *such that* $V \subset \mathcal{S}_{\boldsymbol{P}}^{\gamma+\frac{n}{2}}(\boldsymbol{X}^\wedge, \boldsymbol{E})$ *and the space* W *can be chosen to be a finite dimensional subspace of* $\mathcal{S}_{\boldsymbol{Q}}^{\gamma-\mu+\frac{n}{2}}(\boldsymbol{X}^\wedge, \boldsymbol{F})$ *such that* $\operatorname{im} A + W = \mathcal{K}^{s-\mu,\gamma-\mu+\frac{n}{2}}(\boldsymbol{X}^\wedge, \boldsymbol{F})$ *and* $\operatorname{im} A \cap W = \{0\}$ *for all* $s \in \mathbb{R}$*.*

Theorem 1.18 *Let* $A \in C^\mu(\boldsymbol{X}^\wedge, \boldsymbol{g}; \boldsymbol{v})$ *be an operator such that*

$$A \ : \ \mathcal{K}^{s,\gamma}(\boldsymbol{X}^\wedge, \boldsymbol{g}; \boldsymbol{E}) \to \mathcal{K}^{s-\mu,\delta}(\boldsymbol{X}^\wedge, \boldsymbol{F}) \tag{1.4.9}$$

is an isomorphism for a $s = s_0 \in \mathbb{R}$*. Then* (1.4.9) *is invertible for all* $s \in \mathbb{R}$ *and* $A^{-1} \in C^{-\mu}(\boldsymbol{X}^\wedge, \boldsymbol{g}^{-1}; \boldsymbol{v}^{-1})$*.*

2 The edge symbolic calculus

2.1 Spaces with edges and model cones of different dimensions

Spaces with edges we are talking about can locally be formulated in terms of wedges $X^\Delta \times \Omega$ for a (in simplest cases) closed compact C^∞ manifold X and an open set $\Omega \subseteq \mathbb{R}^q$. Constructions will always be given in a splitting of variables on $(X^\Delta \setminus \{v\}) \times \Omega \cong X^\wedge \times \Omega \ni (r, x, y)$, and we then have to observe invariance under an admitted cocycle of transition maps. A system of diffeomorphisms $\chi \ : \ (X^\Delta \setminus \{v\}) \times \Omega \to X^\wedge \times \Omega$ is said to be a wedge structure on $X^\Delta \times \Omega$ if for every two elements χ_1 and χ_2 of that system the transition map $\chi_2\chi_1^{-1} \ : \ X^\wedge \times \Omega \to X^\wedge \times \Omega$ is the restriction of a diffeomorphism $\mathbb{R} \times X \times \Omega \to \mathbb{R} \times X \times \Omega$ to $\mathbb{R}_+ \times X \times \Omega$. In other words, we also get a cocycle of transitions $\overline{\mathbb{R}}_+ \times X \times \Omega \to \overline{\mathbb{R}}_+ \times X \times \Omega$ (that are smooth up to $r = 0$). Looking at the components of these maps $(r, x, y) \to (\tilde{r}(r, x, y), \tilde{x}(r, x, y), \tilde{y}(r, x, y))$ we then have $\tilde{r}(0, x, y) = 0$ and $\tilde{y}(0, x, y)$ only depends on y, i.e., $y \to \tilde{y}$ induces a diffeomorphism $\Omega \to \Omega$.

A manifold W with edge Y is defined as a topological space, such that $W \setminus Y$ and Y are C^∞ manifolds, and that points $y \in Y$ have neighbourhoods modeled by $X^\Delta \times \Omega$; then Y itself has local coordinates in Ω. Together with the cocycle of transition maps $X^\wedge \times \Omega \to X^\wedge \times \Omega$ for $W \setminus Y$ near Y we also have the maps $\overline{\mathbb{R}}_+ \times X \times \Omega \to \overline{\mathbb{R}}_+ \times X \times \Omega$ that allow us to interpret $W \setminus Y$ as $\operatorname{int} \mathbb{W}$ for a C^∞ manifold $\mathbb{W}$ with boundary $\partial\mathbb{W}$ that is a X bundle over Y. The transition maps for $\partial\mathbb{W}$ are just given by $(x, y) \to (\tilde{x}, \tilde{y})|_{r=0}$. For convenience, in the following we content ourselves with the case that $\partial\mathbb{W}$ is a trivial X bundle, i.e., $\partial\mathbb{W} = X \times Y$, and that the splittings of variables (r, x, y) near $\partial\mathbb{W}$ are chosen in such a way that we have $(r, x, y) \to (\tilde{r}, \tilde{x}, \tilde{y}) = (r, x, \tilde{y})$ for $0 \le r < \varepsilon$ for some $\varepsilon > 0$.

Global operators on $\mathbb{W}$ will be connected with vector bundles $E \in \operatorname{Vect}(\mathbb{W})$ and $J \in \operatorname{Vect}(Y)$, and we want to fix some notation. By definition, $\partial\mathbb{W}$ has a neighbourhood of the form $[0, 1) \times X \times Y$ in the corresponding splitting of variables (r, x, y), and with E we get an associated bundle

$$E|_{[0,1)\times X\times Y} \tag{2.1.1}$$

that can be regarded as a pull-back of $E|_{\{0\}\times X\times Y}$ to $[0,1)\times X\times Y$ under the canonical projection $[0,1)\times X\times Y\to\{0\}\times X\times Y$, $(r,x,y)\to(0,x,y)$. A similar projection $\overline{\mathbb{R}}_+\times X\times Y\to\{0\}\times X\times Y$ gives rise to a pull-back of $E|_{\{0\}\times X\times Y}$ to $\overline{\mathbb{R}}_+\times X\times Y$. For convenience, we employ for these bundles the same letter E; it will be clear from the context where E is given. Moreover, in every $E\in\operatorname{Vect}(\mathbb{W})$ we fix a Hermitian metric in such a way that the induced metric in (2.1.1) does not depend on $r\in[0,1)$, and we then take a similar r-independent Hermitian metric on the pull-back to $\overline{\mathbb{R}}_+\times X\times Y$. In this way, the space $\mathcal{K}^{0,\frac{n}{2}}(X^\wedge,E_y)$, for $n=\dim X$ and $E_y:=E|_{X^\wedge\times\{y\}}$, is equipped with a scalar product that will be taken below in the definition of adjoints, similarly to adjoints in the cone calculus, cf. Section 1.2 above. In local considerations with respect to coordinate neighbourhoods U on Y, we also have restrictions of bundles E on $X^\wedge\times Y$ to $X^\wedge\times U$, and then, analogously to trivialisations of bundles in the usual sense, it will be admitted to regard $E|_{X^\wedge\times U}$ as a pull-back of $E|_{X^\wedge\times\{y\}}$ to $X^\wedge\times U$ under the projection $X^\wedge\times U\to X^\wedge$. Again, we will use the same letter E for the local version of the bundle. It will then make sense to talk about spaces like

$$C^\infty(U\times\mathbb{R}^q,\mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge,E),\mathcal{K}^{\tilde{s},\tilde{\gamma}}(X^\wedge,F)))\tag{2.1.2}$$

for bundles $E,F\in\operatorname{Vect}(\mathbb{W})$, etc., where in (2.1.2) we mean the local versions of the bundles we just described. Our calculus below will have the right invariance properties that justify these conventions.

Let us now pass to configurations with edges where the model cones are of different dimension. The simplest examples are Cartesian products $W=\boldsymbol{X}^\Delta\times Y$, where $\boldsymbol{X}^\Delta$ is as in the introduction a cone with base manifolds $X_1,\dots,X_N$ of different dimension. In general we assume that W is a topological space with a subspace Y such that $W\setminus Y$ is the disjoint union of spaces of the form $\operatorname{int}W_j$, for manifolds W_j, $j=1,\dots,N$, with the same edge Y as described before, and W has, locally near Y, the structure $\boldsymbol{X}^\Delta\times\Omega$ for some open subset $\Omega\subseteq\mathbb{R}^q$ (which corresponds to a chart on Y). We then define the stretched space $\mathbb{W}$ associated with W as the quotient space of the disjoint union $\mathbb{W}_1\cup\ldots\cup\mathbb{W}_N$ that identifies the different copies of Y. In particular, the stretched space of $W=\boldsymbol{X}^\Delta\times Y$ equals $\mathbb{W}=\{\cup_{j=1}^N\overline{\mathbb{R}}_+\times X_j\}\times Y$. For convenience, we also write $\mathbb{W}=(\mathbb{W}_1,\dots,\mathbb{W}_N)$ keeping in mind the mentioned identification map. In the following we assume W to be compact.

As before, for simplicity, from now on we consider the case $N=2$, and we use shortened notation analogous to those used in Section 1, in particular also for diagonal matrix block operators $\langle\eta\rangle^{\frac{\boldsymbol{n}}{2}}:=\operatorname{diag}(\langle\eta\rangle^{\frac{n_1}{2}},\langle\eta\rangle^{\frac{n_2}{2}})$, $\kappa_\lambda^{\boldsymbol{n}}:=\operatorname{diag}(\kappa_\lambda^{n_1},\kappa_\lambda^{n_2})$, $\lambda^{\frac{\boldsymbol{n}}{2}}\kappa_\lambda^{\boldsymbol{n}}:=\operatorname{diag}(\lambda^{\frac{n_1}{2}}\kappa_\lambda^{n_1},\lambda^{\frac{n_2}{2}}\kappa_\lambda^{n_2})$. Moreover $\operatorname{int}\mathbb{W}:=(\operatorname{int}\mathbb{W}_1,\operatorname{int}\mathbb{W}_2)$, while, for any $J_\pm\in\operatorname{Vect}(Y)$, we set $\boldsymbol{v}:=(E_1,F_1;E_2,F_2;J_-,J_+)$ for the bundle data. The abbreviation $\boldsymbol{w}:=(E_1,F_1;E_2,F_2;j_-,j_+)$ for local bundle data will be used for the description of the symbolic structures below, where, making use of the mentioned abuse of notation, the local bundles are obtained from $E_j,F_j\in\operatorname{Vect}(\mathbb{W}_j)$, $j=1,2$, as explained before.

Typical differential operators on $\operatorname{int}\mathbb{W}$ are (because of locality) pairs of independently given differential operators A_j on $\mathrm{int}\mathbb{W}_j$, $j = 1, 2$. They will be assumed to be edge-degenerate, i.e., differential operators with smooth coefficients that, in the splitting of variables $(r, x, y) \in \mathbb{R}_+ \times X_j \times \Omega$ close to the edge, are of the form

$$A_j = r^{-\mu} \sum_{k+|\alpha|\leq\mu} a_{k\alpha}(r,y) \left(-r\frac{\partial}{\partial r}\right)^k (rD_y)^\alpha$$

with coefficients $a_{k\alpha} \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \mathrm{Diff}^{\mu-(k+|\alpha|)}(X_j))$, $j = 1, 2$. Examples are Laplace-Beltrami operators belonging to "wedge"-metrics

$$dr^2 + r^2\, g_{X_j}(r, y) + dy^2$$

with Riemannian metrics g_{X_j} on X_j, smoothly dependent on $(r, y) \in \overline{\mathbb{R}}_+ \times \Omega$, $j = 1, 2$. For the analysis it will be adequate to take the same axial variable r on both sides close to the edge; in fact, r is nothing else than the "distance" variable of a point in W to the edge, regardless of the side.

The program of this section is a calculus of edge-degenerate pseudo-differential operators on W. Because of pseudo-locality, information between both sides is exchanged only on the level of a specific kind of smoothing operators. A main concept in the discussion below is the following notion of operator-valued symbol. For details, the reader can refer, e.g., to [16].

Definition 2.1 *Consider an open set $\Omega \subseteq \mathbb{R}^q$, $\mu \in \mathbb{R}$, and Hilbert spaces $H, \tilde{H}$ endowed with strongly continuous groups of isomorphisms κ_λ, $\tilde{\kappa}_\lambda$, $\lambda \in \mathbb{R}_+$. Then $S^\mu(\Omega \times \mathbb{R}^q; H, \tilde{H})$ denotes the space of all $a \in C^\infty(\Omega \times \mathbb{R}^q, \mathcal{L}(H, \tilde{H}))$ satisfying*

$$\|\tilde{\kappa}^{-1}_{\langle\eta\rangle}\{D^\alpha_y D^\beta_\eta a(y,\eta)\}\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H,\tilde{H})} \leq c_{\alpha\beta K}\, \langle\eta\rangle^{\mu-|\beta|}$$

for all $\alpha, \beta \in \mathbb{N}^q$, $\eta \in \mathbb{R}^q$, $y \in K \subset\subset \Omega$ and suitable constants $c_{\alpha\beta K} \geq 0$.

