The Dirac Operator on Hyperbolic Manifolds of Finite Volume

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Abstract

We study the spectrum of the Dirac operator on hyperbolic manifolds of finite volume. Depending on the spin structure it is either discrete or the whole real line. For link complements in S^3 we give a simple criterion in terms of linking numbers for when essential spectrum can occur. We compute the accumulation rate of the eigenvalues of a sequence of closed hyperbolic 2- or 3-manifolds degenerating into a noncompact hyperbolic manifold of finite volume. It turns out that in three dimensions there is no clustering at all.

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0 Introduction

The aim of this paper is to study the spectrum of the Dirac operator on hyperbolic manifolds with finite volume. Since the corresponding problems for the Laplace-Beltrami operator acting on differential forms have already been examined let us first briefly describe those results. The first natural thing to do is to look at the spectrum of the model space, n-dimensional hyperbolic space H^n. Donnelly [13] computed the spectrum of the Laplace operator Δ_q acting on q-forms on H^n. For the point spectrum he obtains

$$spec_p(\Delta_q) = \begin{cases} \{0\}, & q = n/2 \\ \emptyset, & \text{otherwise} \end{cases}$$

and for the continuous spectrum

$$spec_c(\Delta_q) = \begin{cases} [(n-2q-1)^2/4, \infty), & q \le n/2 \\ [(n-2q+1)^2/4, \infty), & q \ge n/2 \end{cases}$$

The eigenvalue 0 in the case $n = 2q$ occurs with infinite multiplicity. When we pass to quotients of the hyperbolic space we cannot hope to be able to explicitly compute the spectrum anymore. But the essential spectrum which is much more robust than the eigenvalues may still be controlled. Indeed, Mazzeo and Phillips [24] showed that except for the eigenvalue 0 the essential spectrum on a noncompact hyperbolic manifold of finite volume is the same as that of H^n

$$spec_e(\Delta_q) = \begin{cases} [(n-2q-1)^2/4, \infty), & q \le n/2 \\ [(n-2q+1)^2/4, \infty), & q \ge n/2 \end{cases}$$

In dimension 2 and 3 one can approximate hyperbolic manifolds of finite volume by compact ones. In dimension 2 this is clear from Teichmüller theory and it can be done continuously. In three dimensions it follows from Thurston's cusp closing theorem that for any noncompact hyperbolic manifold M of finite volume one can find a sequence of compact hyperbolic manifolds, pairwise nonhomotopic, which converge in a suitable sense to M. What happens to the spectrum under such a degeneration?

Since the spectrum of closed manifolds is discrete we expect that the eigenvalues in the range of the essential spectrum of the limit manifold cumulate. This is true and the rate of clustering has been determined by Ji and Zworski [21] for surfaces, by Chavel and Dodziuk [9] for $n = 3$ and $q = 0$, and by Dodziuk and McGowan [12] for $n = 3$ and $q = 1$. By Hodge duality this covers all cases.

It turns out that each cusp of the limit manifold M contributes to the accumulation rate. This is not surprising because each cusp contributes to the essential spectrum. Let M_i be the approximating sequence of closed hyperbolic manifolds, $M_i \to M$. The cusps of M are approximated by degenerating tubes around short closed geodesics in M_i of length $\ell_{i,j} \stackrel{i\to\infty}{\longrightarrow} 0$, $j = 1, \dots, k$, where k is the number of cusps of M. For an operator L on a manifold N and an interval $I \subset \mathbb{R}$ we introduce the *eigenvalue counting function*

$$\mathcal{N}_{L,N}(I) := \sharp(spec(L) \cap I).$$

Here eigenvalues have to be counted with multiplicity. Then the accumulation rate turns out to be

$$\mathcal{N}_{\Delta_q,M_i}(I) = c(n,q)\frac{x}{\pi}\sum_{j=1}^{k}\log(1/\ell_{i,j}) + \mathrm{O}_x(1)$$

where

$$c(n,q) = \begin{cases} 2, & n=2, \quad k=0,2 \\ 4, & n=2, \quad k=1 \\ 1/2, & n=3, \quad k=0,3 \\ 1, & n=3, \quad k=1,2 \end{cases}$$

and

$$I = \begin{cases} [1/4, 1/4+x^2], & n=2 \\ [1, 1+x^2], & n=3, \quad k=0,3 \\ [0, x^2], & n=3, \quad k=1,2 \end{cases}$$

and $O_x(1)$ denotes an error term bounded as a function of i. Moreover, Colbois and Courtois [10, 11] showed that the eigenvalues below the bottom of the essential spectrum of M are limits of eigenvalues of the M_i.

We want to study the analogous questions for the Dirac operator D acting on spinors, sometimes also called Atiyah-Singer operator, on hyperbolic manifolds. The spectrum of the model space H^n has been computed by Bunke [6]. Note that there is an incorrect statement about the eigenvalue 0 in that paper. See also [7, 8] and the remark after the proof of Lemma 1 in this paper. The result is

$$spec_p(D) = \emptyset, \qquad spec_c(D) = \mathbb{R}.$$

Since D is of first order the spectrum is not semibounded. When we pass to nonsimply connected hyperbolic manifolds a new piece of structure enters the picture for which there is no analog for the Laplace operator. We have to specify a spin structure on the manifold. First of all, this means that we have to restrict our attention to hyperbolic spin manifolds. In particular, the manifolds must be orientable. If the manifold is spin the spin structure is not unique. There are as many spin structures on M as there are elements in the cohomology group $H^1(M; \mathbb{Z}/2\mathbb{Z})$. It turns out that the choice of spin structure has dramatic impact on the Dirac spectrum. We will define in Section 3 what it means that a spin structure is *trivial along a cusp* of a hyperbolic manifold. This is an essentially topological property. Our first result is

Theorem 1. *Let M be a hyperbolic manifold of finite volume equipped with a spin structure.*

If the spin structure is trivial along at least one cusp, then the Dirac spectrum satisfies

$$spec(D) = spec_e(D) = \mathbb{R}.$$

If the spin structure is nontrivial along all cusps, then the spectrum is discrete,

$$spec(D) = spec_d(D).$$

We see already that there is no analog for the eigenvalues below the bottom of the essential spectrum as studied by Colbois and Courtois.

We will see that if M is 2- or 3-dimensional and has only one cusp, then only the second case occurs, the spectrum is always discrete (Corollary 1). If M is a surface with at least two cusps, then both cases occur. The spin structure can be made trivial on any choice of an even number of cusps.

In three dimensions this is not true in general. It can happen that the spectrum is always discrete even if the manifold has more than one cusp. If the hyperbolic manifold is given as the complement of a link in S^3, then there is simple criterion to decide if there is a spin structure such that $spec(D) = \mathbb{R}$.

Theorem 4. *Let $K \subset S^3$ be a link, let $M = S^3 - K$ carry a hyperbolic metric of finite volume.*

If the linking number of all pairs of components (K_i, K_j) of K is even,

$$Lk(K_i, K_j) \equiv 0 \ mod \ 2,$$

$i \neq j$, then the spectrum of the Dirac operator on M is discrete for all spin structures,

$$spec(D) = spec_d(D).$$

If there exist two components K_i and K_j of K, $i \neq j$, with odd linking number, then M has a spin structure such that the spectrum of the Dirac operator satisfies

$$spec(D) = \mathbb{R}.$$

Determining linking numbers modulo 2 is equivalent to counting overcrossings modulo 2, hence extremely simple. See the last section for examples.

Next we study the behavior under the degeneration process in dimension 2 and 3. Of course, we have to assume that the spin structure on M is, in a suitable sense, the limit of the spin structures on the M_i. In two dimensions the result is

Theorem 2. *Let M_i be a sequence of closed hyperbolic surfaces converging to a noncompact hyperbolic surface M of finite volume. Let each M_i have exactly k tubes with trivial spin structure around closed geodesics of length $\ell_{i,j}$ tending to zero. Hence M has exactly $2k$ cusps along which the spin structure is trivial. Let $x > 0$.*

Then the eigenvalue counting function for the Dirac operator satisfies for sufficiently small $\ell_{i,j}$:

$$\mathcal{N}_{D,M_i}(-x, x) = \frac{4x}{\pi} \sum_{j=1}^{k} \log(1/\ell_{i,j}) + \mathrm{O}_x(1).$$

Very recently, Farinelli [17] gave an upper bound on the spectral accumulation of the lower part of the Dirac spectrum of hyperbolic 3-manifolds. However, we will show that in three dimensions there is no clustering at all!

Theorem 3. *Let M_i be a sequence of closed hyperbolic 3-manifolds converging to a noncompact hyperbolic 3-manifold M of finite volume. Let each M_i have exactly k tubes around closed geodesics of length $\ell_{i,j}$ tending to zero. Hence M has exactly k cusps. Let $x > 0$.*

Then the spin structure is nontrivial along all tubes and the eigenvalue counting function for the Dirac operator remains bounded:

$$\mathcal{N}_{D,M_i}(-x, x) = \mathrm{O}_x(1).$$

The reason for this fact, at first glance quite surprising, is of topological nature. The spin structure on the tubes must be nontrivial because the trivial spin structure on the 2-torus is nontrivial in spin cobordism Ω_2^{Spin}. In other words, the spin structures on hyperbolic 3-manifolds of finite volume for which $spec(D) = \mathbb{R}$ do not occur as limits of spin structures on closed hyperbolic 3-manifolds.

We see that the freedom to choose different spin structures leads to new phenomena in the spectral theory of the Dirac operator on hyperbolic manifolds for which there is no analog for the Laplace operator. This also distinguishes the classical Dirac operator acting on spinors from those twisted Dirac operators on locally symmetric spaces which have typically been studied in the context of representation theory [1, 2] and index theory [25].

The paper is organized as follows. In the first section we collect a few facts about hyperbolic manifolds. The structure of the cusps and tubes is important for our purposes. A description of the degeneration process in dimension 2 and 3 is given.

In the second section we present some generalities about the L^2-spectrum of self-adjoint elliptic operators. We give a prove of the so-called decomposition principle which roughly says that modifying the manifold and the operator in a compact region of the manifold does not affect the essential spectrum. This will be extremely useful for us because we can restrict our attention to the cusps of the hyperbolic manifolds. There are many versions of this principle in the literature but we found it convenient to prove it in a quite general form. Our version can e.g. be applied to the Dirac operator on manifolds with boundary with suitable boundary conditions.

In Section 3 we prove Theorem 1. We use a separation of variables along the cusps which reduces the problem to the study of simple Schrödinger operators on an interval.

In the forth section we derive a general version of domain monotonicity. This allows one to estimate eigenvalues by cutting the manifold into pieces. This has been used extensively in the spectral geometry of the Laplace operator. Here we need this tool for the Dirac operator.

We are then able to prove Theorem 2 in Section 5. It is important that tubes in a hyperbolic surface are warped products so that the separation of variables can again be applied.

We would like to do the same thing in three dimensions in Section 7 but we have the problem that tubes are no longer simple warped products. Therefore we include a general formula in Section 6 which relates the square of the Dirac operator on a manifold foliated by hypersurfaces to operators along the leaves and normal derivatives. This way we can regard the square of the Dirac operator on the tube as a Schrödinger operator acting on Hilbert space-valued functions on an interval. We will then be able to prove Theorem 3 in Section 7.

In the last section we discuss the different spin structures which 2- or 3-dimensional hyperbolic manifolds can have. This is more topological in nature. We conclude with a few examples of link complements for which essential spectrum does or does not occur.

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1 Hyperbolic Manifolds of Finite Volume

A *hyperbolic manifold* is a complete connected Riemannian manifold of constant sectional curvature -1. We collect a few well-known facts about such manifolds with special emphasis on the case of finite volume. A thorough introduction to the topic is given in [5].

Every hyperbolic manifold M of finite volume can be decomposed disjointly into a relatively compact M_0 and finitely many cusps $\mathcal{E}_j$,

$$M = M_0 \dot\cup \dot{\bigcup_{j=1}^{k}} \mathcal{E}_j \tag{1}$$

where each $\mathcal{E}_j$ is of the form $\mathcal{E}_j = N_j \times [0, \infty)$. Here N_j denotes a connected compact manifold with a flat metric g_{N_j}, a *Bieberbach manifold*, and $\mathcal{E}_j$ carries the warped product metric $g_{\mathcal{E}_j} = e^{-2t} \cdot g_{N_j} + dt^2$.

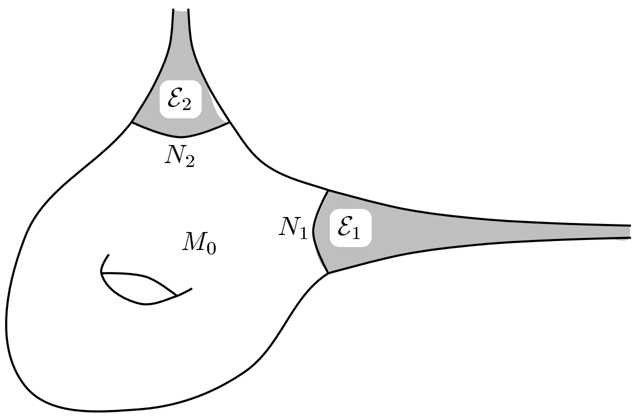


Fig. 1

For example, N_j could be a flat torus, as is always the case if M is 2- or 3-dimensional and orientable. This simple structure of the cusps will allow us to apply a separation of variables technique to the Dirac operator on hyperbolic manifolds of finite volume.

It turns out that very different phenomena occur in hyperbolic geometry depending on the dimension. In dimension 2 there is a whole continuum of hyperbolic structures (hyperbolic metrics modulo isometries) on a given surface. This is known as Teichmüller theory. In particular, if we fix a compact surface M, then there are continuous deformations of hyperbolic metrics on M under which M degenerates to a noncompact hyperbolic surface of finite volume. These deformations correspond to paths in the Teichmüller space converging to the boundary.

In contrast, in dimension $n \geq 3$, we know by Mostow's rigidity theorem that any compact manifold carries at most one hyperbolic structure. Therefore continuous degenerations are not possible.

If $n = 3$ however, the following kind of degeneration still occurs. Thurston's cusp closing theorem says that for any hyperbolic manifold $M = M_0 \dot\cup \dot\bigcup_{j=1}^{k} \mathcal{E}_j$ of finite volume with metric g there are compact hyperbolic manifolds (M_i, g_i) which can be decomposed disjointly into

$$M_i = M_{i,0} \dot\cup \dot\bigcup_{j=1}^{k} T_{i,j}$$

where $T_{i,j}$ is the closed tubular neighborhood of radius $R_{i,j}$ of a simple closed geodesic $\gamma_{i,j} \subset M_i$ of length $\ell_{i,j}$. The boundary $N_{i,j} = \partial T_{i,j}$ is a flat torus. In the degeneration $(i \to \infty)$ the following happens:

- $\ell_{i,j} \to 0$
- $R_{i,j} = \frac{1}{2}\log(1/\ell_{i,j}) + c_0 \to \infty$ where c_0 is some constant independent of i.
- There are diffeomorphisms $\Phi_i : \bar{M}_0 \to \bar{M}_{i,0}$ of compact manifolds with boundary such that the metrics $\Phi_i^*(g_i|_{\bar{M}_{i,0}})$ converge in the C^∞-topology to $g|_{\bar{M}_0}$.
- The pull-backs of the metrics of $N_{i,j}$ converge in the C^∞-topology to the one of N_j.

