COMPACTIFICATION OF HYPERCONCAVE ENDS

GEORGE MARINESCU AND TIEN-CUONG DINH

Abstract. We show that the 'pseudoconcave holes' of some naturally arising class of manifolds, called hyperconcave ends, can be filled in, including the case of complex dimension 2. As a consequence we obtain a stronger version of the compactification theorem of Siu-Yau and extend Nadel's theorems to dimension 2.

1. Introduction

Let X be a relatively compact smooth domain in a complex manifold. We assume that the boundary bX splits in two components Y_1 and Y_2, such that Y_1 is strongly pseudoconvex and Y_2 is strongly pseudoconcave. We say that X can be compactified if there exists a compact manifold $\widehat{X}$, with boundary, such that X is (biholomorphic to) an open set in $\widehat{X}$ and $b\widehat{X} = Y_1$.

A suggestive terminology is to say that X has a pseudoconcave hole at Y_2 and, if the compactification is possible, that the hole can be filled.

By a theorem of Rossi [Ros65, Theorem 3, p. 245] and Andreotti-Siu [AS70, Proposition 3.2] any domain X as above can be compactified, provided $\dim X \geqslant 3$. This is no longer true if $\dim X = 2$, as shown in a counterexample of Grauert [Gra94], Andreotti-Siu [AS70] and Rossi [Ros65].

We consider the following class of manifolds, having a pseudoconcave hole at infinity.

Definition 1.1. A manifold X of dimension $\geqslant 2$ is called a *hyperconcave end* if there exists a proper, smooth function $\varphi : X \longrightarrow (-\infty, a)$, $a \in \mathbb{R} \cup \{+\infty\}$, which is strictly plurisubharmonic on a set of the form $\{\varphi < b\}$, $b \leqslant a$.

The regular part of a variety with isolated singularities is a hyperconcave end. The same is true for the complement of a compact completely pluripolar set (the set where a strongly plurisubharmonic function equals $-\infty$) in a complex manifold. One can also check that the examples of Grauert, Andreotti-Siu and Rossi are not hyperconcave ends.

Date: October 2, 2002.

Partially supported by SFB 288 and the Alexander von Humboldt Stiftung.

Our goal is to compactify hyperconcave ends at $\{\varphi = -\infty\}$ (i.e., to fill the hole at $-\infty$).

Theorem 1.2. (i) *Any hyperconcave end X can be compactified, i.e., there exists a complex manifold $\widehat{X}$ such that X is (biholomorphic to) an open set in $\widehat{X}$ and $(\widehat{X} \smallsetminus X) \cup \{\varphi \leqslant d\}$ is compact for any $d < a$.*

(ii) *More specifically, if φ is strictly plurisubharmonic on the whole X, $\widehat{X}$ can be chosen a normal Stein space with at worst isolated singularities.*

Let us remark that the theorem of Rossi-Andreotti-Siu includes the case of hyperconcave ends of dimension $\geqslant 3$. Theorem 1.2 gives new information only if $\dim X = 2$.

Our proof consists in producing holomorphic functions on the hyperconcave end. For this purpose we solve the L^2 $\bar{\partial}$-Neumann problem for a suitable hermitian metric. To obtain the L^2 estimate, an essential step is to apply a twisting trick of Berndtsson and Siu.

The motivation for the study of the compactification of hyperconcave ends comes from the theory of complex-analytic compactification of quotients $X = M/\Gamma$ of bounded symmetric domains M by arithmetic groups Γ.

Let M be of rank 1, that is M is the unit ball in $\mathbb{C}^n$. Assume that $n \geqslant 2$ and Γ is an arithmetic group. Then the Satake-Baily-Borel compactification $\widehat{X}$ of $X = M/\Gamma$ is obtained by adding a finite set of points which are isolated singularities (so that X turns out to be a hyperconcave end).

From a differential-geometric point of view, arithmetic quotients are Kähler-Einstein manifolds of finite volume and bounded curvature. Siu and Yau [SY82] generalized the compactification of arithmetic quotients of rank 1 by compactifying complete Kähler manifolds X of finite volume and sectional curvature pinched between two negative constants. The first step in their proof is to show that X is a hyperconcave end, using the Busemann function and the lemmas of Margulis and Gromov. Then X can be compactified to a projective manifold by the embedding theorem of Andreotti-Tomassini [AT70]. The second step is to prove that X is a Zariski-open set. We will give a new analytic proof of this step in §4.

The compactification of complete Kähler-Einstein manifolds of finite volume and bounded curvature was thoroughly studied by Mok, see [Mok89] and the references therein.

As the preceding discussion indicates, the arithmetic quotients are, with a few exceptions, pseudoconcave manifolds admitting a positive line bundle. In this respect, Nadel [Nad90] considered the realisation as a quasiprojective manifold of a class of manifolds X with hyperconcave ends, of dimension $\geqslant 3$. One of our goals is to extend to dimension 2 Nadel's theorems, cf. Propositions 5.2-3.

These propositions are consequences of our second main result.

Theorem 1.3. *Let X be a hyperconcave end and let $\widehat{X}$ be a smooth compactification of X. Assume that X can be covered by Zariski-open sets which are uniformized by Stein manifolds. Then $\widehat{X} \smallsetminus X$ is an analytic set which can be blown down to a finite set of points.*

Let us note that the result is natural, in the light of a recent result of Colţoiu-Tibăr [CT02], asserting that the universal cover of a small punctured neighbourhood of an isolated singularity of dimension 2 is Stein, whenever the fundamental group of the link is infinite.

The method of Nadel, in dimension greater than 3, is to compactify the manifold by the theorem of Rossi, and then to apply differential geometric methods, like the existence of Kähler-Einstein metric and the Schwarz-Pick lemma of Yau and Mok-Yau.

Our approach is different. Let $\widehat{\Omega}$ be a Stein neighbourhood of $\widehat{X} \smallsetminus X$. We can choose a finite number of Zariski-open sets $V_1, \cdots, V_k$ of X which are uniformized by Stein manifolds, such that $\cup V_i$ is a neighbourhood of $b\widehat{\Omega}$. Then each V_i satisfies the Kontinuitätssatz. Using Wermer's theorem, we show that $\widehat{\Omega} \smallsetminus V_i$ is a hypersurface of $\widehat{\Omega}$. The set $\widehat{X} \smallsetminus X$ is included in the bounded analytic subset $\widehat{\Omega} \smallsetminus (\cup V_i)$ of $\widehat{\Omega}$, so it must be finite.

As a consequence we get a stronger form of [SY82, Main Theorem]:

Corollary 1.4. *Let X be a complete Kähler manifold of finite volume and bounded negative sectional curvature. If* $\dim X \geqslant 2$, *X is biholomorphic to a quasiprojective manifold which can be compactified by adding finitely many points to a Moishezon space.*

Nadel and Tsuji [NT88] generalized the compactification of arithmetic quotients of any rank, by showing that certain pseudoconcave manifolds are quasiprojective. In dimension 2 their condition coincides with hyperconcavity. Proposition 5.3 yields, in dimension 2, a stronger version of their theorem together with a completely complex-analytic proof of the compactification of arithmetic quotients, cf. Remark 5.5.

The organization of the paper is as follows. In §2 we construct holomorphic functions on a hyperconcave end and then we prove Theorem 1.2 in §3. The proof of Theorem 1.3 occupies §4 and in §5 we extend Nadel's and Andreotti-Siu theorems to dimension 2 manifolds and prove Corollary 1.4.

2. Existence of holomorphic functions

Let X be a hyperconcave end. If X can be compactified, $\{\varphi < b\}$ will be an open set in a Stein space, which has a lot of holomorphic functions. Conversely, in order to prove Theorem 1.2 we shall construct holomorphic functions on X. For the purpose of compactification we can and shall suppose that the function φ in Definition 1.1 is strictly plurisubharmonic everywhere on X and $a > 0$.

Let us first touch briefly the case of dimension $\geqslant 3$ and describe the method of [AS70] and [Ros65]. If $\dim X \geqslant 3$, we can use the Andreotti-Grauert theory [AG62] to show that the natural restrictions $H^1(X, \mathscr{F}) \longrightarrow H^1(\{\varphi < c\}, \mathscr{F})$, $c < a$, are isomorphisms for any coherent analytic sheaf $\mathscr{F}$ (see [AS70, Propositions 1.2-3]). Therefore, for any coherent ideal sheaf $\mathscr{I} \subset \mathscr{O}$ whose zero set is disjoint from $\{\varphi < c\}$, for some $c < a$, the natural restriction $H^0(X, \mathscr{O}) \longrightarrow H^0(X, \mathscr{O}/\mathscr{I})$ is surjective. This leads [AS70, Proposition 1.4] to the existence of many holomorphic functions that separate points, give local coordinates and have a peak at pseudoconvex boundary points of X. (The method also works for domains X as in the beginning of Introduction.)

If $\dim X = 2$, the Andreotti-Grauert theory cannot be applied. However, in the case of hyperconcave ends, we can introduce a good complete metric, in which the L^2 estimate for $(0,1)$-forms holds.

Let X be a hyperconcave end. Let us denote by $\Omega_0 = \{\varphi < 0\}$. Our analysis is based on the existence of a complete Kähler metric on Ω_0, satisfying the condition of Donnelly-Fefferman [DF83]. Denote

$$\chi = -\log(-\varphi), \tag{2.1}$$

which is a smooth function on Ω_0. We set

$$\omega = \sqrt{-1}\partial\bar{\partial}\chi = -\sqrt{-1}\partial\bar{\partial}\log(-\varphi). \tag{2.2}$$

Note that

$$\partial\bar{\partial}\chi = \frac{\partial\bar{\partial}\varphi}{-\varphi} + \frac{\partial\varphi \wedge \bar{\partial}\varphi}{\varphi^2}$$

and

$$\frac{\partial\varphi \wedge \bar{\partial}\varphi}{\varphi^2} = \partial\chi \wedge \bar{\partial}\chi\,.$$

Since $\sqrt{-1}\partial\bar{\partial}\varphi/(-\varphi)$ represents a metric on Ω_0, we get

$$|d\chi|_\omega \leqslant 1. \tag{2.3}$$

This is the Donnelly-Fefferman condition, which will be used soon to obtain the fundamental estimate. Since $\chi : \Omega_0 \longrightarrow \mathbb{R}$ is proper, (2.3) also ensures that ω is complete. Indeed, (2.3) entails that χ is Lipschitz with respect to the geodesic distance induced by ω, so any geodesic ball must be relatively compact.