An element $a \in C^\infty(\Omega \times (\mathbb{R}^q \setminus \{0\}), \mathcal{L}(H, \tilde{H}))$ is called (positively twisted-) homogeneous of order μ if

$$a(y, \lambda\eta) = \lambda^\mu \tilde{\kappa}_\lambda a(y,\eta)\kappa^{-1}_\lambda \tag{2.1.3}$$

for all $\lambda \in \mathbb{R}_+$, $(y, \eta) \in \Omega \times (\mathbb{R}^q \setminus \{0\})$. Note that, for every excision function χ and any a satisfying (2.1.3), $\chi(\eta)a(y, \eta) \in S^\mu(\Omega \times \mathbb{R}^q; H, \tilde{H})$. It is then natural to introduce the following notion of classical operator-valued symbols.

Definition 2.2 *With the same notation of Definition 2.1, we denote by $S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; H, \tilde{H})$ the subset of classical operator-valued symbols of order μ, which consists of all $a \in S^\mu(\Omega \times \mathbb{R}^q; H, \tilde{H})$ that admit an asymptotic expansion $a \sim \sum_j \chi a_{(\mu-j)}$ with $a_{(\mu-j)}$ homogeneous of order $\mu - j$ in the sense of (2.1.3).*

2.2 Edge Sobolev spaces

Definition 2.3 *Let H be a Hilbert space equipped with a strongly continuous group of isomorphisms $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$. The abstract wedge Sobolev space $\mathcal{W}^s(\mathbb{R}^q,H)$ of smoothness $s\in\mathbb{R}$ is the completion of $\mathcal{S}(\mathbb{R}^q,H)$ with respect to the norm*

$$\|u\|_{\mathcal{W}^s(\mathbb{R}^q,H)}=\left\{\int\langle\eta\rangle^{2s}\,\|\kappa^{-1}_{\langle\eta\rangle}(F_{y\to\eta}u)(\eta)\|^2_H d\eta\right\}^{\frac{1}{2}}, \tag{2.2.1}$$

where $F_{y\to\eta}$ is the Fourier transform in $\mathbb{R}^q$.

The construction of $\mathcal{W}^s(\mathbb{R}^q,H)$ will be used also for Fréchet spaces H that are written as projective limits of Hilbert spaces $\{H^j\}_{j\in\mathbb{N}}$, with continuous embeddings $H^{j+1}\hookrightarrow H^j\hookrightarrow\ldots\hookrightarrow H^0$ and a strongly continuous group $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ of isomorphisms on H^0 that restricts to strongly continuous groups of isomorphisms on H^j for every j. Then we have continuous embeddings $\mathcal{W}^s(\mathbb{R}^q,H^{j+1})\hookrightarrow\mathcal{W}^s(\mathbb{R}^q,H^j)$ for all j, and $\mathcal{W}^s(\mathbb{R}^q,H)$ is the projective limit of $\mathcal{W}^s(\mathbb{R}^q,H^j)$, $j\in\mathbb{N}$.

We apply this construction to weighted Sobolev spaces $\mathcal{K}^{s+\frac{n}{2},\gamma+\frac{n}{2}}(X^\wedge,E)$ and Fréchet subspaces $\mathcal{K}_P^{s+\frac{n}{2},\gamma+\frac{n}{2}}(X^\wedge,E)$ with asymptotics of type P; here we apply Remark 1.4 and the action κ_λ defined there. We then obtain the spaces

$$\mathcal{W}_{(P)}^{s+\frac{n}{2},\gamma+\frac{n}{2}}(X^\wedge\times\mathbb{R}^q,E):=\mathcal{W}^{s+\frac{n}{2}}(\mathbb{R}^q,\mathcal{K}_{(P)}^{s+\frac{n}{2},\gamma+\frac{n}{2}}(X^\wedge,E)) \tag{2.2.2}$$

and then, globally, on a corresponding (compact, stretched) manifold $\mathbb{W}$ with edge Y we get the spaces $\mathcal{W}_{(P)}^{s+\frac{n}{2},\gamma+\frac{n}{2}}(\mathbb{W},E)$. In this construction we use several useful properties of the $\mathcal{W}^s$-spaces. In particular, that the spaces (2.2.2) are contained in $H_{\mathrm{loc}}^{s+\frac{n}{2}}(X^\wedge\times\mathbb{R}^q)$ for every s,γ; we then have $\mathcal{W}_{(P)}^{s+\frac{n}{2},\gamma+\frac{n}{2}}(\mathbb{W})\subset H_{\mathrm{loc}}^{s+\frac{n}{2}}(\mathrm{int}\,\mathbb{W})$. This is the construction for a single manifold with edges. For the case $\mathbb{W}=(\mathbb{W}_1,\mathbb{W}_2)$ we simply take the direct sums of the corresponding spaces.

As for the standard Sobolev spaces there are "comp" and "loc" versions of $\mathcal{W}^s$-spaces on open sets $\Omega\subseteq\mathbb{R}^q$, first for the context of Definition 2.3, and then also for the specific spaces on configurations $\mathbb{W}=(\mathbb{W}_1,\mathbb{W}_2)$ with edges when we drop the assumption of compactness.

Definition 2.4 *The space of smoothing operators $\mathcal{Y}^{-\infty}(\mathbb{W},\boldsymbol{g};\boldsymbol{v})$ in the transmission algebra on $\mathbb{W}$ is defined to be the set of all continuous operators*

$$\mathcal{C}\ :\ \begin{array}{c}C_0^\infty(\mathrm{int}\,\mathbb{W},\boldsymbol{E})\\ \oplus\\ C_0^\infty(Y,J_-)\end{array}\ \to\ \begin{array}{c}C^\infty(\mathrm{int}\,\mathbb{W},\boldsymbol{F})\\ \oplus\\ C^\infty(Y,J_+)\end{array}$$

that extend the continuous operators

$$\mathcal{C}\ :\ \begin{array}{c}\mathcal{W}^{s+\frac{n}{2},\gamma+\frac{n}{2}}(\mathbb{W},\boldsymbol{E})\\ \oplus\\ H^{s-\frac{1}{2}}(Y,J_-)\end{array}\ \to\ \begin{array}{c}\mathcal{W}_{\boldsymbol{P}}^{\infty,\gamma-\mu+\frac{n}{2}}(\mathbb{W},\boldsymbol{F})\\ \oplus\\ H^\infty(Y,J_+)\end{array}$$

where the formal adjoint $\mathcal{C}^*$ extends to continuous operators

$$\mathcal{C}^* \ : \quad \begin{array}{c} \mathcal{W}^{s+\frac{\boldsymbol{n}}{2},-\gamma+\mu+\frac{\boldsymbol{n}}{2}}(\mathbb{W},\boldsymbol{F}) \\ \oplus \\ H^{s-\frac{1}{2}}(Y,J_+) \end{array} \quad \rightarrow \quad \begin{array}{c} \mathcal{W}_{\boldsymbol{Q}}^{\infty,-\gamma+\frac{\boldsymbol{n}}{2}}(\mathbb{W},\boldsymbol{E}) \\ \oplus \\ H^{\infty}(Y,J_-) \end{array}$$

for all $s \in \mathbb{R}$, with certain asymptotic types $\boldsymbol{P}$ and $\boldsymbol{Q}$ depending on $\mathcal{C}$. Here, the formal adjoint $\mathcal{C}^*$ is defined by

$$(\mathcal{C}u,v)_{\mathcal{W}^{0,\frac{\boldsymbol{n}}{2}}(\mathbb{W},\boldsymbol{F})\oplus L^2(Y,J_+)} = (u,\mathcal{C}^*v)_{\mathcal{W}^{0,\frac{\boldsymbol{n}}{2}}(\mathbb{W},\boldsymbol{E})\oplus L^2(Y,J_-)}.$$

for all $u \in C_0^\infty(\operatorname{int}\mathbb{W},\boldsymbol{E}) \oplus C_0^\infty(Y,J_-)$, $v \in C_0^\infty(\operatorname{int}\mathbb{W},\boldsymbol{F}) \oplus C_0^\infty(Y,J_+)$.

2.3 Green symbols

We now turn to a first important element of the edge symbolic structure, the so-called Green symbols. They play an analogous role for our calculus as the “singular Green” symbols in standard boundary value problems in the context of Boutet de Monvel [3]. A specific point in the present situation is that Green symbols (and associated Green operators below) transmit information between the different-dimensional parts of our configuration across the edge.

Let $\boldsymbol{g} := (\gamma,\gamma-\mu,\Theta)$, $\Theta = (-(k+1),0]$, $\nu \in \mathbb{R}$, $\mu-\nu \in \mathbb{N}$ and $\boldsymbol{w} := (E_1,F_1;E_2,F_2;j_-,j_+)$.

Definition 2.5 *$\mathcal{R}_G^\nu(\Omega\times\mathbb{R}^q,\boldsymbol{g};\boldsymbol{w})$ for open $\Omega \subseteq \mathbb{R}^q$ is defined to be the set of all families of continuous maps*

$$g(y,\eta) \in C^\infty(\Omega\times\mathbb{R}^q,\mathcal{L}(\mathcal{K}^{s,\gamma+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^\wedge,\boldsymbol{E})\oplus\mathbb{C}^{j_-}),\mathcal{K}^{\infty,\gamma-\mu+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^\wedge,\boldsymbol{F})\oplus\mathbb{C}^{j_+}))$$

such that for

$$g_0(y,\eta) = \operatorname{diag}(\langle\eta\rangle^{\frac{\boldsymbol{n}}{2}},\langle\eta\rangle^{-\frac{1}{2}})\,g(y,\eta)\,\operatorname{diag}(\langle\eta\rangle^{\frac{\boldsymbol{n}}{2}},\langle\eta\rangle^{-\frac{1}{2}})^{-1} \qquad (2.3.3)$$

we have

$$g_0(y,\eta) \in S_{\mathrm{cl}}^\nu(\Omega\times\mathbb{R}^q;\mathcal{K}^{s,\gamma+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^\wedge,\boldsymbol{E})\oplus\mathbb{C}^{j_-},\mathcal{S}_{\boldsymbol{P}}^{\gamma-\mu+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^\wedge,\boldsymbol{F})\oplus\mathbb{C}^{j_+})$$

and

$$g_0^*(y,\eta) \in S_{\mathrm{cl}}^\nu(\Omega\times\mathbb{R}^q;\mathcal{K}^{s,-\gamma+\mu+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^\wedge,\boldsymbol{F})\oplus\mathbb{C}^{j_+},\mathcal{S}_{\boldsymbol{Q}}^{-\gamma}(\boldsymbol{X}^\wedge,\boldsymbol{E})\oplus\mathbb{C}^{j_-})$$

for all $s \in \mathbb{R}$. The elements of $\mathcal{R}_G^\nu(\Omega\times\mathbb{R}^q,\boldsymbol{g};\boldsymbol{w})$ are called Green symbols (of our transmission calculus).