Moreover, if we write

$$T_{i,j}[r_1, r_2] = \{x \in M_i \mid dist(x, \gamma_{i,j}) \in [r_1, r_2]\ \}$$

for the tubular region around $\gamma_{i,j}$, so that $T_{i,j} = T_{i,j}[0, R_{i,j}]$, then we have in addition

- For every $0 < r_1 < r_2 < R_{i,j}$ the tubular region $T_{i,j}[r_1, r_2]$ is isometric to $T^2 \times [r_1, r_2]$ with the metric $g_r + dr^2$ where g_r is the flat metric on the 2-torus given by the lattice $\Gamma_r \subset \mathbb{R}^2$ spanned by the vectors $(2\pi \sinh(r), 0)$ and $(\alpha_{i,j} \sinh(r), \ell_{i,j} cosh(r))$ for some "holonomy angle" $\alpha_{i,j} \in [-\pi, \pi]$.

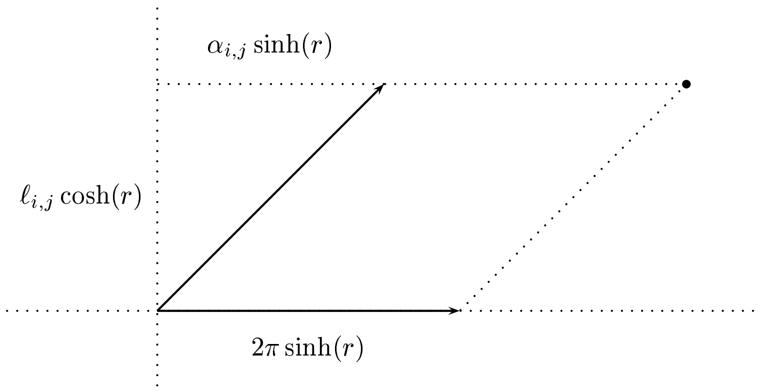


Fig. 2

This description of the degeneration is also valid in the 2-dimensional case, except that the tube $T_{i,j}$ is of the form $T_{i,j} = S^1 \times [-R_{i,j}, R_{i,j}]$ with metric $ds^2 = \ell_{i,j}^2 \cosh(t)^2 d\theta^2 + dt^2$ where $t \in [-R_{i,j}, R_{i,j}]$, $\theta \in S^1 = \mathbb{R}/\mathbb{Z}$ and $R_{i,j} = \log(1/\ell_{i,j}) + c_0$. In particular, the boundary of the tube is of the form $S^0 \times S^1 = S^1 \dot{\cup} S^1$. Hence each tube degenerates into *two* cusps.

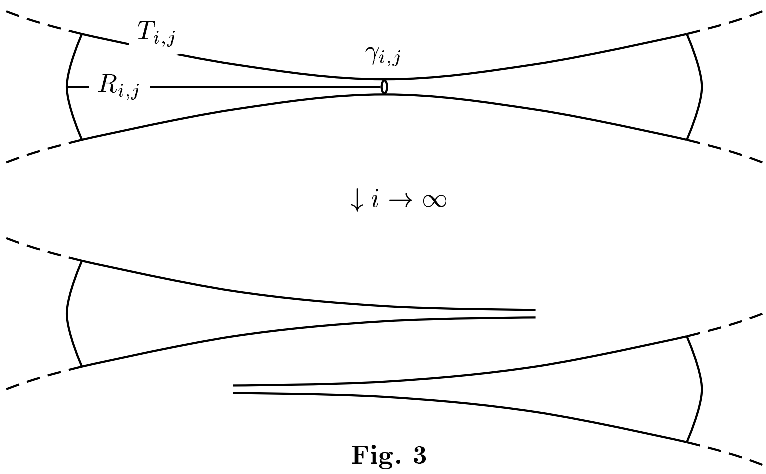


Fig. 3

In order to define the Dirac operator we also need to specify spin structures on our manifolds. In the degeneration we require that the diffeomorphisms $\Phi_i : \bar{M}_0 \to \bar{M}_{i,0}$ can be chosen compatible with the spin structures.

2 Generalities about the L^2-Spectrum

Let H be a complex Hilbert space and let A be a self-adjoint linear operator with dense domain $A : \mathcal{D}(A) \subset H \to H$.

Definition 1. *A number $\lambda \in \mathbb{C}$ is called an* **eigenvalue** *of A if $A - \lambda Id$ is not injective. In this case we call* $\dim(\ker(A - \lambda Id))$ *the* **multiplicity** *of λ. The set of eigenvalues, $spec_p(A)$, is called the* **point spectrum**.

The **essential spectrum***, $spec_e(A)$, is the set of $\lambda \in \mathbb{C}$ for which there exists a sequence $x_i \in \mathcal{D}(A)$ satisfying*

$$\|x_i\| = 1, \qquad (A - \lambda Id)x_i \to 0, \qquad x_i \rightharpoonup 0 \tag{2}$$

for $i \to \infty$. Here "$\rightharpoonup$" denotes weak convergence as opposed to norm convergence "$\to$".

The union of the point spectrum and the essential spectrum is the **spectrum** *of A, $spec(A) = spec_p(A) \cup spec_e(A)$.*

Note that the spectrum of a self-adjoint operator is actually contained in $\mathbb{R}$ and that the point spectrum and the essential spectrum need not be disjoint. Eigenvalues of infinite multiplicity and eigenvalues which are cumulation points of the spectrum are contained in both the point spectrum and the essential spectrum.

Definition 2. *The set*

$$spec_d(A) = spec_p(A) - spec_e(A)$$

is called the **discrete spectrum**. *The set*

$$spec_c(A) = spec_e(A) - spec_p(A)$$

is called the **continuous spectrum**.

Sometimes it will be convenient to look at the square of an operator instead of the operator itself. We will then use that $spec_e(A) = \emptyset$ if and only if $spec_e(A^2) = \emptyset$.

In the definition of the essential spectrum (2) can be replaced by other equivalent conditions. For example, instead of demanding $x_i \rightharpoonup 0$ we could require that there is no convergent subsequence. If the operator A is the closure of an operator L with domain $\mathcal{D}(L)$, then we can as well require $x_i \in \mathcal{D}(L)$. See e.g. [31] for details. A sequence as in (2) is called a *Weyl sequence.*

Let us show that the essential L^2-spectrum of self-adjoint elliptic differential operators on manifolds does not change when one modifies the manifold in a compact region.

In what follows we will denote the space of L^p-sections in a Hermitian vector bundle E over a Riemannian manifold M by $L^p(M,E)$, the Sobolev space of sections whose covariant derivatives up to order k are L^p by $H^{k,p}(M,E)$. The space of k times continuously differentiable sections is denoted by $C^k(M,E)$, $0 \leq k \leq \infty$, and the space of C^k-sections with compact support is denoted by $C_0^k(M,E)$.

Proposition 1. (Decomposition Principle)
Let $\bar{M}$ be a Riemannian manifold, with (possibly empty) compact boundary, $\bar{M} = \overset{\circ}{M} \dot{\cup} \partial M$. Let E be a Hermitian vector bundle over $\bar{M}$. Let L be an essentially self-adjoint linear differential operator of order $d \geq 1$ with domain $\mathcal{D}(L)$, $C_0^\infty(\overset{\circ}{M},E) \subset \mathcal{D}(L) \subset C_0^\infty(\bar{M},E)$. Suppose for every compact $K \subset \bar{M}$ there is an elliptic estimate

$$\|x\|_{H^{d,2}(K,E)} \leq C \cdot \left(\|x\|_{L^2(\bar{M},E)} + \|Lx\|_{L^2(\bar{M},E)}\right) \tag{3}$$

for all $x \in \mathcal{D}(L)$, $C = C(K)$. Denote the closure of L in $L^2(\bar{M},E)$ by $\bar{L}$.

Let $\bar{M}'$ be another Riemannian manifold and let E', L', and $\bar{L}'$ be defined similarly on $\bar{M}'$. We assume there exist compact sets $K \subset \bar{M}$, $K' \subset \bar{M}'$ such that $\bar{M} - K = \bar{M}' - K'$, and $E = E'$, $L = L'$ over $\bar{M} - K$.

Then

$$spec_e(\bar{L}) = spec_e(\bar{L}').$$

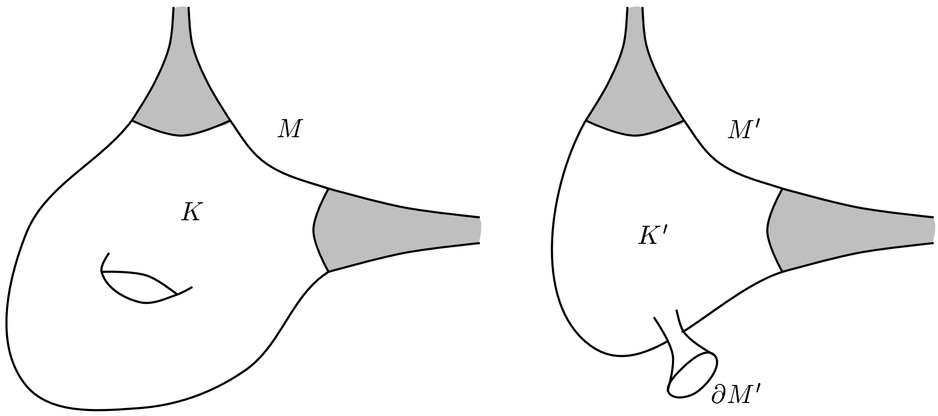


Fig. 4

Note that sections in $C_0^\infty(\bar{M}, E)$ need not vanish on ∂M in contrast to those of $C_0^\infty(\mathring{M}, E)$.

In case $\partial M = \emptyset$ the elliptic estimate (3) holds automatically if L is an elliptic operator [27, p. 379, Thm. 11.1]. In this case the decomposition principle can be found in many places in the literature for various operators (mostly of second order), see e.g. [19, 14, 16].

In the presence of boundary establishing (3) is subtler. It usually follows from *coercive estimates*

$$\|x\|^2_{H^{d,2}(K,E)} \le C \cdot \left(\|x\|^2_{L^2(\bar{M},E)} + \|Lx\|^2_{L^2(\bar{M},E)} + \sum_j \|B_j x\|^2_{H^{d-d_j-1/2,2}(\partial M,E)} \right)$$

where B_j are boundary (pseudo-) differential operators of order $d_j \le d-1$, $x \in C_0^\infty(\bar{M}, E)$. If $B_j x|_{\partial M} = 0$ for all $x \in \mathcal{D}(L)$, then (3) holds. The coercive estimate is automatic if L together with the B_j form a regular elliptic boundary value problem [27, V.11]. For example, a Laplace type operator L together with Dirichlet boundary conditions $x|_{\partial M} = 0$ forms a regular elliptic boundary value problem. We will use Proposition 1 with Dirichlet boundary conditions for the square of the Dirac operator which by the Lichnerowicz formula [23]

$$D^2 = \nabla^*\nabla + \frac{scal}{4} \tag{4}$$

is of Laplace type.

Here a *Laplace type operator* is an operator of the form

$$L = \nabla^*\nabla + \Re$$

where ∇ is a metric connection on a Hermitian vector bundle over a Riemannian manifold, ∇^* is its L^2-adjoint and $\Re$ is a smooth symmetric endomorphism field (zero order term). Laplace type operators are special elliptic operators of second order.

One can also apply the decomposition principle directly to the Dirac operator with suitable boundary conditions. Since we will not use this fact we leave the details to the reader.

Proof of Proposition 1. Since the whole situation is symmetric in $\bar{M}$ and $\bar{M}'$ it is sufficient to show $spec_e(\bar{L}) \subset spec_e(\bar{L}')$. Let $\lambda \in spec_e(\bar{L})$ and let $x_i \in \mathcal{D}(L) \subset L^2(\bar{M}, E)$ be a Weyl sequence as in (2).

Choose a compact subset $K_1 \subset \bar{M}$ whose interior contains K and another compact subset $K_2 \subset \bar{M}$ whose interior contains K_1. By the elliptic estimate (3)

$$\begin{aligned} \|x_i\|_{H^{d,2}(K_2,E)} &\leq C \cdot \left(\|x_i\|_{L^2(\bar{M},E)} + \|Lx_i\|_{L^2(\bar{M},E)}\right) \\ &\leq C \cdot \left(1 + \|Lx_i - \lambda x_i\|_{L^2(\bar{M},E)} + |\lambda|\right) \\ &\overset{(2)}{\leq} C'. \end{aligned}$$

Since $(x_i)_i$ is bounded in the $H^{d,2}$-norm and K_2 is compact we can, by the Rellich's lemma, pass to a subsequence, again denoted $(x_i)_i$, which converges in $H^{d-1,2}(K_2, E)$ to some element $x_\infty \in H^{d-1,2}(K_2, E)$.

To compute x_∞ we pick a cut-off function ψ_1 identical to 1 on K_1 and vanishing outside K_2. On the one hand, since $\psi_1 x_i \to \psi_1 x_\infty$ in $L^2(K_2, E)$,

$$(\psi_1 x_i, \psi_1 x_\infty)_{L^2(\bar{M},E)} = (\psi_1 x_i, \psi_1 x_\infty)_{L^2(K_2,E)} \longrightarrow (\psi_1 x_\infty, \psi_1 x_\infty)_{L^2(K_2,E)}.$$

On the other hand,

$$(\psi_1 x_i, \psi_1 x_\infty)_{L^2(\bar{M},E)} = (x_i, \psi_1^2 x_\infty)_{L^2(\bar{M},E)} \longrightarrow 0$$

because $x_i \rightharpoonup 0$. Hence $\psi_1 x_\infty = 0$ and $x_\infty|_{K_1} = 0 \in H^{d-1,2}(K_1, E)$. Therefore

$$\|x_i\|_{H^{d-1,2}(K_1,E)} \longrightarrow 0. \tag{5}$$

In particular, for i sufficiently large, $\|x_i\|^2_{L^2(K_1,E)} \leq \frac{1}{2}$ and thus

$$\|x_i\|^2_{L^2(\bar{M}-K_1,E)} \geq \frac{1}{2}. \tag{6}$$

Choose a cut-off function $\psi \in C^\infty(\bar{M}, \mathbb{R})$ with $\psi = 0$ on K and $\psi = 1$ on $\bar{M} - K_1$, $0 \leq \psi \leq 1$ everywhere. Let us look at the sequence $y_i \in L^2(\bar{M}', E')$ where $y_i = \psi \cdot x_i$ on $\bar{M} - K = \bar{M}' - K'$ and $y_i \equiv 0$ on K'. First of all, by (6),

$$\|y_i\|^2_{L^2(\bar{M}',E')} \geq \|x_i\|^2_{L^2(\bar{M}-K_1,E)} \geq \frac{1}{2}.$$

Secondly, for any $z \in L^2(\bar{M}', E')$,

$$(y_i, z)_{L^2(\bar{M}',E')} = (x_i, \psi z)_{L^2(\bar{M},E)} \to 0$$

by (2). Hence $y_i \rightharpoonup 0$.

