INVERSE PROBLEM OF THE CALCULUS OF VARIATIONS ON LIE GROUPS

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1. Introduction

On a Lie group G we can introduce three kind of covariant differential closely related to the Lie structure. The canonical left (resp. right) covariant differential, denoted by ∇^L (resp. ∇^R) is characterized by the property that the associated parallel translations are the left (resp. right) translations. A simple way to introduce these operators is to give they value on left-invariant vector-fields X, Y by

$$\nabla^L_X Y = 0, \qquad \nabla^R_X Y = [X, Y]$$

and extend them to arbitrary vector fields by making them tensorial in the X argument and satisfy the Leibniz rule in the Y argument. It is easy to show, using the Bianchi identity, that the curvature tenors are zero, but in general the corresponding torsions are non-vanishing.

The third covariant derivation ∇ is characterized by the properties that it is invariant by the left and right translations and also symmetric i.e. its torsion is zero. ∇ is defined on left invariant vector fields X and Y by

$$\nabla_X Y = \tfrac{1}{2}[X, Y] \tag{1}$$

and then extended to arbitrary vector fields by making ∇ tensorial in the X argument and satisfy the Leibniz rule in the Y argument. It is also called the *canonical covariant derivative* of the Lie group G. An easy computation shows that the curvature of ∇ is

$$R^{\nabla}(X, Y, \alpha) = \nabla_X \nabla_Y \alpha - \nabla_Y \nabla_X \alpha - \nabla_{[X,Y]} \alpha = \tfrac{1}{4}\big[\alpha, [X, Y]\big].$$

Our goal is to study the inverse problem of Lagrangian dynamics for the geodesic SODE associated to the canonical covariant derivative of a Lie group. If G is semi-simple ∇ is the Levi-Civita connection of the Killing. However, if G is not semi-simple, the only result is given by [13] where the author solve the inverse problem for the case of Lie groups up through dimension three.

In this present paper we will concentrate to the higher dimension case and in the main part (Section 4) we push the computation as far as it is possible in the generic case. In Section 5 we consider the example of $\mathrm{ASO_n}$ and compute explicitly the integrability conditions found in Section 4. We also prove that ASO_2 is variational.

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This results was showed also in [13], but the proof is completly different and gives a nice working example for the computation used in Section 4.

2. Connections, covariant differentials, Lagrangians

The natural framework of this problem of the calculus of variations is the tangent bundle, where one can present the obstructions to the integrability of the Euler-Lagrange system in an intrinsic and natural way [10]. We present here the basic objects playing role in the theory.

If M is a manifold TM denotes its tangent space. Let $J : TTM \to TTM$ be the vertical endomorphism and $C; TM \to TTM$ the Liouville vector-fields: if (x) is local coordinates on M, (x, y) is the associate coordinates on TM, then

$$J = dx^\alpha \otimes \frac{\partial}{\partial y^\alpha}, \qquad C = y^\alpha \frac{\partial}{\partial y^\alpha}.$$

Definition 2.1. [8] *A connection on M is a tensor field of type (1-1) Γ on TM such that $J\Gamma = J$ and $\Gamma J = -J$. The connection is called linear if $[C, \Gamma] = 0$, and Γ is $\mathcal{C}^1$ on the 0 section.*

If Γ is a connection, $\Gamma^2 = Id_{TTM}$ and the eigenspace corresponding to the eigenvalue -1 is the vertical space V_z. Then, at any $z \in TM$, we have the splitting

$$T_zTM = H_z \oplus V_z,$$

where H_z is the eigenspace corresponding to $+1$. H_z is called horizontal space. In the sequel we will denote

$$h := \tfrac{1}{2}(I + \Gamma), \qquad v := \tfrac{1}{2}(I - \Gamma),$$

the *horizontal* and *vertical* projectors. Locally we have:

$$h\Big(\frac{\partial}{\partial x^\alpha}\Big) = \frac{\partial}{\partial x^\alpha} - \Gamma^\beta_\alpha(x, y)\frac{\partial}{\partial y^\beta}, \qquad h\Big(\frac{\partial}{\partial y^\alpha}\Big) = 0.$$

where Γ^β_α are the coefficients of the connection. If the connection is linear, the coefficients $\Gamma^\beta_\alpha(x, y)$ are linear in y.

Proposition 2.2. [8] *If S is a spray, than $\Gamma := [J, S]$ is a connection.*

Definition 2.3. *The torsion of a connection Γ is the vectorial 2-form $t := [J, h]$ and the curvature is $R := -\frac{1}{2}[h, h]$.*

For every spray S, the connection $\Gamma = [J, S]$ has zero torsion. Moreover, the vertical distribution is integrable, and therefore $h[vX, vY] = 0$ and so

$$R(X, Y) = -v[hX, hY].$$

Every connection Γ on M determine an almost complex structure F on TM which interchange the horizontal and the vertical spaces. More precisely F is the unique $(1-1)$ tensor field so that $FJ = h$ and $Fh = -J$. In the case when $\Gamma = [J, S]$ we have

$$F = h\,[S, h] - J. \tag{2}$$

Definition 2.4. *Let N and M be two manifolds, $w \in \mathfrak{X}(N)$ and $\xi_z : T_z^v \to T_{\pi(z)}M$ the natural injection for every $z \in TM$. The covariant derivative of z with respect to w is defined by:*

$$D_{w(x)}z = \xi_{z(x)}(v \circ z_* \circ w). \tag{3}$$

A vector field $z \in TM$ along a curve γ is called parallel if $D_{\frac{d}{dt}}z = 0$. A geodesic is a curve $\gamma : [a, b] \to M$ such that $D_{\frac{d}{dt}}\gamma' = 0$.

In particular, for $N = M$ and $w, z \in \mathfrak{X}(M)$, we have

$$D_w z = w^\alpha \left(\frac{\partial z^\beta}{\partial x^\alpha} + \Gamma_\alpha^\beta(x, z(x)) \right) \frac{\partial}{\partial x^\beta}. \tag{4}$$

J. Douglas introduced a tensor which called Jacobi endomorphism and plays an essential role in the theory of the inverse problem of the calculus of variations [4]. His coordinate-free presentation is given by J. Klein in [11]:

Definition 2.5. *The Jacobi endomorphism is the (1-1) tensor Φ on TM defined by*

$$\Phi := v[h, S], \tag{5}$$

where h and v are the horizontal and vertical projectors of the connection $\Gamma = [J, S]$.

It is easy to see that Φ is semi-basic and

$$\Phi = [h, S] + F + J, \tag{6}$$

where F is the almost complex structure associated to Γ. Φ is related to the curvature by the formula

$$R = \tfrac{1}{3}[J, \Phi]. \tag{7}$$

Definition 2.6. *A Lagrangian is a map $E : TM \to \mathbb{R}$ smooth on $TM \setminus \{0\}$ and C^1 on the 0-section. E is called regular if the 2-form*

$$\Omega_E := dd_J E \tag{8}$$

has maximal rank.