By definition, the entries of Green symbols $g(y,\eta) = (g_{ij}(y,\eta))_{i,j=1,2,3}$ are classical operator-valued symbols, acting between the respective components of the involved spaces. Anyway, note that multiplication by powers of $\langle\eta\rangle$ does not preserve the symbol classes $\mathcal{R}_{M+G}^\nu(\Omega\times\mathbb{R}^q,\boldsymbol{g};\boldsymbol{w})$ and $\mathcal{R}^\nu(\Omega\times\mathbb{R}^q,\boldsymbol{g};\boldsymbol{w})$ that we introduce below. A similar phenomenon already occurs in the usual situation, where one has $L^{\mu}(\mathbb{R}^n_x \times \Omega_y) \cdot L^{\mu}(\Omega_y) \not\subset L^{\mu+\nu}(\mathbb{R}^n_x \times \Omega_y)$. The reason for introducing this operation in the definitions concerns the administration of the order in a "unified" way, in the spirit of Douglis and Nirenberg. In fact, we have DN-homogeneity in the following sense. First, $g_0(y,\eta)$ is classical of order ν, i.e., there is a homogeneous principal part $g_{0,(\nu)}(y,\eta)$, such that

$$g_{0,(\nu)}(y,\lambda\eta) = \lambda^{\nu}\operatorname{diag}(\kappa^{\boldsymbol{n}}_{\lambda},\mathrm{id})\, g_{0,(\nu)}(y,\eta)\operatorname{diag}(\kappa^{\boldsymbol{n}}_{\lambda},\mathrm{id})^{-1} \tag{2.3.4}$$

for all $(y,\eta) \in \Omega \times (\mathbb{R}^q \setminus \{0\})$, $\lambda \in \mathbb{R}_+$. For $g(y,\eta)$ itself the entries have different orders, that are immediate from (2.3.3). We get the matrix of orders

$$\boldsymbol{\nu} := \begin{pmatrix} \nu & \nu + \frac{n_2-n_1}{2} & \nu - \frac{n_1+1}{2} \\ \nu - \frac{n_2-n_1}{2} & \nu & \nu - \frac{n_2+1}{2} \\ \nu + \frac{n_1+1}{2} & \nu + \frac{n_2+1}{2} & \nu \end{pmatrix} \tag{2.3.5}$$

Let $g_{(\boldsymbol{\nu})}(y,\eta) = (g_{ij,(\boldsymbol{\nu}_{ij})}(y,\eta))_{i,j=1,2,3}$ denote the matrix of homogeneous principal components of $g(y,\eta)$. Then, DN-homogeneity of $g_{(\boldsymbol{\nu})}(y,\eta)$ itself means

$$g_{(\boldsymbol{\nu})}(y,\lambda\eta) = \lambda^{\nu}\operatorname{diag}(\lambda^{\frac{\boldsymbol{n}}{2}}\kappa^{\boldsymbol{n}}_{\lambda},\lambda^{-\frac{1}{2}}\mathrm{id})\, g_{(\boldsymbol{\nu})}(y,\eta)\operatorname{diag}(\lambda^{\frac{\boldsymbol{n}}{2}}\kappa^{\boldsymbol{n}}_{\lambda},\lambda^{-\frac{1}{2}}\mathrm{id})^{-1} \tag{2.3.6}$$

for all $(y,\eta) \in \Omega \times (\mathbb{R}^q \setminus \{0\})$, $\lambda \in \mathbb{R}_+$.

2.4 Mellin transmission symbols

Another specific part of the symbolic structure of transmission operators (with information being exchanged between the $X_1^{\wedge}$ and $X_2^{\wedge}$-sides of the configuration) are symbols with values in the C_{M+G} algebra, cf. Definition 1.7. In the following, with $[.]$ we will denote a positive smooth function such that $[\eta] = |\eta|$ for $|\eta| \geq c$ and some fixed constant $c > 0$.

Definition 2.6 *Let* $\mathcal{R}^{\nu}_{M+G}(\Omega \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})$ *for* $\boldsymbol{g} := (\gamma, \gamma-\mu, \Theta)$ *as in the beginning of Section* 2.3 *and* $\boldsymbol{w} := (E_1, F_1; E_2, F_2; j_-, j_+)$ *defined to be the set of all operator families*

$$(m+g)(y,\eta) \;:\; \begin{matrix} \mathcal{K}^{s,\gamma+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^{\wedge},\boldsymbol{E}) \\ \oplus \\ \mathbb{C}^{j_-} \end{matrix} \to \begin{matrix} \mathcal{K}^{\infty,\gamma-\mu+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^{\wedge},\boldsymbol{F}) \\ \oplus \\ \mathbb{C}^{j_+} \end{matrix},$$

$s \in \mathbb{R}$, *where* $g(y,\eta) \in \mathcal{R}^{\nu}_{G}(\Omega \times \mathbb{R}^q, \boldsymbol{g}, \boldsymbol{w})$, *cf. Definition* 2.5, *while* $m(y,\eta) := (m_{ij}(y,\eta))_{i,j=1,2,3}$ *for* $i,j = 1,2$ *is given by*

$$\begin{aligned} m_{ij}(y,\eta) &= r^{-\nu}\omega(r[\eta])\sum_{l=0}^{k} r^l \sum_{|\alpha|\leq l+(\mu-\nu)} \{\mathrm{op}^{\beta}_M(f_{l\alpha,ij}) + \mathrm{op}^{\tilde{\beta}}_M(\tilde{f}_{l\alpha,ij})\}\eta^{\alpha}\,\tilde{\omega}(r[\eta]) \\ &\quad : \mathcal{K}^{s,\gamma+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^{\wedge},\boldsymbol{E}) \to \mathcal{K}^{\infty,\delta+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^{\wedge},\boldsymbol{F}) \end{aligned} \tag{2.4.1}$$

with arbitrary Mellin symbols $f_{l\alpha,ij}(y,z) \in C^{\infty}(U,\ M^{-\infty}_{W_{l\alpha,ij}}(X_j, X_i; E_j, F_i))$, $\tilde{f}_{l\alpha,ij}(y,z) \in C^{\infty}(U, M^{-\infty}_{\widetilde{W}_{l\alpha,ij}}(X_j, X_i; E_j, F_i))$, *cut-off functions* ω, $\tilde{\omega}$, *and weights*

$\beta = \beta(l)$, $\tilde{\beta} = \tilde{\beta}(l)$, such that $W_{l\alpha,ij} \cap \Gamma_{\frac{1}{2}-\beta} = \widetilde{W}_{l\alpha,ij} \cap \Gamma_{\frac{1}{2}-\tilde{\beta}} = \emptyset$ for all l, α, i, j. Concerning the weights β and $\tilde{\beta}$ we can (and will) choose them in the same normalised way as in Remark 1.8. *Finally, $m_{ij}(y,\eta) := 0$ for $i = 3$ or $j = 3$.*

Without loss of generality we set $\tilde{f}_{00,ij} \equiv 0$; then, the principal conormal symbol of $(m+g)(y,\eta)$ of (conormal) order ν is defined as

$$\sigma_{\mathrm{M}}(m+g)(y,z,\eta) = \left(\sum_{|\alpha| \le \mu - \nu} (f_{0\alpha,ij}(y,z) + \tilde{f}_{0\alpha,ij}(y,z))\eta^{\alpha} \right)_{i,j=1,2}, \quad (2.4.2)$$

which, in the case $\mu = \nu$, reduces to

$$\sigma_{\mathrm{M}}(m+g)(y,z,\eta) = (f_{00,ij}(y,z))_{i,j=1,2}. \quad (2.4.3)$$

Remark 2.7 *Similarly to Remark* 1.8, *the choice of weights or cut-off functions only affects $m(y,\eta)$ modulo a Green symbol in the sense of Definition* 2.5. *The same is true if we change the function $\eta \mapsto [\eta]$. Observe that terms appearing in the sum* (2.4.1) *become automatically Green symbols for $l > k$, cf. the analogous remark for the cone after Definition* 1.7.

Proposition 2.8 *Every operator in $\mathcal{R}^{\nu}_{M+G}(\Omega \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})$ with $\boldsymbol{g}$ and $\boldsymbol{w}$ as in Definition* 2.6 *above is a classical operator-valued symbol. We have, for all $\alpha \in \mathbb{N}^q$, $\beta \in \mathbb{N}^q$, $D^{\alpha}_y D^{\beta}_{\eta} \mathcal{R}^{\nu}_{M+G}(\Omega \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w}) \subset \mathcal{R}^{\nu-|\beta|}_{M+G}(\Omega \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})$, and $\nu < \gamma - \delta - k$ implies $\mathcal{R}^{\nu}_{M+G}(\Omega \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w}) \subset \mathcal{R}^{\nu}_{G}(\Omega \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})$.*

In fact, setting $a(y,\eta) := (m+g)(y,\eta)$ and

$$a_0(y,\eta) = \mathrm{diag}(\langle\eta\rangle^{\frac{\boldsymbol{n}}{2}}, \langle\eta\rangle^{-\frac{1}{2}})\, a(y,\eta)\, \mathrm{diag}(\langle\eta\rangle^{\frac{\boldsymbol{n}}{2}}, \langle\eta\rangle^{-\frac{1}{2}})^{-1}, \quad (2.4.4)$$

we have the relation

$$a_0(y,\eta) \in S^{\nu}_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma+\frac{\boldsymbol{n}}{2}}_{(\boldsymbol{P})}(\boldsymbol{X}^{\wedge}, \boldsymbol{E}) \oplus \mathbb{C}^{j_-}, \mathcal{K}^{\infty,\gamma-\mu+\frac{\boldsymbol{n}}{2}}_{(\boldsymbol{Q})}(\boldsymbol{X}^{\wedge}, \boldsymbol{F}) \oplus \mathbb{C}^{j_+}) \quad (2.4.5)$$

for all $s \in \mathbb{R}$ (the interpretation with asymptotics is that for every pair of asymptotic types $\boldsymbol{P}$ there is a pair of asymptotic type $\boldsymbol{Q}$ depending on $a(y,\eta)$ as well as on $\boldsymbol{P}$, and that the corresponding relation holds). To see (2.4.5), because of Definition 2.5 it suffices to consider the finite sum of Mellin expressions (2.4.1) that are smooth in (y,η) as operator functions and homogeneous for large $|\eta|$: the latter property just implies that the Mellin part of $a_0(y,\eta)$ is also a classical operator-valued symbol. If $u(y,\eta)$ denotes a summand of (2.4.1) containing $r^l\eta^{\alpha}$ we have

$$u(y,\lambda\eta) = \lambda^{\nu-l+|\alpha|}\mathrm{diag}(\lambda^{\frac{\boldsymbol{n}}{2}}\kappa^{\boldsymbol{n}}_{\lambda}, \lambda^{-\frac{1}{2}}\mathrm{id})\, u(y,\eta)\, \mathrm{diag}(\lambda^{\frac{\boldsymbol{n}}{2}}\kappa^{\boldsymbol{n}}_{\lambda}, \lambda^{-\frac{1}{2}}\mathrm{id})^{-1}$$

for all $\lambda \ge 1$, $|\eta| \ge c$ with some sufficiently large $c > 0$. Analogously to (2.3.4), we have a homogeneous principal symbol $a_{0,(\nu)}(y,\eta)$ of $a_0(y,\eta)$ of order ν (clearly, with actions in the spaces involved in (2.4.5)). Moreover, returning from $a_0(y,\eta)$ to the original symbol $a(y,\eta)$ via relation (2.4.4) we get, according to the scheme (2.3.5) of DN-orders, a matrix $a_{(\boldsymbol{\nu})}(y,\eta) = (a_{ij,(\boldsymbol{\nu}_{ij})}(y,\eta))_{i,j=1,2,3}$ of homogeneous principal components of $a(y,\eta)$. Then, DN-homogeneity of $a_{(\boldsymbol{\nu})}(y,\eta)$ means

$$a_{(\boldsymbol{\nu})}(y,\lambda\eta) = \lambda^{\nu}\,\mathrm{diag}(\lambda^{\frac{\boldsymbol{n}}{2}}\kappa_\lambda^{\boldsymbol{n}}, \lambda^{-\frac{1}{2}}\mathrm{id})\, a_{(\boldsymbol{\nu})}(y,\eta)\,\mathrm{diag}(\lambda^{\frac{\boldsymbol{n}}{2}}\kappa_\lambda^{\boldsymbol{n}}, \lambda^{-\frac{1}{2}}\mathrm{id})^{-1} \qquad (2.4.6)$$

for all $(y,\eta) \in \Omega \times (\mathbb{R}^q \setminus \{0\})$, $\lambda \in \mathbb{R}_+$.

2.5 Edge amplitude functions

We describe here a specific space of operators-valued amplitude functions that will produce below our transmission operators modulo smoothing operators in the transmission operator algebra.

We first return to the reformulation of pseudo-differential actions on cones in terms of the Mellin transform in axial direction, cf. Theorem 1.2. We start from operator functions $p_l(r,y,\varrho,\eta)$ of analogous structure as (1.1.6) where here (because of the full symbolic calculus) we take them of order ν in place of μ. We then consider elements $\tilde{h}_l(r,y,z,\eta) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M_{\mathcal{O}}^{\nu}(X_l;\mathbb{R}^q))$ such that $h_l(r,y,z,\eta) := \tilde{h}_l(r,y,z,r\eta)$ satisfies relation

$$\mathrm{op}_r(p_l)(y,\eta) = \mathrm{op}_M^{\beta}(h_l)(y,\eta) \bmod C^\infty(\Omega, L^{-\infty}(X_l^\wedge;\mathbb{R}^q)) \qquad (2.5.1)$$

for every β. Similarly, we have a corresponding version of Remark 1.3. Let us set $\boldsymbol{g} := (\gamma, \gamma-\mu, \Theta)$ for $\Theta = (-(k+1), 0]$, and $\boldsymbol{w} := (E_1, F_1; E_2, F_2; j_-, j_+)$.