Thirdly,

$$L' y_i = L(\psi x_i) = \psi L x_i + Q x_i$$

where $Q = [L, \psi]$ is a differential operator of order $d-1$. Moreover, Q vanishes outside K_1 because $\nabla\psi$ does. There is a constant $C_2 > 0$ such that

$$\|Qx_i\|_{L^2(\bar{M}',E')} \leq C_2 \cdot \|x_i\|_{H^{d-1,2}(K_1,E)}.$$

Therefore (5) implies $\|Qx_i\|_{L^2(\bar{M}',E')} \to 0$. We conclude

$$\begin{aligned} \|L'y_i - \lambda y_i\|_{L^2(\bar{M}',E')} &\leq \|\psi(Lx_i - \lambda x_i)\|_{L^2(\bar{M},E)} + \|Qx_i\|_{L^2(\bar{M}',E')} \\ &\leq \|Lx_i - \lambda x_i\|_{L^2(\bar{M},E)} + \|Qx_i\|_{L^2(\bar{M}',E')} \longrightarrow 0. \end{aligned}$$

Thus the sequence $(y_i / ||y_i||_{L^2(\bar{M}',E')})_i$ is a Weyl sequence for the operator $\bar{L}'$. Hence $\lambda \in spec_e(\bar{L}')$. $\square$

The proposition will be very useful for the study of the essential spectrum of hyperbolic manifolds because it tells us that we only need to consider the operator on the cusps and those have a very simple form.

3 The Dirac Operator on Hyperbolic Manifolds

In this section we will study the type of the spectrum of the Dirac operator on hyperbolic manifolds of finite volume. Studying the type means finding out if the spectrum is e.g. purely discrete or purely essential or contains both components. Our Dirac operator will always be the classical Dirac operator, sometimes also called Atiyah-Singer operator, acting on spinors. For definitions see [22].

If M is an n-dimensional Riemannian spin manifold and $N \subset M$ is an oriented hypersurface, then every spin structure on M canonically induces a spin structure on N. If n is odd, then the restriction to N of the spinor bundle ΣM of M is precisely the spinor bundle of N, $\Sigma M|_N = \Sigma N$. If n is even, then $\Sigma M|_N$ is isomorphic to $\Sigma N \oplus \Sigma N$.

Let H denote the mean curvature function of N with respect to the unit normal field ν. Let D^M be the Dirac operator of M. Let D^N be the Dirac operator of N in case n is odd. If n is even let D^N be the direct sum of the Dirac operator of N and its negative. In either case D^N acts on sections of $\Sigma M|_N$. The two operators D^M and D^N are related by the formula

$$-\nu \cdot D^M \sigma = D^N \sigma - \frac{n-1}{2} H\sigma + \nabla^M_\nu \sigma, \qquad (7)$$

see e.g. [4, 29]. Here σ is a section of ΣM defined in a neighborhood of N, "$\cdot$" denotes Clifford multiplication with respect to the manifold M and ∇^M is the Levi-Civita connection of ΣM.

The case of a warped product will be of special importance. Let N be an $(n-1)$-dimensional Riemannian spin manifold, let $I \subset \mathbb{R}$ be an interval. We give $M = N \times I$ the product spin structure and the warped product metric

$$ds^2(x,t) = \rho(t)^2 g_N(x) + dt^2$$

where $\rho : I \to \mathbb{R}$ is a fixed positive smooth function. For example, cusps of a hyperbolic manifold are of this form with $I = [0, \infty)$ and $\rho(t) = e^{-t}$. Let $\nu = \frac{\partial}{\partial t}$ be the unit vector field along I. The mean curvature of $N \times \{t\}$ in M is now given by $H(t) = -\frac{\dot{\rho}(t)}{\rho(t)}$.

Lemma 1. *Let M be a warped product as above. Suppose there is a subspace X of the kernel of D^N such that*

$$\ker(D^N) = X \oplus \nu \cdot X, \qquad X \perp \nu \cdot X.$$

Write $d = \dim(X) = \dim(\ker(D^N))/2$. Let $0 < \mu_1 \leq \mu_2 \leq \mu_3 \cdots \to \infty$ be the positive eigenvalues of D^N, each eigenvalue repeated according to its multiplicity.

Then there is a unitary equivalence

$$L^2(M, \Sigma M) \to \bigoplus_{\mu \in spec(D^N)} L^2(I, \mathbb{C}, dt) = \bigoplus_{j=1}^{d} L^2(I, \mathbb{C}^2, dt) \oplus \bigoplus_{j=1}^{\infty} L^2(I, \mathbb{C}^2, dt)$$

under which the Dirac operator D^M is transformed into

$$D^M \to \bigoplus_{j=1}^{d} D_0 \oplus \bigoplus_{j=1}^{\infty} D_{\mu_j}$$

where

$$D_\mu = \begin{pmatrix} 0 & -\frac{d}{dt} + \frac{\mu}{\rho(t)} \\ \frac{d}{dt} + \frac{\mu}{\rho(t)} & 0 \end{pmatrix}.$$

Similarly, the square of the Dirac operator is transformed into

$$(D^M)^2 \to \bigoplus_{\mu \in spec(D^N)} L_\mu$$

where

$$L_\mu = -\frac{d^2}{dt^2} + \frac{\mu \dot{\rho}(t)}{\rho(t)^2} + \frac{\mu^2}{\rho(t)^2}$$

on $L^2(I, \mathbb{C}, dt)$.

Proof. We decompose

$$L^2(N, \Sigma M|_N) = \mathcal{H}^+ \oplus X \oplus \nu \cdot X \oplus \mathcal{H}^-,$$

where $\mathcal{H}^\pm$ is the sum of eigenspaces of D^N for positive or negative eigenvalues respectively. Let $\varphi_1, \varphi_2, \varphi_3, \ldots$ be orthonormal eigenvectors corresponding to the positive eigenvalues $0 < \mu_1 \le \mu_2 \le \mu_3 \cdots \to \infty$, $\varphi_j \in L^2(N, \Sigma M|_N)$. Then we have the Hilbert space decomposition

$$\mathcal{H}^+ = \bigoplus_{j=1}^{\infty} \mathbb{C} \cdot \varphi_j.$$

Since Clifford multiplication with ν anticommutes with D^N we see that $\nu \cdot \varphi_j$ is an eigenvector for the eigenvalue $-\mu_j$ and hence

$$\mathcal{H}^- = \bigoplus_{j=1}^{\infty} \mathbb{C} \cdot \nu \cdot \varphi_j.$$

Write $\mathcal{H}_j := \mathbb{C} \cdot \varphi_j \oplus \mathbb{C} \cdot \nu \cdot \varphi_j$. Similarly, let $\psi_1, \ldots, \psi_d$ be an orthonormal basis of X and put $\tilde{\mathcal{H}}_j := \mathbb{C} \cdot \psi_j \oplus \mathbb{C} \cdot \nu \cdot \psi_j$. Then

$$L^2(N, \Sigma M|_N) = \bigoplus_{j=1}^{d} \tilde{\mathcal{H}}_j \oplus \bigoplus_{j=1}^{\infty} \mathcal{H}_j.$$

By (7) the Dirac operator D^M leaves the Hilbert space decomposition

$$L^2(M, \Sigma M) = \bigoplus_{j=1}^{d} L^2(I, \tilde{\mathcal{H}}_j, \rho(t)^{n-1} dt) \oplus \bigoplus_{j=1}^{\infty} L^2(I, \mathcal{H}_j, \rho(t)^{n-1} dt)$$

invariant and $D^M(\alpha_j \varphi_j + \alpha_{-j} \nu \varphi_j) = (\alpha_{-j} \mu_j + \frac{n-1}{2} H \alpha_{-j} - \dot{\alpha}_{-j}) \varphi_j + (\alpha_j \mu_j - \frac{n-1}{2} H \alpha_j + \dot{\alpha}_j) \nu \varphi_j$. The map

$$\alpha_j \varphi_j + \alpha_{-j} \nu \varphi_j \mapsto \rho(t)^{\frac{n-1}{2}} \begin{pmatrix} \alpha_j \\ \alpha_{-j} \end{pmatrix}$$

yields a unitary equivalence

$$L^2(I, \mathcal{H}_j, \rho(t)^{n-1} dt) \to L^2(I, \mathbb{C}^2, dt)$$

under which the Dirac operator is transformed into

$$D_{\mu_j} = \begin{pmatrix} 0 & -\frac{d}{dt} + \frac{\mu_j}{\rho(t)} \\ \frac{d}{dt} + \frac{\mu_j}{\rho(t)} & 0 \end{pmatrix}$$

and similarly for the zero eigenvalues. The formula for the square of the Dirac operator follows immediately. $\square$

Remark. The assumption $\ker(D^N) = X \oplus \nu \cdot X$, $X \perp \nu \cdot X$, is necessary only for the decomposition of the Dirac operator D^M itself, not for its square $(D^M)^2$. This assumption is automatically satisfied if M has even dimension. In this case $\Sigma M|_N = \Sigma^+ M|_N \oplus \Sigma^- M|_N \cong \Sigma N \oplus \Sigma N$ and one can simply take $X = \ker(D^N) \cap C^\infty(N, \Sigma^+ M|_N)$. If $\dim(M)$ is odd, then the assumption is equivalent to $\hat{\mathcal{A}}(N) = 0$.

Remark. Lemma 1 together with Proposition 1 is already enough to give a simple computation of the Dirac spectrum of hyperbolic n-space H^n. After removing a point o from H^n the space is isometric to a warped product $S^{n-1} \times (0, \infty)$ where S^{n-1} carries its standard metric of constant sectional curvature 1 and $\rho(t) = \sinh(t)$. By Lemma 1 the square of the Dirac operator on $H^n - \{o\}$ is unitarily equivalent to $\bigoplus_{\mu \in spec(S^{n-1})} L_\mu$ where $L_\mu = -\frac{d^2}{dt^2} + V_\mu(t)$, $V_\mu(t) = \frac{\mu \cosh(t) + \mu^2}{\sinh(t)^2}$. All Dirac eigenvalues μ of S^{n-1} are nonzero.

Since $V_\mu(t) \to 0$ for $t \to \infty$ we know $spec_e(L_\mu) = [0, \infty)$ where L_μ is acting on the Hilbert space $L^2([1, \infty), \mathbb{C}, dt)$ with say Dirichlet boundary conditions at $t = 1$, see [15, p. 1448, Thm. 16(b)]. By Proposition 1 we have that $spec_e((D^{H^n})^2) = spec_e((D^{H^n - B_1(o)})^2) = [0, \infty)$.

One checks [3] that D_μ does not have any square integrable eigenfunctions on $(0, \infty)$, see also [20, 30, 32]. In particular, there are no L^2-eigenspinors for the Dirac operator on H^n, $spec_p(D^{H^n}) = \emptyset$.

We conclude $spec((D^{H^n})^2) = spec_c((D^{H^n})^2) = [0, \infty)$. Finally, since H^n is a simply connected symmetric space the spectrum of the Dirac operator is symmetric about 0. In even dimensions this is automatic. In odd dimensions the geodesic reflection about o can be used to map an eigenspinor or a Weyl sequence for $\lambda \in spec(D^{H^n})$ into one for $-\lambda$. We obtain

$$spec(D^{H^n}) = spec_c(D^{H^n}) = \mathbb{R}.$$

See [6] for a computation of this spectrum using harmonic analysis. Note that there is an incorrect statement about the eigenvalue 0 in that paper. See also [7, 8].

Definition 3. *Let M be a hyperbolic manifold of finite volume. Let $\mathcal{E} = N \times [0, \infty)$ be a cusp of M. A spin structure of M will be called* **trivial along** $\mathcal{E}$ *if the induced operator D^N on N has a nontrivial kernel, i.e. there exist $\varphi \in C^\infty(N, \Sigma M|_N)$, $\varphi \neq 0$, but $D^N \varphi = 0$.*

This terminology is justified by the fact that in the most prominent case when N is a flat torus, the trivial (biinvariant) spin structure of N is the only one among its 2^{n-1} spin structures for which the Dirac operator has a nontrivial kernel.

The following theorem is our first main result. It tells us that only two extremal cases can occur for the type of spectrum of the Dirac operator on a hyperbolic manifold of finite volume. It can only be purely discrete spectrum or the whole real line. It is the spin structure which is responsible for which of the two cases occurs.

Theorem 1. *Let M be a hyperbolic manifold of finite volume equipped with a spin structure.*

If the spin structure is trivial along at least one cusp, then the Dirac spectrum satisfies

$$spec(D) = spec_e(D) = \mathbb{R}.$$

If the spin structure is nontrivial along all cusps, then the spectrum is discrete,

$$spec(D) = spec_d(D).$$

PROOF. Recall decomposition (1) of M into a relatively compact part and finitely many cusps,

$$M = M_0 \dot{\cup} \dot{\bigcup_{j=1}^{k}} \mathcal{E}_j.$$

We start with the case that the spin structure is trivial along at least one end. Hence M has a cusp $\mathcal{E}_1 = N_1 \times [0,\infty)$ with metric $g_{\mathcal{E}_1} = e^{-2t} g_{N_1} + dt^2$ where g_{N_1} is a flat metric and $\ker(D^{N_1}) \neq 0$. Choose $\varphi \in \ker(D^{N_1})$, $\varphi \neq 0$, such that $\nu \cdot \varphi = i \cdot \varphi$ or $\nu \cdot \varphi = -i \cdot \varphi$. This is possible since $\nu^2 = -1$. W.l.o.g. let $\nu \cdot \varphi = i \cdot \varphi$.

Let $\lambda \in \mathbb{R}$. We look at spinors σ on $\mathcal{E}_1$ of the form

$$\sigma = \alpha\varphi \tag{8}$$

where $\alpha : [0,\infty) \to \mathbb{C}$ is a smooth function.