We may assume in the sequel that -1 is a regular value of φ. We define $\Omega = \{\varphi < -1\}$. The metric ω is complete at the pseudoconcave end of Ω and extends smoothly over the boundary $b\Omega$.

Let us denote by $\mathscr{C}_c^{0,q}(\Omega)$ the space of smooth $(0,q)$-forms with compact support in Ω. We denote $\vartheta = -\ast\,\partial\,\ast$, the formal adjoint of $\bar\partial$ with respect to the scalar product $(u,v) = \int_\Omega \langle u,v\rangle\, dV_\omega$, where $\langle u,v\rangle = \langle u,v\rangle_\omega$ and $dV_\omega = \omega^n/n!$.

The first step of the proof of Theorem 1.2 is to establish the fundamental estimate near $-\infty$.

Lemma 2.1. *For any $v \in \mathscr{C}_c^{0,1}(\Omega)$ we have*

$$\|v\|^2 \leqslant 8\big(\|\bar\partial v\|^2 + \|\vartheta v\|^2\big)\,. \tag{2.4}$$

Proof. The proof is based on the Berndtsson-Siu trick [Ber92], [Siu96], used to prove the vanishing theorem of Donnelly-Fefferman. On the trivial bundle $E = \Omega\times\mathbb{C}$ we introduce the auxilliary hermitian metric $e^{\chi/2}$. We denote by ϑ_χ the formal adjoint of $\bar\partial$ with respect to the scalar product

$$(u,v)_\chi = \int_\Omega \langle u,v\rangle\, e^{\chi/2}\, dV_\omega\,.$$

Then $\vartheta_\chi = e^{-\chi/2}\,\vartheta\, e^{\chi/2}$. We apply the Bochner-Kodaira-Nakano formula for $u \in \mathscr{C}_c^{0,1}(\Omega)$:

$$\int_\Omega \big\langle[\sqrt{-1}\partial\bar\partial(-\chi/2),\Lambda_\omega]\,u,u\big\rangle\, e^{\chi/2}\, dV_\omega \leqslant \int_\Omega \big(|\bar\partial u|^2 + |\vartheta_\chi u|^2\big)\, e^{\chi/2}\, dV_\omega\,, \tag{2.5}$$

where Λ_ω represents the contraction with ω and $[A,B] = AB - (-1)^{\deg A\cdot\deg B}BA$ is the graded commutator of the operators A, B. The idea is to substitute $v = u\, e^{\chi/4}$. It is readily seen that

$$|\bar\partial u|^2\, e^{\chi/2} \leqslant 2|\bar\partial v|^2 + \tfrac18|\bar\partial\chi|^2|v|^2\,, \tag{2.6}$$

$$|\vartheta_\chi u|^2\, e^{\chi/2} \leqslant 2|\vartheta v|^2 + \tfrac18|\partial\chi|^2|v|^2\,. \tag{2.7}$$

Moreover

$$\big\langle[\sqrt{-1}\partial\bar\partial(-\chi/2),\,\Lambda_\omega]u,u\big\rangle\, e^{\chi/2} = \big\langle[\sqrt{-1}\partial\bar\partial(-\chi/2),\,\Lambda_\omega]v,v\big\rangle\,.$$

In general, for a (p,q)–form α we have the identity $\langle[\omega,\Lambda_\omega]\alpha,\alpha\rangle = (p+q-n)|\alpha|^2$, where $n = \dim X$. Taking into account that $\omega = \sqrt{-1}\partial\bar\partial\chi$ and that v is a $(0,1)$–form, we obtain

$$\big\langle[\sqrt{-1}\partial\bar\partial(-\chi/2),\Lambda_\omega]u,u\big\rangle\, e^{\chi/2} = \tfrac{n-1}{2}|v|^2 \geqslant \tfrac12|v|^2\,. \tag{2.8}$$

Thus, by (2.5), (2.6), (2.7), (2.8) and by Donnelly-Fefferman condition (2.3),

$$\tfrac12\int_\Omega |v|^2\, dV_\omega \leqslant 2\int_\Omega \big(|\bar\partial v|^2 + |\vartheta v|^2\big)\, dV_\omega + \tfrac14\int_\Omega |v|^2\, dV_\omega\,. \tag{2.9}$$

This immediately implies (2.4) for elements $v \in \mathscr{C}_c^{0,1}(\Omega)$. □

The next step is to combine the estimate (2.4) with the standard estimate near the strongly pseudoconvex boundary $b\Omega = \{\varphi = -1\}$.

Let $\eta : (-\infty, 0) \longrightarrow \mathbb{R}$ be a smooth function such that $\eta(t) = 0$ on $(-\infty, -2]$, $\eta'(t) > 0$, $\eta''(t) > 0$ on $(-2, 0)$. Let us introduce the scalar product

$$(u, v)_\eta = \int_\Omega \langle u, v\rangle \, e^{-\eta(\varphi)} \, dV_\omega \,, \tag{2.10}$$

the coresponding norm $\|\cdot\|_\eta$ and L_2 space, denoted $L_2^{0,q}(\Omega, \eta(\varphi))$. Note that $L_2^{0,q}(\Omega, \eta(\varphi)) = L_2^{0,q}(\Omega)$ and that the two norms are equivalent.

Let us denote by $\mathscr{C}_c^{0,q}(\bar\Omega)$ the space of smooth $(0,q)$-forms with compact support in $\bar\Omega$. Consider $\bar\partial : \mathscr{C}_c^{0,q}(\bar\Omega) \longrightarrow \mathscr{C}_c^{0,q+1}(\bar\Omega)$ and its maximal closed extension in $L_2^{0,q}(\Omega, \eta(\varphi))$, whose domain $\operatorname{Dom}\bar\partial$ consists of elements u such that $\bar\partial u$, calculated in the sense of distributions, belong to $L_2^{0,q+1}(\Omega, \eta(\varphi))$.

We denote by ϑ_η the formal adjoint of $\bar\partial$ with respect to the scalar product (2.10). Then $\vartheta_\eta = \vartheta + i\big(\partial\eta(\varphi)\big)$, where $i(\,\cdot\,)$ represents the interior product. Let $\sigma(\vartheta, df) = *\partial f \wedge *$ be the symbol of ϑ, calculated on the cotangent vector df. It is clear that $\sigma(\vartheta_\eta, df) = \sigma(\vartheta, df)$ does not depent on η. We introduce the spaces

$$B^{0,q} = \{\, \alpha \in \mathscr{C}_c^{0,q}(\bar\Omega) : \sigma(\vartheta, d\varphi)\alpha = 0 \text{ on } b\Omega\}. \tag{2.11}$$

Integration by parts [FK72, Propositions 1.3.1-2] yields

$$\operatorname{Dom}\bar\partial_\eta^* \cap \mathscr{C}_c^{0,q}(\bar\Omega) = B^{0,q}, \quad \bar\partial_\eta^* = \vartheta_\eta \text{ on } B^{0,q}, \tag{2.12}$$

where $\bar\partial_\eta^*$ is the Hilbert space adjoint of $\bar\partial$ on $L_2^{0,q}(\Omega, \eta(\varphi))$.

Lemma 2.2. *The space $B^{0,q}$ is dense in $\operatorname{Dom}\bar\partial \cap \operatorname{Dom}\bar\partial_\eta^*$ in the graph norm $u \longmapsto \big(\|u\|_\eta^2 + \|\bar\partial u\|_\eta^2 + \|\bar\partial^* u\|_\eta^2\big)^{1/2}$.*

Proof. We use first the idea from [AV65, Lemma 4, p. 92–3] in order to reduce the proof to the case of a compactly supported form u. The completeness of the metric ω implies the existence of a sequence $\{a_\nu\}_\nu \subset \mathscr{C}_c^\infty(\bar\Omega)$, such that $0 \leqslant a_\nu \leqslant 1$, $a_{\nu+1} = 1$ on $\operatorname{supp} a_\nu$, $|da_\nu| \leqslant 1/\nu$ for every $\nu \geqslant 1$ and $\{\operatorname{supp} a_\nu\}_\nu$ exhaust $\bar\Omega$. Indeed, consider a smooth function $\rho : \mathbb{R} \longrightarrow [0,1]$ such that $\rho = 0$ on a neighbourhood of $(-\infty, -2]$, $\rho = 1$ on a neighbourhood of $[-1, \infty)$ and $0 \leqslant \rho' \leqslant 2$. Then $a_\nu = \rho(\chi/2^{\nu+1})$ satisfies the conditions above.

Let $u \in \operatorname{Dom}(\bar\partial) \cap \operatorname{Dom}(\bar\partial^*)$. Then $a_\nu u \in \operatorname{Dom}(\bar\partial) \cap \operatorname{Dom}(\bar\partial^*)$ and

$$\|\bar\partial(a_\nu u) - a_\nu \bar\partial u\|_\eta = O(1/\nu)\|u\|_\eta \,,$$
$$\|\bar\partial^*(a_\nu u) - a_\nu \bar\partial^* u\|_\eta = O(1/\nu)\|u\|_\eta \,.$$

Hence $\{a_\nu u\}$ converges to u in the graph norm. So to prove the assertion we can start with a form u having compact support in $\bar\Omega$. But then the approximation in the graph norm follows from the Friedrichs theorem on the identity of weak and strong derivatives (see [Hör65, Proposition 1.2.4]). □

We confine next our attention to the fundamental estimate on Ω.

Lemma 2.3. *If η grows sufficiently fast, there exists a constant $C > 0$ such that*

$$\|u\|_\eta^2 \leqslant C\Big(\|\bar\partial u\|_\eta^2 + \|\bar\partial_\eta^* u\|_\eta^2 + \int_K |u|^2\, e^{-\eta(\varphi)}\, dV_\omega\Big)\,, \tag{2.13}$$
$$u \in \operatorname{Dom}\bar\partial \cap \operatorname{Dom}\bar\partial_\eta^* \subset L_2^{0,1}(\Omega, \eta(\varphi))\,,$$

where $K = \{-3 \leqslant \varphi \leqslant -3/2\}$.

Proof. We give the trivial line bundle $E = \Omega \times \mathbb{C}$ the metric $e^{-\eta(\varphi)}$. Let $u \in B^{0,1}$, $\operatorname{supp} u \subset \{-3 \leqslant \varphi\}$.