Definition 2.9 *The space $\mathcal{R}^\nu(\Omega \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})$ for $\mu - \nu \in \mathbb{N}$ is defined to be the set of all operator functions of the form*

$$a(y,\eta) := \begin{pmatrix} a_1(y,\eta) & 0 & 0 \\ 0 & a_2(y,\eta) & 0 \\ 0 & 0 & 0 \end{pmatrix} + (m+g)(y,\eta) \qquad (2.5.2)$$

where

(i) *the elements of the diagonal in the first term are of the form $a_l(y,\eta) := \theta(r)\{r^{-\nu}\omega(r[\eta])\mathrm{op}_M^{\gamma}(h_l)(y,\eta)\tilde{\omega}(r[\eta]) + r^{-\nu}(1-\omega(r[\eta]))\mathrm{op}_r(p_l)(y,\eta)(1-\tilde{\tilde{\omega}}(r[\eta]))\}\tilde{\theta}(r)$ for operator functions p_l and h_l linked to each other in the sense of relations (2.5.1), $l=1,2$, where $\theta, \tilde{\theta}, \omega, \tilde{\omega}, \tilde{\tilde{\omega}}$ are arbitrary cut-off functions satisfying $\omega\tilde{\omega} = \omega$, $\omega\tilde{\tilde{\omega}} = \tilde{\tilde{\omega}}$;*

(ii) *$(m+g)(y,\eta) \in \mathcal{R}^\nu_{M+G}(\Omega \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})$.*

Remark 2.10 *As it was proved in* [6], *the edge-amplitudes a_l can equivalently be written in the form*

$$a_l(y,\eta) = \theta(r) r^{-\nu} \mathrm{op}_M^{\gamma}(h_l)(y,\eta)\tilde{\theta}(r) + g_l(y,\eta)$$

where g_l is a suitable Green symbol of order ν in the edge algebra on $\mathbb{W}_l$, $l = 1,2$.

The following proposition completes Proposition 2.8 by a corresponding property of the diagonal elements $a_j(y,\eta)$ in (2.5.2), which are operator-valued symbols in the edge calculus on W_j, cf. [16].

Proposition 2.11 *Elements $a(y,\eta) \in \mathcal{R}^\nu(\Omega \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})$ are operator-valued symbols in the sense that*

$$a_0(y,\eta) := \mathrm{diag}(\langle\eta\rangle^{\frac{\boldsymbol{n}}{2}}, \langle\eta\rangle^{-\frac{1}{2}})a(y,\eta)\mathrm{diag}(\langle\eta\rangle^{\frac{\boldsymbol{n}}{2}}, \langle\eta\rangle^{-\frac{1}{2}})^{-1}$$

belong to the spaces

$$S^\nu(\Omega \times \mathbb{R}^q; \mathcal{K}^{s+\frac{\boldsymbol{n}}{2},\gamma+\frac{\boldsymbol{n}}{2}}_{(\boldsymbol{P})}(\boldsymbol{X}^\wedge, \boldsymbol{E}) \oplus \mathbb{C}^{j_-}, \mathcal{K}^{s+\frac{\boldsymbol{n}}{2}-\mu,\gamma+\frac{\boldsymbol{n}}{2}-\mu}_{(\boldsymbol{Q})}(\boldsymbol{X}^\wedge, \boldsymbol{F}) \oplus \mathbb{C}^{j_+})$$

for all $s \in \mathbb{R}$ (with interpretation analogous to the one given after (2.4.5)*).*

Let us now introduce DN-homogeneous principal edge symbols for elements $a(y,\eta) \in \mathcal{R}^\nu(\Omega\times\mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})$. The smoothing Mellin plus Green part $(m+g)(y,\eta)$ has been discussed in Sections 2.3, 2.4. Concerning the target spaces we take those which are suitable for the remaining entries $a_i(y,\eta)$, $i = 1,2$, in representation (2.5.2). We set

$$\begin{aligned} a_{l,(\nu)}(y,\eta) \;&:=\; r^{-\nu}\,\omega(r|\eta|)\,\mathrm{op}^\gamma_M(h_{l,0})(y,\eta)\,\tilde{\omega}(r|\eta|) \\ &\;+\; r^{-\nu}(1-\omega(r|\eta|))\,\mathrm{op}_r(p_{l,0})(y,\eta)\,(1-\tilde{\tilde{\omega}}(r|\eta|)) \end{aligned}$$

where subscripts 0 at h_l and p_l have the same meaning as in Remark 1.3.

We then finally define

$$a_{(\boldsymbol{\nu})}(y,\eta) := \mathrm{diag}(a_{1,(\nu)}(y,\eta), a_{2,(\nu)}(y,\eta), 0) + (m+g)_{(\boldsymbol{\nu})}(y,\eta),$$

which is regarded as a family of operators

$$a_{(\boldsymbol{\nu})}(y,\eta) \;:\; \begin{array}{c} \mathcal{K}^{s+\frac{\boldsymbol{n}}{2},\gamma+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^\wedge, \boldsymbol{E}) \\ \oplus \\ \mathbb{C}^{j_-} \end{array} \;\to\; \begin{array}{c} \mathcal{K}^{s-\nu+\frac{\boldsymbol{n}}{2},\gamma-\nu+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^\wedge, \boldsymbol{F}) \\ \oplus \\ \mathbb{C}^{j_+} \end{array},$$

$(y,\eta) \in T^*\Omega \setminus 0$, DN-homogeneous in the sense of relation (2.4.6). According to the notation in Section 1.1 after Remark 1.3, we have homogeneous principal symbols $\sigma_\psi(\mathrm{op}_r(p_l))(r,x,y,\varrho,\xi,\eta)$, $(\varrho,\xi,\eta) \neq 0$, and we set

$$\sigma_\psi(a_l)(r,x,y,\varrho,\xi,\eta) := \theta(r)\,r^{-\nu}\sigma_\psi(p_l)(r,x,y,\varrho,\xi,\eta),$$

$l = 1,2$. For purposes below we also introduce the compressed variants

$$\tilde{\sigma}_\psi(a_l)(r,x,y,\varrho,\xi,\eta) := \theta(r)\,\sigma_\psi(p_l)(r,x,y,r^{-1}\varrho,\xi,r^{-1}\eta) \tag{2.5.3}$$

that are smooth up to $r = 0$. For $a(y,\eta) \in \mathcal{R}^\nu(\Omega \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})$, then we have altogether

$$\sigma_\psi(a) := \mathrm{diag}(\sigma_\psi(a_1), \sigma_\psi(a_2))$$

and $\tilde{\sigma}_\psi(a) := \mathrm{diag}(\tilde{\sigma}_\psi(a_1), \tilde{\sigma}_\psi(a_2))$.

Remark 2.12 *The construction of the symbol classes could be repeated with no changes in the case where* $\Omega \subseteq \mathbb{R}^{q'}$ *with* $q' > q$. *This allows us to consider also the so-called "double symbols"* $a(y, y', \eta)$, *taking values in the same spaces as those described above. In particular, one can introduce, in this case, left and right symbols of operators initially formed via double symbols. Note that, owing to Proposition* 2.8, *the asymptotic summations involved in such constructions for smoothing Mellin operators would produce, after a finite number of derivatives, Green operators.*

3 Edge problems

3.1 Edge transmission operators

Let us fix a cut-off function $\theta(r) \in C_0^\infty(\overline{\mathbb{R}}_+)$ and a partition of unity $\{\varphi_j\}_{j=1,\dots,N}$ belonging to a finite atlas on the edge Y. Moreover, let $\{\psi_j\}_{j=1,\dots,N}$ be C^∞ functions with compact support in the charts such that $\psi_j \equiv 1$ on $\operatorname{supp}\varphi_j$ for all j. For convenience, operators of multiplication by 3×3-diagonal matrices containing θ or φ_j, ψ_j in the diagonal will be denoted by the same symbols θ, etc. Let $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$, $\nu \in \mathbb{R}$, $\mu - \nu \in \mathbb{N}$, $\boldsymbol{v} = (E_1, F_1; E_2, F_2; J_-, J_+)$ and $\boldsymbol{w} = (E_1, F_1; E_2, F_2; j_-, j_+)$, where the local bundles in $\boldsymbol{w}$ are related to the global ones in $\boldsymbol{v}$ as explained in Section 2.1 (and denoted by the same letters).

Definition 3.1 *The space* $\mathcal{Y}^\nu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ *of edge transmission operators of order* ν *on* $\mathbb{W}$ *associated with weight data* $\boldsymbol{g}$ *is defined to be the set of all operators*

$$\mathcal{A} : \begin{array}{c} C_0^\infty(\operatorname{int}\mathbb{W}, \boldsymbol{E}) \\ \oplus \\ C_0^\infty(Y, J_-) \end{array} \to \begin{array}{c} C^\infty(\operatorname{int}\mathbb{W}, \boldsymbol{F}) \\ \oplus \\ C^\infty(Y, J_+) \end{array}$$

of the form

$$\mathcal{A} = \theta \left(\sum_{j=1}^{N} \varphi_j \mathcal{A}_j \psi_j \right) \tilde{\theta} + (1 - \theta)\mathcal{A}_{\mathrm{int}}(1 - \tilde{\tilde{\theta}}) + \mathcal{C} \tag{3.1.1}$$

where

(i) $\mathcal{A}_j = \mathrm{Op}(a_j)$ *for some* $a_j(y, \eta) \in \mathcal{R}^\nu(\Omega \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w})$, $j = 1, \dots, N$;

(ii) $\mathcal{A}_{\mathrm{int}} = \mathrm{diag}(A_{1,\mathrm{int}}, A_{2,\mathrm{int}}, 0)$ *for operators* $A_{k,\mathrm{int}} \in L_{\mathrm{cl}}^\nu(\operatorname{int}\mathbb{W}_k; E_k, F_k)$, $k = 1, 2$;

(iii) $\mathcal{C} \in \mathcal{Y}^{-\infty}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$.

Let $\mathcal{Y}_{M+G}^\nu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ or $\mathcal{Y}_G^\nu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ denote the subsets where $A_{j,\mathrm{int}}$ vanishes, $j = 1, 2$ and $a_j(y, \eta) \in \mathcal{R}_{M+G}^\nu$ or $\mathcal{R}_G^\nu$ for all $j = 1, \dots, N$.

For the case that the fiber dimensions of $J_\pm$ are zero we use the same notation $\mathcal{Y}^\nu$, $\mathcal{Y}_{M+G}^\nu$ and $\mathcal{Y}_G^\nu$ for the respective classes; clearly, the $J_\pm$-components in $\boldsymbol{v}$ then disappear.

Theorem 3.2 *Every $\mathcal{A} \in \mathcal{Y}^{\nu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ extends to continuous operators*

$$\mathcal{A} \ : \ \begin{array}{c} \mathcal{W}^{s+\frac{\boldsymbol{n}}{2},\gamma+\frac{\boldsymbol{n}}{2}}(\mathbb{W}, \boldsymbol{E}) \\ \oplus \\ H^{s-\frac{1}{2}}(Y, J_-) \end{array} \rightarrow \begin{array}{c} \mathcal{W}^{s-\nu+\frac{\boldsymbol{n}}{2},\gamma-\nu+\frac{\boldsymbol{n}}{2}}(\mathbb{W}, \boldsymbol{F}) \\ \oplus \\ H^{s-\frac{1}{2}-\nu}(Y, J_+) \end{array} \tag{3.1.2}$$

for all $s \in \mathbb{R}$. Moreover, for every pair $\boldsymbol{P}$ of asymptotic types there is a pair $\boldsymbol{Q}$ such that (3.1.2) *restricts to continuous operators*

$$\mathcal{A} \ : \ \begin{array}{c} \mathcal{W}_{\boldsymbol{P}}^{s+\frac{\boldsymbol{n}}{2},\gamma+\frac{\boldsymbol{n}}{2}}(\mathbb{W}, \boldsymbol{E}) \\ \oplus \\ H^{s-\frac{1}{2}}(Y, J_-) \end{array} \rightarrow \begin{array}{c} \mathcal{W}_{\boldsymbol{Q}}^{s-\nu+\frac{\boldsymbol{n}}{2},\gamma-\nu+\frac{\boldsymbol{n}}{2}}(\mathbb{W}, \boldsymbol{F}) \\ \oplus \\ H^{s-\frac{1}{2}-\nu}(Y, J_+) \end{array} \tag{3.1.3}$$

for all $s \in \mathbb{R}$.