For $\alpha_\lambda(t) = e^{((n-1)/2 - \lambda i)t}$, $\sigma_\lambda = \alpha_\lambda \varphi$, we see using (7) and $H = 1$

$$\begin{aligned}
D\sigma_\lambda &= \nu \left\{ D^{N_1}\sigma_\lambda - \frac{n-1}{2} H \sigma_\lambda + \nabla^M_\nu \sigma_\lambda \right\} \\
&= \nu \left\{ 0 - \frac{n-1}{2} \alpha_\lambda \varphi + \dot{\alpha}_\lambda \varphi \right\} \\
&= -\lambda i \nu \cdot \sigma_\lambda \\
&= \lambda \sigma_\lambda.
\end{aligned}$$

For $0 < a < b < \infty$ we denote $\mathcal{E}_{1,a,b} := N_1 \times [a,b] \subset \mathcal{E}_1 \subset M$. For a spinor $\sigma = \alpha\varphi$ of the form (8) one easily computes the L^2-norm

$$||\sigma||^2_{L^2(\mathcal{E}_{1,a,b})} = ||\varphi||^2_{N_1} \cdot \int_a^b |\alpha(t)|^2 e^{-(n-1)t} dt.$$

W.l.o.g. assume $||\varphi||_{N_1}^2 = 1$. Then we obtain for $\sigma = \sigma_\lambda$

$$||\sigma_\lambda||^2_{L^2(\mathcal{E}_{1,a,b})} = b - a.$$

Now choose smooth functions $\psi_m : \mathbb{R} \to \mathbb{R}$ such that

- $\psi_m \equiv 0$ on $\mathbb{R} - [m-2, 2m+2]$,
- $\psi_m \equiv \frac{1}{\sqrt{m}}$ on $[m, 2m]$,
- $0 \leq \psi_m \leq \frac{1}{\sqrt{m}}$ everywhere,
- $\left|\frac{d\psi_m}{dt}\right| \leq \frac{1}{\sqrt{m}}$ everywhere.

We extend $\sigma_m := \psi_m \sigma_\lambda$ by zero to all of M and compute

$$||\sigma_m||^2_{L^2(M)} \geq \frac{1}{m} ||\sigma_\lambda||^2_{L^2(\mathcal{E}_{1,m,2m})} = 1,$$

$$\begin{aligned} ||(D-\lambda)\sigma_m||^2_{L^2(M')} &= ||\nabla\psi_m \cdot \sigma_\lambda||^2_{L^2(M)} \\ &= ||\nabla\psi_m \cdot \sigma_\lambda||^2_{L^2(\mathcal{E}_{1,m-2,m} \cup \mathcal{E}_{1,2m,2m+2})} \\ &\leq ||\nabla\psi_m||^2_{L^\infty} \cdot ||\sigma_\lambda||^2_{L^2(\mathcal{E}_{1,m-2,m} \cup \mathcal{E}_{1,2m,2m+2})} \\ &\leq \frac{4}{m}. \end{aligned}$$

Hence $(D-\lambda)\sigma_m \to 0$. For arbitrary square-integrable χ on M we have

$$\begin{aligned} |(\chi, \sigma_m)_{L^2(M)}| &= |(\chi, \sigma_m)_{L^2(\mathcal{E}_{1,m-2,2m+2})}| \\ &\leq ||\chi||_{L^2(\mathcal{E}_{1,m-2,2m+2})} \cdot ||\psi_m||_{L^\infty} \cdot ||\sigma_\lambda||_{L^2(\mathcal{E}_{1,m-2,2m+2})} \\ &\leq \overbrace{||\chi||_{L^2(\mathcal{E}_{1,m-2,2m+2})}}^{\to 0} \cdot \overbrace{\frac{1}{\sqrt{m}}\sqrt{m+4}}^{\to 1}. \end{aligned}$$

Thus

$$\sigma_m \rightharpoonup 0.$$

This shows $\lambda \in spec_e(D)$ and hence

$$spec_e(D) = \mathbb{R}.$$

Let us now put $M' := \dot{\bigcup}_{j=1}^{k} \mathcal{E}_j$ and turn to the case that the spin structure is nontrivial along all cusps, i.e. $\ker(D^{\partial M'}) = 0$.

By Lemma 1 the square of the Dirac operator on M' is unitarily equivalent to $\bigoplus_{\mu \in spec(D^{\partial M'})} L_\mu$ where $L_\mu = -\frac{d^2}{dt^2} + \mu e^t + \mu^2 e^{2t}$ is a Schrödinger operator on $L^2([0,\infty),\mathbb{C},dt)$ with potential $V_\mu(t) = \mu e^t + \mu^2 e^{2t}$. We impose Dirichlet boundary conditions.

Note that all $\mu \in spec(D^{\partial M'})$ are nonzero. Since $V_\mu \to \infty$ for $t \to \infty$ the classical theory of Weyl and Titchmarsh [15, p. 1448, Thm. 16(a)] tells us that the spectrum of L_μ is purely discrete, $spec_e(L_\mu) = \emptyset$.

To see $spec_e(\bigoplus_\mu L_\mu) = \emptyset$ we show that only finitely many $\mu \in spec(D^{\partial M'})$ contribute to the spectrum in a given compact interval $[-C, C] \subset \mathbb{R}$.

Only finitely many $\mu \in spec(D^{\partial M'})$ satisfy $-1/2 < \mu < 0$. For all $\mu \in spec(D^{\partial M'}) - (-1/2, 0)$ we see that $V_\mu(t) = \mu e^t + \mu^2 e^{2t} \geq \mu^2 - |\mu|$, hence $spec(L_\mu) \subset [\mu^2 - |\mu|, \infty)$. Since the $\mu \in spec(D^{\partial M'})$ form a discrete set with $\mu^2 \to \infty$ there are only finitely many μ for which $spec(L_\mu) \cap [-C, C] \neq \emptyset$. Thus $spec(\bigoplus_\mu L_\mu)$ is discrete.

By Proposition 1

$$spec_e(D^2) = spec_e((D^{M'})^2) = spec_e(\bigoplus_{\mu \in spec(D^{\partial M'})} L_\mu) = \emptyset.$$

Thus $spec_e(D) = \emptyset$ and the theorem is proven. □

Corollary 1. *Let M be a 2- or 3-dimensional hyperbolic manifold of finite volume equipped with a spin structure. Let M have exactly one cusp.*

Then the spectrum of the Dirac operator is discrete,

$$spec(D) = spec_d(D).$$

Proof. Decomposition (1) of M is in this case

$$M = M_0 \dot{\cup} \mathcal{E}.$$

Here M_0 is a compact manifold with boundary S^1 or T^2 respectively. It is well-known that the trivial spin structure on S^1 and T^2 do not bound a spin structure on a compact manifold. Indeed, they generate spin cobordism Ω_1^{Spin} and Ω_2^{Spin} respectively, see e.g. [22, p. 91]. Hence the spin structure must be nontrivial along $\mathcal{E}$. Theorem 1 yields the assertion. □

Corollary 2. *Every 2- or 3-dimensional oriented hyperbolic manifold of finite volume has a spin structure such that the spectrum of the Dirac operator is discrete,*

$$spec(D) = spec_d(D).$$

Proof. Again we look at decomposition (1). Chopping off the cusps yields the compact manifold $\bar{M}_0$ with boundary. The boundary is a disjoint union of S^1s or 2-tori. Gluing in disks or solid tori we obtain an oriented closed manifold M'. In dimension 2 and 3 all orientable manifolds are spin. In three dimensions this follows from triviality of the tangent bundle. Pick a spin structure on M' and restrict it to $\bar{M}_0$. Since the trivial spin structures on S^1 and T^2 do not bound, the induced spin structure must be nontrivial on all boundary components. Extending the spin structure to M yields a spin structure which is nontrivial along all cusps. Hence Theorem 1 yields the statement. $\square$

As we shall see in the last section a surface of finite volume with at least two cusps can always be given a spin structure such that $spec(D) = \mathbb{R}$. Hence both cases in Theorem 1 occur. In three dimensions this depends on the manifold. It can happen that the spectrum is always discrete even if the manifold has more than one cusp. If the hyperbolic manifold is given as the complement of a link in S^3, then there is simple criterion to decide if there is a spin structure such that $spec(D) = \mathbb{R}$. This involves counting of overcrossings (Theorem 4). See the last section for examples.

4 Domain Monotonicity

In order to study spectral degeneration in the next section we need a tool known as domain monotonicity in the spectral theory of the Laplace-Beltrami operator. We have to find a version for the Dirac operator. It will allow us to estimate the spectrum by decomposing the manifold into pieces and controlling the spectrum of the individual pieces. When doing this new boundary components appear and we have to exhibit suitable boundary conditions.

Domain monotonicity can be conveniently expressed in terms of eigenvalue counting functions. Let $\bar{M}$ be an n-dimensional compact Riemannian manifold with smooth boundary ∂M. Let L be a formally self-adjoint elliptic differential operator acting on sections of a Hermitian or Riemannian vector bundle defined over $\bar{M}$. Let the domain of L, specified by boundary conditions $\mathcal{B}$, be such that L becomes essentially self-adjoint. Denote the corresponding self-adjoint extension by $\bar{L}$. For any interval $I \subset \mathbb{R}$ we introduce the *eigenvalue counting function*

$$\mathcal{N}^{\mathcal{B}}_{L,\bar{M}}(I) := \sharp(spec(\bar{L}) \cap I).$$

By passing to n-dimensional submanifolds of $\bar{M}$ it is possible to estimate $\mathcal{N}^{\mathcal{B}}_{L,\bar{M}}(I)$ from above and from below. The two estimates are quite different in nature. Let us start with the simpler one, the estimate from below. Recall that an operator d is called *overdetermined elliptic* if its principal symbol $\sigma_d(\xi)$ is injective for all nonzero covectors $\xi \in T^*M$.

Proposition 2. (Domain Monotonicity I)
Let $\bar{M}$ be a compact Riemannian manifold with smooth boundary ∂M. Let $d : C^\infty(M, E) \to C^\infty(M, F)$ be an overdetermined elliptic linear differential

operator of first order, defined on Hermitian vector bundles E and F. Put $L = d^*d$. Let $\bar{N} \subset \bar{M}$ be a compact submanifold with smooth boundary ∂N, $\dim(\bar{N}) = \dim(\bar{M})$. We impose Dirichlet boundary conditions, $\mathcal{D}(L) = \{\varphi \in C^\infty(\bar{M}) \mid \varphi|_{\partial M} = 0\}$, similarly for $\bar{N}$.

Then for any $x > 0$

$$\mathcal{N}^{\text{Dirichlet}}_{L,\bar{M}}[0,x) \geq \mathcal{N}^{\text{Dirichlet}}_{L,\bar{N}}[0,x).$$

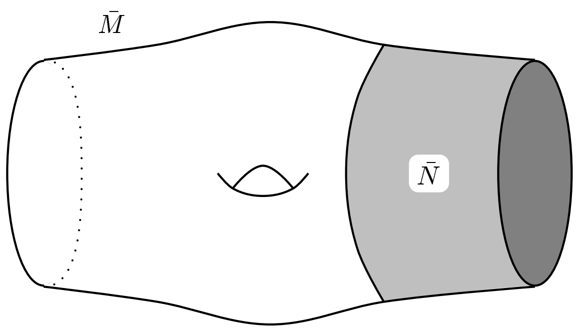


Fig. 5

Proof. The operator $\bar{L}$ is the self-adjoint operator associated with the closed semi-bounded quadratic form

$$q(\varphi) := (d\varphi, d\varphi)_{L^2}$$

with a form core given by $C_0^\infty(M, \Sigma M)$, cf. [26, VIII.6].

Extension by zero yields an embedding $C_0^\infty(N, \Sigma N) \hookrightarrow C_0^\infty(M, \Sigma M)$ and the quadratic form for $\bar{N}$ is simply the restriction of the quadratic form for $\bar{M}$. The variational characterization of eigenvalues yields the proposition. $\square$

Example. If $d : C^\infty(M, \mathbb{R}) \to C^\infty(M, T^*M)$ is exterior differentiation, then the proposition yields the standard domain monotonicity for the Laplace operator $L = \Delta$.

Example. If $d = D : C^\infty(M, \Sigma M) \to C^\infty(M, \Sigma M)$ is the Dirac operator on a Riemannian spin manifold, then we obtain a monotonicity principle for the square of the Dirac operator $L = D^2$. We will use this in the next section.

For the reverse estimate we assume for simplicity that M is a closed manifold. By a *decomposition* of M we mean finitely many submanifolds $\bar{M}_1, \ldots, \bar{M}_k$ of M with smooth boundaries ∂M_j and $\dim(M_j) = \dim(M)$ such that

- The interiors $\overset{\circ}{M}_j$ are pairwise disjoint,
- $M = \bigcup_{j=1}^{k} \bar{M}_j$.

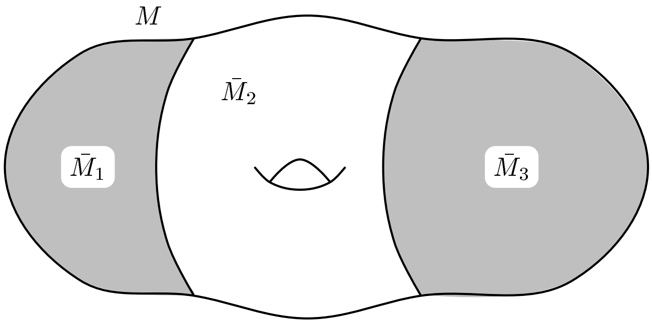


Fig. 6

If $\bar{M}$ is a compact Riemannian manifold with smooth boundary ∂M and exterior unit normal field ν and $L = d^*d$ is as above, then we call the boundary condition

$$(\sigma_{d^*}(\nu)d\varphi)|_{\partial M} = 0$$

the *natural boundary conditions* for L.

Proposition 3. (Domain Monotonicity II)
Let M be a closed Riemannian manifold. Let $d : C^\infty(M, E) \to C^\infty(M, F)$ be an overdetermined elliptic linear differential operator of first order, defined on Hermitian vector bundles E and F. Put $L = d^*d$. Let $M = \bigcup_{j=1}^{k} \bar{M}_j$ be a decomposition of M as explained above.

We impose natural boundary conditions for the $\bar{M}_j$. Let L together with the natural boundary conditions form a regular elliptic boundary value problem.

