K-THEORY OF BOUNDARY VALUE PROBLEM

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1 Introduction

Let X be a closed manifold. Suppose it is decomposed into a sum of two manifolds X_+, X_- glued along the common boundary

$$\partial X_+ = \partial X_- = M\,.$$

Let

$$D : C^\infty(X,\xi) \to C^\infty(X,\eta)$$

be an elliptic operator of the first order. We assume that it possesses the unique extension property: if $Df = 0$ and $f_{|M} = 0$ then $f = 0$. In what follows we will consider only elliptic operators of the first order such that D and D^* have the unique extension property. This assumption is made for simpicity of exposition. Later we will describe the general case. Set $\xi_0 = \xi_{|M}$. It is isomorphic to $\eta_{|M}$. The isomorphism is given by the symbol map applied at the unit vector normal to M.

One defines the spaces $H_i(D) \subset L^2(M,\xi_0)$ for $i \in \{+,-\}$, which are the closures of the spaces of boundary values of solutions on the manifolds $X_\pm$ with boundary $\partial X_\pm = M$. The spaces $H_\pm(D)$ are defined to be the closures of :

$$\{f \in C^\infty(M,\xi_0)\ :\ \exists \tilde f \in C^\infty(X_\pm, \xi_{|X_\pm}),\ f = \tilde f_{|M},\ \ D(\tilde f) = 0\ \}$$

in $L^2(M,\xi_0)$. The pair of spaces $H_\pm(D)$ is a Fredholm pair, [Bo1]. There are associated Calderón projectors $P_+(D)$ and $P_-(D)$. For Dirac type operators they are given by an explicit construction, see [BW] §12. In general, Calderón projectors may depend on the parametrix. One can work in L^2-space since Calderón projectors are continuous in the L^2-norm.

On the level of K-theory the operator D defines an element $[D]\in K_0(X)$ via Kasparov's construction. We have Mayer-Vietoris exact sequence

$$\begin{array}{ccccccc}
K_0(M) & \to & K_0(X_+)\oplus K_0(X_-) & \to & K_0(X) & \ni & [D] \\
\uparrow\delta & & & & \downarrow\delta & & \\
K_1(X) & \leftarrow & K_1(X_+)\oplus K_1(X_-) & \leftarrow & K_1(M) & \ni & \delta[D].
\end{array}$$

Problem 1.1 *Can one recover $\delta[D]\in K_1(M)$ from the pair $H_\pm(D)$?*

The answer is positive (modulo torsion for the moment).

To organize somehow the set of possible Cauchy data we introduce two algebraic objects. The first one is a K-theory constructed from *good Fredholm pairs*. The generators are the Fredholm pairs which (roughly speaking) remain to be Fredholm after conjugation with functions. They act naturally on $K_1(M)$. On the other hand we have pairs of geometric origin. We call them *admissible*. They are the pairs of subspaces which are images of projectors which almost commute with an action of the algebra of functions. From such pairs we derive a K-theory as well. We do not know how close are these K-theories to each other. The application $D\mapsto\delta[D]$ factors through both of them.

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2 K-homology and pairings

We sumarize the elements of Kasparov construction of K-homology, [Ka], which we will use later. We will need only a simplified version as presented e.g. in [Co2], pp 287-289.

2.1 Even Fredholm modules

We recall that $K_0(X)$ is generated by so-called *even Fredholm modules.* These are the quadruples

$$(H, F, \rho, \epsilon)\,,$$

where H is a super Hilbert space with grading given by ϵ, F is a Fredholm operator of odd degree, ρ is a representation of an algebra of functions $\mathcal{B} = C(X)$ (or, in general, an arbitrary C^*-algebra). We assume that

- $F^2 \sim 1$,
- $[F, \rho(f)] \sim 0$, for $f \in \mathcal{B}$.

Here "$\sim$"denotes equality up to compact operators.

Let $\Phi : H_0 \to H_1$ be an elliptic pseudodifferential operator on a compact manifold. The Hilbert spaces H_1 and H_2 are approproate completions of $C^\infty(X, \xi)$ and $C^\infty(X, \eta)$. Let Ψ be a parametrix of Φ. Then the quadruple

$$\left(H_0 \oplus H_1, \begin{bmatrix} 0 & \Psi \\ \Phi & 0 \end{bmatrix}, \rho, \epsilon\right)$$

is an even Fredholm module.

2.2 Odd Fredholm modules

The triples

$$(H, F, \rho)\,,$$

where H is without grading, are called *odd Fredholm modules.* We assume the same relations as above. Odd Fredholm modules are the cycles representing the elements of $K_1(X)$.