Proof. It is enough to prove the assertion for the local situation near the edge. The result then follows by Proposition 2.11 and by the continuity of pseudo-differential operators defined by operator-valued symbols in abstract wedge Sobolev spaces. In fact, $\mathrm{Op}(a) \ : \ \mathcal{W}^s(\mathbb{R}^q, E) \rightarrow \mathcal{W}^{s-\mu}(\mathbb{R}^q, \tilde{E})$ for $a \in S^{\mu}(\Omega \times \mathbb{R}^q; E, \tilde{E})$, and, in particular, $\mathrm{Op}(\langle\eta\rangle^{\delta}) \ : \ \mathcal{W}^s(\mathbb{R}^q, E) \rightarrow \mathcal{W}^{s-\delta}(\mathbb{R}^q, E)$, are continuous maps for arbitrary $\mu, s, \delta \in \mathbb{R}$, see [16], Proposition 1.3.24. $\square$

By definition, the elements $\mathcal{A} \in \mathcal{Y}^{\nu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ can be viewed as operator block matrices $\mathcal{A} = (A_{ij})_{i,j=1,2,3}$.

By the conditions $A_{ij} = 0$ when $i \neq k$ or $j \neq k$, $k = 1, 2$, we get subspaces of $\mathcal{Y}^{\nu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$, called $\mathcal{Y}^{\nu}(\mathbb{W}_k, \boldsymbol{g}; \boldsymbol{v}_k)$ for $\boldsymbol{v}_k = (E_k, F_k)$. These are nothing else than the edge operator spaces on a (stretched) manifold $\mathbb{W}_k$ with edge Y in the sense of [16] (with some abuse of notation in the definition of weight data). Conversely, every $\mathcal{A}_k \in \mathcal{Y}^{\nu}(\mathbb{W}_k, \boldsymbol{g}; \boldsymbol{v}_k)$ can be embedded as an element $\mathcal{A}_k^0 \in \mathcal{Y}^{\nu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ by filling up it by zeros to a corresponding 3×3-block matrix. Then every $\mathcal{A} \in \mathcal{Y}^{\nu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ has the form

$$\mathcal{A} = \mathcal{A}_1^0 + \mathcal{A}_2^0 + \mathcal{M} + \mathcal{G}, \tag{3.1.4}$$

where $\mathcal{A}_k \in \mathcal{Y}^{\nu}(\mathbb{W}_k, \boldsymbol{g}; \boldsymbol{v}_k))$, $k = 1, 2$, and $\mathcal{M} + \mathcal{G} \in \mathcal{Y}^{\nu}_{M+G}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$.

Remark 3.3 *$\mathcal{A} \in \mathcal{Y}^{\nu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ and $\mathcal{A}_k^0|_{\mathrm{int}\,\mathbb{W}_k} \in L^{-\infty}(\mathrm{int}\,\mathbb{W}_k; \boldsymbol{v}_k)$ for $k = 1, 2$ implies $\mathcal{A} \in \mathcal{Y}^{\nu}_{M+G}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$.*

Let us now introduce the principal symbolic structure (in fact, symbolic hierarchy) of elements in our transmission operator spaces. For every $\mathcal{A} \in \mathcal{Y}^{\nu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ we define

$$\sigma(\mathcal{A}) := (\sigma_{\psi}(\mathcal{A}), \sigma_{\Lambda}(\mathcal{A})),$$

where $\sigma_{\psi}(\mathcal{A})$ is the pair of homogeneous principal interior symbols in the edge algebras on $\mathbb{W}_k$

$$\sigma_{\psi}(\mathcal{A}_k) \ : \ \pi^*_{\mathrm{int}\,\mathbb{W}_k} E_k \rightarrow \pi^*_{\mathrm{int}\,\mathbb{W}_k} F_k, \qquad k = 1, 2, \tag{3.1.5}$$

$\pi_{\mathrm{int}\,\mathbb{W}_k} : T^*(\mathrm{int}\,\mathbb{W}_k) \setminus 0 \to \mathrm{int}\,\mathbb{W}_k$, for the operators $\mathcal{A}_k$ belonging to $\mathcal{A}_k^0$ in formula (3.1.4). It remains to specify $\sigma_\Lambda(\mathcal{A})$. First, from the calculus for $\mathcal{A}_k \in \mathcal{Y}^\nu(\mathbb{W}_k, \boldsymbol{g}; \boldsymbol{v}_k)$, we have associated homogeneous principal edge symbols

$$\sigma_\Lambda(\mathcal{A}_k) : \pi_Y^* \begin{pmatrix} \mathcal{K}^{s+\frac{n_k}{2},\gamma+\frac{n_k}{2}}(X_k^\wedge, E_k) \\ \oplus \\ J_- \end{pmatrix} \to \pi_Y^* \begin{pmatrix} \mathcal{K}^{s-\mu+\frac{n_k}{2},\gamma-\mu+\frac{n_k}{2}}(X_k^\wedge, F_k) \\ \oplus \\ J_+ \end{pmatrix}, \tag{3.1.6}$$

$k = 1, 2$, $\pi_Y : T^*Y \setminus 0 \to Y$, recall (2.1.2) and the description of the local bundles in Section 2.1. The principal edge symbols $\sigma_\Lambda(\mathcal{A}_k)$ give rise to corresponding families $\sigma_\Lambda(\mathcal{A}_k^0)$ by filling up block matrices by zero entries. Moreover, we have the local homogeneous principal edge symbols of $\mathcal{M} + \mathcal{G}$ that are the principal parts of classical operator-valued symbols, and we then get $\sigma_\Lambda(\mathcal{M}+\mathcal{G})$ globally on $T^*Y \setminus 0$. Then

$$\sigma_\Lambda(\mathcal{A}) := \sigma_\Lambda(\mathcal{A}_1^0) + \sigma_\Lambda(\mathcal{A}_2^0) + \sigma_\Lambda(\mathcal{M} + \mathcal{G})$$

is a family of maps

$$\sigma_\Lambda(\mathcal{A}) : \pi_Y^* \begin{pmatrix} \mathcal{K}^{s+\frac{\boldsymbol{n}}{2},\gamma+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^\wedge, \boldsymbol{E}) \\ \oplus \\ J_- \end{pmatrix} \to \pi_Y^* \begin{pmatrix} \mathcal{K}^{s-\mu+\frac{\boldsymbol{n}}{2},\gamma-\mu+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^\wedge, \boldsymbol{F}) \\ \oplus \\ J_+ \end{pmatrix}. \tag{3.1.7}$$

By virtue of the DN-homogeneity of the ingredients of $\sigma_\Lambda(\mathcal{A})$, formulated above, we have

$$\sigma_\Lambda(\mathcal{A})(y, \lambda\eta) = \lambda^\nu \mathrm{diag}(\lambda^{\frac{\boldsymbol{n}}{2}} \kappa_\lambda^{\boldsymbol{n}}, \lambda^{-\frac{1}{2}}\mathrm{id})\, \sigma_\Lambda(\mathcal{A})(y, \eta)\, \mathrm{diag}(\lambda^{\frac{\boldsymbol{n}}{2}} \kappa_\lambda^{\boldsymbol{n}}, \lambda^{-\frac{1}{2}}\mathrm{id})^{-1} \tag{3.1.8}$$

for all $(y, \eta) \in T^*Y \setminus 0$, $\lambda \in \mathbb{R}_+$.

Remark 3.4 *Because of the edge-degenerate behaviour of the upper left corner of an operator $\mathcal{A}$, cf. Definition* 2.9, *locally near the edge in the splitting of variables into $(r, x_{(k)}, y)$ with covariables $(\varrho, \xi_{(k)}, \eta)$, the symbols* (3.1.6) *have the form*

$$\sigma_\psi(\mathcal{A}_k)(r, x_{(k)}, y, \varrho, \xi_{(k)}, \eta) = r^{-\nu} \tilde{\sigma}_\psi(\mathcal{A}_k)(r, x_{(k)}, y, \tilde{\varrho}, \xi_{(k)}, \tilde{\eta})_{\tilde{\varrho}=r\varrho, \tilde{\eta}=r\eta},$$

where $\tilde{\sigma}_\psi(\mathcal{A}_k)$ is smooth up to $r = 0$, cf. (2.5.3) *and the end of Section* 2.5.

Remark 3.5 *Note that, together with the map $\sigma : \mathcal{A} \mapsto \sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_\Lambda(\mathcal{A}))$ defined above, we could define a map* $\mathrm{op} : p = (p_\psi, p_\Lambda) \mapsto \mathrm{op}(p) = \mathcal{P}$ *such that $\sigma(\mathcal{P}) = p$.*

3.2 Composition and adjoint

We now discuss the algebra property for the edge operators introduced in Definition 3.1. As in Section 1, we concentrate on the composition, and only state the next theorem about adjoints.

Theorem 3.6 *Let* $\mathcal{A} \in \mathcal{Y}^{\nu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ *with* $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$, $\mu - \nu \in \mathbb{N}$, $\Theta = (-(k+1), 0]$, $\boldsymbol{v} = (F_1, G_1; F_2, G_2; J_-, J_+)$. *Then, the formal adjoint* $\mathcal{A}^*$, *formed in the same sense of Definition* 2.4, *satisfies* $\mathcal{A}^* \in \mathcal{Y}^{\nu^*}(\mathbb{W}, \boldsymbol{g}^*; \boldsymbol{v}^*)$, *where* $\boldsymbol{g}^* = (-\gamma+\mu, -\gamma, \Theta)$ *and* $\boldsymbol{v}^* = (F_1, E_1; F_2, E_2; J_+, J_-)$, *while* ν^* *refers to the fact that the matrix of orders for the entries of* $\mathcal{A}^*$ *is the transpose of* (2.3.5). *Moreover,* $\sigma(\mathcal{A}^*) = \sigma(\mathcal{A})^*$, *with an obvious meaning of* * *on the right.*

Theorem 3.7 $\mathcal{A} \in \mathcal{Y}^{\nu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ *and* $\mathcal{B} \in \mathcal{Y}^{\tilde{\nu}}(\mathbb{W}, \boldsymbol{h}; \boldsymbol{w})$, *for* $\boldsymbol{g} = (\beta - \tilde{\mu}, \beta - \mu - \tilde{\mu}, \Theta)$, $\boldsymbol{h} = (\beta, \beta - \tilde{\mu}, \Theta)$, $\Theta = (-(k+1), 0]$, $\boldsymbol{v} = (F_1, G_1; F_2, G_2; \tilde{J}, J_+)$ *and* $\boldsymbol{w} = (E_1, F_1; E_2, F_2; J_-, \tilde{J})$ *implies* $\mathcal{A}\mathcal{B} \in \mathcal{Y}^{\nu+\tilde{\nu}}(\mathbb{W}, \boldsymbol{g} \circ \boldsymbol{h}; \boldsymbol{v} \circ \boldsymbol{w})$ *with* $\boldsymbol{g} \circ \boldsymbol{h} = (\beta, \beta - \mu - \tilde{\mu}, \Theta)$, $\boldsymbol{v} \circ \boldsymbol{w} = (E_1, G_1; E_2, G_2; J_-, J_+)$ (*cf. the notation of Theorem* 1.12), *and we have*

$$\sigma(\mathcal{A}\mathcal{B}) = \sigma(\mathcal{A})\, \sigma(\mathcal{B}) \tag{3.2.1}$$

with componentwise composition. If one factor belongs to the subclass with subscript $M + G$ *or* G, *then the same is true of the composition.*

Proof. We assume, for simplicity, that the bundles in $\boldsymbol{v}$ and $\boldsymbol{w}$ are trivial, and omit them from the notation from now on.