Let us use a form of the Bochner-Kodaira formula introduced by Andreotti-Vesentini [AV65] and Griffiths [Gri66]. The curvature of the hermitian bundle $(E, e^{-\eta(\varphi)})$ is denoted $\Theta(E)$. It is a $(1,1)$-form on X, $\Theta(E) = \sum \theta_{\alpha\beta}\, dz^\alpha \wedge d\bar z^\beta$, where $\theta_{\alpha\beta} = \partial_{z^\alpha}\partial_{\bar z^\beta}\eta(\varphi)$. Let θ^μ_β be the curvature tensor with the first index raised. Let $u = \sum u_\lambda\, d\bar z^\lambda$ be a $(0,1)$–form on Ω. We define the $(0,1)$–form $\Theta(E)u = \sum \theta^\mu_\lambda u_\mu\, d\bar z^\lambda$. We also introduce the Ricci curvature $\operatorname{Ric}_\omega = -\Theta(K_X)$, where K_X is the canonical bundle of X. For a E–valued $(0,1)$-form u, we set $\operatorname{Ric}_\omega u = -\Theta(K_X)u$.

For any $u \in B^{0,1}$ we have:

$$\|\bar\partial u\|_\eta^2 + \|\bar\partial^* u\|_\eta^2 = \big\|\overline\nabla u\big\|_\eta^2 + \big(\Theta(E)u, u\big)_\eta + \big(\operatorname{Ric}_\omega u, u\big)_\eta + \int_{b\Omega} \mathcal{L}(u,u) e^{-\eta(\varphi)}\, dS \tag{2.14}$$

where $\mathcal{L}$ is the Levi operator (see [Gri66, p.418]) and $\overline\nabla$ denotes the covariant derivative in the $(0,1)$–direction. This formula was given by Griffiths [Gri66, p. 429, (7.14)].

Since $b\Omega$ is pseudoconvex, $\mathcal{L}(u,u) \geqslant 0$ for all $u \in B^{0,1}$. On $\{-3 \leqslant \varphi \leqslant -1\}$, $\operatorname{Ric}_\omega$ is bounded and independent of η, so there exists a constant $R > 0$ such that $\langle \operatorname{Ric}_\omega u, u\rangle \geqslant -R|u|^2$, pointwise, for any u with $\operatorname{supp} u \subset \{-3 \leqslant \varphi\}$. On the other hand, we can use the strict plurisubharmonicity of φ to choose a sufficiently increasing η (replace η with $\tau\eta$ for $\tau \gg 1$) such that $\langle \Theta(E)u, u\rangle \geqslant (R+1)|u|^2$, pointwise on $\{(-3/2) \leqslant \varphi\}$, for any u. Since $\langle \Theta(E)u, u\rangle \geqslant 0$ everywhere, we obtain from (2.14)

$$\|u\|_\eta^2 \leqslant \|\bar\partial u\|_\eta^2 + \|\bar\partial_\eta^* u\|_\eta^2 + (R+1)\int_{\{-3\leqslant\varphi\leqslant(-3/2)\}} |u|^2\, e^{-\eta(\varphi)}\, dV_\omega\,, \tag{2.15}$$
$$u \in B^{0,1}\,,\ \operatorname{supp} u \subset \{-3 \leqslant \varphi\}\,.$$

Let $u \in B^{0,1}$. We choose a cut-off function $\rho_1 \in \mathscr{C}^\infty(\Omega)$ such that $\operatorname{supp}\rho_1 = \{-3 \leqslant \varphi\}$, $\rho_1 = 1$ on $\{-2 \leqslant \varphi\}$. Set $\rho_2 = 1 - \rho_1$. On $\operatorname{supp}\rho_2$, η vanishes, therefore $\bar\partial^*_\eta(\rho_2 u) = \vartheta(\rho_2 u)$. Upon applying (2.4) for $\rho_2 u$ we get

$$\|\rho_2 u\|^2_\eta \leqslant 8\big(\|\bar\partial(\rho_2 u)\|^2_\eta + \|\bar\partial^*_\eta(\rho_2 u)\|^2_\eta\big)\,, \quad u \in B^{0,1}\,. \tag{2.16}$$

The estimate (2.15) for $\rho_1 u$ and (2.16) together with standard inequalities deliver (2.13) for elements $u \in B^{0,1}$. By Lemma 2.2, estimate (2.13) holds for all forms $u \in \operatorname{Dom}\bar\partial \cap \operatorname{Dom}\bar\partial^*_\eta \subset L^{0,1}_2(\Omega, \eta(\varphi))$. □

From now on, we fix the function η and we write $(\cdot,\cdot) = (\cdot,\cdot)_\eta$ and $\|\cdot\| = \|\cdot\|_\eta$, that is, we omit the index η.

Consider the complex of closed, densely defined operators

$$L^{0,0}_2(\Omega,\eta(\varphi)) \xrightarrow{T=\bar\partial} L^{0,1}_2(\Omega,\eta(\varphi)) \xrightarrow{S=\bar\partial} L^{0,2}_2(\Omega,\eta(\varphi))\,,$$

and the closed, densely defined operator

$$\operatorname{Dom}\Delta'' = \{u \in \operatorname{Dom}S \cap \operatorname{Dom}T^* \,:\, Su \in \operatorname{Dom}S^*\,,\ T^*u \in \operatorname{Dom}T\}\,,$$
$$\Delta'' u = S^*Su + TT^*u \quad \text{for } u \in \operatorname{Dom}\Delta''\,.$$

Remark that Δ'' is an extension of the operator $\bar\partial\vartheta_\eta + \vartheta_\eta\bar\partial$ defined on $\mathscr{C}^{0,1}_c(\Omega)$.

Theorem 2.4. (i) *The operators T and Δ'' have closed range and we have the strong Hodge decomposition*

$$L^{0,1}_2(\Omega,\eta(\varphi)) = \operatorname{Range}(TT^*) \oplus \operatorname{Range}(S^*S) \oplus \mathscr{H}^{0,1}\,.$$

(ii) *There exists a bounded operator N on $L^{0,1}_2(\Omega,\eta(\varphi))$ such that $\Delta''N = N\Delta'' = \operatorname{Id} - P_h$, $P_hN = NP_h = 0$, where P_h is the orthogonal projection on $\mathscr{H}^{0,1}$.*

(iii) *If $f \in \operatorname{Range}T$, the unique solution $u \perp \operatorname{Ker}T$ of the equation $Tu = f$ is given by $u = T^*Nf$.*

(iv) *The operator N maps $L^{0,1}_2(\Omega,\eta(\varphi)) \cap \mathscr{C}^{0,1}(\Omega)$ into itself.*

Proof. By [Hör65, Theorems 1.1.1–2] we know that a necessary and sufficient condition for $\operatorname{Range}T$ and $\operatorname{Range}S^*$ to be closed is that there exists a positive constant C such that

$$\|f\|^2 \leqslant C(\|Sf\|^2 + \|T^*f\|^2), \quad f \in \operatorname{Dom}S \cap \operatorname{Dom}T^*,\ f \perp \operatorname{Ker}S \cap \operatorname{Ker}T^* \tag{2.17}$$

We denote in the sequel $\mathscr{H}^{0,1} = \operatorname{Ker}S \cap \operatorname{Ker}T^*$. By [Hör65, Theorem 1.1.3], if from every sequence $\{f_k\} \subset \operatorname{Dom}S \cap \operatorname{Dom}T^*$ with $\{\|f_k\|\}$ bounded and $\|T^*f_k\| + \|Sf_k\| \longrightarrow 0$, for $k \longrightarrow \infty$, one can select a strongly convergent subsequence, (2.17) holds and $\mathscr{H}^{0,1}$ is finite dimensional. Take any sequence $\{f_k\}$ as above.

By Lemma 2.2 we may assume that $\{f_k\} \subset B^{0,1}$. Let ξ be a smooth, compactly supported function on Ω, such that $\xi = 1$ on K. Hence

$$\big((\Delta'' + \mathrm{Id})(\xi f_k), \xi f_k\big) = \|\xi f_k\|^2 + \|S(\xi f_k)\|^2 + \|T^*(\xi f_k)\|^2 \tag{2.18}$$

is also bounded. Set $M = \operatorname{supp}\xi$. Denote by $W^1_0(M)$ the Sobolev space based on $(0,1)$–forms, and by $\|\cdot\|_1$ the norm on this space.

By Gårding's inequality, there exists a constant $C > 0$ such that

$$\|v\|_1^2 \leqslant C\big((\Delta'' v, v) + \|v\|\big), \qquad v \in \mathscr{C}^{0,1}_c(M). \tag{2.19}$$

Upon applying (2.19) for $v = \xi f_k$ we have by (2.18) that $\{\xi f_k\}$ is a bounded sequence in $(W^1_0(M), \|\cdot\|_1)$.

By the Rellich Lemma, the inclusion $\big(W^1_0(M), \|\cdot\|_1\big) \hookrightarrow \big(L^{0,1}_2(\Omega, \eta(\varphi)), \|\cdot\|\big)$ is compact. We can select therefore a convergent subsequence in $L^{0,1}_2(\Omega, \eta(\varphi))$, denoted also $\{\xi f_k\}$. Since $\eta = 1$ on K, it follows that $\{f_k|_K\}$ converges in $\|\cdot\|$. By estimate (2.13), this entails that $\{f_k\}$ converges on Ω in $\|\cdot\|$. We have thus proved (2.17).

Observe that $\operatorname{Ker}\Delta'' = \mathscr{H}^{0,1}$. From (2.17) we infer that

$$\|f\| \leqslant C\|\Delta'' f\|\,, \quad f \in \operatorname{Dom}\Delta'',\ f \perp \operatorname{Ker}\Delta''\,. \tag{2.20}$$

Therefore Δ'' has closed range. We know from a theorem of Gaffney [FK72, Proposition 1.3.8], that Δ'' is self-adjoint. Hence the decomposition

$$L^{0,1}_2(\Omega, \eta(\varphi)) = \operatorname{Range}\Delta'' \oplus \operatorname{Ker}\Delta'' = \operatorname{Range}(T\,T^*) \oplus \operatorname{Range}(S^*S) \oplus \mathscr{H}^{0,1}\,.$$

By (2.20) there exists a bounded inverse N of Δ'' on $\operatorname{Range}\Delta''$. We extend N to $L^{0,1}_2(\Omega, \eta(\varphi))$ by setting $N = 0$ on $\mathscr{H}^{0,1}$. We obtain thus a bounded operator N on $L^{0,1}_2(\Omega, \eta(\varphi))$, called the Neumann operator, bounded by the constant C from (2.20), satisfying $\operatorname{Ker}N = \mathscr{H}^{0,1}$ and $\operatorname{Range}N = \operatorname{Range}\Delta''$. It is now easy to check that (i), (ii) and (iii) hold true.