The Lagrangian E is regular if and only if $\det\left(\frac{\partial^2 E}{\partial y^\alpha \partial y^\beta}\right) \neq 0$. If $E : TM \to \mathbb{R}$ is a regular Lagrangian, then the vector field S on TM defined by the equation

$$i_S \Omega_E = d(E - \mathcal{L}_C E) \tag{9}$$

is a spray and the paths of S are the solutions to

$$\frac{d}{dt}\frac{\partial E}{\partial \dot{x}^\alpha} - \frac{\partial E}{\partial x^\alpha} = 0, \qquad \alpha = 1 \ldots n.$$

the Euler-Lagrange equations [6].

Definition 2.7. *A spray S is called variational if there exists a smooth regular Lagrangian E which satisfies (9), the Euler-Lagrange equation.*

To every Lagrangian E and spray S a scalar 1-form ω_E can be associated by

$$\omega_E := i_S \Omega_E + d\mathcal{L}_C E - dE. \tag{10}$$

which is called *Euler-Lagrange* form. It is easy to see that ω_E is semi-basic, and the local expression in the standard coordinate system on TM is

$$\omega_E = \sum_{i=1}^{n} \left(S\Big(\frac{\partial E}{\partial y^i}\Big) - \frac{\partial E}{\partial x^i} \right) \, dx^i.$$

Therefore along a curve $\gamma = x(t)$ associated with S we have

$$\omega_E\big|_{\gamma} = \sum_{i=1}^{n} \left(\frac{d}{dt} \frac{\partial E}{\partial \dot{x}^i} - \frac{\partial E}{\partial x^i} \right) \, dx^i,$$

where d/dt denotes the derivation along γ. So, in order to find the solution to the inverse problem of the calculus of variations for a given second order differential equation (SODE), we have to look for a regular Lagrangian such that $\omega_E \equiv 0$. Of course, for this purpose we must study the local integrability of the second order partial differential operator $P_1 : \mathcal{C}^\infty(TM) \longrightarrow \mathcal{S}ec\, T_v^*$ called the *Euler-Lagrange operator* defined by

$$P_1 := i_S dd_J + d\mathcal{L}_C - d. \tag{11}$$

3. The geometry of the Lie groups

In this section we will describe the connection and geometric structures associated to the canonical covariant derivation ∇ of a Lie group G. Since our goal is to study the invers problem locally, we will suppose that G is a subset of $\subset Gl(n, \mathbb{R})$.

In the sequel we will denote by L_x the left translation by the group element x. Let (x) be coordinates on G, (x, y) be the standard associated coordinate system on TG. We will also use the "left-invariant" coordinates (x, α) on $TG \simeq T \times \mathfrak{g}$, where $\alpha = L_{x^{-1}} y$ is the Maurer-Cartan form. The corresponding coordinates on TTG are (x, α, X, A) i.e.

$$(x, \alpha, X, A) = X \frac{\partial}{\partial x}\Big|_{(x,\alpha)} + A \frac{\partial}{\partial \alpha}\Big|_{(x,\alpha)}.$$

Being α (and therefore A also) invariant coordinates with respect to the left translations, we find that a left translation by a group element g induces on TTG

$$L_g(x, \alpha, X, A) = (gx, \alpha, gX, A) = gX \frac{\partial}{\partial x}\Big|_{(gx,\alpha)} + A \frac{\partial}{\partial \alpha}\Big|_{(gx,\alpha)}$$

The canonical projection $\pi : TG \to G$ is $(x, \alpha) \to x$, therefore $\pi_* : TTG \to TG$ is given by $(x, \alpha, X, A) \to (x, x^{-1}X)$ and the vertical subspace on $(x, \alpha) \in TG$ is

$$V_{(x,\alpha)}TM := \operatorname{Ker} \pi_* = \big\{ (x, \alpha, 0, b) \,|\, b \in \mathfrak{g} \big\}.$$

3.1. Connection asoociated to the canonical covariant differential.

A vector-field $X \in \mathfrak{X}(G)$ can be represented as a map $X : G \to G \times \mathfrak{g}$, $x \to \big(x, a(x)\big)$, where $a(x) \in \mathfrak{g}$. X is a left-invariant vector-field, if and only if $a(x) = a$ is a constant map.

Let $X, Y : G \to G \times \mathfrak{g}$ be two left-invariant vector-field on G, $X : x \to (x, a)$, $Y : x \to (x, b)$. The integral curves of X passing through a point $x \in G$ is $L_x(exp(tX)) = x(1 + ta + \frac{t^2}{2}a^2 + ...)$, and therefore

$$\begin{aligned} Y_*X(x) &= \tfrac{d}{dt}\big|_{t=0} Y_{L_x(exp(tX))} = \tfrac{d}{dt}\big|_{t=0} L_x(exp(tX))_* Y \\ &= \big(x, b, \tfrac{d}{dt}\big|_{t=0}\big(x(1 + ta + \frac{t^2}{2}a^2 + ...), \tfrac{d}{dt}\big|_{t=0} b\big)\big) = (x, b, xa, 0). \end{aligned}$$

By the definition of ∇, we have

$$(\nabla_X Y)_x = (x, \tfrac{1}{2}[a, b]) \tag{12}$$

so from (3) taking $D = \nabla$ we obtain that

$$v(x, b, xa, 0) = (x, b, 0, \tfrac{1}{2}[a, b]) = \tfrac{1}{2}[a, b]\frac{\partial}{\partial \alpha}\Big|_{(x,b)}, \tag{13}$$

and

$$h(x, b, xa, 0) = (x, b, xa, -\tfrac{1}{2}[a, b]) = xa\frac{\partial}{\partial x}\Big|_{(x,b)} - \tfrac{1}{2}[a, b]\frac{\partial}{\partial \alpha}\Big|_{(x,b)}. \tag{14}$$

Therefore the horizontal and vertical lifts of a left-invariant vector-field $X = (x, a)$ are

$$\begin{aligned} X^h_{(x,\alpha)} &= h(x, \alpha, xa, 0) = (x, \alpha, xa, -\tfrac{1}{2}[a, \alpha]) = xa\frac{\partial}{\partial x}\Big|_{(x,\alpha)} - \tfrac{1}{2}[a, \alpha]\frac{\partial}{\partial \alpha}\Big|_{(x,\alpha)}, \\ X^v_{(x,\alpha)} &= JX^h_{(x,\alpha)} = (x, \alpha, 0, a) = a\frac{\partial}{\partial \alpha}\Big|_{(x,\alpha)}. \end{aligned}$$

We can remark that at each point (x, α) of TG, the horizontal subspace $H_{(x,\alpha)}$ associated to ∇ is the average of the horizontal space determined by ∇^L and ∇^R.