Let us write $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1 + \mathcal{C}$, $\mathcal{B} = \mathcal{B}_0 + \mathcal{B}_1 + \mathcal{D}$, where the decomposition refers to (3.1.1), with finite sums of operators $\mathcal{A}_0$, $\mathcal{B}_0$ referring to edge-amplitude functions, interior operators $\mathcal{A}_1$, $\mathcal{B}_1$ localised far from the edge, and smoothing operators $\mathcal{C}$, $\mathcal{D}$, respectively. The nature of compositions containing $\mathcal{C}$ or $\mathcal{D}$ as factors is clear from Definition 2.4 and Theorems 3.2, 3.6. Moreover, the composition $\mathcal{A}_1\mathcal{B}_1$ of interior operators entirely refers to the standard pseudo-differential calculus and yields again an operator of the required structure. From the products $\mathcal{A}_0\mathcal{B}_1$ and $\mathcal{A}_1\mathcal{B}_0$ we also get interior operators plus smoothing operators in the calculus because of pseudo-locality, that gives us smoothing operators whenever an operator in our class is composed from both sides by C^∞ functions of disjoint support. Thus, there remains to consider $\mathcal{A}_0\mathcal{B}_0$. Without loss of generality we assume that our open covering on Y is chosen in such a way that whenever open sets have a non-empty intersection, their union is contained in a coordinate neighbourhood. Then, using again pseudo-locality of summands in $\mathcal{A}_0$ or $\mathcal{B}_0$, the essential contributions are of the form $\operatorname{op}(a) \operatorname{op}(b)$ for local amplitude functions $a(y, y', \eta)$ and $b(y, y', \eta)$ in an open set $\Omega \subseteq \mathbb{R}^q$, cf. Remark 2.12. In this construction, because of involved factors of compact supports in y or y', the amplitude functions may be regarded as elements of $\mathcal{R}^{\nu}(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q, \boldsymbol{g})$ and $\mathcal{R}^{\nu}(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q, \boldsymbol{h})$ with variables/covariables (y, y', η) and compact support in (y, y'). This meets the standard scenario of pseudo-differential calculus globally on $\mathbb{R}^q$ with uniform symbol estimates, in the variant of operator-valued symbols. The general calculus allows us to pass from double symbols $a(y, y', \eta)$ to left or right symbols $a_L(y, \eta)$ or $a_R(y', \eta)$ on the level of operators, modulo operators of order $-\infty$. In other words, we have

$$\operatorname{op}(a) \sim \operatorname{op}(a_L) \text{ and } \operatorname{op}(b) \sim \operatorname{op}(b_R), \tag{3.2.2}$$

where $\sim$ indicates equality modulo operators of order $-\infty$. A technical point is to verify that these remainders are even smoothing in our calculus. Looking at the structure of our amplitude functions, there are non-smoothing summands only referring to $\mathbb{W}_1$ or $\mathbb{W}_2$. The corresponding considerations for those summands are known from the usual edge calculus, see also the scheme of Theorem 1.12 for the pointwise behaviour of operator functions on the respective model cones. Therefore, to characterise remainders in (3.2.2) we have to deal only with the case of amplitude functions belonging to $\mathcal{R}^{\nu}_{M+G}$ and $\mathcal{R}^{\tilde{\nu}}_{M+G}$, respectively. In the finite weight interval case we are considering, those are finite sums of Mellin terms plus Green terms. By Definition 2.5, the Green symbols are completely covered by the abstract scheme of operator-valued symbols. So there remain amplitude functions consisting of finitely many summands of expressions of the form

$$r^j \omega(r[\eta]) \operatorname{op}_M^{\beta}(h)(y,y')\eta^{\alpha}\, \widetilde{\omega}(r[\eta]), \tag{3.2.3}$$

$|\alpha| \leq j$, for smoothing Mellin symbols $h(y,y',z)$ and certain weights β. Owing to the structure of the amplitude functions (3.2.3), we can pass to left or right symbol representations directly, by a finite Taylor expansion in y' at the diagonal of $\Omega \times \Omega$, modulo Green remainders as treated before, cf. Proposition 2.8 and [16], Theorems 1.1.30 and 1.1.54. In other words, we arrive at the composition $\operatorname{op}(a_L)\operatorname{op}(b_R) = \operatorname{op}(a_L b_R)$ after ignoring terms with smoothing factors that yield smoothing operators in our calculus.

Concerning $(a_L b_R)(y,y',\eta)$, we have to verify $a_L b_R \in \mathcal{R}^{\nu}(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q, \boldsymbol{g} \circ \boldsymbol{h})$. There are again some summands known from the theory of standard manifolds with edges. The main contribution of the latter category comes from the non-smoothing terms with holomorphic Mellin symbols. They are treated thoroughly in the paper [6]. Concerning compositions where one factor is of Green type we get again Green symbols, similarly to the corresponding calculations for the standard edge algebra. Thus there remain terms in the composition of the 2×2-upper left corner of a_L and b_R. After the observations before, all of them are treated, except when one factor is of the form (3.2.3). If both factors are of this type, we get an operator of type $\mathcal{R}^{\nu}_{M+G}$ (again, see the analogous result for the pointwise composition in the proof of Theorem 1.12). So the last kind of terms is the one involving one factor like (3.2.3) and one non-smoothing. When composing with symbols localised in the interior, the presence of factors of the type $\varphi(t) = \tilde{\omega}(t)(1-\omega)(t)$, compactly supported in $(0,1)$, immediately gives Green operators (note also that cut-off functions evaluated in $r[\eta]$ turn out to be classical operator-valued symbols). The other type of composition of Mellin operators gives expressions similar to those appearing in the standard edge calculus, namely

$$\omega(r[\eta])\, r^{-\nu} \operatorname{op}_M^{\beta}(f)(y)\, \widetilde{\omega}(r[\eta])\, r^{-\tilde{\nu}} \operatorname{op}_M^{\beta}(h)(y',\eta)\, \widetilde{\widetilde{\omega}}(r[\eta]). \tag{3.2.4}$$

They are formally treated as the similar terms examined in the composition of operators on the cones, and give symbols of Mellin plus Green type. This completes the proof of $a_L b_R \in \mathcal{R}^{\nu}(\mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^q, \boldsymbol{g} \circ \boldsymbol{h})$. The last step is now to pass again to a left symbol $(a_L b_R)_L$ modulo a smoothing remainder on the level of operators, which is possible, by the considerations above.

The symbolic rule for the $\psi-$ component of $\sigma(\mathcal{AB})$ is clear. That also the principal edge symbols are multiplied is again a consequence of the analogous rule for the standard edge algebra and of the following observation: since the Mellin symbols appearing in the products (3.2.4) can be written as $\tilde{h}(r, y, z, r\eta) = \tilde{h}(0, y, z, r\eta) + r\tilde{h}_{(1)}(r, y, z, r\eta)$ with a smooth remainder $\tilde{h}_{(1)}$, only the product of the principal edge symbols of $\mathcal{A}$ and $\mathcal{B}$ can contribute to $\sigma_\Lambda(\mathcal{AB})$, due to the presence of at least one r factor in the other three terms. $\square$

3.3 Ellipticity

Definition 3.8 *An element $\mathcal{A} \in \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ for $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$ and $\boldsymbol{v} = (E_1, F_1; E_2, F_2; J_-, J_+)$, is said to be elliptic if*

(i) *both bundle homomorphisms* (3.1.5) *are isomorphisms, where also the "compressed variants"* (*locally near the edge*) $\tilde{\sigma}_\psi(\mathcal{A}_k)$, $k = 1, 2$, *are isomorphisms up to $r = 0$;*

(ii) *the family of maps* (3.1.7) *is a bundle isomorphism for an $s = s_0 \in \mathbb{R}$.*

Remark 3.9 *Similarly to the "usual" edge calculus, condition* (ii) *implies that* (3.1.7) *is an isomorphism for all $s \in \mathbb{R}$.*

Remark 3.10 *Condition* (ii) *is an analogue of the classical Shapiro-Lopatinskij condition for boundary value problems: here they have the shape of transmission conditions.*

Note that the values of $\sigma_\Lambda(\mathcal{A})(y, \eta)$ for $(y, \eta) \in T^*Y \setminus 0$ are uniquely determined by the restriction to S^*Y, the unit cosphere bundle induced by T^*Y (recall that we have fixed a Riemannian metric on Y). In particular, the relation

$$\begin{aligned} \sigma_\Lambda(\mathcal{A})(y, \eta) \quad &= \quad |\eta|^\nu \mathrm{diag}(|\eta|^{\frac{n_1}{2}} \kappa^{n_1}_{|\eta|}, |\eta|^{\frac{n_2}{2}} \kappa^{n_2}_{|\eta|}, \lambda^{-\frac{1}{2}}\mathrm{id})^{-1}\, \sigma_\Lambda(\mathcal{A}) \left(y, \frac{\eta}{|\eta|} \right) \\ &\qquad \mathrm{diag}(|\eta|^{\frac{n_1}{2}} \kappa^{n_1}_{|\eta|}, |\eta|^{\frac{n_2}{2}} \kappa^{n_2}_{|\eta|}, \lambda^{-\frac{1}{2}}\mathrm{id}) \qquad (3.3.1) \end{aligned}$$

defines the extension of $\sigma_\Lambda(\mathcal{A})(y, \eta)|_{S^*Y}$ by homogeneity to $T^*Y \setminus 0$. Clearly, $\sigma_\Lambda(\mathcal{A})(y, \eta)|_{S^*Y}$ is a family of isomorphisms if and only if so is $\sigma_\Lambda(\mathcal{A})(y, \eta)$ for all $(y, \eta) \in T^*Y \setminus 0$.

Let us now draw some further conclusions from the ellipticity condition on $\sigma_\Lambda(\mathcal{A})$. Write $\sigma_\Lambda(\mathcal{A})(y, \eta)|_{S^*Y} =: a(y, \eta) = (a_{ij}(y, \eta))_{i,j=1,2,3}$, $b(y, \eta) := (a_{ij}(y, \eta))_{i,j=1,2}$. Then, if $a(y, \eta)$ is invertible,

$$b(y, \eta) \ : \ \mathcal{K}^{s+\frac{\boldsymbol{n}}{2}, \gamma+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^\wedge, \boldsymbol{E}) \to \mathcal{K}^{s-\mu+\frac{\boldsymbol{n}}{2}, \gamma-\mu+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^\wedge, \boldsymbol{F}) \qquad (3.3.2)$$

is a family of Fredholm operators, belonging to the cone transmission algebra of Section 1. As such, there is the symbolic structure of cone operators from that calculus. In particular, we have the principal conormal symbol

$$\sigma_{\mathrm{M}}(b)(y, z) \ : \ H^{s+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}, \boldsymbol{E}) \to H^{s-\mu+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}, \boldsymbol{F}) \qquad (3.3.3)$$

which has the form

$$\sigma_{\mathrm{M}}(b)(y,z) = (\sigma_{\mathrm{M}}(b_{ij})(y,\eta))_{i,j=1,2}\,,$$

$$\sigma_{\mathrm{M}}(b_{ij})(y,z) = \mathrm{diag}(h_{1,0}(0,y,z,0), h_{2,0}(0,y,z,0)) + (f_{00,ij}(y,z))_{i,j=1,2},$$

cf. the notation in Definition 2.9 (i), Remark 1.3, and formula (2.4.3) (for $\mu = \nu$). From the cone calculus on $\boldsymbol{X}^\wedge$ we know that the Fredholm property of (3.3.2) at a point $(y,\eta) \in S^*Y$ implies that (3.3.2) is a family of bijections, for all $y \in Y$, $z \in \Gamma_{\frac{1}{2}-\gamma}$.

Remark 3.11 *The ellipticity of the operator $\mathcal{A}$ with respect to the interior symbol $\sigma_\psi(\mathcal{A})$, cf. Definition* 3.8 (i)*, implies that* (3.3.2) *is elliptic in the sense of the σ_ψ- and σ_{e}- components of principal symbols from the cone theory, for every $(y,\eta) \in S^*Y$, in particular, $b_{jj}(y,\eta)$ is σ_{e}-elliptic (i.e., exit elliptic for $r \to +\infty$), $j = 1,2$. The ellipticity with respect to $\sigma_{\mathrm{M}}(\mathcal{A})$ is not automatic. If we require that* (3.3.3) *is a family of isomorphisms for all $y \in Y$, $z \in \Gamma_{\frac{1}{2}-\gamma}$ (for any fixed s, which implies the same for all $s \in \mathbb{R}$), then (if $(\mathcal{A}_{ij})_{i,j=1,2}$ denotes the 2×2-upper left corner of $\mathcal{A}$) the operators*

$$b(y,\eta) := \sigma_\Lambda((\mathcal{A}_{ij})_{i,j=1,2})(y,\eta) : \mathcal{K}^{s+\frac{n}{2},\gamma+\frac{n}{2}}(\boldsymbol{X}^\wedge,\boldsymbol{E}) \to \mathcal{K}^{s-\mu+\frac{n}{2},\gamma-\mu+\frac{n}{2}}(\boldsymbol{X}^\wedge,\boldsymbol{F})$$

form a family of Fredholm operators, parametrised by $(y,\eta) \in S^*Y$, cf. Theorem 1.16.