Then for any $x > 0$

$$\mathcal{N}_{L,M}[0, x) \leq \sum_{j=1}^{k} \mathcal{N}^{\text{Natural}}_{L,\bar{M}_j}[0, x).$$

Proof. If $\bar{N}$ is a compact Riemannian manifold and $L = d^*d$, then L with domain $\mathcal{D}(L) = \{\varphi \in C^\infty(\bar{N}, \Sigma N) \mid (\sigma_{d^*}(\nu)d\varphi)|_{\partial N} = 0\}$ is essentially self-adjoint [27, V.12]. Look at the closed semi-bounded quadratic form

$$q(\varphi) := (d\varphi, d\varphi)_{L^2}$$

with form core $C^\infty(\bar{N}, \Sigma N)$. The Green's formula

$$(d^* d\varphi, \psi)_{L^2(\bar{N},\Sigma\bar{N})} - (d\varphi, d\psi)_{L^2(\bar{N},\Sigma\bar{N})} = \int_{\partial M} \langle \sigma_{d^*}(\nu) d\varphi, \psi\rangle$$

shows that the self-adjoint A operator associated with q has domain

$$\begin{aligned}
\mathcal{D}(A) &= \{\varphi \in \mathcal{D}(q) \mid \exists \chi \in L^2(\bar{N}, \Sigma\bar{N}) : (\psi, \chi)_{L^2} = (d\psi, d\varphi)_{L^2} \\
&\quad\ \forall \psi \in \mathcal{D}(q)\} \\
&\supset \{\varphi \in H^2(\bar{N}, \Sigma\bar{N}) \mid (\sigma_{d^*}(\nu) d\varphi)|_{\partial N} = 0\} \\
&= \mathcal{D}(\bar{L}).
\end{aligned}$$

Hence $\bar{L} \subset A$ and since both operators are self-adjoint $\bar{L} = A$. Therefore the eigenvalues of $L = d^* d$ can be computed using the quadratic form q with $C^\infty(\bar{N}, \Sigma\bar{N})$ as space of admissible test sections.

Returning to our closed manifold M with the decomposition $M = \bar{M}_1 \cup \ldots \cup \bar{M}_k$ we look at the isometric embedding

$$C^\infty(M, \Sigma M) \subset L^2(M, \Sigma M) \to \bigoplus_{j=1}^{k} C^\infty(\bar{M}_j, \Sigma\bar{M}_j) \subset \bigoplus_{j=1}^{k} L^2(\bar{M}_j, \Sigma\bar{M}_j),$$

$$\varphi \mapsto (\varphi|_{\bar{M}_1}, \ldots, \varphi|_{\bar{M}_k}).$$

Under this embedding the quadratic form q corresponding to $L = d^* d$ on M is the restriction of the orthogonal sum $q_1 \oplus \ldots \oplus q_k$ of the forms for $\bar{M}_j$. Now the variational characterization of eigenvalues completes the proof. $\square$

Example. If $d : C^\infty(M, \mathbb{R}) \to C^\infty(M, T^*M)$ is exterior differentiation, then we obtain Neumann boundary conditions for the Laplace operator $L = \Delta$. More generally, let $d = \nabla : C^\infty(M, E) \to C^\infty(M, T^*M \otimes E)$ be a Riemannian connection. Then the above monotonicity principle holds for the operator $L = \nabla^*\nabla$ with *Neumann boundary conditions*:

$$0 = \sigma_{\nabla^*}(\nu) \nabla\varphi = -\nabla_\nu \varphi.$$

Example. If $d = D : C^\infty(M, \Sigma M) \to C^\infty(M, \Sigma M)$ is the Dirac operator on a Riemannian spin manifold, then the monotonicity principle for the square of the Dirac operator $L = D^2$ uses boundary conditions

$$\nu \cdot D\varphi|_{\partial M} = 0$$

or equivalently

$$D\varphi|_{\partial M} = 0.$$

5 Degeneration in Two Dimensions

Now we study the behavior of the spectrum under the degeneration process described in the first section. If the limit manifold has continuous spectrum we expect that the eigenvalues of the compact manifolds accumulate in the degeneration. We will see that this is true and compute the accumulation rate. We start with the 2-dimensional case.

Theorem 2. *Let M_i be a sequence of closed hyperbolic surfaces converging to a noncompact hyperbolic surface M of finite volume. Let each M_i have exactly k tubes with trivial spin structure around closed geodesics of length $\ell_{i,j}$ tending to zero. Hence M has exactly $2k$ cusps along which the spin structure is trivial. Let $x > 0$.*

Then the eigenvalue counting function for the Dirac operator satisfies for sufficiently small $\ell_{i,j}$:

$$\mathcal{N}_{D,M_i}(-x,x) = \frac{4x}{\pi}\sum_{j=1}^{k}\log(1/\ell_{i,j}) + \mathrm{O}_x(1).$$

Here $\mathrm{O}_x(1)$ denotes an error term bounded as a function of i where the bound is allowed to depend on x.

Proof. To keep the notation simple we restrict ourselves to the case that the M_i have exactly one degenerating tube with either trivial or nontrivial spin structure, hence $k = 0$ or $k = 1$. Recall from Section 1 that the tube T_i is isometric to $S^1 \times [-R_i, R_i]$ with warped product metric $ds^2 = \ell_i^2\cosh(t)^2 d\theta^2 + dt^2$, $\theta \in S^1 = \mathbb{R}/\mathbb{Z}$, $t \in [-R_i, R_i]$, $R_i = \log(1/\ell_i) + c_0$.

Choose a constant $c_1 = c_1(x)$ such that for all nonzero eigenvalues μ of D^{S^1} we have

$$e^{c_1-c_0}\cdot|\mu|\cdot(e^{c_1-c_0}\cdot|\mu| - 1) > x^2. \tag{9}$$

Put

$$\tilde{T}_i := S^1 \times [-R_i + c_1, R_i - c_1] \subset T_i \subset M_i$$

and

$$\tilde{M}_{i,0} := M_{i,0} \cup \left(S^1 \times [-R_i, -R_i + c_1]\right) \cup \left(S^1 \times [R_i - c_1, R_i]\right) \subset M_i.$$

Then $M_i = \tilde{M}_{i,0} \cup \tilde{T}_i$.

The Lichnerowicz formula (4) says in our case

$$D^2 = \nabla^*\nabla + \frac{scal}{4} = \nabla^*\nabla - \frac{1}{2}.$$

Proposition 3 yields

$$\begin{array}{rcl}\mathcal{N}_{D,M_i}(-x,x) & = & \mathcal{N}_{D^2,M_i}[0,x^2) \\ & = & \mathcal{N}_{\nabla^*\nabla-1/2,M_i}[0,x^2) \\ & = & \mathcal{N}_{\nabla^*\nabla,M_i}[1/2,x^2+1/2) \\ & \le & \mathcal{N}_{\nabla^*\nabla,M_i}[0,x^2+1/2) \\ & \le & \mathcal{N}^{\text{Neumann}}_{\nabla^*\nabla,\tilde{T}_i}[0,x^2+1/2)+\mathcal{N}^{\text{Neumann}}_{\nabla^*\nabla,\tilde{M}_{i,0}}[0,x^2+1/2) \\ & = & \mathcal{N}^{\text{Neumann}}_{D^2,\tilde{T}_i}[-1/2,x^2)+\mathcal{N}^{\text{Neumann}}_{D^2,\tilde{M}_{i,0}}[-1/2,x^2).\end{array}$$

All $\tilde{M}_{i,0}$ are diffeomorphic and the metrics converge in the C^∞-topology to the metric of the limit surface. Thus the eigenvalues also converge and therefore $\mathcal{N}^{\text{Neumann}}_{D^2,\tilde{M}_{i,0}}[-1/2,x^2)=\mathrm{O}_x(1)$.

Using Lemma 1 we obtain

$$\mathcal{N}^{\text{Neumann}}_{D^2,\tilde{T}_i}[-1/2,x^2)=\sum_{\mu\in spec(D^{S^1})}\mathcal{N}^{\text{Neumann}}_{L_\mu,[-R_i+c_1,R_i-c_1]}[-1/2,x^2)$$

where $L_\mu=-\frac{d^2}{dt^2}+V_\mu$, $V_\mu(t)=\mu\cdot\frac{\ell_i\sinh(t)+\mu}{\ell_i^2\cosh(t)^2}$. We can estimate the potential V_μ on $[-R+c_1,R-c_1]=[-\log(1/\ell)-c_0+c_1,\log(1/\ell)+c_0-c_1]$ for nonzero $\mu\in spec(D^{S^1})$ as follows:

$$\begin{array}{rcl}V_\mu(t) & \ge & |\mu|\dfrac{|\mu|-\ell|\sinh(t)|}{\ell^2\cosh(t)^2} \\ & \ge & |\mu|\left(\dfrac{|\mu|}{\ell^2\cosh(t)^2}-\dfrac{1}{\ell\cosh(t)}\right) \\ & = & \dfrac{|\mu|}{\ell\cosh(t)}\left(\dfrac{|\mu|}{\ell\cosh(t)}-1\right) \\ & \ge & \dfrac{|\mu|}{\ell e^{R-c_1}}\left(\dfrac{|\mu|}{\ell e^{R-c_1}}-1\right) \\ & = & e^{c_1-c_0}|\mu|(e^{c_1-c_0}|\mu|-1) \\ & > & x^2\end{array}$$

by (9). Hence for nonzero $\mu\in spec(D^{S^1})$ all eigenvalues of L_μ with Neumann boundary conditions are bigger than x^2, i.e. $\mathcal{N}^{\text{Neumann}}_{L_\mu,[-R_i+c_1,R_i-c_1]}[-1/2,x^2)=0$. Denote the multiplicity of the eigenvalue 0 in $spec(D^{S^1})$ by $mult(0)$. We have shown

$$\begin{array}{rcl}\mathcal{N}_{D,M_i}(-x,x) & \le & \mathcal{N}^{\text{Neumann}}_{D^2,\tilde{T}_i}[-1/2,x^2)+\mathcal{N}^{\text{Neumann}}_{D^2,\tilde{M}_{i,0}}[-1/2,x^2) \\ & = & \mathcal{N}^{\text{Neumann}}_{D^2,\tilde{T}_i}[-1/2,x^2)+\mathrm{O}_x(1) \\ & = & mult(0)\cdot\mathcal{N}^{\text{Neumann}}_{-\frac{d^2}{dt^2},[-R_i+c_1,R_i-c_1]}[-1/2,x^2)+\mathrm{O}_x(1)\end{array}$$

$$
\begin{aligned}
&= mult(0) \cdot \mathcal{N}^{\text{Neumann}}_{-\frac{d^2}{dt^2},[-R_i+c_1,R_i-c_1]}[0,x^2) + \mathrm{O}_x(1) \\
&= mult(0) \cdot \frac{x \cdot 2(R_i - c_1)}{\pi} + \mathrm{O}_x(1) \\
&= mult(0) \cdot \frac{2xR_i}{\pi} + \mathrm{O}_x(1).
\end{aligned}
$$

In case the spin structure is nontrivial along T_i we have $mult(0) = 0$ and

$$
\mathcal{N}_{D,M_i}(-x,x) = \mathrm{O}_x(1).
$$

The theorem is proven in this case. If the spin structure is trivial, we have $mult(0) = 2$, hence

$$
\begin{aligned}
\mathcal{N}_{D,M_i}(-x,x) &\leq \frac{4xR_i}{\pi} + \mathrm{O}_x(1) \\
&= \frac{4x}{\pi}(\log(1/\ell_i) + c_0) + \mathrm{O}_x(1) \\
&= \frac{4x}{\pi}\log(1/\ell_i) + \mathrm{O}_x(1).
\end{aligned}
$$

In this case we also need a lower bound which is easily obtained by applying Proposition 2 and Lemma 1:

$$
\begin{aligned}
\mathcal{N}_{D,M_i}(-x,x) &= \mathcal{N}_{D^2,M_i}[0,x^2) \\
&\geq \mathcal{N}^{\text{Dirichlet}}_{D^2,T_i}[0,x^2) \\
&= \sum_{\mu \in spec(D^{S^1})} \mathcal{N}^{\text{Dirichlet}}_{L_\mu,[-R_i,R_i]}[0,x^2). \\
&\geq mult(0) \cdot \mathcal{N}^{\text{Dirichlet}}_{L_0,[-R_i,R_i]}[0,x^2) \\
&= 2 \cdot \frac{x \cdot 2R_i}{\pi} + \mathrm{O}(1) \\
&= \frac{4x}{\pi}\log(1/\ell_i) + \mathrm{O}_x(1).
\end{aligned}
$$

$\square$

6 Manifolds Foliated by Hypersurfaces

In three dimensions we have the problem that the degenerating tube is not a warped product so that the simple separation of variables of Lemma 1 does not apply. But the tube is foliated by flat 2-tori as described in Section 1. In order to take advantage of this we derive a formula relating the square of the Dirac operator on a manifold foliated by hypersurfaces to normal derivatives and operators acting on the leaves.

In this paper we will only need Corollary 4 at the end of this section. The reader may skip this section at a first reading and only come back to it when needed.

We hope that the general formula in Proposition 4 will also be useful in other contexts.

Let M be a Riemannian spin manifold of dimension n. Let M be foliated by oriented (hence spin) hypersurfaces $\{N\}$. Denote the unit normal field to the foliation by ν, its shape operator by B, $B(X) = -\nabla_X \nu$, and its mean curvature function by $H := \frac{1}{n-1}\mathbf{Tr}B$.

Let ΣM be the spinor bundle on M. Recall that $\Sigma M|_N$ is the spinor bundle of N if n is odd. If n is even, then $\Sigma M|_N$ coincides with the sum of two copies of the spinor bundle of N. Clifford multiplication with respect to N is given by

$$X \otimes \varphi \to X \cdot \nu \cdot \varphi$$

where the dot "$\cdot$" denotes Clifford multiplication with respect to M. Recall equation (7)

$$-\nu \cdot D^M = D^N - \frac{n-1}{2}H + \nabla^M_\nu .$$

One can also relate ∇^M to the spinorial Levi-Civita connection ∇^N for N by

$$\nabla^M_X \varphi = \nabla^N_X \varphi + \frac{1}{2}B(X) \cdot \nu \cdot \varphi \tag{10}$$

see e.g. [4, Prop. 2.1].

We need one more piece of notation. Define

$$\mathfrak{D}^B := \sum_{i=1}^{n-1} e_i \cdot \nu \cdot \nabla^N_{B(e_i)} = \sum_{i=1}^{n-1} B(e_i) \cdot \nu \cdot \nabla^N_{e_i}$$

If B happens to be a multiple of the identity, $B = c{\cdot}\mathrm{Id}$, then

$$\mathfrak{D}^B = c\sum_{i=1}^{n-1} e_i \cdot \nu \cdot \nabla^N_{e_i} = cD^N .$$

Proposition 4. *Let M be an n-dimensional Riemannian spin manifold with Ricci curvature Ric. Let M be foliated by oriented hypersurfaces $\{N\}$ as described above. Then*

$$\begin{aligned} (D^M)^2 \quad = \quad & (D^N)^2 - (\nabla^M_\nu)^2 + (n-1)H\nabla^M_\nu + \nabla^M_{\nabla_\nu \nu} - \mathfrak{D}^B \\ & -\frac{n-1}{2}(\nabla^N H)\cdot\nu - \frac{(n-1)^2}{4}H^2 + \frac{1}{2}|B|^2 - \frac{1}{2}\nu \cdot Ric(\nu). \end{aligned}$$

Here $|B|$ denotes the Hilbert-Schmidt norm of B, i.e. $|B|^2 = \sum_j \lambda_j^2$ where $\lambda_1, \ldots, \lambda_{n-1}$ are the eigenvalues of B.