3.2. The spray and the geodesics.

A second order differential equations are described by vector-fields $\tilde{S}$ on TG so that $J\tilde{S} = C$. In the local coordinate system (x, α), $\tilde{S}$ is a spray if and only if it has the form $\tilde{S}_{(x,\alpha)} = (x, \alpha, x\alpha, \tilde{f})$ i.e.

$$\tilde{S}_{(x,\alpha)} = x\alpha\frac{\partial}{\partial x}\Big|_{(x,\alpha)} + \tilde{f}\frac{\partial}{\partial \alpha}\Big|_{(x,\alpha)}$$

for some functions $\tilde{f} = (f^i_j)$.

The spray S associated to the connection Γ can be computed as $S = h\tilde{S}$, where $\tilde{S}$ is an arbitrary spray on G. It has the special property that the projection of its integral curves are the geodesics with respect to the associated covariant derivative ∇. In order to compute S we can chose for $\tilde{S}$ simply the spray corresponding to the functions $\tilde{f} = 0$, and we find that

$$S = h\tilde{S} = h(x, \alpha, x\alpha, 0) = (x, \alpha, x\alpha, \tfrac{1}{2}[\alpha, \alpha]) = (x, \alpha, x\alpha, 0)$$

i.e. S and $\tilde{S}$ coincide. The equation of the geodesics is $S_{\dot{\gamma}} = \ddot{\gamma}$, where $\gamma = x_t$ is a path on G. Its first and second derivatives are $\dot{\gamma} = (x_t, \alpha_t)$ and $\ddot{\gamma} = (x, \alpha, x\alpha, \dot{\alpha}) =$

$(x, \alpha, x\alpha, -x^{-1}\dot{x}x^{-1}\dot{x}+x^{-1}\ddot{x})$, since $\alpha = x^{-1}\dot{x}$. Therefore the SODE corresponding to ∇ is given by the system

$$\ddot{x} = \dot{x}x^{-1}\dot{x} \tag{15}$$

In order to solve the inverse problem we have to find necessary and sufficient conditions for a system of second order ODE's to be the Euler-Lagrange equations of a regular Lagrangian function. In the practice this consist of find conditions for the integrability of the Euler-Lagrange system. The Euler-Lagrange system is largely over-determined partial differential system. The first compatibility conditions are describe by the curvature tensor R, the Jacobi endomorphism Φ and its derivatives ([?], [?]). In this section we will examine them in the case of the Lie groups.

3.3. **The curvature.** The curvature R of a connection Γ is described by the Nijenhuis torsion of the associated horizontal projection: $R := -\frac{1}{2}[h, h]$. Let us compute $R(X^h, Y^h)$, where the X^h and Y^h are horizontal lifts of the left-invariant vector fields $X = (x, a)$ and $Y = (x, b)$. Using the formula

$$M\frac{\partial[\alpha, N]}{\partial\alpha}\frac{\partial}{\partial\alpha} = [M, N]\frac{\partial}{\partial\alpha},$$

where N denotes a constant matrix, we obtain that

$$\begin{aligned}
[X^h, Y^h] &= \left[xa\frac{\partial}{\partial x} - \tfrac{1}{2}[a, \alpha]\frac{\partial}{\partial\alpha},\ xb\frac{\partial}{\partial x} - \tfrac{1}{2}[b, \alpha]\frac{\partial}{\partial\alpha}\right] \\
&= x[a, b]\frac{\partial}{\partial x} + \tfrac{1}{4}\left(\big[[\alpha, a], b\big] - \big[[\alpha, b], a\big]\right)\frac{\partial}{\partial\alpha} \\
&= x[a, b]\frac{\partial}{\partial x} + \tfrac{1}{4}\big[\alpha, [a, b]\big]\frac{\partial}{\partial\alpha} = \left(x, \alpha, x[a, b], \tfrac{1}{4}\big[\alpha, [a, b]\big]\right)
\end{aligned}$$

Now, using (13) we find that

$$\begin{aligned}
R_{(x,\alpha)}(X^h, Y^h) &= -\tfrac{1}{2}[h, h](X^h, Y^h) = -v[X^h, Y^h] \\
&= -\left(x, \alpha, 0, \tfrac{1}{2}\big[[a, b], \alpha\big] + \tfrac{1}{4}\big[\alpha, [a, b]\big]\right) \\
&= \quad \left(x, \alpha, 0, \tfrac{1}{4}\big[\alpha, [a, b]\big]\right) \\
&= \quad R^{\nabla}(X, Y, \alpha)^v
\end{aligned}$$

i.e. roughly speaking the curvature $R_{(x,\alpha)}$ is the vertical lift of $R^{\nabla}_x(\ ,\ ,\alpha)$.

The Jacobi endomorphism. Let Φ be the Jacobi endomorphism, $\Phi := v\circ[h, S]\circ h$, and $X := (x, \alpha)$ a left-invariant field. Then

$$\Phi(X^h) = v\left[xa\frac{\partial}{\partial x} - \tfrac{1}{2}[a, \alpha]\frac{\partial}{\partial\alpha}, x\alpha\frac{\partial}{\partial x}\right] = v\left(\tfrac{1}{2}x[a, \alpha]\frac{\partial}{\partial x}\right) = \tfrac{1}{4}\big[[a, \alpha], \alpha\big]\frac{\partial}{\partial\alpha}$$

and so

$$\Phi(X^h) = R^{\nabla}(\alpha, X, \alpha)^v.$$

The derivatives of the curvature and the Jacobi endomorphism.
The derivatives of R and Φ are describe by induction with the following formula:

$$R^{(k+1)} := h^*v[S, R^{(k)}], \qquad \Phi^{(k+1)} := h^*v[S, \Phi^{(k)}].$$

(The bracket in the above formula is the Frölicher-Nijenhuis bracket. In the case when one of the arguments is a vector-fields it means simply the Lie derivatives with respect to the vector-fields.)