Finally, assertion (iv) follows from the interior regularity for the elliptic operator $\bar\partial\vartheta_\eta + \vartheta_\eta\bar\partial$ (see e.g. [FK72, Theorem 2.2.9]). □

Remark 2.5. By using the estimates in local Sobolev norms near the boundary points, we can prove as in Folland-Kohn [FK72] that N maps $L^{0,1}_2(\Omega, \eta(\varphi)) \cap \mathscr{C}^{0,1}(\bar\Omega)$ into itself. We could repeat then the solution of the Levi problem as given in [FK72, Theorem 4.2.1], in order to find holomorphic peak functions, for each boundary point. However, we propose in Corollary 2.6, a simpler proof for the existence of peak functions, which doesn't involve the regularity up to the boundary of the $\bar\partial$-Neumann problem.

Corollary 2.6. *Let $p \in b\Omega$ and f be a holomorphic function on a neigbourhood of p such that $\{f = 0\} \cap \bar\Omega = \{p\}$. Then for every m big enough, there is a* *function* $g \in \mathscr{O}(\Omega) \cap \mathscr{C}^\infty(\bar{\Omega} \setminus \{p\})$, *a smooth function* Φ *on a neigbourhood* V *of* p *and constants* $a_1, \cdots, a_{m-1}$ *such that*

$$g = \frac{1 + a_{m-1} f + \cdots + a_1 f^{m-1}}{f^m} + \Phi$$

on $V \cap \Omega$. *In particular, we have* $\lim_{z \to p} |g(z)| = \infty$.

Proof. We will apply the last theorem for a domain $\Omega_\delta = \{\varphi < -1 + \delta\}$ with $\delta > 0$ small enough. Let U be a small neighbourhood of p where f is defined. Pick $\psi \in \mathscr{C}_c^\infty(U)$ such that $\psi = 1$ on a neighbourhood V of p. Set

$$h_m = \psi / f^m \text{ on } U \text{ and } 0 \text{ on } X \setminus U$$

and

$$v_m = 0 \text{ on } V \text{ and } \bar{\partial} h_m \text{ on } X \setminus V.$$

Observe that v_m belongs to $\mathscr{C}_c^{0,1}(\bar{\Omega}_\delta)$ for δ small enough. Moreover, we have $\bar{\partial} v_m = 0$ on Ω_δ. Fix a such δ and now apply Theorem 2.4 for $\Omega' = \Omega_\delta$. By this theorem, the codimension of $\operatorname{Range} T$ in $\operatorname{Ker} S$ is finite. For every m big enough, there are constants $a_1, \cdots, a_{m-1}$ such that $v = v_m + a_{m-1} v_{m-1} + \cdots + a_1 v_1$ belongs to $\operatorname{Range} T$. Then there is $\Phi' \in \mathscr{C}^{0,0}(\Omega')$ such that $\bar{\partial} \Phi' = -v$. Set $h = h_m + a_{m-1} h_{m-1} + \cdots + a_1 h_1$ and $g = h + \Phi'$. We have $\bar{\partial} g = 0$ on $\Omega' \setminus \{f = 0\}$. Then $g \in \mathscr{O}(\Omega) \cap \mathscr{C}^\infty(\bar{\Omega} \setminus \{p\})$. The function Φ in the corollary is equal to Φ' on V. Thus it is smooth on V. The proof is completed. □

3. The embedding

In this section we prove Theorem 1.2, using the results from Section 2 and the method of [AS70].

It is clear that the conclusions of Section 2 hold true for any set of the form $\{\varphi < c\}$, where $c \leqslant b$ is a regular value of φ. To see this, we only have to replace the function χ from (2.1) with $\chi = -\log(c + \varepsilon - \varphi)$, for appropriate $\varepsilon > 0$. In the sequel we shall adhere to the following notations: for $d < c < a$ set $X_c = \{\varphi < c\}$ and $X_d^c = \{d < \varphi < c\}$.

Proposition 3.1. *Let* c *be a regular value of* φ. *Then for* $\delta > 0$ *small enough we have:*

(a) *The holomorphic functions on* X_c *separate points on* $X_{c-\delta}^c$,
(b) *The holomorphic functions on* X_c *give local coordinates on* $X_{c-\delta}^c$, *and*
(c) *for any* $d \in (c - \delta, c)$ *there exists* $d^* \in (d, c)$, *such that the holomorphically convex hull of* X_d, *with respect to the algebra of holomorphic functions on* X_c, *is contained in* X_{d^*}.

Remark 3.2. If $\dim X \geqslant 3$, an analogous statement to Proposition 3.1 was proved in [AS70, Proposition 1.4], using the Andreotti-Grauert theory, as explained at the beginning of §2.

Proposition 3.1 will be the consequence of the following two lemmas. We can assume that $bX_{c-\varepsilon}$ is smooth for $\varepsilon > 0$ small enough. Choose a projection π from a neighbourhood of bX_c into bX_c. We will denote by (x, ε) the point of $bX_{c-\varepsilon}$ whose projection is $x \in bX_c$.

Lemma 3.3. *Let x_1, x_2 be two different points in bX_c. Then there are two neighbourhoods V_1, V_2 of x_1, x_2 and $\nu = \nu(x_1, x_2) > 0$ such that the holomorphic functions of X_c separate $V_1 \times (0, \nu]$ and $V_2 \times (0, \nu]$.*

Proof. This is a direct corollary from the existence of a function holomorphic in X_c, and $\mathcal{C}^\infty$ in $\bar{X}_c \setminus \{x_1\}$ which tends to ∞ at x_1. □

Lemma 3.4. *Let x be a point of bX_c. Then there are a neighbourhood V of x and $\tau = \tau(x) > 0$ such that the holomorphic functions in X_c give local coordinates for $V \times (0, \tau]$.*

Proof. Without loss of generality and in order to simplify the notations, we consider the case $n = 2$. Choose a local coordinates system such that $x = 0$ and locally $X_c \subset \{|z_1 - 1/2|^2 + |z_2|^2 < 1/4\}$. We now apply Corollary 2.6 for functions $f_1(z) = z_1$ and $f_2(z) = z_1(1 - z_2)$. Denote by g_1, g_2 the holomorphic functions constructed by this corollary for a number m big enough. We can also construct the analogue functions if we replace m by $m + 1$. Denote by g_1' and g_2' these new functions.

Let $G : X_c \longrightarrow \mathbb{C}^4$ given by $G = (g_1, g_2, g_1', g_2')$. We will prove that G gives local coordinates. Set

$$I(z) = \left(\frac{z_1}{z_3}, 1 - \frac{z_2 z_3}{z_1 z_4}\right).$$

Let W be a small neigbourhood of 0. By Corollary 2.6, the map $I \circ G$ is defined on $W \cap X_c$ and can be extended to a smooth function on W. Moreover, on W we have

$$I \circ G(z) = \Big(z_1 + O(z_1^2), z_2 + O(z_1)\Big).$$

Then $I \circ G$ gives an immersion of $W \cap X_c$ in $\mathbb{C}^2$, whenever W is small enough. In consequence, G gives coordinates of $W \cap X_c$. □

Proof of Proposition 3.1. We cover $bX_c \times bX_c$ by a finite family of open sets of the form $V_1 \times V_2$ (from Lemma 3.3) and the form $V \times V$ (from Lemma 3.4). We have a finite family of ν and τ. Then properties (a) and (b) hold for every δ smaller than these τ and ν. Property (c) is an immediate consequence of Corollary 2.6. □

Proof of Theorem 1.2. First let us remark that the assertion (i) is a consequence of (ii), so we shall prove only the latter. We assume therefore that the function $\varphi : X \longrightarrow (-\infty, a)$ is strictly plurisubharmonic everywhere.

The proof of the compactification statement for $\dim X \geqslant 3$ in [AS70, Proposition 3.2] uses only the assertions (a), (b) and (c) of Proposition 3.1, so we just have to follow it. For the readers' convenience we give here the details. The main tool is the Hartogs extension phenomenon.

Let c and δ as in Proposition 3.1 and choose $d \in (c-\delta, c)$. By Proposition 3.1, (c), the holomorphically convex hull of $\{\varphi \leqslant d\}$, with respect to $\mathscr{O}(X_c)$, is contained in X_{d^*}, for some $d^* \in (d, c)$. We can therefore find a finite number of open sets U_i, and functions $f_i \in \mathscr{O}(X_c)$, $1 \leqslant i \leqslant k$, such that

$$\{\varphi = d^*\} \subset \cup_{i=1}^{k} U_i\,, \quad |f_i\,1_{U_i}| > 1\,, \quad |f_i\,1_{\{\varphi\leqslant d\}}| < 1/2\,. \tag{3.1}$$

By Proposition 3.1, (a) and (b), we can find $f_{k+1}, \cdots, f_l \in \mathscr{O}(X_c)$ which separate points and give local coordinates on $\{d \leqslant \varphi \leqslant d^*\}$. Whithout lost of generality we can assume that $|f_i\,1_{\{\varphi\leqslant d\}}| < 1/2$.

Consider the map $\alpha : X_c \longrightarrow \mathbb{C}^l$, $\alpha(x) = (f_1(x), \cdots, f_l(x))$. We shall use the notation

$$P_\varepsilon = \left\{ z \in \mathbb{C}^l \;:\; |z_i| < \varepsilon\,,\; 1 \leqslant i \leqslant l \right\}\,, \quad 0 < \varepsilon \leqslant 1\,.$$

From relation (3.1) we deduce

$$\alpha\big(\{\varphi \leqslant d\}\big) \subset P_{1/2}\,, \quad \alpha\big(\{\varphi = d^*\}\big) \cap P_1 = \emptyset\,. \tag{3.2}$$

Denote $G = P_1 \smallsetminus \overline{P}_{1/2}$ and set $H = \alpha^{-1}(G) \cap X_d^{d^*}$. It is clear that $\alpha(H)$ is a complex submanifold of G, for α is a proper injective immersion. Since $\alpha(H)$ has dimension at least 2, it follows from the Hartogs phenomenon [GR65, Theorem VII, D.6] that we can find an $\varepsilon \in [1/2, 1]$, such that $\alpha(H) \cap (P_1 \smallsetminus \overline{P}_\varepsilon)$ can be extended to an analytic subset V of P_1.

We can glue the topological spaces $X_c \smallsetminus \alpha^{-1}(\overline{P}_\varepsilon)$ and V along $H \smallsetminus \alpha^{-1}(\overline{P}_\varepsilon)$ using the identification given by the holomorphic map α. Hence, we obtain a complex space $\widetilde{X}_c$, such that $X_c \smallsetminus \alpha^{-1}(\overline{P}_\varepsilon)$ and V are open subsets of $\widetilde{X}_c$.