2.3 Pairing $K^0(X)$ with $K_0(X)$

The generators of K-theory $K^0(X)$ are represented by complex vector bundles. By general reasons there is a perfect pairing

$$K^0(X) \otimes K_0(X) \xrightarrow{\langle\,\cdot\,,\,\cdot\,\rangle} \mathbf{Z}\,.$$

Suppose that there is given an element of $K_0(X)$ represented by an elliptic pseudodifferential operator D and ζ is a bundle. Then

$$\langle[\zeta], [D]\rangle = ind\left(D \otimes \zeta : C^\infty(X, \xi \otimes \zeta) \to C^\infty(X, \eta \otimes \zeta)\right)\,.$$

2.4 Pairing $K^1(X)$ with $K_1(X)$

We have

$$K^1(X) = [X, \Omega BU] = [X, \lim_{n\to\infty} U(n)] = \pi_0 \left(\lim_{n\to\infty} Gl_n(\mathcal{B}) \right) .$$

Suppose $A \in Gl_n(\mathcal{B})$ represents an element in $K^1(X)$ and $\mathcal{F} = (H, F, \rho)$ is an odd Fredholm module. We can assume by [Co1], Appendix 2, p.88, that $F^2 = 1$ strictly. Then F is a symmetry. Let $P = \frac{F+1}{2}$ be the projector on the fixed subspace. Then

$$\langle [A], [\mathcal{F}] \rangle = ind(P^{\oplus n} \circ \widetilde{A} : P(H)^n \to P(H)^n) .$$

Here $\widetilde{A}$ is the authomorphism of H^n induced by A and the representation ρ

$$\widetilde{A}(x_1, \ldots, x_n) = (y_1, \ldots, y_n) , \quad \text{where} \quad y_k = \sum_{i=1}^{n} \rho(A_{ki}) x_i .$$

2.5 Differential in Mayer-Vietoris sequence

Suppose $X = X_+ \cup_M X_-$. The differential in Mayer-Vietoris exact sequence for K-homology is hard to grasp. One of our goals is to describe

$$\delta : K_0(X) \to K_1(M) .$$

On the other hand, the cohomological differential

$$d : K^1(M) \to K^0(X)$$

is easy. Let $A \in Gl_n(C(M)) = Map(M, Gl_n(\mathbf{C}))$ represents an element in $K^1(M)$. Define a vector bundle Θ^X_A on X obtained from the trivial ones on $X_\pm$ glued along M by the transition function given by the matrix A. Then

$$d[A] = [\Theta^X_A] - [\Theta^X_{I_n}] ,$$

where I_n is $n \times n$ identity matrix.

3 Fredholm pairs

Let us first summarize some facts about Fredholm pairs. We will follow [Bo1]-[Bo3]. Suppose that H_+ and H_- are two closed subspaces of a Hilbert space, such that $H_+ + H_-$ is also closed and

- $H_+ \cap H_-$ is of finite dimension,
- $H_+ + H_-$ is of finite codimension.

We assume that both spaces have infinite dimension. Then we define an index of the Fredholm pair

$$Ind(H_\pm) = \dim(H_+ \cap H_-) - \operatorname{codim}(H_+ + H_-)\,.$$

EXAMPLE 3.1 Let D be an elliptic operator on $X = X_+ \cup_M X_-$. Then the pair of boundary value spaces defined in the introduction is a Fredholm pair. Moreover $\ker(D) \simeq H_+(D) \cap H_-(D)$ and by [Bo1] (see [BW], §24 for Dirac type operators)

$$Ind(H_\pm(D)) = ind(D)\,.$$

Here Ind denotes the index of a pair, whereas ind stands for the index of an operator.

Suppose that H is decomposed into a direct sum

$$H = X \oplus Y\,.$$

Let P and Q be the corresponding projectors. Let $\mathcal{I} \subset L(H)$ be the ideal of finite rank operators. Define $Gl(P,\mathcal{I}) \subset Gl(H)$ to be the set of invertible authomorphisms of H commuting with P up to the ideal $\mathcal{I}$. We have the following theorem.

THEOREM 3.2 ([Bo1]) *Let $H_\pm$ be a Fredholm pair with $H_+ = X$. Then there exists $\phi \in Gl(P,\mathcal{I})$, such that $H_- = \phi(Y)$. Moreover the operator $P + \phi Q$ is Fredholm and*

$$ind(P + \phi Q) = Ind(H_\pm)\,.$$

The map

$$\widetilde{ind} : Gl(P,\mathcal{I}) \to \mathbf{Z}$$

$$\widetilde{ind}(\phi) = ind(P + \phi Q)$$

is a group homomorphism.

The ideal $\mathcal{I}$ can be replaced by the ideal of compact operators.