Let $X = (x, a)$ and $Y = (x, b)$ left-invariant vector-fields. We have $R' = R^{(1)} = h^*v[R, S]$, i.e.

$$R'(X^h, Y^h) = v\big([R(X^h, Y^h), S] - R([X^h, S], Y^h) - R(X^h, [Y^h, S])\big).$$

Or

$$\begin{aligned}
[X^v, S] &= \left[a\frac{\partial}{\partial\alpha}, x\alpha\frac{\partial}{\partial x}\right] = xa\frac{\partial}{\partial x},\\
[X^h, S] &= \left[xa\frac{\partial}{\partial x} - \tfrac{1}{2}[a,\alpha]\frac{\partial}{\partial\alpha}, x\alpha\frac{\partial}{\partial x}\right] = \tfrac{1}{2}x[a,\alpha]\frac{\partial}{\partial x},
\end{aligned}$$

so using the fact that R is semi-basic we find

$$\begin{aligned}
R'(X^h, Y^h) =& v\left[\tfrac{1}{4}[\alpha,[a,b]]\frac{\partial}{\partial\alpha},\, x\alpha\frac{\partial}{\partial x}\right] - R\Big(\tfrac{1}{2}x[a,\alpha]\frac{\partial}{\partial x}, xb\frac{\partial}{\partial x}\Big) - R\Big(xa\frac{\partial}{\partial x}, \tfrac{1}{2}x[b,\alpha]\frac{\partial}{\partial x}\Big)\\
=& v\Big(\tfrac{1}{4}x[\alpha,[a,b]]\frac{\partial}{\partial x}\Big) - \tfrac{1}{8}[\alpha,[[a,\alpha],b]\frac{\partial}{\partial\alpha} - \tfrac{1}{8}[\alpha,[a,[b,\alpha]]]\frac{\partial}{\partial\alpha}\\
=& \tfrac{1}{8}[[\alpha,[a,[b,\alpha]]],\alpha]\frac{\partial}{\partial\alpha} - \tfrac{1}{8}[\alpha,[[a,\alpha],b]\frac{\partial}{\partial\alpha} - \tfrac{1}{8}[\alpha,[a,[b,\alpha]]]\frac{\partial}{\partial\alpha} =\\
=& \tfrac{1}{8}\Big[\alpha,\, [[a,b],\alpha] + [[b,\alpha],a] + [[\alpha,a],b]\Big] = 0.
\end{aligned}$$

Of course, the higher order derivatives $R^{(k)}$ vanish also. Moreover

$$\begin{aligned}
\Phi'(X^h) =& v\circ[\Phi, S](hX^h) = v[\Phi X^h, S] - v\Phi[X^h, S]\\
=& v\left[\tfrac{1}{4}\left[[a,\alpha],\alpha\right]\frac{\partial}{\partial\alpha}, x\alpha\frac{\partial}{\partial x}\right] - \Phi\left(\tfrac{1}{2}x[a,\alpha]\frac{\partial}{\partial x}\right)\\
=& v\left(\tfrac{1}{4}x\left[[a,\alpha],\alpha\right]\frac{\partial}{\partial x}\right) - \tfrac{1}{8}[[[a,\alpha],\alpha],\alpha]\frac{\partial}{\partial\alpha}\\
=& \tfrac{1}{8}\Big[\left[[a,\alpha],\alpha\right],\alpha\Big]\frac{\partial}{\partial\alpha} - \tfrac{1}{8}[[[a,\alpha],\alpha],\alpha]\frac{\partial}{\partial\alpha} = 0
\end{aligned}$$

and also, the higher order derivatives $\Phi^{(k)}$ vanish.

4. The inverse problem of the calculus of variations on Lie groups

As we explained at the end of Section 2, in order to solve the inverse problem of the Lagrange dynamics we have to study the integrability of the Euler-Lagrange differential operator P_1 - eventually enlarged with some compatibility conditions, and to show the existence of a *regular* solution. For the first, we use the theory of formal integrability of partial differential systems (see [3], [10]), while the proof of the second remains a simple linear algebraic computation.

The computations made in [9] or [10] show that the first compatibility conditions for P_1 is given by the equation

$$i_\Gamma \Omega_E = 0, \tag{16}$$

where $\Omega_E := dd_J E$. The geometric meaning of this condition is the following: if ∇ is variational and E is a regular Lagrangian associated to it, then the horizontal distribution associated to ∇ must be Lagrangian with respect to the symplectic 2-form Ω_E. If the dimension of the basic manifold is greater than one, the above condition is not satisfied by all the second order solution and therefore P_1 is not integrable. In order to take into the consideration the condition (16) we introduce the operator

$$P_\Gamma : C^\infty(TM) \longrightarrow \Lambda^2 T^*, \qquad P_\Gamma(E) = i_\Gamma dd_J E, \tag{17}$$

and define the system $P_2 := (P_1, P_\Gamma)$.

We refer to [9] and [10] once again, where the compatibility condition of P_2 is calculated. Namely, a second order formal solution E can be lifted into a third order formal solution if and only if the equations

$$i_\Phi \Omega_E = 0, \tag{18}$$

$$i_R \Omega_E = 0, \tag{19}$$

hold, where Φ is the Jacobi endomorphism and R is the curvature introduced in Section 2. The operator $P_2 = (P_1, P_\Gamma)$ is regular and involutive (see [10]). Therefore in the cases where the curvature is zero or flat (i.e. $\Phi = \lambda J$ with some scalar function λ), the system is formally integrable. In particular we find the

Proposition 4.1. *The canonical connection of commutative Lie group is variational.*

Indeed, in the commutative case R and therefore Φ are zero, so the equation (18) and (19) hold.

If ∇ is non flat, then the equations (18) and (19) are not satisfied. Therefore we must to study the integrability of the system

$$\begin{cases} i_S \Omega_E + d\mathcal{L}_C E - dE = 0 \\ i_\Gamma \Omega_E = 0 \\ i_\Phi \Omega_E = 0 \\ i_R \Omega_E = 0 \end{cases} \tag{20}$$

where $\Omega_E := dd_J E$. The differential operator corresponding to the system (20) is

$$P_3 : C^\infty(TG) \longrightarrow F_3,$$

defined by

$$P_3 := \big(P_1,\, P_\Gamma,\, P_\Phi,\, P_R\big),$$

where $F_3 := T^* \oplus \Lambda^2 T^* \oplus \Lambda^2 T^* \oplus \Lambda^3 T^*$ and

$$P_\Phi : C^\infty(TG) \longrightarrow \Lambda^2 T^*, \qquad P_\Phi(E) = i_\Phi dd_J E, \tag{21}$$

$$P_R : C^\infty(TG) \longrightarrow \Lambda^3 T^*, \qquad P_R(E) = i_R dd_J E, \tag{22}$$

are also second order linear differential operators.