We show next that $\widetilde{X}_c$ is a Stein space. It is enough to construct a strictly plurisubharmonic exhaustion function. For this purpose, observe first that f_i, $1 \leqslant i \leqslant l$, extend naturally to functions $\widetilde{f}_i$, by setting $\widetilde{f}_i = z_i$ on V. Choose $0 < \delta_1 < \delta_2 < \delta$ and a cut-off function ϱ which equals 0 on $\{\varphi \leqslant c - \delta_2\}$ and 1 on $\{\varphi \geqslant c - \delta_1\}$. Consider moreover a smooth increasing convex function λ defined on $(-\infty, c)$, such that $\lim_{t\to c} \lambda(t) = +\infty$. Then for a sufficiently large constant $A > 0$, the function

$$\psi = A \sum_{i=1}^{l} |\widetilde{f}_i|^2 + \varrho\lambda(\varphi)$$

is a strictly plurisubharmonic exhaustion function on $\widetilde{X}_c$.

We have thus found for every $c < a$ a Stein space $\widetilde{X}_c$, such that the strip X_c^a is (biholomorphic to) an open subset of $\widetilde{X}_c$. By passing to the normalization, we may assume that $\widetilde{X}_c$ is normal. From [AS70, Corollary 3.2], we deduce that for any $c^* < a$ the normal Stein spaces $\widetilde{X}_c$ and $\widetilde{X}_{c^*}$ are biholomorphic and the biholomorphism is the identity on the smaller strip. We can drop the subscript c and denote by $\widetilde{X}$ the common Stein completion. By letting $c \longrightarrow -\infty$ we obtain that X itself is biholomorphic to an open set of $\widetilde{X}$. The proof is complete. □

Remark 3.5. We can use another argument than Proposition 3.1 to show Theorem 1.2. Firstly, we can find a CR embedding $\Psi : bX_c \longrightarrow \mathbb{C}^N$, using Sarkis [Sar02, Corollaire 4.13]. The latter result asserts that a compact strictly pseudoconvex 3-dimensional CR manifold is embeddable in the euclidian space provided it is embeddable in the projective space and possesses a non-constant CR function. In our case these conditions are fullfiled. Using the complete metric (2.2) and the positivity of the trivial line bundle, it is easy to see that the $(n, 0)$-forms embed bX_c in the projective space. By Corollary 2.6 and Sarkis' theorem the embedability of bX_c in some $\mathbb{C}^N$ follows.

Secondly, we apply the Harvey-Lawson theorem [HL75] to find a Stein space $S \subset \mathbb{C}^N$ which bounds $\Psi(bX_c)$, show that Ψ extends to a holomorphic map $\widehat{\Psi} : X_c \longrightarrow S$, injective near bX_c, and finally infer from here that X can be compactified.

The existence of peak holomorphic functions in Corollary 2.6 affords however the simpler and more elementary proof based on Proposition 3.1.

Remark 3.6 (Generalization of Theorem 1.2). Theorem 1.2 holds also for normal complex spaces with isolated singularities. These are the only allowed normal singularities in dimension 2.

Indeed, let X be a hyperconcave end with isolated normal singularities. Note that Definition 1.1 makes sense also for complex spaces. Let a_i denote the singular points and choose functions φ_i with pairwise disjoint compact supports, such that φ_i is strictly plurisubharmonic in a neighbourhood of a_i and $\lim_{z\to a_i} = -\infty$. Using the function $\widehat{\varphi} = \varphi + \sum \varepsilon_i \varphi_i$, with ε_i small enough, we see that $\operatorname{Reg} X$ is a hyperconcave end. By Theorem 1.2 we get a normal Stein compactification Y of $\operatorname{Reg} X$.

Take $\{V_i\}$ pairwise disjoint Stein neighbourhoods of $\{a_i\}$. Then $V_i \smallsetminus \{a_i\} \subset \operatorname{Reg} X$ and a normal Stein compactification of $V_i \smallsetminus \{a_i\}$ is V_i. Using the uniqueness of a normal Stein compactification [AS70, Corollary 3.2] we infer that the V_i are disjointly embedded in Y. Therefore, Y is also a compactification of X.

In particular, the singular set of a hyperconcave end with only isolated singularities must be finite.

We want now to compare the compactification result with the examples of Grauert, Andreotti-Siu and Rossi. We call henceforth a non-ramified covering simply covering. An immediate consequence of Theorem 1.2 is the following.

Corollary 3.7. *Let V be a Stein manifold of $\dim V \geqslant 2$. Let K be a compact completely pluripolar set, $K = \varphi^{-1}(-\infty)$ where φ is a strictly plurisubharmonic function defined on a neighbourhood U of K, smooth on $U \smallsetminus K$. Then any finite covering of $V \smallsetminus K$ can be compactified to a strongly pseudoconvex space.*

Proof. $V \smallsetminus K$ is a hyperconcave end and any finite covering of a hyperconcave end is also a hyperconcave end. □

Corollary 3.8 is in stark contrast to the examples of non-compactifiable pseudoconcave ends of Grauert [Gra94], Andreotti-Siu [AS70] and Rossi [Ros65]. They are obtained as finite coverings of small neighbourhoods of the boundaries of Stein manifolds of dimension 2. Such coverings have 'big' holes which cannot be filled, whereas 'small', i.e. completely pluripolar holes can always be. The basic lemma for construction of non-fillable holes is [AS70, Proposition 7.1, p. 263].

Lemma 3.8. *Let V be a relatively compact simply connected Stein domain in a complex manifold of dimension $\geqslant 2$. Let U be a neighbourhood of bV and let $p : W \longrightarrow U$ be a nontrivial finite cover of U. Then W cannot be compactified. If $\pi_1(bV)$ has proper subgroups of finite index, such coverings do exist.*

Proof. Indeed, assume that $\widehat{W}$ is a completion of W. Then $\widehat{W} \smallsetminus W$ has a strongly pseudoconvex boundary. By the Hartogs' extension theorem [GR65, Theorem VII, D 4] we obtain an extension $\widehat{p} : \widehat{W} \longrightarrow V$. The map $\widehat{p}$ must be ramified, otherwise $\widehat{p}$ would be a nontrivial cover of the simply connected manifold V. The ramification set R is analytic and contained in $V \smallsetminus U$, thus a finite set. Since V is simply connected, non–singular and of dimension $\geqslant 2$, $V \smallsetminus R$ is simply connected. Therefore $\widehat{W} \smallsetminus \widehat{p}^{-1}(R)$ is an irreducible covering of $V \smallsetminus R$. But this is possible only if p is trivial. This contradiction shows that $\widehat{W}$ cannot possibly exist.

Now, if $\pi_1(bV)$ contains non-trivial subgroups of finite index, this remains true for $\pi_1(U)$ where U is any small neighbourhood which retracts on bV. This means that U has non-trivial finite coverings. □

Note that, if $n \geqslant 3$, $\pi_1(bV) = \pi_1(V) = \{0\}$, by a Morse theoretic argument of Andreotti-Frankel, so examples when Lemma 3.8 is not empty can occur only for $\dim V = 2$. Moreover, if $\dim V = 2$ and $K \Subset V$ is completely pluripolar, it follows from Corollary 3.7 and Lemma 3.8 that $\pi_1(V \smallsetminus K)$ doesn't have proper subgroups of finite index.

Example 3.9 (Andreotti-Siu [AS70, p. 267-70]). Let $K \subset \mathbb{P}^3$ be a Kummer surface with 16 isolated, non-degenerate, canonical singular points. K is isomorphic to a quotient $T/\{id, \tau\}$, where T is an algebraic torus and τ is an involution ($\tau^2 = id$) with 16 fixed points. Therefore $\mathrm{Reg}\,(K)$ admits as double covering the torus T minus 16 points.

There exists a 1-real-parameter family of algebraic surfaces $\{K_\varepsilon\}$ such that for $\varepsilon \neq 0$, K_ε is non–singular and $K_0 = K$. There exists a manifold $\mathscr{K} \subset \mathbb{R} \times \mathbb{P}^3$ such that $K_\varepsilon = \mathrm{pr}_{\mathbb{R}}^{-1}(\varepsilon)$, where $\mathrm{pr}_{\mathbb{R}}$ is the projection on the first factor.

If $G \subset \mathbb{P}^3$ is the union of 16 small neighbourhoods of the singular points in $\mathbb{P}^3$, $\mathscr{K} \smallsetminus \mathrm{pr}_{\mathbb{P}^3}^{-1}(G)$ is differentiably trivial near 0. This implies that $K_\varepsilon \smallsetminus \mathrm{pr}_{\mathbb{P}^3}^{-1}(G)$ is diffeomorphic to $K_0 \smallsetminus \mathrm{pr}_{\mathbb{P}^3}^{-1}(G)$ for small ε.

Consider now a singular point p of the Kummer surface K and B be a small ball around p in $\mathbb{P}^3$. Then $V_\varepsilon = K_\varepsilon \cap \mathrm{pr}_{\mathbb{P}^3}^{-1}(B)$ are simply connected Stein spaces, which for $\varepsilon \neq 0$ are non-singular. From the preceding paragraph if follows that, from small $\varepsilon \neq 0$, ∂V_ε is diffeomorphic to ∂V, so that small neighbourhoods U_ε of ∂V_ε have differentiable double cover small concentric shells W_ε in the neighbourhood of a fixed point of the involution $\tau : T \longrightarrow T$.

On W_ε we take the induced complex structure. By Lemma 3.8 the holes of W_ε cannot be filled. It is obvious that W_ε one not hyperconcave ends.

Example 3.10. This example appeared in Rossi [Ros65, p. 252–6] (being attributed to Andreotti), Andreotti-Siu [AS70, p. 262–70] (where credit is given to Grauert) and Grauert [Gra94, p. 273]. It is constructed by the same principle as above, but it is more spectacular. It provides complex structures on a ball minus a point, actually on $\mathbb{P}^2 \smallsetminus \{[1:0:0]\}$, which are not fillable.

Let Q_ε be the family of quadrics in $\mathbb{P}^3$ given in the homogeneous coordinates $[w_0 : w_1 : w_2 : w_3]$ by the equation $w_3(w_3 + \varepsilon w_0) = w_1 w_2$. For $\varepsilon \neq 0$ they are non-singular. There exists an application $\Phi : \mathbb{P}^2 \smallsetminus \{[1:0:0]\} \longrightarrow V \smallsetminus A$, see [AS70, (1), p. 265], where A is a real analytic sphere, such that Φ is a two-sheeted differentiable ramified covering. We can use Φ to induce a new complex structure on $\mathbb{P}^2 \smallsetminus \{[1:0:0]\}$, so that Φ becomes holomorphic. By a variant of Lemma 3.8 for ramified coverings, we see that $\mathbb{P}^2 \smallsetminus \{[1:0:0]\}$ with the new structure cannot be compactified.