Proof. Let $e_1, \dots, e_{n-1}$ be a local orthonormal tangent frame to one leaf of the foliation. We locally solve the following linear ordinary differential equation in the normal direction:

$$\nabla_\nu e_j = -\langle \nabla_\nu \nu, e_j \rangle \nu. \tag{11}$$

Here ∇ denotes the Levi-Civita connection on TM. We claim that this extends the frame to an orthonormal frame $e_1, \dots, e_{n-1}, \nu$ on an open subset of M.

Namely, we compute

$$\begin{aligned}
\partial_\nu \langle \nu, e_j \rangle &= \langle \nabla_\nu \nu, e_j \rangle + \langle \nu, \nabla_\nu e_j \rangle \\
&\overset{(11)}{=} \langle \nabla_\nu \nu, e_j \rangle + \langle \nu, -\langle \nabla_\nu \nu, e_j \rangle \nu \rangle \\
&= \langle \nabla_\nu \nu, e_j \rangle - \langle \nabla_\nu \nu, e_j \rangle \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\partial_\nu \langle e_i, e_j \rangle &= \langle \nabla_\nu e_i, e_j \rangle + \langle e_i, \nabla_\nu e_j \rangle \\
&\overset{(11)}{=} \langle -\langle \nabla_\nu \nu, e_i \rangle \nu, e_j \rangle + \langle e_i, -\langle \nabla_\nu \nu, e_j \rangle \nu \rangle \\
&= 0.
\end{aligned}$$

Let R^Σ be the curvature tensor on ΣM. Recall [4, Prop. 2.3] that Clifford multiplication by ν anticommutes with D^N

$$D^N(\nu \cdot \varphi) = -\nu \cdot D^N \varphi. \tag{12}$$

Now let us start the computation of $(D^M)^2$. Squaring (7) we obtain, using (12) and $D^N(f\varphi) = (\nabla^N f) \cdot \nu \cdot \varphi + f D^N \varphi$,

$$\begin{aligned}
(D^M)^2 &= \left(\nu \cdot D^N - \frac{n-1}{2} H\nu + \nu \cdot \nabla^M_\nu \right) \left(\nu \cdot D^N - \frac{n-1}{2} H\nu + \nu \cdot \nabla^M_\nu \right) \\
&= (D^N)^2 - \frac{n-1}{2} \nu \cdot (\nabla^N H) \cdot \nu \cdot \nu - \frac{n-1}{2} H D^N + D^N \nabla^M_\nu \\
&\quad + \frac{n-1}{2} H D^N - \frac{(n-1)^2}{4} H^2 + \frac{n-1}{2} H \nabla^M_\nu \\
&\quad + \nu \cdot (\nabla_\nu \nu) \cdot D^N - \nabla^M_\nu D^N \\
&\quad + \frac{n-1}{2} \cdot \partial_\nu H - \frac{n-1}{2} H\nu \cdot (\nabla_\nu \nu) + \frac{n-1}{2} H \nabla^M_\nu \\
&\quad + \nu \cdot (\nabla_\nu \nu) \cdot \nabla^M_\nu - (\nabla^M_\nu)^2 \\
&= (D^N)^2 - (\nabla^M_\nu)^2 - \frac{n-1}{2} (\nabla^N H) \cdot \nu + [D^N, \nabla^M_\nu] \\
&\quad + (n-1) H \nabla^M_\nu + \nu \cdot (\nabla_\nu \nu) \cdot \nabla^M_\nu + \nu \cdot (\nabla_\nu \nu) \cdot D^N \\
&\quad - \frac{(n-1)^2}{4} H^2 + \frac{n-1}{2} \cdot \partial_\nu H - \frac{n-1}{2} H\nu \cdot (\nabla_\nu \nu)
\end{aligned} \tag{13}$$

A simple computation yields the well-known formula

$$\sum_{j=1}^{n-1} e_j \cdot R^{\Sigma}(e_j,\nu) = \frac{1}{2}Ric(\nu). \tag{14}$$

We get

$$\begin{aligned}
[D^N,\nabla^M_\nu] &= D^N\nabla^M_\nu - \nabla^M_\nu D^N \\
&= D^N\nabla^M_\nu - \nabla^M_\nu \sum_{j=1}^{n-1} e_j\nu\nabla^N_{e_j} \\
&\overset{(11)}{=} \sum_{j=1}^{n-1}\langle\nabla_\nu\nu,e_j\rangle\nu\nu\nabla^N_{e_j} - \sum_{j=1}^{n-1} e_j(\nabla_\nu\nu)\nabla^N_{e_j} + \sum_{j=1}^{n-1} e_j\nu[\nabla^N_{e_j},\nabla^M_\nu] \\
&\overset{(10)}{=} -\nabla^N_{\nabla_\nu\nu} + \sum_{j=1}^{n-1}(\nabla_\nu\nu)e_j\nabla^N_{e_j} + 2\sum_{j=1}^{n-1}\langle e_j,\nabla_\nu\nu\rangle\nabla^N_{e_j} \\
&\quad + \sum_{j=1}^{n-1} e_j\nu[\nabla^M_{e_j} - \frac{1}{2}B(e_j)\nu, \nabla^M_\nu] \\
&= (\nabla_\nu\nu)\nu D^N + \nabla^N_{\nabla_\nu\nu} + \sum_{j=1}^{n-1} e_j\nu(\nabla^M_{[e_j,\nu]} + R^{\Sigma}(e_j,\nu)) \\
&\quad + \frac{n-1}{2}H\nabla^M_\nu + \frac{1}{2}\sum_{j=1}^{n-1} e_j\nu\nabla^M_\nu\circ B(e_j)\nu \\
&= (\nabla_\nu\nu)\nu D^N + \nabla^N_{\nabla_\nu\nu} \\
&\quad + \sum_{j=1}^{n-1} e_j\nu\nabla^M_{-B(e_j)-\nabla^M_\nu e_j} - \nu\sum_{j=1}^{n-1} e_jR^{\Sigma}(e_j,\nu) + \frac{n-1}{2}H\nabla^M_\nu \\
&\quad + \nabla^M_\nu\circ\frac{1}{2}\sum_{j=1}^{n-1} e_j\nu B(e_j)\nu - \frac{1}{2}\sum_{j=1}^{n-1}(\nabla^M_\nu e_j)\nu B(e_j)\nu \\
&\quad - \frac{1}{2}\sum_{j=1}^{n-1} e_j(\nabla_\nu\nu)B(e_j)\nu \\
&\overset{(10)(11)(14)}{=} (\nabla_\nu\nu)\nu D^N + \nabla^N_{\nabla_\nu\nu} \\
&\quad - \mathfrak{D}^B + \sum_{j=1}^{n-1} e_j\nu\cdot\frac{1}{2}B(-B(e_j))\nu + \sum_{j=1}^{n-1} e_j\nu\nabla^M_{\langle\nabla_\nu\nu,e_j\rangle\nu} - \frac{1}{2}\nu Ric(\nu) \\
&\quad + \frac{n-1}{2}H\nabla^M_\nu - \nabla^M_\nu\circ\frac{n-1}{2}H + \frac{1}{2}\sum_{j=1}^{n-1}\langle\nabla_\nu\nu,e_j\rangle\nu\nu B(e_j)\nu \\
&\quad + \frac{1}{2}\sum_{j=1}^{n-1}(\nabla_\nu\nu)e_jB(e_j)\nu + \sum_{j=1}^{n-1}\langle\nabla_\nu\nu,e_j\rangle B(e_j)\nu
\end{aligned}$$

$$
\begin{aligned}
= \quad & (\nabla_\nu \nu)\nu D^N + \nabla^N_{\nabla_\nu \nu} - \mathfrak{D}^B + \frac{1}{2}|B|^2 + (\nabla_\nu \nu)\nu\nabla^M_\nu - \frac{1}{2}\nu Ric(\nu) \\
& + \frac{n-1}{2} H\nabla^M_\nu - \frac{n-1}{2}\partial_\nu H - \frac{n-1}{2} H\nabla^M_\nu - \frac{1}{2}B(\nabla_\nu \nu)\nu \\
& - \frac{n-1}{2} H(\nabla_\nu \nu)\nu + B(\nabla_\nu \nu)\nu \\
= \quad & (\nabla_\nu \nu)\nu D^N + \nabla^N_{\nabla_\nu \nu} - \mathfrak{D}^B + \frac{1}{2}|B|^2 + (\nabla_\nu \nu)\nu\nabla^M_\nu - \frac{1}{2}\nu Ric(\nu) \\
& - \frac{n-1}{2}\partial_\nu H + \frac{1}{2}B(\nabla_\nu \nu)\nu - \frac{n-1}{2} H(\nabla_\nu \nu)\nu \qquad (15)
\end{aligned}
$$

Plugging (15) into (13) yields

$$
\begin{aligned}
(D^M)^2 \quad = \quad & (D^N)^2 - (\nabla^M_\nu)^2 - \frac{n-1}{2}(\nabla^N H)\nu \\
& + (n-1)H\nabla^M_\nu + \nu(\nabla_\nu \nu)\nabla^M_\nu + \nu(\nabla_\nu \nu)D^N \\
& - \frac{(n-1)^2}{4}H^2 + \frac{n-1}{2}\partial_\nu H - \frac{n-1}{2}H\nu(\nabla_\nu \nu) \\
& + (\nabla_\nu \nu)\nu D^N + \nabla^N_{\nabla_\nu \nu} - \mathfrak{D}^B + \frac{1}{2}|B|^2 + (\nabla_\nu \nu)\nu\nabla^M_\nu - \frac{1}{2}\nu Ric(\nu) \\
& - \frac{n-1}{2}\partial_\nu H + \frac{1}{2}B(\nabla_\nu \nu)\nu - \frac{n-1}{2}H(\nabla_\nu \nu)\nu \\
= \quad & (D^N)^2 - (\nabla^M_\nu)^2 - \frac{n-1}{2}(\nabla^N H)\nu + (n-1)H\nabla^M_\nu \\
& - \frac{(n-1)^2}{4}H^2 + \nabla^N_{\nabla_\nu \nu} - \mathfrak{D}^B + \frac{1}{2}|B|^2 - \frac{1}{2}\nu Ric(\nu) + \frac{1}{2}B(\nabla_\nu \nu)\nu \\
\overset{(10)}{=} \quad & (D^N)^2 - (\nabla^M_\nu)^2 + (n-1)H\nabla^M_\nu + \nabla^M_{\nabla_\nu \nu} - \mathfrak{D}^B \\
& - \frac{n-1}{2}(\nabla^N H)\cdot\nu - \frac{(n-1)^2}{4}H^2 + \frac{1}{2}|B|^2 - \frac{1}{2}\nu\cdot Ric(\nu)
\end{aligned}
$$

□

Now let us specialize to the situation $M = N \times I$ where N is a closed $(n-1)$-dimensional spin manifold, $I \subset \mathbb{R}$ is an interval, and M carries a metric of the form

$$ds^2 = g_r + dr^2$$

where g_r is a 1-parameter family of metrics on N. The foliation is given by the leaves $N \times \{r\}$. Then $\nabla_\nu \nu = 0$ and the formula in Proposition 4 simplifies to

$$
\begin{aligned}
(D^M)^2 \quad = \quad & (D^N)^2 - (\nabla^M_\nu)^2 + (n-1)H\nabla^M_\nu - \mathfrak{D}^B \\
& - \frac{n-1}{2}(\nabla^N H)\cdot\nu - \frac{(n-1)^2}{4}H^2 + \frac{1}{2}|B|^2 - \frac{1}{2}\nu\cdot Ric(\nu).
\end{aligned}
$$

We fix $r_0 \in I$. Let P_r denote parallel transport from $N \times \{r_0\}$ to $N \times \{r\}$ along ν. It is easy to see that

$$U : L^2(M, \Sigma M) \to L^2(I, L^2(N, g_{r_0}, \Sigma M|_{N\times\{r_0\}}), dt),$$

$$(U\sigma)(r)(x) = \sqrt{\frac{dvol_{g_r}}{dvol_{g_{r_0}}}} P_r^{-1}\sigma(x,r),$$

is a Hilbert space isometry.

Corollary 3. *The square of the Dirac operator on M, $(D^M)^2$, transforms under U into the following Schrödinger operator acting on functions with values in $L^2(I, L^2(N, g_{r_0}, \Sigma M|_{N\times\{r_0\}}), dt)$:*

$$U(D^M)^2U^{-1} = -\frac{d^2}{dr^2} + V$$

where

$$V = -\frac{n-1}{2}\frac{\partial H}{\partial r} + \frac{|B|^2}{2} + P_r^{-1}\left((D^N)^2 - \mathfrak{D}^B - \frac{n-1}{2}(\nabla^N H)\nu - \frac{1}{2}\nu Ric(\nu)\right) P_r$$

Proof. The first variation formula for the volume element of hypersurfaces tells us

$$\begin{aligned}
\frac{d}{dr}\log\sqrt{\frac{dvol_{g_r}}{dvol_{g_{r_0}}}} &= \frac{1}{2}\frac{\frac{d}{dr}dvol_{g_r}}{dvol_{g_r}} \\
&= -\frac{1}{2}\mathbf{Tr}(B) \\
&= -\frac{n-1}{2}H. \qquad (16)
\end{aligned}$$

We compute

$$\begin{aligned}
U\nabla_\nu^M U^{-1}(v) &= U\nabla_\nu^M\left(\sqrt{\frac{dvol_{g_{r_0}}}{dvol_{g_r}}}P_r v\right) \\
&= U\left(\frac{n-1}{2}H\sqrt{\frac{dvol_{g_{r_0}}}{dvol_{g_r}}}P_r v + \sqrt{\frac{dvol_{g_{r_0}}}{dvol_{g_r}}}P_r\dot{v}\right) \\
&= \frac{n-1}{2}Hv + \dot{v} \qquad (17)
\end{aligned}$$

and

$$\begin{aligned}
U(\nabla_\nu^M)^2U^{-1}(v) &= U\nabla_\nu^M U^{-1}\left(\frac{n-1}{2}Hv + \dot{v}\right) \\
&= \frac{n-1}{2}H\left(\frac{n-1}{2}Hv + \dot{v}\right) + \frac{n-1}{2}\frac{\partial H}{\partial r}v + \frac{n-1}{2}H\dot{v} + \ddot{v} \\
&= \ddot{v} + (n-1)H\dot{v} + \left(\frac{(n-1)^2}{4}H^2 + \frac{n-1}{2}\frac{\partial H}{\partial r}\right)v. \qquad (18)
\end{aligned}$$

Equations (17) and (18) yield

$$
\begin{aligned}
& U\left(-(\nabla_\nu^M)^2+(n-1)H\nabla_\nu^M-\frac{(n-1)^2}{4}H^2\right)U^{-1}(v) \\
= \quad & -\ddot v-(n-1)H\dot v-\left(\frac{(n-1)^2}{4}H^2+\frac{n-1}{2}\frac{\partial H}{\partial r}\right)v \\
& +\frac{(n-1)^2}{2}H^2v+(n-1)H\dot v-\frac{(n-1)^2}{4}H^2 \\
= \quad & -\ddot v-\frac{n-1}{2}\frac{\partial H}{\partial r}v. \qquad (19)
\end{aligned}
$$

The corollary now follows from Proposition 4. □

Example. Let us now look at the example of main interest in this paper, the tube around a closed geodesic in a hyperbolic 3-manifold. Recall from Section 1 that $M = T[1,R]$ is isometric to $T^2 \times [1,R]$ with Riemannian metric $ds^2 = g_r + dr^2$ where g_r is the flat metric on T^2 given by the lattice $\Gamma_r \subset \mathbb{R}^2$ spanned by the vectors $(2\pi\sinh(r), 0)$ and $(\alpha_{i,j}\sinh(r), \ell_{i,j}cosh(r))$.