P_3 is a second order linear partial differential operator. Its symbols is the map $\sigma_2(P_3) : S^2T^* \longrightarrow F_3$ given by the product of the symbols of the operators building it:

$$\sigma_2(P_3) = \sigma_2(P) \times \sigma_2(P_\Gamma) \times \sigma_2(P_\Phi) \times \sigma_2(P_R)$$

where

$$\begin{aligned}
(\sigma_2(P_1)B)(X) &= B(S, JX),\\
(\sigma_2(P_\Gamma)B)(X,Y) &= 2\big(B(hX, JY) - B(hY, JX)\big),\\
(\sigma_2(P_\Phi)B)(X,Y) &= B(AX, JY) - B(AY, JX),\\
(\sigma_2(P_R)B)(X,Y,Z) &= B(R(X,Y), JZ) + B(R(Y,Z), JX) + B(R(Z,X), JY)
\end{aligned}$$

for every $B \in S^2T^*$, and $X, Y, Z \in T$.

Remark 4.2. *Although the curvature tensor R^∇ is constant on the Lie group G, the rank of the Jacobi endomorphism Φ and the curvature tensor R on TG can be changing depending on the point chosen in TG. Therefore the differential operator P_3 is regular only in some neighborhood of generic points.*

For the symbol of the first prolongation $\sigma_3(P_3)$ of P_3 we find that $\sigma_3(P_3) : S^3T^* \longrightarrow T^* \otimes F_3$, where $\sigma_3(P_3) = \sigma_3(P) \times \sigma_3(P_\Gamma) \times \sigma_3(P_\Phi) \times \sigma_3(P_R)$ and the component maps are

$$\begin{aligned}
(\sigma_3(P_1)B)(X,Y) &= B(X, S, JY),\\
(\sigma_3(P_\Gamma)B)(X,Y,Z) &= 2\big(B(X, hY, JZ) - B(X, hZ, JY)\big),\\
(\sigma_3(P_\Phi)B)(X,Y,Z) &= B(X, AY, JZ) - B(X, AZ, JY),\\
(\sigma_3(P_R)B)(X,Y,Z,W) &= \sum_{YZW}^{\text{cycl}} B(X, R(Y,Z), JW).
\end{aligned}$$

Let us consider the map $\tau : T^* \otimes F_3 \longrightarrow K$ with $\tau := \tau_\Gamma \oplus \tau_A \oplus \tau_R \oplus \tau_{[J,J]} \oplus \tau_{A'} \oplus \tau_{[h,A]} \oplus \tau_{[A,A]} \oplus \tau_{[J,A]} \oplus \tau_{R'} \oplus \tau_{[J,R]} \oplus \tau_{[h,R]} \oplus \tau_{[A,R]}$ where the corresponding maps are defined as follows: for every $\xi := (B, C_\Gamma, C_\Phi, D_R) \in T^* \otimes F_3 \equiv (T^* \otimes T^*) \oplus$

$(T^* \otimes \Lambda^2 T^*) \oplus (T^* \otimes \Lambda^2 T^*) \oplus (T^* \otimes \Lambda^3 T^*)$ we set

$$
\begin{aligned}
&(\tau_{\Gamma}\xi)(X,Y) := B(JX,Y) - B(JY,X),\\
&(\tau_{A}\xi)(X,Y) := B(hX,Y) - B(hY,X) - \tfrac{1}{2}C_\Gamma(S,X,Y)\\
&(\tau_{R}\xi)(X,Y,Z) := C_\Gamma(hX,Y,Z) + C_\Gamma(hY,Z,X) + C_\Gamma(hZ,X,Y)\\
&(\tau_{[J,J]}\xi)(X,Y,Z) := C_\Gamma(JX,Y,Z) + C_\Gamma(JY,Z,X) + C_\Gamma(JZ,X,Y)\\
&(\tau_{A'}\xi)(X,Y) := B(AX,Y) - B(AY,X) - C_A(S,X,Y)\\
&(\tau_{[h,A]}\xi)(X,Y,Z) := \sum_{XYZ}^{\text{cyc}} C_A(hX,Y,Z) - \tfrac{1}{2}\sum_{XYZ}^{\text{cyc}} C_\Gamma(AX,Y,Z)\\
&(\tau_{[A,A]}\xi)(X,Y,Z) := \sum_{XYZ}^{\text{cyc}} C_A(AX,Y,Z)\\
&(\tau_{[J,A]}\xi)(X,Y,Z) := \sum_{XYZ}^{\text{cyc}} C_A(JX,Y,Z)\\
&(\tau_{R'}\xi)(X,Y,Z) := \sum_{XYZ}^{\text{cyc}} B(R(X,Y),W) - C_R(S,X,Y,W)\\
&(\tau_{[J,R]}\xi)(X,Y,Z,W) := \sum_{XYZ}^{\text{cyc}} C_R(JX,Y,Z,W)\\
&(\tau_{[h,R]}\xi)(X,Y,Z,W) := \sum_{XYZ}^{\text{cyc}} C_R(hX,Y,Z,W)\\
&(\tau_{[A,R]}\xi)(X,Y,Z,W) := \sum_{XYZ}^{\text{cyc}} C_A(R(X,Y),Z,W) - \sum_{XYZ}^{\text{cyc}} C_R(AX,Y,Z,W)
\end{aligned}
$$

A simple computation shows that $\operatorname{Im}\sigma_3(P_3) \subset \operatorname{Ker}\tau$. Taking an arbitrary linear connection D on TM we can compute the map $\varphi := \tau \circ D \circ p_0(P_3)$ defined on the space of second order solutions of P_3: it give compatibility conditions of P_3 (cf. [3], [10]).

Remark 4.3. *In general, $\operatorname{Im}\sigma_3(P_3) \neq \operatorname{Ker}\tau$, but τ is the map with the smallest kernel that can be construct in this general situation, i.e. without any restriction on G.*

It is not difficult to show that if E is a second order solution of P_3 at a point $v \in TG$, i.e. $P_3E(v) = 0$, then

$$
\begin{aligned}
(\varphi_E)_v = (&i_\Gamma\Omega_E, i_A\Omega_E, i_R\Omega_E,\, i_{[J,J]}\Omega_E,\, i_{\Phi'}\Omega_E,\, i_{[h,\Phi]}\Omega_E,\\
&i_{[\Phi,\Phi]}\Omega_E,\, i_{[J,\Phi]}\Omega_E,\, i_{R'}\Omega_E,\, i_{[J,\Phi]}\Omega_E,\, i_{[h,R]}\Omega_E,\, i_{[\Phi,R]}\Omega_E),
\end{aligned}
$$

and the new conditions arising are the following: if a second order solution E can be lifted into a third order solution then :

$$i_{\Phi'}\omega_E = 0 \tag{23}$$
$$i_{R'}\Omega_E = 0 \tag{24}$$
$$i_{[h,R]}\Omega_E = 0 \tag{25}$$
$$i_{[R,R]}\Omega_E = 0 \tag{26}$$
$$i_{[J,R]}\Omega_E = 0 \tag{27}$$
$$i_{[h,\Phi]}\Omega_E = 0 \tag{28}$$
$$i_{[\Phi,R]}\Omega_E = 0 \tag{29}$$
$$i_{[\Phi,\Phi]}\Omega_E = 0 \tag{30}$$