Also, $\mathbb{P}^2 \smallsetminus \{[1:0:0]\}$ with the new complex structure cannot be a hyperconcave end, for if it were, $V \smallsetminus A$ would be one too (we average the values of a defining function on the two sheets). So A would be completely pluripolar, which is a plain contradiction.

Remark 3.11 (Complex cobordism). Another point of view on Theorem 1.2 is to consider the sets $Y_c = \{\varphi = c\}$, which, for regular values c of φ, are compact strongly pseudoconvex CR manifolds of dimension 3. Following Epstein and

Henkin [EH01], we call two CR manifolds Y_1 and Y_2 of dimension 3 strictly complex cobordant, if there exists a complex manifold X with boundary, such that $bX = Y_1 \cup Y_2$ and there exists a strictly plurisubharmonic function ρ on X so that Y_1 and Y_2 are two non-critical level sets of ρ. They show then [EH01, Theorem 1] that if Y_1 bounds a complex manifold, also the components of Y_2 bound complex manifolds.

Theorem 1.2 can be rephrased by saying that, if a compact strongly pseudoconvex CR manifold Y is strictly complex cobordant to $-\infty$, the manifold Y bounds a strongly pseudoconvex compact manifold. In particular, Y is embeddable in $\mathbb{C}^N$, for some N.

Note that, by the example of Grauert, Andreotti-Siu and Rossi, there exist compact strongly pseudoconvex CR manifolds of dimension 3 which do not bound a complex manifold and are not embeddable in $\mathbb{C}^N$. This is in contrast to a theorem of Boutet de Monvel [dM75, p. 5] asserting that all compact strongly pseudoconvex CR manifolds of dimension $\geqslant 5$ are embeddable.

4. Compactification by adding finitely many points

The present section is devoted to proving sufficient conditions for the set $\widehat{X} \smallsetminus X$ to be analytic. If this is the case, it can be actually blown down to a finite set, due to the existence of a strongly pseudoconvex neighbourhood.

In order to prove Theorem 1.3 we consider first the particular case when the compactification $\widehat{X}$ is a Stein space.

We begin with some preparations. Let V by a complex manifold. We say that V satisfies the *Kontinuitätssatz* if for any smooth family of closed holomorphic discs $\overline{\Delta}_t$ in V indexed by $t \in [0, 1)$ such that $\cup b\Delta_t$ lies on a compact subset of V, then $\cup\overline{\Delta}_t$ lies on a compact subset of V. It is clear that every Stein manifold satisfies the Kontinuitätssatz, using the strictly plurisubharmonic exhaustion function and the maximum principle. Moreover, if the universal cover of V is Stein then V satisfies Kontinuitätssatz since we can lift the family of discs to the universal cover.

Let F be a closed subset of V. We say that F is *pseudoconcave* if $V \setminus F$ satisfies the local Kontinuitätssatz in V, i.e. for every $x \in F$ there is a neigbourhood W of x such that $W \setminus F$ satisfies the Kontinuitätssatz. Observe that the finite union of pseudoconcave subsets is pseudoconcave and every complex hypersurface is pseudoconcave.

We have the following proposition which implies the Theorem 1.3.

Proposition 4.1. *Let $\widehat{\Omega}$ be a Stein space with isolated singularities S and K a completely pluripolar compact subset of $\widehat{\Omega}$ which contains S. Assume that $\Omega =$*

$\widehat{\Omega}\setminus K$ can be covered by Zariski-open sets which satisfy the local Kontinuitätssatz in $\widehat{\Omega}\setminus S$. Then K is a finite set.

Proof. We can suppose that $\widehat{\Omega}$ is a subvariety of a complex space $\mathbb{C}^N$. Let B be a ball containing K such that $bB\cap\widehat{\Omega}$ is transversal. By hypothesis, we can choose a finite family of Zariski-open sets V_1, ..., V_k which are uniformized by Stein manifolds and $\cap F_i$ is empty near bB, where $F_i=\Omega\setminus V_i$. Observe that F_i is an analytic subset of Ω, $\overline{F}_i\subset F_i\cup K$. Since $F_i\cup(K\setminus S)$ is pseudoconcave in $\widehat{\Omega}\setminus S$, F_i have no component of codimension $\geqslant 2$. By Hartogs theorem, if $n=\dim X>2$, there is a complex subvariety $\widehat{F_i}$ of $\widehat{\Omega}$ which contains F_i. This is also a consequence of Harvey-Lawson theorem [HL75]. We will prove this property for the case $n=2$. Set $F=\cup F_i$.

Observe that $\Gamma=F\cap bB$ is an analytic real curve. The classical Wermer theorem [Wer58] says that $\mathrm{hull}(\Gamma)\setminus\Gamma$ is an analytic subset of pure dimension 1 of $\mathbb{C}^N\setminus\Gamma$ where $\mathrm{hull}(\Gamma)$ is the polynomial hull of Γ. By uniqueness theorem, $\mathrm{hull}(\Gamma)\subset\widehat{\Omega}$. Since S is finite, we have $\mathrm{hull}(\Gamma\cup S)=\mathrm{hull}(\Gamma)\cup S$. Set $F'=(F\cup K)\cap\overline{B}$ and $F''=\mathrm{hull}(\Gamma)\cup S$.

Lemma 4.2 (In the case $n=2$). *We have $F'\subset F''$.*

Proof. Assume that $F'\not\subset F''$. Then there are a point $p\in F'$ and a polynomial h on U such that $\sup_{F''}|h|<\sup_{F'}|h|=|h(p)|$. Set $r=h(p)$. By maximum principle, we have $h^{-1}(r)\cap F'\subset K\setminus S$. In particular, we have $p\in K\setminus S$. Recall that $F'\setminus S$ is pseudoconcave in $\widehat{\Omega}'=\widehat{\Omega}\cap B\setminus S$. We will construct a smooth family of discs which does not satisfy the Kontinuitätssatz. This gives a contradiction. The construction is trivial if p is isolated in F'. We assume that p is not isolated. By using a small perturbation of h, we can suppose that $h(p)$ is not isolated in $h(F')$.

Set $\Sigma'=h(F')$ and $\Sigma''=h(F'')$. Then Σ' (resp. Σ'') is included in the closed disk (resp. open disk) of center 0 and of radius $|r|$. The holomorphic curves $\{h=\mathrm{const}\}$ define a holomorphic foliation, possibly singular, of $\widehat{\Omega}'$. The difficulty is that the fibre $\{h=r\}$ can be singular at p. Denote by T the set of critical values of h in $h(B)$. Then T is finite.

Denote also Θ the unbounded component of $\mathbb{C}\setminus(\Sigma''\cup T)$. It is clear that Σ' meets Θ. This property is stable for every small pertubation of the polynomial h. Since K is a completely pluripolar, $K\cap h^{-1}(a)$ is a polar subset of $h^{-1}(a)$ for every $a\in\mathbb{C}$.

Choose a point $b\in\Theta$ such that $0<\mathrm{dist}(b,\Sigma')<\mathrm{dist}(b,\Sigma''\cup T)$ and $a\in\Sigma'$ such that $\mathrm{dist}(a,b)=\mathrm{dist}(b,\Sigma')$. We have $a\notin\Sigma''\cup T$. Replacing b by a point of the interval (a,b) we can suppose that $\mathrm{dist}(a,b)<\mathrm{dist}(a',b)$ for every $a'\in\Sigma''\setminus\{a\}$.

Fix a point $q \in F'$ such that $h(q) = a$. Set $\delta_1 = |a-b|$. Since $a \notin T$, we can choose a local coordinates system (z_1, z_2) of an open neigbourhood W of q in $\widehat{\Omega}'$ such that $z_1 = h(z) - b$, $q = (a-b, 0)$ and $\{(z_1, z_2),\ |z_1| < \delta_1 + \delta_2, |z_2| < 2\} \subset W$ with $\delta_2 > 0$ small enough. We can choose a W which does not meet F'' and is small as we want.

Let L be the complex line $\{z_1 = a - b\}$. By maximum principle, $K' = F' \cap L$ is equal to $K \cap L$. Then K' is a polar subset of L. This implies that the length of K' is equal to 0. Thus, for almost every $s \in (0, 2)$ the circle $\{|z_2| = s\} \cap L$ does not meet K'. Without lost of generality, we can suppose that K' does not meet $\{|z_2| = 1\} \cap L$. Now we define the disk $\overline{\Delta}_t$ by

$$\overline{\Delta}_t = \{z_1 = (a-b)t, |z_2| \leq 1\}$$

for $t \in [0, 1)$. This smooth familly of discs does not verify the Kontinuitätssatz for $W \setminus F'$. $\square$

Now, denote by $\widehat{F}_i$ the smaller hypersuface of $\widehat{\Omega}$ which contains F_i. Set $\widehat{F} = \cup \widehat{F}_i$. If $n = 2$ we have $F \cup K \subset \widehat{F}$. This is also true for $n > 2$. It is sufficient to apply the last lemma for linear slices of $\widehat{\Omega}$.

Lemma 4.3. *Let L be a pseudoconcave subset of a complex manifold V. If L is included in a hypersurface L' of V then L is itself a hypersurface of V.*

Proof. Observe that L is not included in a subvariety of codimension ≥ 2 of V. Assume that L is not a hypersurface of V. Then there is a point p in $\operatorname{Reg} L'$ which belongs to the boundary of L in L'. Choose a local coordinates system $(z_1, \ldots, z_n)$ of a neigbourhood W of p such that W contains the unit polydisk Δ^n, $p \in \Delta^n$ and $L' \cap W = \{z_1 = 0\} \cap W$. We can suppose that $0 \notin L$ and we can choose W small as we want.