The shape operator B has the eigenvalues $\tanh(r)$ and $\coth(r)$. Hence $|B|^2 = \tanh(r)^2 + \coth(r)^2$, the mean curvature $H = \frac{1}{2}(\tanh(r) + \coth(r))$ is constant along the leaves and $\nabla^N H = 0$. Since the sectional curvature of $T[r_1, r_2]$ is -1 we have $Ric = -2 \cdot \mathrm{Id}$. Therefore Corollary 3 gives

$$U(D^M)^2U^{-1} = -\frac{d^2}{dr^2}+\tanh(r)^2+\coth(r)^2-2+P_r^{-1}\left((D^N)^2-\mathfrak{D}^B\right)P_r.$$

It will be important to estimate the potential

$$V = \tanh(r)^2+\coth(r)^2-2+P_r^{-1}\left((D^N)^2-\mathfrak{D}^B\right)P_r$$

of this Schrödinger operator from below. Note that $\mathfrak{D}^B$ is formally self-adjoint because B is parallel along the leaves. If φ is a spinor field along a leaf, then

$$
\begin{aligned}
|\mathfrak{D}^B\varphi| \quad &= \quad \left|\sum_{j=1}^{2} B(e_j)\cdot\nu\cdot\nabla_{e_j}^N\varphi\right| \\
&\le \quad 2\cdot|B|\cdot|\nabla^N\varphi|,
\end{aligned}
$$

hence

$$
\begin{aligned}
||\mathfrak{D}^B\varphi||^2_{L^2(N,\Sigma M|_N)} \quad &\le \quad 4\cdot|B|^2\cdot||\nabla^N\varphi||^2_{L^2(N,\Sigma M|_N)} \\
&= \quad 4\cdot|B|^2\cdot((\nabla^N)^*\nabla^N\varphi,\varphi)_{L^2(N,\Sigma M|_N)} \\
&= \quad 4\cdot|B|^2\cdot\left(\left((D^N)^2-\frac{scal_N}{4}\right)\varphi,\varphi\right)_{L^2(N,\Sigma M|_N)} \\
&= \quad 4(\tanh(r)^2+\coth(r)^2)\cdot||D^N\varphi||^2_{L^2(N,\Sigma M|_N)} \\
&\le \quad 4(1+\coth(1)^2)\cdot||D^N\varphi||^2_{L^2(N,\Sigma M|_N)} \\
&\le \quad 16\cdot||D^N\varphi||^2_{L^2(N,\Sigma M|_N)}.
\end{aligned}
$$

Thus if $||D^N\varphi||^2_{L^2(N,\Sigma M|_N)} \geq \mu^2 \cdot ||\varphi||^2_{L^2(N,\Sigma M|_N)}$, $\mu \geq 4$, we get

$$\begin{aligned}\Big(((D^N)^2 - \mathfrak{D}^B)\varphi, \varphi\Big)_{L^2(N,\Sigma M|_N)} &\geq ||D^N\varphi||^2_{L^2(N,\Sigma M|_N)} \\ &\quad -4 \cdot ||D^N\varphi||_{L^2(N,\Sigma M|_N)} \\ &\geq \mu(\mu-4)||\varphi||^2_{L^2(N,\Sigma M|_N)}.\end{aligned}$$

Hence on each Hilbert subspace of $L^2(N, g_{r_0}, \Sigma M|_{N\times\{r_0\}})$ which is left invariant by $P_r^{-1}(D^N)^2 P_r$ and by $P_r^{-1}\mathfrak{D}^B P_r$ on which $P_r^{-1}(D^N)^2 P_r \geq \mu^2$, $\mu \geq 4$, we know that $P_r^{-1}((D^N)^2 - \mathfrak{D}^B)P_r \geq \mu(\mu-4)$ and hence

$$V \geq \tanh(r)^2 + \coth(r)^2 - 2 + \mu(\mu-4) > \mu(\mu-4) - 1 \tag{20}$$

In order to proceed we need to control the eigenvalues of $(D^N)^2$.

Lemma 2. *The smallest nonzero eigenvalue of $(D^N)^2$ on $N \times \{r\}$ is monotonically decreasing in $r \in [1, R]$.*

Proof. The Dirac eigenvalues of a flat torus $T^2 = \mathbb{R}^2/\Gamma$ can be computed in terms of the dual lattice Γ^*. A spin structure corresponds to a pair $\delta = (\delta_1, \delta_2)$, $\delta_j = 0, 1$, and the square of the Dirac operator for the corresponding spin structure has the eigenvalues

$$4\pi^2 \left| v - \frac{1}{2}(\delta_1 v_1 + \delta_2 v_2) \right|^2$$

where v_1, v_2 are a basis of Γ^* and v runs through Γ^*, cf. [18]. In our case Γ has the basis

$$w_1 = \begin{pmatrix} 2\pi\sinh(r) \\ 0 \end{pmatrix}, w_2 = \begin{pmatrix} \alpha\sinh(r) \\ \ell\cosh(r) \end{pmatrix},$$

compare Section 1. Hence a basis for Γ^* is given by

$$v_1 = \begin{pmatrix} \frac{1}{2\pi\sinh(r)} \\ -\frac{\alpha}{2\pi\ell\cosh(r)} \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ \frac{1}{\ell\cosh(r)} \end{pmatrix}.$$

Thus the eigenvalues

$$\frac{(k_1 - \delta_1/2)^2}{\sinh(r)^2} + \frac{(2\pi(k_2 - \delta_2/2) - \alpha(k_1 - \delta_1/2))^2}{\ell^2\cosh(r)^2},$$

$k_1, k_2 \in \mathbb{Z}$, are monotonically decreasing functions. $\square$

The lemma together with (20) immediately implies

Corollary 4. *If $M = T[1, R]$ carries a nontrivial spin structure and if the smallest eigenvalue μ^2 of $(D^N)^2$ on $N \times \{R\}$ satisfies $\mu \geq 4$, then $(D^M)^2$ is unitarily equivalent to a Schrödinger operator*

$$-\frac{d^2}{dr^2} + V,$$

acting on Hilbert space-valued functions on $[1, R]$ *with*

$$V \geq \mu(\mu - 4) - 1.$$

7 Degeneration in Three Dimensions

With the preparations in the previous section we are able to modify the proof of Theorem 2 such that it also works in three dimensions. In contrast to the 2-dimensional case there is no accumulation of eigenvalues.

Theorem 3. *Let* M_i *be a sequence of closed hyperbolic 3-manifolds converging to a noncompact hyperbolic 3-manifold* M *of finite volume. Let each* M_i *have exactly* k *tubes around closed geodesics of length* $\ell_{i,j}$ *tending to zero. Hence* M *has exactly* k *cusps. Let* $x > 0$.

Then the spin structure is nontrivial along all tubes and the eigenvalue counting function for the Dirac operator remains bounded:

$$\mathcal{N}_{D,M_i}(-x, x) = \mathrm{O}_x(1).$$

Proof. Again we restrict ourselves to the case that there is exactly one degenerating tube, i.e. $k = 1$. Recall the decomposition of the manifolds

$$M_i = M_{i,0} \dot{\cup} T_i[0, R_i],$$

$R_i = \frac{1}{2}\log(1/\ell_i) + c_0$. Since $\partial M_{i,0} = \partial T_i[0, R_i]$ bounds the solid 2-torus $T_i[0, R_i]$ the induced spin structure on $\partial T_i[0, R_i]$ must be nontrivial.

Look at decomposition (1) of the limit manifold

$$M = M_0 \dot{\cup} \mathcal{E}$$

where the cusp $\mathcal{E} = N \times [0, \infty)$ carries the warped product metric $e^{-2t} \cdot g_N + dt^2$. Since we assumed compatibility of the spin structures of the M_i and of M, the spin structure of M must also be nontrivial along $\mathcal{E}$.

Let μ_0 be the smallest positive eigenvalue of the Dirac operator on (N, g_N). Choose a constant $c_1 = c_1(x)$ such that

$$e^{c_1}\mu_0(e^{c_1}\mu_0 - 4) - 1 > x^2 \tag{21}$$

and

$$e^{c_1}\mu_0 > 4.$$

The number $e^{c_1} \cdot \mu_0$ is the smallest positive eigenvalue of the Dirac operator on $(N, e^{-2c_1}g_N)$. Put

$$\tilde{M}_{i,0} := M_{i,0} \cup T_i[R_i - c_1, R_i].$$

This yields the following decomposition of the manifolds M_i:

$$M_i = \tilde{M}_{i,0} \cup T_i[1, R_i - c_1] \cup T_i[0,1].$$

Using Proposition 3 and the Lichnerowicz formula (4) we get as in the proof of Theorem 2

$$\begin{aligned} \mathcal{N}_{D,M_i}(-x,x) \quad &\leq \quad \mathcal{N}^{\text{Neumann}}_{D^2,\tilde{M}_{i,0}}[-3/2,x^2) + \mathcal{N}^{\text{Neumann}}_{D^2,T_i[1,R_i-c_1]}[-3/2,x^2) \\ &\quad\quad + \mathcal{N}^{\text{Neumann}}_{D^2,T_i[0,1]}[-3/2,x^2). \end{aligned} \tag{22}$$

The same argument as in the proof of Theorem 2 gives

$$\mathcal{N}^{\text{Neumann}}_{D^2,\tilde{M}_{i,0}}[-3/2,x^2) = \mathrm{O}_x(1). \tag{23}$$

The universal covering of the 1-tube $T_i[0,1]$ around the closed geodesic γ_i is the 1-tube $\mathcal{T}$ around a geodesic γ in hyperbolic 3-space. The group of deck transformations is isomorphic to $\mathbb{Z}$ and is generated by a shift of length $\ell = \ell_i$ along γ while rotating about the angle $\alpha = \alpha_i$. Denote this isometry of $\mathcal{T}$ by $A_{\ell,\alpha}$. Note that α takes values in the compact interval $[-\pi,\pi]$. As long as ℓ also takes values in a compact interval, say $\ell \in [1/2,1]$, all eigenvalues vary in a bounded range and $\mathcal{N}^{\text{Neumann}}_{D^2,\mathcal{T}/A_{\ell,\alpha}}[-3/2,x^2) = \mathrm{O}_x(1)$.

Now if $0 < \ell < \frac{1}{2}$ choose $m \in \mathbb{N}$ such that $m\ell \in [1/2,1]$. Then $\mathcal{T}/A_{\ell,\alpha}$ is covered by $\mathcal{T}/A_{m\ell,\alpha'}$. Hence every eigenvalue of $\mathcal{T}/A_{\ell,\alpha}$ is also an eigenvalue of $\mathcal{T}/A_{m\ell,\alpha'}$. Therefore $\mathcal{N}^{\text{Neumann}}_{D^2,\mathcal{T}/A_{\ell,\alpha}}[-3/2,x^2) \leq \mathcal{N}^{\text{Neumann}}_{D^2,\mathcal{T}/A_{m\ell,\alpha'}}[-3/2,x^2) = \mathrm{O}_x(1)$. This shows $\mathcal{N}^{\text{Neumann}}_{D^2,\mathcal{T}/A_{\ell,\alpha}}[-3/2,x^2) = \mathrm{O}_x(1)$ for all $\alpha \in [-\pi,\pi]$ and $\ell \in (0,1]$. Hence

$$\mathcal{N}^{\text{Neumann}}_{D^2,T_i[0,1]}[-3/2,x^2) = \mathrm{O}_x(1). \tag{24}$$

It remains to estimate $\mathcal{N}^{\text{Neumann}}_{D^2,T_i[1,R_i-c_1]}[-3/2,x^2)$. By Corollary 4 the operator D^2 on $T_i[1,R_i-c_1]$ is unitarily equivalent to a Schrödinger operator $-\frac{d^2}{dr^2} + V_i$. For sufficiently large i the potential V_i is bounded from below by $e^{c_1}\mu_0(e^{c_1}\mu_0 - 4) - 1$. This follows from (21) because the eigenvalues of $\partial\tilde{M}_{i,0}$ converge to those of $\partial\tilde{M}_i$. We conclude

$$\mathcal{N}^{\text{Neumann}}_{D^2,T_i[1,R_i-c_1]}[-3/2,x^2) = 0 \tag{25}$$

for sufficiently large i. Plugging (23), (24), and (25) into (22) we obtain

$$\mathcal{N}_{D,M_i}(-x,x) = \mathrm{O}_x(1).$$

□

8 Spin Structures on Hyperbolic Manifolds

The previous discussion has shown that the spectrum of the Dirac operator depends in a crucial way on the spin structure. This is true for the degeneration as well as for the L^2-spectrum of a hyperbolic manifold of finite volume. The fact that there is no spectral accumulation in three dimensions has a topological reason. Tubes necessarily carry a nontrivial spin structure because the trivial one on the 2-torus does not bound. In this last section we will discuss the question which kind of spin structures are actually carried by 2- or 3-dimensional hyperbolic manifolds of finite volume.

All (co-)homology groups in this section are to be taken with coefficients $\mathbb{Z}/2\mathbb{Z}$. Recall that $H^1(M)$ acts simply transitively on the set of spin structures of a spin manifold M.

The 2-dimensional case. Let M be an oriented surface with k ends. Topologically M is a closed surface $\bar{M}$ with k points $p_1, \ldots, p_k$ removed. Let D_j denote small disks around p_j. The Mayer-Vietoris sequence for the pair $(M, \dot{\bigcup}_{j=1}^k D_j)$ yields an exact sequence

$$0 \longrightarrow H^1(\bar{M}) \longrightarrow H^1(M) \longrightarrow \bigoplus_{j=1}^{k} H^1(D_j - \{p_j\}) \longrightarrow H^2(\bar{M}) \longrightarrow 0. \qquad (26)$$

Pick a spin structure on $\bar{M}$ to identify spin structures with elements of $H^1(\bar{M})$. Take the restriction of this spin structure to M and identify the spin structures on M with elements of $H^1(M)$. The unique spin structure on D_j induces the nontrivial spin structure on $D_j - \{p_j\} \simeq S^1$. Hence (26) tells us that the restriction mapping from spin structures on $\bar{M}$ to M is injective and a spin structure on M extends to $\bar{M}$ if and only if it is nontrivial along all ends.