The equation (4) and (24) hold since every derivatives of R and Φ vanish. From the Bianchi identity ($[h,[h,h]] = 0$) we have also $[R,R] = 0$, and therefore $i_{[h,R]}\Omega = 0$ and $i_{[R,R]}\Omega = 0$. For the other we have:

- $[J,R] = -\big[J, \frac{1}{2}[h,h]\big] = \big[h,[J,h]\big] \overset{t=0}{=} 0$, so (27) holds.
- The vectorial 2-form $[h,A]$ is semi-basic, hence $d_{[A,h]}d_J E = i_{[A,h]}\Omega_E$. Also the torsion $t = \frac{1}{2}[J,\Gamma]$ is zero, so

$$\begin{aligned}[h,A] &= \big[h,[h,S] - h[h,S]\big] = \big[h,[h,S]\big] + \big[h, F + J\big] = \\ &= \big[h,[h,S]\big] + [h,F] + [h,J] \overset{t=0}{=} -[R,S] + FR - R\barwedge F = R'.\end{aligned} \tag{31}$$

  Since $R' = 0$, the equation (28) is satisfied.
- $[A,R] = \big[A, \frac{1}{2}[h,h]\big] = -\frac{1}{2}\big[h,[h,A]\big] - \frac{1}{2}\big[h,[A,h]\big] = \big[h,[A,h]\big] = [h,R'] = 0.$ Using once again that $R' = 0$ we find that (29) is satisfied.
- In order to compute (30) we remark that if N is a constant matrix, then

$$M\frac{\partial[[N,x],x]}{\partial x}\frac{\partial}{\partial x} = \Big([[N,M],x] + [[N,x],M]\Big)\frac{\partial}{\partial x}.$$

  Indeed,

$$\begin{aligned}M\frac{\partial[[N,x],x]}{\partial x}\frac{\partial}{\partial x} &= M^i_j\frac{\partial}{\partial x^i_j}(N^\alpha_k x^\beta_\alpha x^l_\beta - 2x^\alpha_k N^\beta_\alpha x^l_\beta + x^\alpha_k x^\beta_\alpha N^l_\beta)\frac{\partial}{\partial x^l_k} \\ &= M^i_j(N^\alpha_k \delta^j_\alpha \delta^i_\beta x^l_\beta + N^\alpha_k x^\beta_\alpha \delta^j_\beta \delta^i_l - 2\delta^j_k \delta^i_\alpha N^\beta_\alpha x^l_\beta - 2x^\alpha_k N^\beta_\alpha \delta^j_\beta \delta^i_l \\ &+ \delta^j_k \delta^i_\alpha x^\beta_\alpha N^l_\beta + x^\alpha_k \delta^j_\alpha \delta^i_\beta N^l_\beta)\frac{\partial}{\partial x^l_k} = \Big([[N,M],x] + [[N,x],M]\Big)\frac{\partial}{\partial x}\end{aligned}$$

  Now let $X = (x,a)$ and $Y = (x,b)$ be two left-invariant vector fields on G and X^h and Y^h its horizontal lifts. Using the fact that Φ is semi-basic we have $\Phi \circ \Phi = 0$ and also

$$\begin{aligned}\Phi[\Phi X^h, Y^h]_{(x,\alpha)} =& \Phi[(x,\alpha,0,\tfrac{1}{4}[[a,\alpha],\alpha]),(x,\alpha,xb,\tfrac{1}{2}[b,\alpha])] \\ =& \Phi\Big[\tfrac{1}{4}[[a,\alpha],\alpha]\frac{\partial}{\partial\alpha}, xb\frac{\partial}{\partial x} + \tfrac{1}{2}[b,\alpha]\frac{\partial}{\partial\alpha}\Big] = 0,\end{aligned}$$

we have $[\Phi,\Phi](X^h,Y^h)=[\Phi X^h,\Phi Y^h]$. Therefore

$$\begin{aligned}
[\Phi,\Phi](X^h,Y^h) &= [\Phi X^h,\Phi Y^h] = \Big[\tfrac{1}{4}[[a,\alpha],\alpha]\frac{\partial}{\partial\alpha},\tfrac{1}{4}[[b,\alpha],\alpha]\frac{\partial}{\partial\alpha}\Big]\\
&= \tfrac{1}{16}\Big([[\Phi(b),a],\alpha]-[[\Phi(a),b],\alpha]-[[[a,\alpha],[b,\alpha]],\alpha]\Big)\frac{\partial}{\partial\alpha}\\
&= \tfrac{1}{16}\big(R(\Phi(X^h),Y^h,\alpha)-R(\Phi(Y^h),X^h,\alpha)+R([X^h,\alpha],[Y^h,\alpha],\alpha)\big)\frac{\partial}{\partial\alpha}
\end{aligned}$$

In the generic case the vectorial 2-form $[\Phi,\Phi]$ is not linearly related to R and Φ and therefor the equation (30) represents a new compatibility condition for the system. As we will see in the next section, a nice nontrivial example where this condition is satisfied too is given by $\mathrm{ASO_n}$, the group of Euclidean transformations of $\mathbb{R}^n$.

5. An example: The Euclidean group

In this section we shall apply the theory developed in the previous sections to the Euclidean group $\mathrm{ASO_n}$ of $\mathbb{R}^n$ i.e. the group of rotations and translations. It consists of maps of the form:

$$\begin{array}{ccc}\mathbb{R}^n & \longrightarrow & \mathbb{R}^n\\ x & \longmapsto & Ax+t\end{array}$$

where $A\in SO(n)$ is a rotation matrix. $\mathrm{ASO_n}$ can be represented as a matrix Lie group, in fact as a subgroup of $GL((n+1),\mathbb{R})$ by

$$\mathrm{ASO_n}=\left\{\left(\begin{array}{cc}A & t\\ 0 & 1\end{array}\right)\,\Big|\;t\in\mathbb{R}^n,\quad A\in SO(n)\right\},$$

and the action on $\mathbb{R}^n$ is given as a matrix action when we write an element of $\mathbb{R}^n$ as $x={}^t(x^1,...,x^n,1)$. Therefore the Lie algebra of $\mathrm{ASO_n}$ can be present as

$$\mathfrak{aso}_n=\left\{\left(\begin{array}{cc}M & t\\ 0 & 0\end{array}\right)\,\Big|\;t\in\mathbb{R}^n,\quad M\in Mat(n)\right\}.$$