Let $\pi : \Delta^n \longrightarrow \Delta^{n-1}$ be the projection on the last $n-1$ coordinates. Let $q \in L^* = \pi(L \cap \Delta^n)$ such that $\operatorname{dist}(0, L^*) = \operatorname{dist}(0, q)$. Consider the smooth family of discs given by $\overline{\Delta}_t = \{z = (z_1, z'') : |z_1| < 1/2\, , z'' = tq\}$. This family does not verify the Kontinuitätssatz in $W \setminus L$. $\square$

We can end the proof of Proposition 4.1. We know that $(F_i \cup K) \setminus S$ is pseudoconcave in $\widehat{\Omega} \setminus S$ and $F_i \cup K \subset \widehat{F}$. By Lemma 4.3, $(F_i \cup K) \setminus S$ is a hypersurface of $\widehat{\Omega} \setminus S$. By Remmert-Stein theorem, any analytic set can be extended through a point, so $F_i \cup K$ is a hypersurface of $\widehat{\Omega}$. Then $F_i \cup K \subset \widehat{F}_i$ since $\cap F_i = \varnothing$. We deduce that K is included in $\cap \widehat{F}_i$ which is analytic and bounded subset of $\mathbb{C}^N$. Therefore K must be a finite set. $\square$

Remark 4.4. The Proposition 4.1 holds for K not pluripolar. For this case, the proof is more complicated. Using another submersion of $\widehat{\Omega}$ given by the map

$z \longmapsto (h(z), h(z)+\varepsilon z_1, \cdots, h(z)+\varepsilon z_N)$, we can suppose that $R = \max_{K\smallsetminus S}|z| > \max_{F''}|z|$. Let $q \in bB_R \cap (K \smallsetminus S)$ where B_R is is a ball of center 0 and radius R. Using a small affine change of coordinates, we can suppose that $bB_R \cap \widehat{\Omega}$ is transversal at q. We then construct easily a family of discs close to $T_q(bB_R) \cap \widehat{\Omega}$, which does not satisfy the Kontinuitätssatz, where $T_q(bB_R)$ is the complex tangent space of bB_R at q.

Proof of Theorem 1.3. Let X be a hyperconcave end such that the exhaustion function φ is overall plurisubharmonic. Let $\widehat{X}$ be a manifold which compactifies X. Then $\widehat{X} \smallsetminus X$ has a strictly pseudoconvex neighbourhood V. Based on Remmert's reduction theory, Grauert [Gra62, Satz 3, p.338] showed that there exists a maximal analytic set A of V. Moreover, by [Gra62, Satz 5, p.340] there exists a normal Stein space V' with at worst isolated singularities, a discrete set $D \subset V'$ and a proper holomorphic map $\pi : V \longrightarrow V'$, biholomorphic between $V \smallsetminus A$ and with $V' \smallsetminus D$ and $\pi(A) = D$. That is, A can be blown down to the finite set D.

The maximum principle for φ implies $A \subset \widehat{X} \smallsetminus X$. Let $\psi : V' \longrightarrow [-\infty, \infty)$ be given by $\psi = \varphi \circ \pi^{-1}$ on $V' \smallsetminus D$ and $\psi = -\infty$ on $\pi(\widehat{X} \smallsetminus X)$. Then ψ is a strictly plurisubharmonic function on V' and $\pi(\widehat{X} \smallsetminus X)$ is its pluripolar set. By Proposition 4.1, $\pi(\widehat{X} \smallsetminus X)$ is a finite set. Therefore $\widehat{X} \smallsetminus X$ consists of A and possibly a finite set, thus an analytic set. $\square$

Remark 4.5. If in Theorem 1.3 we suppose only that X admits a Zariski-open dense set which is uniformized by a Stein manifold, we can prove in the same way, that $\widehat{X} \smallsetminus X$ is included in a hypersurface of $\widehat{X}$, i.e. X contains a Zariski-open dense set of $\widehat{X}$.

5. Extension of Nadel's theorems

We are in the position to extend the theorems of Nadel [Nad90] to the case of dimension 2.

As preparation it is necessary to introduce some notions. If in Definition 1.1 we may take $a = +\infty$ and φ bounded from above, the manifold X is said to be *hyper* 1-*concave*. In particular, $\{\varphi \geqslant c\}$ is compact for every $c \in \mathbb{R}$. This is a particular case of strongly 1-concave manifolds in the sense of Andreotti-Grauert [AG62].

If X is a hyper 1-concave manifold, it follows from [And63] that the meromorphic function field $\mathscr{K}(X)$ has transcendence degree over $\mathbb{C}$ less or equal than $\dim X$. If the transcendence degree equals $\dim X$, that is, if there exist $\dim X$ algebraically independent meromorphic functions over $\mathbb{C}$, we say that X is *Moishezon*. We have the following characterization.

Proposition 5.1. *A hyper 1-concave manifold X is Moishezon if and only if X is biholomorphic to an open set of a compact Moishezon space $\widehat{X}$. A sufficient condition for X to be Moishezon is to admit a semipositive line bundle which is positive at one point.*

Proof. By Theorem 1.2 (i), there exist a compact complex space $\widehat{X}$, such that $X \subset \widehat{X}$. Moreover $\widehat{X} \smallsetminus X$ is a pluripolar set. Let us remark that, due to the existence of a Stein neighbourhood of the set $\widehat{X} \smallsetminus X$, all meromorphic functions on X extend uniquely to $\widehat{X}$. Hence $\mathscr{K}(\widehat{X}) = \mathscr{K}(X)$, which implies the first part of the proposition. The second part is the content of [TCM01, Corollary 3.2]. □

We extend now Nadel's main result [Nad90, Theorem 0.1] to dimension 2.

Proposition 5.2. *Let X be a connected manifold of dimension $n \geqslant 2$. Assume that:*

(i) *X is hyper 1-concave.*
(ii) *X is Moishezon.*
(iii) *X can be covered by Zariski-open sets which can be uniformized by Stein manifolds.*

Then X can be compactified by adding finitely many points to a compact Moishezon space.

Proof. By conditions (i) and (iii) and Theorem 1.3, we can find a compact complex space $\widehat{X}$, with at worst isolated singularities, such that X is an open set of $\widehat{X}$ and $\widehat{X} \smallsetminus X$ is finite. Since $\mathscr{K}(\widehat{X}) = \mathscr{K}(X)$, $\widehat{X}$ is itself Moishezon. □

Note that Proposition 5.2 implies that a manifold satisfying (i)-(iii) has finite topological type.

The next result characterizes, along the lines of Kodaira, those non-compact manifolds of dimension $n \geqslant 2$ that can be compactified by adding finitely many points and that admit quasiprojective algebraic structure. It corresponds to [Nad90, Theorem 0.2], where the case $n \geqslant 3$ is considered.

Proposition 5.3. *Let X be a connected manifold of dimension $n \geqslant 2$. The following conditions are necessary and sufficient for X to be a quasiprojective manifold which can be compactified to a Moishezon space by adding finitely many points.*

(i) *X is hyper 1-concave.*
(ii) *X admits a positive line bundle E.*
(iii) *X can be covered by Zariski-open sets which can be uniformized by Stein manifolds.*

Proof. The necessity of conditions (i) and (ii) is obvious, while the necessity of (iii) follows from a theorem of Griffiths [Gri71, Theorem I].

For the sufficiency, we need a variant of the embedding theorem of Andreotti-Tomassini [AT70, Theorem 2, p. 97] (see also [AS70, Theorem 4.1]).

Lemma 5.4. *Let X be a hyper 1-concave manifold and E be a positive line bundle on X. Then X is biholomorphic to an open set of a projective algebraic manifold.*

Proof. If φ denotes the exhaustion function of X, $\sqrt{-1}\big(A\Theta(E)+\partial\bar{\partial}(-\log(-\varphi))\big)$, $A \gg 1$, is a complete Kähler metric on X. Using the L^2 estimates with singular weights for positive line bundles (see Demailly [Dem82]) we obtain that the ring $\oplus_{k>0}H^0(X, E^k \otimes K_X)$ separates points and gives local coordinates everywhere on X. Repeating the proof of [NT88, Lemma 2.1] we obtain the conclusion. □

Let $\widehat{X}$ be a projective compactification given by Lemma 5.4. Then Theorem 1.3 implies that $\widehat{X} \smallsetminus X$ is an analytic set. Proposition 5.3 is proved. □

Remark 5.5. We can obtain a stronger version of [NT88, Theorem 0.1] in dimension 2. Nadel and Tsuji use the following terminology. A manifold X of dimension n is called very strongly $(n-2)$-concave if there exists a $\mathscr{C}^2$ function $\psi : X \longrightarrow \mathbb{R}$ such that $\{\psi > c\} \Subset X$, for all $c \in \mathbb{R}$, and outside a compact set ψ is plurisubharmonic and $\sqrt{-1}\partial\bar{\partial}\psi$ has at least 2 positive eigenvalues. If $\dim X = 2$, this notion coincides with hyperconcavity. Nadel and Tsuji prove the following theorem: a complete Kähler manifold (X, ω), $\dim X = n$, satisfying the conditions

(i) $\mathrm{Ric}(\omega) < 0$,
(ii) X is very strongly $(n-2)$-concave,
(iii) The universal covering of X is Stein,

is biholomorphic to a quasiprojective manifold. By Proposition 5.3 we can, if $\dim X = 2$, remove the requirement that X is complete Kähler and replace (i) with the existence of a positive line bundle.

Let $X = M/\Gamma$ be an irreducible arithmetic quotient of dimension $n \geqslant 2$. The proof of Borel [Bor70] shows that X is very strongly $(n-2)$-concave. Thus, Proposition 5.3 gives, in dimension 2, a generalization of the fact that arithmetic quotients can be compactified, with a completely complex-analytic proof.

Remark 5.6. A compact Moishezon space $\widehat{X}$ with isolated singularities such that $\widehat{X} \smallsetminus X$ is finite needs not be either projective or algebraic in the sense of Weil. See the example of Grauert [Gra62, p. 365–6].

Remark 5.7. We can obtain the following version of Proposition 5.2. Namely, a manifold X, $\dim X = 2$, satisfying the two conditions, (i)' X is hyper 1-concave and

(ii)' X can be covered by Zariski-open sets which can be uniformized by bounded domains of holomorphy in $\mathbb{C}^n$,

can be compactified by adding finitely many points to a compact Moishezon space. Indeed, we show first as in [Nad90, p. 187] that X is Moishezon, using the result of Mok-Yau on the existence of complete Kähler-Einstein metrics on bounded domains of holomorphy, as well as the L^2-estimates for $\bar{\partial}$. Then we resort to Proposition 5.2.

Note that in [Nad90, Theorem 0.3] it is shown that a manifold X as above, with $\dim X > 2$, has moreover the stucture of an abstract algebraic variety.

We come now to the proof of Corollary 1.4. The statement was also noted by Nadel [Nad90], in dimension $\geqslant 3$. Our proof is based on Proposition 5.3, so it gives a complex-analytic proof of the analyticity of the pluripolar set, also for the Siu-Yau theorem.