From the regularity properties of solutions to elliptic transmission equations on the infinite stretched cone $X^\wedge$ we know that $\ker b(y,\eta)$ and $\operatorname{coker} b(y,\eta)$ of the Fredholm operators (3.3.2) are independent of s. Let us assume for simplicity that S^*Y is connected. Then, $\operatorname{ind} b(y,\eta)$ is constant, i.e., independent of $(y,\eta) \in S^*Y$.

From standard construction of K-theory in connection with families of Fredholm operators parametrised by a compact topological space we have an index element

$$\operatorname{ind}_{S^*Y} b \in K(S^*Y),$$

where $K(.)$ denotes the K-group of the space in brackets. The canonical projection $\pi_1 : S^*Y \to Y$ gives rise to a pull-back $\pi_1^* K(Y) \to K(S^*Y)$. In the present case, from the fact that (3.1.7) is an isomorphism, we can read off the index element of b explicitly, namely,

$$\operatorname{ind}_{S^*Y} b = [\pi_1^* J_+] - [\pi_1^* J_-],$$

which belongs to $\pi_1^* K(Y)$. In this connection we have the following theorem, that extends a topological criterion of Atiyah and Bott [2] about the existence of Shapiro-Lopatinskij elliptic conditions to an elliptic operator on a manifold with boundary.

Theorem 3.12 *Let $A = (A_{ij})_{i,j=1,2} \in \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{w})$ for $\boldsymbol{g} = (\gamma, \gamma-\mu, \Theta)$ and $\boldsymbol{w} = (E_1, F_1; E_2, F_2)$ be an operator that is σ_ψ-elliptic in the sense of Definition* 3.8 (i) *and such that* (3.3.2) *is a family of Fredholm operators. Then,* *there exists an elliptic operator* $\mathcal{A} = (\mathcal{A}_{ij})_{i,j=1,2,3} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ *for* $\boldsymbol{v} := (E_1, F_1; E_2, F_2; J_-, J_+)$ *and suitable* $J_{\pm} \in \mathrm{Vect}\,(Y)$ *with* $A = (\mathcal{A}_{ij})_{i,j=1,2}$ *if and only if* $\mathrm{ind}_{\,S^*Y} b \in \pi_1^* K(Y)$.

Proof. Let us first assume that $\mathcal{A} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ is an elliptic operator. Then, in particular, $\sigma_{\Lambda}(\mathcal{A})$ is a family of isomorphisms (3.1.7). Since the entries $\sigma_{\Lambda}(\mathcal{A}_{ij})$ for $i = 3$ or $j = 3$ are of finite rank, it follows that

$$b := \sigma_{\Lambda}(\mathcal{A}_{ij})_{i,j=1,2} \ : \ \pi_1^* \mathcal{K}^{s+\frac{\boldsymbol{n}}{2}, \gamma+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^{\wedge}, \boldsymbol{E}) \to \pi_1^* \mathcal{K}^{s-\mu+\frac{\boldsymbol{n}}{2}, \gamma-\mu+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^{\wedge}, \boldsymbol{F}) \tag{3.3.4}$$

is a family of Fredholm operators parametrised by $(y, \eta) \in S^*Y$. Moreover, (3.1.7) implies

$$\mathrm{ind}_{\,S^*Y} b = [\pi_1^* J_+] - [\pi_1^* J_-] \in K(S^*Y), \tag{3.3.5}$$

where here [.] denotes the equivalence class of the bundle in brackets in the K-group on S^*Y. Since $[\pi_1^* J_+] - [\pi_1^* J_-] = \pi_1^*([J_+] - [J_-])$, the index element (3.3.5) is the pull-back of $[J_+] - [J_-] \in K(Y)$ under $\pi_1 \ : \ S^*Y \to Y$, i.e., it follows that (3.3.5) belongs to $\pi_1^* K(Y)$.

On the other hand, assuming that (3.3.4) is a family of isomorphisms such that $\mathrm{ind}_{\,S^*Y} b \in \pi_1^* K(Y)$ holds, we can find an $N_- \in \mathbb{N}$ and a map $k \in C^{\infty}(S^*Y, \mathcal{L}(\mathbb{C}^{N_-}, C_0^{\infty}(\boldsymbol{X}^{\wedge}, \boldsymbol{F})))$ such that

$$(\, b(y,\eta) \quad k(y,\eta) \,) \ : \ \begin{array}{c} \mathcal{K}^{s+\frac{\boldsymbol{n}}{2}, \gamma+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^{\wedge}, \boldsymbol{E}) \\ \oplus \\ \mathbb{C}^{N_-} \end{array} \to \mathcal{K}^{s-\mu+\frac{\boldsymbol{n}}{2}, \gamma-\mu+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^{\wedge}, \boldsymbol{F}) \tag{3.3.6}$$

is surjective for every $(y, \eta) \in S^*Y$ and every $s \in \mathbb{R}$. The existence of k is an easy consequence of the elliptic regularity in the calculus on the cone $\boldsymbol{X}^{\wedge}$ and of the fact that $C_0^{\infty}(\boldsymbol{X}^{\wedge}, \boldsymbol{F})$ is dense in the space $\mathcal{K}^{s-\mu+\frac{\boldsymbol{n}}{2}, \gamma-\mu+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^{\wedge}, \boldsymbol{F})$ for every $s, \gamma \in \mathbb{R}$. From general properties of families of Fredholm operators parametrised by a compact C^{∞} manifold, here S^*Y, we know that $\widetilde{G} := \ker(\, b(y,\eta) \quad k(y,\eta) \,)$ is isomorphic to a C^{∞} vector bundle G on S^*Y. Thus, we can fill up the family (3.3.6) to a C^{∞} family of isomorphisms

$$\begin{pmatrix} b(y,\eta) & k(y,\eta) \\ t(y,\eta) & p(y,\eta) \end{pmatrix} : \begin{array}{c} \mathcal{K}^{s+\frac{\boldsymbol{n}}{2}, \gamma+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^{\wedge}, \boldsymbol{E}) \\ \oplus \\ \mathbb{C}^{N_-} \end{array} \to \begin{array}{c} \mathcal{K}^{s-\mu+\frac{\boldsymbol{n}}{2}, \gamma-\mu+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^{\wedge}, \boldsymbol{F}) \\ \oplus \\ G_{(y,\eta)} \end{array} \tag{3.3.7}$$

(where $G_{(y,\eta)}$ denotes the fiber of G over the point (y, η)). The second row of (3.3.7) can be obtained by composing any isomorphism $\widetilde{G} \to G$ from the left by a projection $\mathcal{K}^{s+\frac{\boldsymbol{n}}{2}, \gamma+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^{\wedge}, \boldsymbol{E}) \oplus \mathbb{C}^{N_-} \to \widetilde{G}_{(y,\eta)}$. Because of elliptic regularity in the cone calculus, $\widetilde{G}$ is independent of s (it is, in fact, a subbundle of $\mathcal{S}^{\gamma+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^{\wedge}, \boldsymbol{E}) \oplus \mathbb{C}^{N_-}$ for $\mathcal{S}^{\gamma+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}, \boldsymbol{E}) = \cap \langle r \rangle^{-N} \mathcal{K}^{N, \gamma+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^{\wedge}, \boldsymbol{E})$. Now, $\mathrm{ind}_{\,S^*Y} b = [G] - [\mathbb{C}^{N_-}]$ is just the definition of the index element of a family of Fredholm operators b that employs a family of isomorphisms (3.3.7), though, as it is known, $\mathrm{ind}_{\,S^*Y} b$ is independent of the choice of (3.3.7). Moreover, we have $G = \pi_1^* J_+$ for some $J_+ \in \mathrm{Vect}\,(Y)$ as soon as N_- is chosen sufficiently large. The entries $k(y,\eta)$, $t(y,\eta)$ and $p(y,\eta)$ can be extended to $T^*Y \setminus 0$ by homogeneity with respect to the group actions, so that the extension of (3.3.7) to $T^*Y \setminus 0$ satisfies (3.1.8) with μ in place of ν. Note that then, locally, $\chi(\eta)k(y,\eta)$, $\chi(\eta)t(y,\eta)$ and $\chi(\eta)p(y,\eta)$ are classical operator-valued symbols. Setting $\boldsymbol{w} = (E_1, F_1; E_2, F_2; \mathbb{C}^{N_-}, \mathbb{C}^{N_+})$, where N_+ is the fiber dimension of J_+, we have

$$c := \begin{pmatrix} 0 & \chi(\eta)k(y,\eta) \\ \chi(\eta)t(y,\eta) & \chi(\eta)p(y,\eta) \end{pmatrix} \in \mathcal{R}^{\mu}_{G}(\Omega \times \mathbb{R}^q, \boldsymbol{g}; \boldsymbol{w}), \tag{3.3.8}$$

with 0 denoting a 2×2 vanishing matrix. For operators defined globally on $\mathbb{W}$ we write $\boldsymbol{v} = (E_1, F_1; E_2, F_2; J_-, J_+)$ for $J_- = \mathbb{C}^{N_-}$. By hypotheses,

$$D = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v}) \text{ and } \sigma_\Lambda(D) = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$$

We get the claimed result setting $\mathcal{A} := C + D$, where C is the Green operator defined by the local amplitude function c. $\square$

3.4 Parametrices and regularity with asymptotics

Definition 3.13 *Let $\mathcal{A} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ be an operator in the notation of Definition* 3.8. *Then an operator $\mathcal{P} \in \mathcal{Y}^{-\mu}(\mathbb{W}, \boldsymbol{g}^{-1}; \boldsymbol{v}^{-1})$ for $\boldsymbol{g}^{-1} := (\gamma - \mu, \gamma, \Theta)$, $\boldsymbol{v}^{-1} := (F_1, E_1; F_2, E_2; J_+, J_-)$ is said to be a parametrix of $\mathcal{A}$ if*

$$\mathcal{A}\mathcal{P} - \mathcal{I} \in \mathcal{Y}^{-\infty}(\mathbb{W}, \boldsymbol{g}_r; \boldsymbol{v}_r) \text{ and } \mathcal{P}\mathcal{A} - \mathcal{I} \in \mathcal{Y}^{-\infty}(\mathbb{W}, \boldsymbol{g}_l; \boldsymbol{v}_l)$$

for $\boldsymbol{g}_r = (\gamma - \mu, \gamma - \mu, \Theta)$, $\boldsymbol{v}_r = (F_1, F_1; F_2, F_2; J_+, J_+)$ and $\boldsymbol{g}_l = (\gamma, \gamma, \Theta)$, $\boldsymbol{v}_l = (E_1, E_1; E_2, E_2; J_-, J_-)$.