The system $\mathfrak{B}:=\{E_{i,j},\,E_k\mid j<i,\ i,j,k=1...n,\}$ give us a basis of $\mathfrak{aso}_n$, where $E_{i,j}$ denotes a skew-symmetric matrix with 1 in the i-th row and j-th column, -1 in the j-th row and i-th column, and 0 elsewhere, and E_i denotes the matrix with 1 in the i-th row and n+1-th column and 0 elsewhere. The bracket of the Lie algebra is:

$$\begin{aligned}
[E_i,E_j] &= 0\\
[E_{ij},E_k] &= -\delta_{ik}E_j+\delta_{jk}E_i\\
[E_{ij},E_{kl}] &= \delta_{jk}E_{il}+\delta_{il}E_{jk}-\delta_{jl}E_{ik}-\delta_{ik}E_{jl}
\end{aligned}$$

Let us compute the components of the curvature tensor R_v at a tangent vector $v=(x,\alpha)\in T_x\mathrm{ASO_n}$, where $\alpha=\alpha^{mn}E_{m,n}+\alpha^mE_m$. In order to make the computation simplest, we use the notation $E_{ji} = -E_{ij}$ for $i > j$.

$$\begin{aligned}
R_v(E_{ij}, E_{kl}) = R^\nabla(E_{ij}, E_{kl}, v) &= \tfrac{1}{4}\delta_k^j(\alpha^{lm}E_{mi} - \alpha^{im}E_{ml} + \alpha^i E_l - \alpha^l E_i) \\
&+ \tfrac{1}{4}\delta_l^i(\alpha^{km}E_{mj} - \alpha^{jm}E_{mk} + \alpha^j E_k - \alpha^k E_j) \\
&- \tfrac{1}{4}\delta_k^i(\alpha^{lm}E_{mj} - \alpha^{jm}E_{ml} + \alpha^j E_l - \alpha^l E_j) \\
&- \tfrac{1}{4}\delta_l^j(\alpha^{km}E_{mi} - \alpha^{im}E_{mk} + \alpha^i E_k - \alpha^k E_i)
\end{aligned}$$

$$R_v(E_{ij}, E_j) = R^\nabla(E_{ij}, E_j, v) = \tfrac{1}{4}\delta_j^k(\alpha^{mi} - \alpha^{im})E_m - \tfrac{1}{4}\delta_i^k(\alpha^{mj} - \alpha^{jm})E_m$$
$$R_v(E_i,\, E_j) = R^\nabla(E_i,\, E_j, v) = 0.$$

Remark 5.1. *It is not difficult to check that the Nijenhuis bracket of the Jacobi endomorphism is proportional to the curvature: $[\Phi, \Phi] = \frac{1}{4}R$. Since from the integrability conditions (4) - (30) only the last one can be non-trivial, we obtain that all compatibility condition found in the previeus section is satisfied.*

In the sequel we focus on the thre dimensional group ASO_2 and we show the

Theorem 5.2. *The canonical connection of ASO_2 is variational.*

To prove the theorem we have to examine the integrability of the system (20) which contains the Euler-Lagrange equations and its first compatibilities conditions. We denote by e_1, e_2 and e_{12} the left-invariant vector-fields on ASO_2 corresponding to the element E_1, E_2 and E_{12} of $\mathfrak{aso}_2$.

Remark 5.3. *Let v be an arbitrary vector of $T\mathrm{ASO}_2$ represented as (x, α) in $\mathrm{ASO}_2 \times \mathfrak{aso}_2$. From the formulae of the curvature above we obtain that*

$$R_v = \omega \wedge J$$

where $\omega_v = \alpha_{12}e^{12}$, e^{12} being the dual of e_{12}.

From this reason the operator P_R can be removed from P_3 because the equation represented by it is identically satisfied. Indeed, for every Lagrangian E we have

$$i_R\Omega_E = i_{\omega\wedge J}\Omega_E = \omega \wedge i_J dd_J E = \omega \wedge d_J d_J E = \omega \wedge d_{[J,J]}E = 0$$

because $[J, J] = 0$. So it is sufficient to consieder only the operator

$$P_{\mathrm{ASO}_2} : C^\infty(T\mathrm{ASO}_2) \longrightarrow F_{\mathrm{ASO}_2} \tag{32}$$

where $P_{\mathrm{ASO}_2} := (P_1, P_\Gamma, P_\Phi)$ and $F_{\mathrm{ASO}_2} := T^* \oplus \Lambda^2 T^* \oplus \Lambda^2 T^*$.

Therefore, to prove Theorem 5.2 is is sufficient to show that there exists second order regular solution for P_{ASO_2} (Lemma 5.4), and P_{ASO_2} is formally integrable, i.e. every second order solution can be lift into an infinit order solution (Lemma 5.5 and Lemma 5.6).

Lemma 5.4. *At every $v \neq 0$ in $T\mathrm{ASO}_2$ there exists second order regular formal solution to P_{ASO_2}.*

Indeed, let (x^i) be a local coordinate system on ASO_2, (x^i, y^i) the associated coordinate system on $T\mathrm{ASO}_2$ in the neighborhood of v. If E is a function on $T\mathrm{ASO}_2$ its second order jet at v will be denoted by

$$(j_2E)_x = (x^i, v^i, p, p_j, p_{\underline{j}}, p_{jk}, p_{j\underline{k}}, p_{\underline{jk}})$$

where $p = E(v)$, and $p_i = \frac{\partial E}{\partial x^i}$, $p_{\underline{i}} = \frac{\partial E}{\partial y^i}$, $p_{ij} = \frac{\partial^2 E}{\partial x^i \partial x^j}$, $p_{i\underline{j}} = \frac{\partial^2 E}{\partial x^i \partial y^j}$, $p_{\underline{ij}} = \frac{\partial^2 E}{\partial y^i \partial y^j}$ are the corresponding derivatives computed at v. $(j_2E)_x$ is a second order regular solution of P_{ASO_2} at v if and only if

$$\det(p_{\underline{ij}}) \neq 0, \tag{33}$$

and $(P_1E)_v = 0$, $(P_\Gamma E)_v = 0$ and $(P_{\mathrm{ASO}_2}E)_v = 0$ are satisfied, i.e. if we have (33) and the *linear system*:

$$v^\alpha p_{\alpha\underline{i}} + f^\alpha p_{\underline{\alpha i}} - p_i = 0, \tag{34}$$

$$p_{j\underline{i}} - p_{i\underline{j}} + \Gamma_i^\alpha p_{\underline{\alpha j}} - \Gamma_j^\alpha p_{\underline{\alpha i}} = 0, \tag{35}$$

$$\Phi_i^\alpha p_{\underline{\alpha j}} - \Phi_j^\alpha p_{\underline{\alpha i}} = 0, \tag{36}$$