Proof of Corollary 1.4. Let X be a complete Kähler manifold of finite volume and bounded negative sectional curvature and $\dim X \geqslant 2$. Upon using the same argument as in Siu-Yau [SY82] or [NT88, §3] we show, with the help of the Busemann function, that X is hyper 1-concave. Moreover, X has negative Ricci curvature, hence K_X is a positive line bundle over X. Finally, by Wu's theorem [Wu67], any simply connected complete Kähler manifold of nonpositive sectional curvature is Stein. Hence the universal covering of X is Stein.

We proved that X fulfils the conditions of Proposition 5.3. Therefore, by the conclusion of this Proposition, the proof of Corollary 1.4 is complete. $\square$

We close the section with one more result about embedding of hyper 1-concave manifolds. Generalizing the Andreotti-Tomassini theorem, Andreotti-Siu [AS70, Theorem 7.1] show that a strongly 1–concave manifold X of $\dim X \geqslant 3$ can be embedded in the projective space if it admits a line bundle E such that $\oplus_{k>0} H^0(X, E^k)$ gives local coordinates on a sufficiently large compact then X. The proof is based on techniques of extending analytic sheaves. Moreover, the result breaks down in dimension 2 as the following example shows.

Example 5.8. We use Example 3.9 and its notations. The strongly 1–concave manifold $Y_\varepsilon = K_\varepsilon \smallsetminus \mathrm{pr}_{\mathbb{P}^3}^{-1}(G)$ has as differential double covering the torus minus 16 small balls around the fixed points of the involution. Denote X_ε this new manifold. With the induced complex structure from Y_ε, X_ε is a strongly 1-concave, non-compactifiable manifold. If on X_ε we consider the pull-back E_ε of the hyperplane line bundle on Y_ε, $\oplus_{k>0} H^0(X_\varepsilon, E_\varepsilon^k)$ gives local coordinates everywhere. But X_ε is not embeddable in the projective space, for if it were, we could compactify it by [And63].

We show in the next proposition that if we impose the condition of hyperconcavity such phenomenon cannot occur. Here ϕ and b have the same meaning as in Definition 1.1.

Proposition 5.9. *Let X be a hyper 1-concave manifold of dimension $n \geqslant 2$. Let c be a real number such that $c < b$. Assume there is a line bundle E over $X' = \{\varphi > c\}$ such that the ring $\oplus_{k>0} H^0(X', E^k)$ gives local coordinates on X'. Then X is biholomorphic to an open subset of a projective manifold.*

Proof. By Theorem 1.2, X is an open subset of a variety $\widehat{X}$ with isolated singularities. Moreover $\widehat{X} \smallsetminus X'$ is a Stein space.

Replacing c by a c' such that $c < c' < b$ we can suppose that there are holomorphic sections $s_0, \ldots, s_m$ of $H^0(X', E^k)$ which give local coordinates of X' where k is big enough. We can define a holomorphic map $\pi : X' \longrightarrow \mathbb{P}^m$ by

$$\pi(z) = [s_0(z) : \cdots : s_m(z)].$$

Then π gives a local immersion of X' in $\mathbb{P}^m$. Since $\widehat{X} \smallsetminus X'$ is embeddable in an euclidian space, a theorem of Dolbeault-Henkin-Sarkis [DH97], [Sar99] implies that π can be extended to a meromorphic map from $\widehat{X}$ into $\mathbb{P}^m$.

Denote by Z the set consisting of the singular points of $\widehat{X}$, the points of indeterminacy of π and the critical points of π. Then Z is a compact analytic subset of $\widehat{X} \smallsetminus X'$. Since $\widehat{X} \smallsetminus X'$ is Stein space, Z is a finite set. The map π gives local immersion of $\widehat{X} \smallsetminus Z$ in $\mathbb{P}^m$.

Let H be the canonical line bundle of $\mathbb{P}^m$ and set $L = \pi^*(H)$. Then L is a positive line bundle of $\widehat{X} \smallsetminus Z$. In particular L is positive on $X \smallsetminus Z$ and by a theorem of Shiffman [Shi71] extends to a positive line bundle on X. By Lemma 5.4, X is biholomorphic to an open subset of a projective manifold. $\square$

Aknowledgements. The first-named author thanks Professors Mihnea Colţoiu and Bo Berndtsson for discussions and constant encouragement.

The second-named author would like to thank the Alexander von Humboldt Foundation for its support and Professor Jürgen Leiterer for his hospitality during the preparation of this paper.

References

[AG62] A. Andreotti and H. Grauert, *Théorème de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France **90** (1962), 193–259.

[And63] A. Andreotti, *Théorèmes de dépendance algébrique sur les espaces complexes pseudo-concaves*, Bull. Soc. Math. France **91** (1963), 1–38.

[AS70] A. Andreotti and Y. T. Siu, *Projective embeddings of pseudoconcave spaces*, Ann. Sc. Norm. Sup. Pisa **24** (1970), 231–278.

[AT70] A. Andreotti and G. Tomassini, *Some remarks on pseudoconcave manifolds*, Essays on Topology and Related Topics dedicated to G. de Rham (A. Haeflinger and R. Narasimhan, eds.), Springer-Verlag, 1970, pp. 85–104.

[AV65] A. Andreotti and E. Vesentini, *Carleman estimates for the Laplace-Beltrami equation on complex manifolds*, Inst. Hautes Etudes Sci. Publ. Math. **25** (1965), 81–130.

[Ber92] B. Berndtsson, *A simple proof of an L^2-estimate for $\bar\partial$ on complete Kähler manifolds*, Preprint, 1992.

[Bor70] A. Borel, *Pseudo-concavité et groupes arithmetiques*, Essays on Topology and Related Topics dedicated to G. de Rham (A. Haeflinger and R. Narasimhan, eds.), Springer-Verlag, 1970, pp. 70–84.

[CT02] M. Colţoiu and M. Tibăr, *On Steiness of the universal covering of the complement of a 2 dimensional complex singularity*, Preprint, 2002.

[Dem82] J.-P. Demailly, *Estimations L^2 pour l'opérateur $\bar\partial$ d'un fibré holomorphe semipositif au–dessus d'une variété kählérienne complète*, Ann. Sci. École Norm. Sup. **15** (1982), 457–511.

[DF83] H. Donnelly and C. Fefferman, *L^2–cohomology and index theorem for the Bergman metric*, Ann. Math. **118** (1983), 593–618.

[DH97] P. Dolbeault and G. Henkin, *Chaînes holomorphes de bord donné dans $\mathbb{CP}^n$*, Bull. Soc. Math. de France **118** (1997), 383–445.

[dM75] L. Boutet de Monvel, *Intégration des equations de Cauchy–Riemann induites formelles*, Séminaire Goulaouic–Lions–Schwartz, 1974–75, Exposé No.9.

[EH01] C. L. Epstein and G. M. Henkin, *Can a good manifold come to a bad end ?*, Proc. Steklov Math. Inst., vol. 235, 2001.

[FK72] G. B. Folland and J. J. Kohn, *The Neumann Problem for the Cauchy–Riemann Complex*, Ann. Math. Studies 75, Princeton University Press, New York, 1972.

[GR65] R. C. Gunning and H. Rossi, *Analytic functions of Several Complex Variables*, Prentice Hall, Englewood Cliffs, N. J., 1965.

[Gra62] H. Grauert, *Über Modifikationen und exceptionelle analytische Mengen*, Math. Ann. **146** (1962), 331–368.

[Gra94] H. Grauert, *Theory of q-convexity and q-concavity*, Several Complex Variables VII (H. Grauert, Th. Peternell, and R. Remmert, eds.), Encyclopedia of mathematical sciences, vol. 74, Springer Verlag, 1994.

[Gri66] Ph. A. Griffiths, *The extension problem in complex analysis: Embeddings with positive normal bundle*, Amer. J. Math. **88** (1966), 366–446.

[Gri71] Ph. A. Griffiths, *Complex–analytic properties of certain Zariski open sets on algebraic varieties*, Ann. of Math. **94** (1971), 21–51.

[HL75] R. Harvey and B. Lawson, *On boundaries of complex varieties I*, Ann. of Math.. **102** (1975), 233–290.

[Hör65] L. Hörmander, *L^2-estimates and existence theorem for the $\bar\partial$-operator*, Acta Math. **113** (1965), 89–152.

[Mok89] N. Mok, *Compactification of complete Kähler-Einstein manifolds of finite volume*, Recent developments in geometry, Comtemp. Math., vol. 101, AMS, 1989, pp. 287–301.

[Nad90] A. Nadel, *On complex manifolds which can be compactified by adding finitely many points*, Invent. Math. **101** (1990), no. 1, 173–189.

[NT88] A. Nadel and H. Tsuji, *Compactification of complete Kähler manifolds of negative Ricci curvature*, J. Diff. Geom. **28** (1988), no. 3, 503–512.

[Ros65] H. Rossi, *Attaching analytic spaces to an analytic space along a pseudoconcave boundary*, Proc. Conf. Complex. Manifolds (Minneapolis), Springer–Verlag, New York, 1965, pp. 242–256.

[Sar99] F. Sarkis, *CR-meromorphic extension and the nonembeddability of the Andreotti-Rossi CR structure in the projective space*, Int. J. Math. **10** (1999), 897–915.

[Sar02] F. Sarkis, *Problème de Plateau complexe dans les variétés kählériennes*, Bull. Soc. Math. France **130** (2002), 169–209.

[Shi71] B. Shiffman, *Extension of positive holomorphic line bundles*, Bull. Am. Math. Soc. **77** (1971), 1091–1093.

[Siu96] Y. T. Siu, *The Fujita conjecture and the extension theorem of Ohsawa-Takegoshi*, Geometric Complex Analysis (J. Noguchi et al., ed.), World Scientific Publishing Co., 1996, pp. 577–592.

[SY82] Y. T. Siu and S. T. Yau, *Compactification of negatively curved complete Kähler manifolds of finite volume*, Ann. Math. Stud., vol. 102, Princeton Univ. Press, 1982, pp. 363–380.

[TCM01] R. Todor, I. Chiose, and G. Marinescu, *Morse inequalities for covering manifolds*, Nagoya Math. J. **163** (2001), 145–165.

[Wer58] J. Wermer, *The hull of a curve in* $\mathbb{C}^n$, Ann. of Math. **68** (1958), 45–71.

[Wu67] H. Wu, *Negatively curved Kähler manifolds*, Notices. Amer. Math. Soc. **14** (1967), 515.

Humboldt-Universität zu Berlin, Institut für Mathematik, Unter den Linden 6,10099 Berlin, Germany

E-mail address: george@mathematik.hu-berlin.de

Université Paris-Sud, Bâtiment 425-Mathématique, 91405 Orsay, France

E-mail address: Tiencuong.Dinh@math.u-psud.fr