Theorem 3.14 *Let $\mathcal{A} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ be elliptic in the sense of Definition* 3.8. *Then, $\mathcal{A}$ has a parametrix $\mathcal{P} \in \mathcal{Y}^{-\mu}(\mathbb{W}, \boldsymbol{g}^{-1}; \boldsymbol{v}^{-1})$.*

Proof. For convenience, we consider the case where E_k and F_k, $k = 1, 2$, are trivial bundles of fiber dimension 1 and omit the bundle data from the notation. Moreover, for every 3×3 matrix $\mathcal{B}$ we will write $\widetilde{\mathcal{B}}$ for its 2×2 upper left corner $(\mathcal{B}_{ij})_{i,j=1,2}$.
As in relation (3.1.4), the operator $\mathcal{A}$ can be written in the form $\operatorname{diag}(\mathcal{A}_1, \mathcal{A}_2, 0)$ $+ \mathcal{M} + \mathcal{G}$, for $\mathcal{A}_k \in \mathcal{Y}^{\mu}(\mathbb{W}_k, \boldsymbol{g})$ and $\mathcal{M} + \mathcal{G} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g})$. By virtue of Definition 3.8 (i), applied to $\mathcal{A}_k$ for $k = 1, 2$, from the elliptic theory in the edge algebra on $\mathbb{W}_k$ we find operators $\mathcal{B}_k \in \mathcal{Y}^{-\mu}(\mathbb{W}_k, \boldsymbol{g}^{-1})$ such that $\mathcal{B}_k \mathcal{A}_k = \mathcal{I} \bmod \mathcal{Y}^{0}_{M+G}(\mathbb{W}_k, (\gamma, \gamma, \Theta))$, cf. Remark 3.3. Setting $\mathcal{P}_0 := \operatorname{diag}(\mathcal{B}_1, \mathcal{B}_2, 0)$ we then obtain

$$\mathcal{P}_0 \mathcal{A} = \mathcal{I} \bmod \mathcal{Y}^{0}_{M+G}(\mathbb{W}, \boldsymbol{g}_l).$$

On the level of principal conormal symbols this yields the identity

$$\sigma_{\mathrm{M}}(\mathcal{P}_0)(y, z + \mu)\, \sigma_{\mathrm{M}}(\mathcal{A})(y, z) = 1 + f(y, z)$$

for a function $f \in C^{\infty}(Y, \mathcal{M}^{-\infty}_{R})$ where $\mathcal{M}^{-\infty}_{R}$ is the space of 2×2 matrices of smoothing operator-valued Mellin symbols $f_{ij} \in M^{-\infty}_{R_{ij}}(X_i, X_j)$ with asymptotic types R_{ij}, $j = 1, 2$ (recall that, by Definition 2.6, only $\widetilde{\mathcal{A}}$ contributes to the Mellin principal symbol). Since $(1+f(y,z))^{-1} = 1+\widetilde{f}(y,z)$ for another element $\widetilde{f}(y,z) \in C^\infty(Y, \mathcal{M}_S^{-\infty})$ and some matrix S of asymptotic types, we get

$$\sigma_{\mathrm{M}}^{-1}(\mathcal{A})(y,z) = (1+\widetilde{f}(y,z))\,\sigma_{\mathrm{M}}(\mathcal{P}_0)(y,z+\mu) = \sigma_{\mathrm{M}}(\mathcal{P}_0)(y,z+\mu) + l(y,z+\mu)$$

for some $l(y,z) \in C^\infty(Y, \mathcal{M}_P^{-\infty})$ with a resulting asymptotic type P, where the carrier of P does not intersect $\Gamma_{\frac{1}{2}-(\gamma-\mu)}$. The next step in the construction of the parametrix is to pass to $\widetilde{\mathcal{P}}_1 := \widetilde{\mathcal{P}}_0 + \mathcal{M}_0$, where $\mathcal{M}_0 = r^\mu \omega(r[\eta])\mathrm{op}_M^{\gamma-\mu}(l)(y)\widetilde{\omega}(r[\eta])$ is an operator such that

$$\sigma_{\mathrm{M}}(\widetilde{\mathcal{P}}_1)(y,z+\mu) = \sigma_{\mathrm{M}}(\mathcal{A})^{-1}(y,z). \tag{3.4.1}$$

We thus obtain $\widetilde{\mathcal{P}}_1\widetilde{\mathcal{A}} = \mathcal{I} - \mathcal{M}_1$ where, because of (3.4.1), the highest conormal symbol of $\mathcal{M}_1$ vanishes. Thus, setting $\widetilde{\mathcal{P}}_2 := \sum_{j=0}^{k} \mathcal{M}_1^j$, we get $\widetilde{\mathcal{P}}_2\widetilde{\mathcal{A}} = \mathcal{I} - \mathcal{G}$ for a 2×2 matrix $\mathcal{G}$ of Green operators of order 0.

The σ_Λ-ellipticity of the operator $\mathcal{A}$ shows that

$$\sigma_\Lambda(\widetilde{\mathcal{A}})(y,\eta)\ :\ \mathcal{K}^{s+\frac{\boldsymbol{n}}{2},\gamma+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^\wedge) \to \mathcal{K}^{s-\mu+\frac{\boldsymbol{n}}{2},\gamma-\mu+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^\wedge) \tag{3.4.2}$$

is a family of Fredholm operators, parametrised by $(y,\eta) \in S^*Y$, where

$$\mathrm{ind}\,_{S^*Y}\sigma_\Lambda(\widetilde{\mathcal{A}}) = [\pi_1^* J_+] - [\pi_1^* J_-].$$

(3.4.2) is a family of elliptic operators in the sense of the cone algebra on $\boldsymbol{X}^\wedge$ (cf. Section 1), and $\sigma_\Lambda(\widetilde{\mathcal{P}}_2)(y,\eta)$ is a family of parametrices in that algebra, which implies

$$\mathrm{ind}\,_{S^*Y}\sigma_\Lambda(\widetilde{\mathcal{P}}_2) = [\pi_1^* J_-] - [\pi_1^* J_+].$$

Now, similarly to the considerations in the proof of Theorem 3.12, we find a family of isomorphisms

$$p(y,\eta) := \begin{pmatrix} \sigma_\Lambda(\widetilde{\mathcal{P}}_2) & \sigma_\Lambda(K) \\ \sigma_\Lambda(T) & \sigma_\Lambda(Q) \end{pmatrix}(y,\eta) : \begin{array}{c} \mathcal{K}^{s+\frac{\boldsymbol{n}}{2},\gamma+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^\wedge) \\ \oplus \\ (J_+ \oplus \mathbb{C}^N)_y \end{array} \to \begin{array}{c} \mathcal{K}^{s-\mu+\frac{\boldsymbol{n}}{2},\gamma-\mu+\frac{\boldsymbol{n}}{2}}(\boldsymbol{X}^\wedge) \\ \oplus \\ (J_+ \oplus \mathbb{C}^N)_y \end{array} \tag{3.4.3}$$

for some (sufficiently large) N. The entries $\sigma_\Lambda(K)$, $\sigma_\Lambda(T)$, $\sigma_\Lambda(Q)$ may be extended by homogeneity $-\mu$ (with respect to the group actions, cf. relation (3.3.1)) to a family that has the structure of a homogeneous Green symbol of order $-\mu$ in our edge symbolic algebra. In order to invert $\sigma_\Lambda(\mathcal{A})(y,\eta)$, we compose (from the right) $a = \mathrm{diag}(\sigma_\Lambda(\mathcal{A})|_{S^*Y}, \mathrm{id}_{\mathbb{C}^N})$ with (3.4.3) and get a family of operators that has the form $b(y,\eta) = (b_{ij}(y,\eta))_{i,j=1,2}$ where b_{11} takes values in $C^0_{M+G}(\boldsymbol{X}^\wedge, \boldsymbol{g})$ and (by construction) satisfies $\sigma_{\mathrm{M}}(b_{11}) = 1$ for all $(y,\eta) \in S^*Y$, while the other entries $b_{ij}(y,\eta)$ are of finite rank. Since the involved factors in the composition are invertible, $b(y,\eta)$ is a family of invertible operators as well.

We now obtain

$$b(y,\eta)\,a(y,\eta) = 1 + g(y,\eta) \tag{3.4.4}$$

for an element $g(y,\eta) \in \sigma_\Lambda(\mathcal{R}^0_G(\Omega \times \mathbb{R}^q, \boldsymbol{g}))|_{S^*Y}$. The invertibility of $a(y,\eta)$ and $b(y,\eta)$ implies that also (3.4.4) is invertible. Using the fact that there is an element $h \in \sigma_\Lambda(\mathcal{R}^0_G(\Omega \times \mathbb{R}^q, \boldsymbol{g}))|_{S^*Y}$ such that $1 + h(y,\eta) = (1 + g(y,\eta))^{-1}$ we can pass to $(1 + h(y,\eta))\,b(y,\eta) = a^{-1}(y,\eta)$. Since $a(y,\eta)$ is a block matrix with $\mathrm{id}_{\mathbb{C}^N}$ in the right lower corner, the same is true of $a^{-1}(y,\eta)$, i.e., the latter expression gives us $\sigma_\Lambda(\mathcal{A})^{-1}(y,\eta)|_{S^*Y}$ itself. Since the above multiplications preserve the nature of operator families on S^*Y that belong to $\sigma_\Lambda(\mathcal{R}^0(\Omega \times \mathbb{R}^q, \boldsymbol{g}))|_{S^*Y}$, we get this same property for $\sigma_\Lambda(\mathcal{A})^{-1}(y,\eta)|_{S^*Y}$. Thus, by an extension by homogeneity $-\mu$, we can produce

$$\sigma_\Lambda(\mathcal{A})^{-1}(y,\eta) \in \sigma_\Lambda(\mathcal{R}^{-\mu}(\Omega \times \mathbb{R}^q, \boldsymbol{g})).$$

Using Remark 3.5, we get an operator $\mathcal{P}_3 \in \mathcal{Y}^{-\mu}(\mathbb{W}, \boldsymbol{g}^{-1})$ such that $\sigma(\mathcal{P}_3) = \sigma(\mathcal{A})^{-1}$. This gives $\mathcal{P}_3\mathcal{A} = \mathcal{I} + \mathcal{C}$ for $\mathcal{C} \in \mathcal{Y}^{-1}(\mathbb{W}, \boldsymbol{g}^{-1})$. Since in the spaces of our specific operator-valued symbols it is possible to perform asymptotic summations, a formal Neumann series argument gives the desired result. $\square$

Theorem 3.15 *If $\mathcal{A} \in \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ is elliptic, the conditions*

(i) $Au = f \in \mathcal{W}^{s-\mu+\frac{n}{2},\gamma-\mu+\frac{n}{2}}_{(\boldsymbol{Q})}(\mathbb{W}, \boldsymbol{F}) \oplus H^{s-\frac{1}{2}-\mu}(Y, J_+)$,

(ii) $u \in \mathcal{W}^{-\infty,\gamma-\mu+\frac{n}{2}}(\mathbb{W}, \boldsymbol{E}) \oplus H^{-\infty}(Y, J_-)$

imply $u \in \mathcal{W}^{s+\frac{n}{2},\gamma+\frac{n}{2}}_{(\boldsymbol{P})}(\mathbb{W}, \boldsymbol{E}) \oplus H^{s-\frac{1}{2}}(Y, J_+)$ for every $s \in \mathbb{R}$. Here $\boldsymbol{Q}$ is any asymptotic type and $\boldsymbol{P}$ depends on $\boldsymbol{Q}$ and $\mathcal{A}$ (not on s).

Theorem 3.16 *Let $\mathcal{A} \in \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; \boldsymbol{v})$ be elliptic. Then the operator* (3.1.2) *is Fredholm for every $s \in \mathbb{R}$. Moreover,* $\ker \mathcal{A}$ *is a finite-dimensional subspace $V \subset \mathcal{W}^{\infty,\gamma+\frac{n}{2}}_{\boldsymbol{P}}(\mathbb{W}, \boldsymbol{E}) \oplus H^\infty(Y, J_-)$ for some asymptotic type $\boldsymbol{P}$, and there is a finite-dimensional subspace $W \subset \mathcal{W}^{\infty,\gamma-\mu+\frac{n}{2}}(\mathbb{W}, \boldsymbol{F}) \oplus H^\infty(Y, J_+)$ such that* $\mathrm{im}\,\mathcal{A} \cap W = \{0\}$ *and* $\mathrm{im}\,\mathcal{A} + W = \mathcal{W}^{s-\mu+\frac{n}{2},\gamma-\mu+\frac{n}{2}}(\mathbb{W}, \boldsymbol{F}) \oplus H^{s-\frac{1}{2}-\mu}(Y, J_+)$. *This is valid for all $s \in \mathbb{R}$ with s-independent V and W. Finally, there is a parametrix $\mathcal{P} \in \mathcal{Y}^{-\mu}(\mathbb{W}, \boldsymbol{g}^{-1}; \boldsymbol{v}^{-1})$ such that $\mathcal{I} - \mathcal{P}\mathcal{A}$ and $\mathcal{I} - \mathcal{A}\mathcal{P}$ are projections to V and W respectively.*

Theorem 3.15 above expresses elliptic regularity of solutions in weighted edge Sobolev spaces and subspaces with asymptotics. The proof is based on Theorem 3.14 and employs $\mathcal{P}$ as a left parametrix, together with Theorems 3.2 and 3.7. The scheme of the argument is standard.

The proof of Theorem 3.16 employs Theorem 3.15 together with $\mathcal{P}$ as a right parametrix. Generalities of Fredholm operators acting in scales of spaces in the present situation then tell us that $\mathcal{A}$ admits a parametrix in the asserted special form.

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