where f^α are the components of the spray and Γ_β^α are the coefficients of the connection $\Gamma = [J, S]$. Computing he Jacobi endomorphism Φ with the help of the formula $\Phi = i_S R$ we find that in the basis $\mathcal{B} = \{e_1^h, e_2^h, S, e_1^v, e_2^v, C\}$ at $v = (x, \alpha)$ the matrix of Φ is $\begin{pmatrix} 0 & 0 \\ \Phi_i^j & 0 \end{pmatrix}$, where 0 denots the 3×3 zero matrix and

$$(\Phi_i^j) = \begin{pmatrix} \alpha_{12} & 0 & 0 \\ 0 & \alpha_{12} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now let g be a scalar product of T_x^v so that the basis $\{e_1^v, e_2^v, C\}$ is orthogonal. If $(p_{\underline{ij}})$ is the matrix of g with respect to the basis $\{\frac{\partial}{\partial y^i}\}_{i=1,\dots,n}$, we find that (33) and (36) are satisfied. Solving the system (34) and (35) with respect to the pivot terms p_i and $p_{j\underline{i}}$ we arrive at a regular second order formal solution of P_{ASO_2} at v.

Lemma 5.5. *Every second order solution of P_{ASO_2} can be lifted into a 3rd order solution.*

Let $\tau_{\mathrm{ASO}_2} := (\tau_\Gamma \oplus \tau_A \oplus \tau_{[J,J]} \oplus \tau_{A'} \oplus \tau_{[h,A]} \oplus \tau_{[A,A]} \oplus \tau_{[J,A]} \oplus \rho)$ defined on $T^* \otimes F_{\mathrm{ASO}_2}$, where the maps τ_Γ, τ_A, $\tau_{[J,J]}$, $\tau_{A'}$, $\tau_{[h,A]}$, $\tau_{[A,A]}$ and $\tau_{[J,A]}$ are defined on page 11 with the corresponding formula, and let

$$(\rho\xi)(C_1, C_\Gamma, C_\Phi)(X, Y) := C_\Phi(hX, Y, S) - \tfrac{1}{2}C_\Gamma(X, Y, S) - C_1(AY, X)$$

for every $X, Y \in \mathcal{H} := \{e_1^h, e_2^h\}$: the space generated by the horizontal lift of the infinitesimal translations e_1 and e_2. It is easy to show that $\operatorname{Im}\sigma_3(P_{\mathrm{ASO}_2}) = \operatorname{Ker}\tau_{\mathrm{ASO}_2}$. Therefore, taking an arbitrary linear connection D on TM we can compute the map $\varphi := \tau_{\mathrm{ASO}_2} \circ D \circ p_0(P_{\mathrm{ASO}_2})$ defined on the space of second order solutions of P_3: it give all the compatibility conditions of P_3 (cf. [10]).

Using the computation of the previous section and the Remark 5.1 we obtain, that

$$\varphi_E = (0, 0, 0, 0, 0, 0, 0, \rho(DP_{\mathrm{ASO}_2}E))$$

for every second order solution $j_2(E)$ of P_{ASO_2}. Let us compute $\rho(DP_{\mathrm{ASO}_2}E)$: if $X, Y \in \mathcal{H}$, then

$$\begin{aligned}\rho(DP_{\mathrm{ASO}_2}E)(X,Y) &= X\big(\Omega_E(\Phi Y, S)\big) - \tfrac{1}{2}\Phi Y\big(i_\Gamma \Omega_E(X,S)\big) - \Phi Y\big(\omega_E(X)\big)\\ &= X\Omega_E(\Phi Y, S) - \Phi Y \Omega_E(X,S) - d\omega_E(\Phi Y, X).\end{aligned}$$

Since

$$d\omega_E = d(i_S dd_J E + d\mathcal{L}_C E - dE) = di_S dd_J E = \mathcal{L}_S \Omega_E,$$

we obtain

$$\begin{aligned}\rho(DP_{\mathrm{ASO}_2}E)(X,Y) &= \Omega_E([S,\Phi Y], X) + \Omega_E(\Phi Y, [S,X]) + \sum_{X,\Phi Y,S} X\Omega_E(\Phi Y, S)\\ &= d\Omega_E(X, \Phi Y, S) + \Omega_E([X,\Phi Y], S) = \Omega_E([X,\Phi Y], S)\end{aligned}$$

because $d\Omega_E = d^2 d_J E = 0$. Replacing X and Y by the generators e_i^h and e_j^h in the above formula and using the fact that $[e_i^h, e_j^v] = 0$ we obtain that at v we have

$$\begin{aligned}\rho(DP_{\mathrm{ASO}_2}E)(e_i^h, e_j^h) &= \Omega_E([e_i^h, \Phi e_j^h], S) = \Omega_E([e_i^h, \alpha_{12} e_j^v], S)\\ &= (e_i^h \alpha_{12})\Omega_E(e_j^v, S) = \tfrac{(e_i^h \alpha_{12})}{\alpha_{12}} i_A \Omega_E(e_j^v, S) = 0,\end{aligned}$$

because E being a second order solution at v, $i_A\Omega_E$ vanishes on $T_v\mathrm{ASO}_2$. This means that for every second order solution E the map φ_E is zero and so every second order solution can be lifted into a third order solution. This prove Lemma 5.5.

Lemma 5.6. *The symbol of P_{ASO_2} is involutive.*

To show the lemma we have to find a quasi-regular basis ([3], [10]) i.e. a basis $\{v_i\}_{i=1}^6$ of T^* so that

$$\dim g_2(P_{\mathrm{ASO}_2}) + \sum_{k=1}^{6} \dim g_2(P_{\mathrm{ASO}_2})_{v_1\ldots v_k} = \dim g_3(P_{\mathrm{ASO}_2}), \tag{37}$$

where $g_2(P_{\mathrm{ASO}_2}) = \operatorname{Ker}\sigma_2(P_{\mathrm{ASO}_2})$ is the kernel of the symbol of P_{ASO_2}, $g_3(P_{\mathrm{ASO}_2}) := \operatorname{Ker}\sigma_3(P_{\mathrm{ASO}_2})$ is the kernel of the symbol of the frist prolongation, and $g_2(P_{\mathrm{ASO}_2})_{v_1\ldots v_k} := \{B \in g_2(P_{\mathrm{ASO}_2}) \mid B(v_i, .) = 0, i = 1...k\}$.

An easy computation shows that the basis $\{v_i\}_{i=1}^6$ satisfies the condition (37), where

$$v_1 := e_1^h + C, \quad v_2 := e_2^h + e_1^v, \quad v_3 := S + e_2^v, \quad v_4 := e_1^v, \quad v_5 := e_2^v, \quad v_6 := C.$$

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