If we identify $H^1(D_j - \{p_j\}) \cong \mathbb{Z}/2\mathbb{Z}$ and $H^2(\bar{M}) \cong \mathbb{Z}/2\mathbb{Z}$, then the map $\bigoplus_{j=1}^k H^1(D_j - \{p_j\}) \to H^2(\bar{M})$ corresponds to $(\mathbb{Z}/2\mathbb{Z})^k \xrightarrow{(1,\ldots,1)} \mathbb{Z}/2\mathbb{Z}$. Hence any spin structure on M must be trivial along an even number of ends.

To summarize, *a spin structure on M corresponds uniquely to a spin structure on $\bar{M}$ together with a choice of an even number of ends along which the spin structure is trivial.*

$$\left\{ \begin{array}{c} \text{spin structures} \\ \text{on } M \end{array} \right\} \stackrel{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{spin structures} \\ \text{on } \bar{M} \end{array} \right\} \times \left\{ \begin{array}{c} \text{choices of an even} \\ \text{number of ends of } M \end{array} \right\}$$

In particular, on hyperbolic surfaces of finite area with more than one cusp both cases in Theorem 1 do occur.

To discuss Theorem 2 let now M be a closed oriented surface. Let $T \approx S^1 \times I$ be a tube around around a closed geodesic γ. The tube T may carry the trivial or the nontrivial spin structure. Is it possible to "flip" the spin structure, i.e. are both spin structures on T induced by some spin structure on M?

One can flip the spin structure if and only if there exists a cohomology class in $H^1(M)$ acting nontrivially on $[\gamma] \in H_1(M)$, i.e. if and only if the homology class $[\gamma]$ is nonzero in $H_1(M)$. This is the case if and only if removing γ does not decompose the surface into two connected components.

If $M - \gamma$ is disconnected which spin structure does T carry? Both connected components can be given a spin structure which is nontrivial along all ends, cf. the discussion above. These spin structures can be glued together to give a spin structure on the original surface M. Hence T must carry the nontrivial spin structure in this case. We note:

The tube can carry both spin structures if and only if cutting along γ does not decompose the surface into two connected components. In this case spectral accumulation in Theorem 2 may or may not occur depending on the choice of spin structure. If $M - \gamma$ disconnects, then the tube carries the nontrivial spin structure and does not contribute to the spectral accumulation.

The 3-dimensional case. The proof of Theorem 3 has shown that all tubes in a closed hyperbolic 3-manifold carry the nontrivial spin structure. This is responsible for the fact that there is no spectral accumulation in three dimensions. Spin structures on hyperbolic 3-manifolds of finite volume which are trivial along some cusps do not occur as limits of spin structures on closed hyperbolic 3-manifolds. Do they exist at all?

Let us first show that just like in two dimensions *any spin structure on a hyperbolic 3-manifold of finite volume is trivial along an even number of cusps.*

Let $M = M_0 \dot\cup \dot{\bigcup}_{j=1}^{k} \mathcal{E}_j$ be a hyperbolic spin 3-manifold with k cusps. Let the spin structure be trivial along k_1 cusps and nontrivial along k_2 cusps, $k = k_1 + k_2$. Chop off the ends to obtain the compact manifold $\bar{M}_0$ with boundary. Two of the three nontrivial spin structures on the 2-torus bound spin structures on the solid torus S, $T^2 = \partial S$. The third one can be transformed by some automorphism of T^2 into one which bounds a spin structure on the solid torus. Hence using appropriate gluing maps we can glue in solid tori to the boundary components of $\bar{M}_0$ on which the spin structure is nontrivial and extend the spin structure. We obtain a compact spin manifold M_0' whose boundary consists of k_1 tori. The induced spin structure is trivial on all these boundary components.

Assume k_1 were odd, $k_1 = 2m + 1$. Choose m pairs of boundary tori and identify them. Since the spin structures on the tori are all trivial they can be glued together. We obtain a compact spin manifold M_0'' whose boundary consists of the one remaining 2-torus. The induced spin structure on this torus is trivial. This contradicts the fact that the trivial spin structure on T^2 does not bound. $\square$

Here is a criterion for when a boundary torus can inherit the trivial spin structure.

Lemma 3. *Let M be an oriented 3-manifold with boundary. Let T be a connected component of the boundary diffeomorphic to a 2-torus. Then the following* *two assertions are equivalent:*

- *M carries a spin structure inducing the trivial spin structure on T.*
- *The inclusion map $T \hookrightarrow M$ induces an* **injective** *map on the first homology*
$$H_1(T) \to H_1(M).$$

Proof. A solid torus S induces exactly two of the three nontrivial spin structures on its boundary $\partial S = T^2$. Denote the three nontrivial spin structures on T^2 by $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$. Choose a spin structure on $M \cup_{(\mathrm{Id}, T^2)} S$. Since the induced spin structure on T^2 bounds a spin structure on S it is nontrivial, say $\mathcal{S}_1$.

The automorphisms of T^2 act transitively on $\{\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3\}$. Choose an automorphism Φ of T^2 such that $\Phi^*\mathcal{S}_1$ is the nontrivial spin structure on T^2 which is **not** induced by one on S. Pick a spin structure on $M \cup_{(\Phi, T^2)} V$. The induced spin structure on T^2 is again nontrivial but $\neq \mathcal{S}_1$.

We have found two spin structures on M inducing two different nontrivial spin structures on T.

Case 1. $H_1(T) \to H_1(M)$ is injective, i.e. $H^1(M) \to H^1(T)$ is surjective.

In this case $H^1(M)$ acts transitively on the spin structures of T and in particular the trivial spin structure occurs.

Case 2. $H_1(T) \to H_1(M)$ is not injective, i.e. $H^1(M) \to H^1(T)$ is not surjective.

In this case $\dim ImH^1(M) \leq 1$ and hence $\sharp ImH^1(M) \leq 2$ where $ImH^1(M)$ denotes the image of $H^1(M)$ in $H^1(T)$. Therefore at most two spin structures are induced on T. But as we have seen above there are two nontrivial spin structures which do occur. Hence $\dim ImH^1(M) = 1$ and T inherits exactly two spin structures both nontrivial. □

A main source of hyperbolic manifolds of finite volume is given by complements of links in S^3. For such manifolds Lemma 3 can be translated in a very simple criterion.

Theorem 4. *Let $K \subset S^3$ be a link, let $M = S^3 - K$ carry a hyperbolic metric of finite volume.*

If the linking number of all pairs of components (K_i, K_j) of K is even,
$$Lk(K_i, K_j) \equiv 0 \ mod\ 2,$$
$i \neq j$, then the spectrum of the Dirac operator on M is discrete for all spin structures,
$$spec(D) = spec_d(D).$$

If there exist two components K_i and K_j of K, $i \neq j$, with odd linking number, then M has a spin structure such that the spectrum of the Dirac operator satisfies

$$spec(D) = \mathbb{R}.$$

PROOF. Each component K_j of K corresponds to one cusp of M. Let $K_1, \ldots, K_k$ be the components of K and let $S_1, \ldots, S_k$ denote thin solid tori around the link components. The solid tori have to be pairwise disjoint. Denote the boundary tori by $T_j = \partial S_j$.

The Mayer-Vietoris sequence for the pair $(M, \dot{\bigcup}_{j=1}^{k} S_j)$ yields an exact sequence

$$0 \longrightarrow \bigoplus_{j=1}^{k} H_1(T_j) \longrightarrow H_1(M) \oplus \bigoplus_{j=1}^{k} H_1(S_j) \longrightarrow 0 \qquad (27)$$

Choose a basis α_j, β_j of $H_1(T_j)$ such that α_j generates the kernel of $H_1(T_j) \to H_1(S_j)$ and β_j is represented by curve unlinked to the soul of S_j. From (27) we see that the map $\bigoplus_{j=1}^{k} H_1(T_j) \to H_1(M)$ restricted to the span of $\alpha_1, \ldots, \alpha_k$ is injective.

Now let c_j denote the linking numbers of K_1 and K_j, $j \geq 2$. Then β_1 is homologous to $\sum_{j=2}^{k} c_j \alpha_j \in H_1(M)$, see Figure 7.

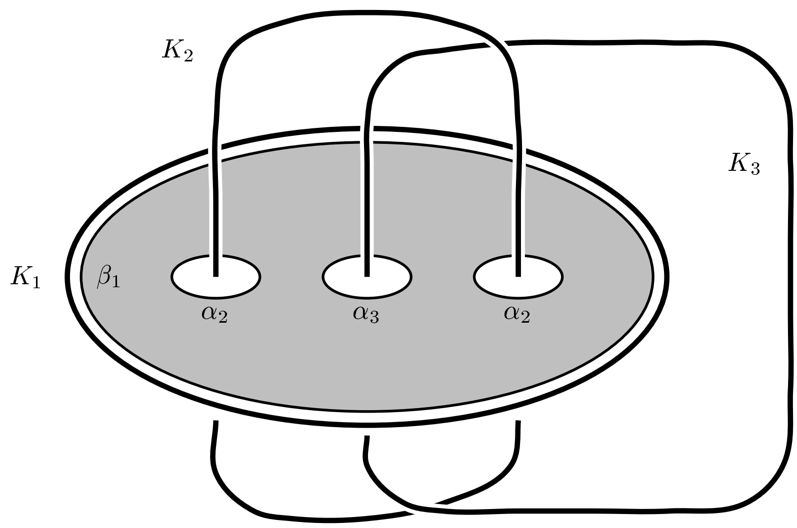


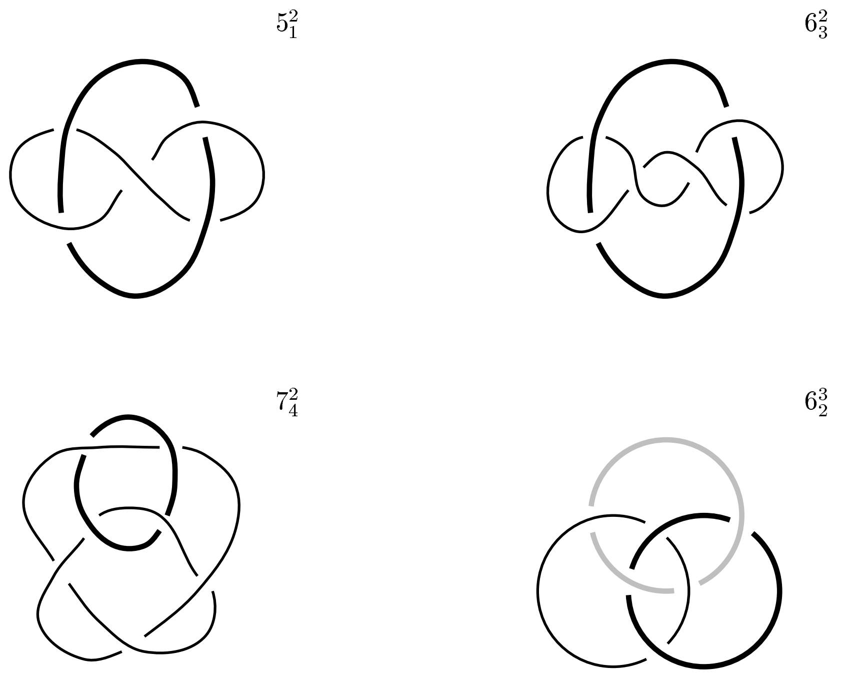
Fig. 7

Thus β_1 maps under $H_1(T_1) \to H_1(M)$ to 0 if and only if all c_j are even. Otherwise it maps to an element linearly independent from the image of α_1.

Hence $H_1(T_1) \to H_1(M)$ is injective if and only if there is a link component K_j such that $Lk(K_1, K_j)$ is odd. Lemma 3 and Theorem 1 finish the proof. $\square$

The proof shows that pairs of link components with odd linking number correspond to those pairs of ends along which the spin structure can be made trivial. Note that the condition on the linking numbers is extremely easy to verify in given examples. Since we compute modulo 2 orientations of link components are irrelevant. If the link is given by a planar projection, then modulo 2, $Lk(K_i, K_j)$ is the same as the number of over-crossings of K_i over K_j.

Examples. The complements of the following links possess a hyperbolic structure of finite volume. All linking numbers are even. Hence the Dirac spectrum on those hyperbolic manifolds is discrete for all spin structures.

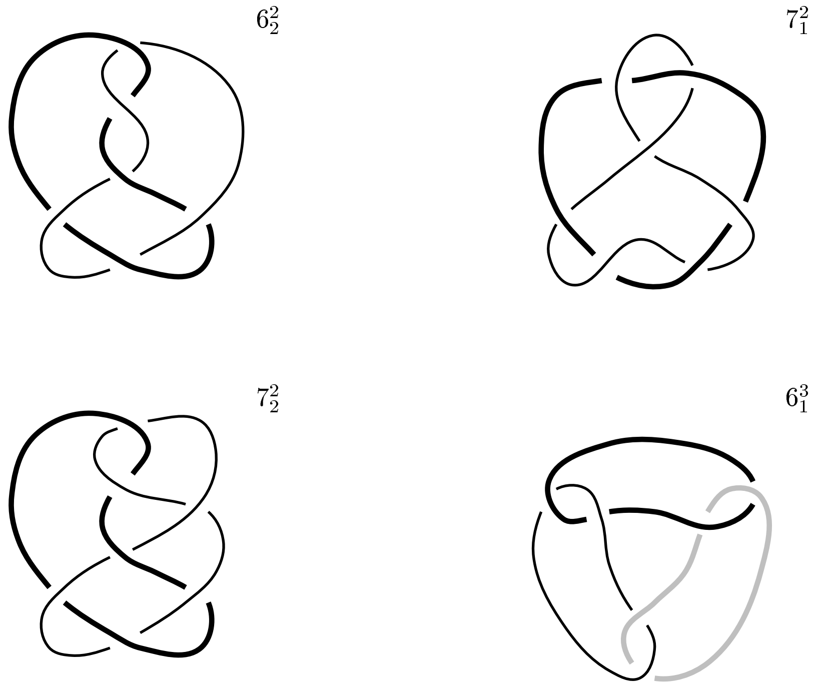


Spectrum of the Dirac operator is discrete.

Fig. 8

Note that the links 5_1^2 (Whitehead link) and 6_2^3 (Borromeo rings) are among the first ones for whose complements Thurston constructed hyperbolic structures [28].

Examples. The complements of the following links possess a hyperbolic structure of finite volume. There are odd linking numbers. Hence those hyperbolic manifolds have a spin structure for which the Dirac spectrum is the whole real line.



For some spin structures $spec(D) = \mathbb{R}$.

Fig. 9

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