One can show by simple linear algebra argument that

$$ind(P + \phi Q) = ind(Q\phi : Y \to Y) .$$

4 Good Fredholm pairs

Suppose there is given an algebra $\mathcal{B}$ and its representation in a Hilbert space H. For a Fredholm pair $H_\pm$ in H and an invertible matrix $A \in GL(n;\mathcal{B})$ we define new Fredholm pair $A \bowtie H_\pm$ in $H^n = \mathbf{C}^n \otimes H$. We set

$$(A \bowtie H_\pm)_+ = \widetilde{A}(H^n_+) \qquad (A \bowtie H_\pm)_- = H^n_- .$$

Definition 4.1 Let $\mathcal{B}$ be a C^*-algebra. A *good Fredholm pair* is a quadruple

$$(H, H_+, H_-, \rho) ,$$

where H is a Hilbert space, ρ is a representation of $\mathcal{B}$ in H, such that for any invertible matrix $A \in GL(n;\mathcal{B})$ the pair $A \bowtie H_\pm$ is a Fredholm pair. Usually we will drop ρ in the notation.

Denote by $GFP(\mathcal{B})$ the set of good Fredholm pairs divided by the equivalence relation generated by homotopies and stabilization with respect to the direct sum. We also consider as trivial the pairs of the form $(H_1 \oplus 0, 0 \oplus H_2)$, where H_1 and H_2 are two representations of $\mathcal{B}$.

Lemma 4.2 *The semigroup* $(GFP(\mathcal{B}), \oplus)$ *is a group.*

Proof. Let $H_\pm \in GFP(\mathcal{B})$. Then

$$(H_+, H_-) \oplus (H_+^\perp, H_-^\perp) = (H_+ \oplus H_+^\perp, H_- \oplus H_-^\perp) \sim_{htp} (H \oplus 0, 0 \oplus H) .$$

□

For a topological space M set $GFP(M) = GFP(\mathcal{B})$, where $\mathcal{B} = C(M)$ is the algebra of continuous complex-valued bounded functions on M. It is a covariant functor from spaces to abelian groups with involution (interchanging spaces in the pairs).

Example 4.3 $GFP(pt) \simeq \mathbf{Z}$ with the trivial involution.

Exercise 4.4 Compute $GFP(S^0)$.

Proposition 4.5 *For an elliptic operator D the pair $H_\pm(D) \subset L^2(M, \xi_0)$ is a good Fredholm pair.*

Proof. Let $A \in GL(n, C(M)) = Map(M, Gl_n(\mathbf{C}))$. We construct a vector bundle Θ_A^X, see 2.5. Then the pair $A \bowtie H_\pm(D)$ is isomorphic to the pair obtained from the twisted operator

$$D \otimes \Theta_A^X : C^\infty(X, \xi \otimes \Theta_A^X) \to C^\infty(X, \eta \otimes \Theta_A^X)\,,$$

$$A \bowtie H_\pm(D) \simeq H_\pm(D \otimes \Theta_A^X)\,.$$

Thus it it a Fredholm pair by [Bo1]. □

Theorem 4.6 *There exists a map*

$$\alpha : GFP(M) \longrightarrow K_1(M) \otimes \mathbf{Q}$$

such, that for an elliptic operator D

$$\alpha(H_\pm(D)) = \delta[D]\,.$$

Proof. We are going to define the map α by duality. Since

$$K_1(M) \otimes \mathbf{Q} \simeq Hom(K^1(M), \mathbf{Q})$$

it is enough to construct an action of $GFP(M)$ on

$$K^1(M) = \pi_0 \left(\lim_{n \to \infty} Map(M, U(n)) \right)\,.$$

Suppose

$$A : M \to U(n)\,.$$

Then for $H_\pm \in GFP(M)$ set

$$\langle \alpha(H_\pm), [A] \rangle := Ind(A \bowtie H_\pm) - Ind(I_n \bowtie H_\pm) = Ind(A \bowtie H_\pm) - n\, Ind(H_\pm)\,.$$

The value of this pairing does not change after homotopy nor stabilization.

Suppose the pair comes from an elliptic operator D. Then

$$Ind(A \bowtie H_\pm(D)) = ind(D \otimes \Theta^X_A)\,.$$

Thus

$$\begin{aligned}
\langle H_\pm(D), [A] \rangle &= Ind(A \bowtie H_\pm(D)) - Ind(I_n \bowtie H_\pm(D)) \\
&= ind(D \otimes \Theta^X_A) - ind(D \otimes \Theta^X_{I_n}) \\
&= \langle [D], [\Theta^X_A] - [\Theta^X_{I_n}] \rangle \\
&= \langle [D], d[A] \rangle = \langle \delta[D], [A] \rangle\,.
\end{aligned}$$

□

Note that the element $H_\pm(D)$ encodes more information than $\alpha(H_\pm(D))$. One can recover the index of the original operator.

PROBLEM 4.7 *Can one construct a map $\omega : K_0(X) \to GFP(\mathcal{B})$ such, that for a elliptic operator $\omega([D]) = H_\pm(D)$?*

5 Admissible Fredholm pairs.

The following is motivated by the paper of Birman and Solomyak, [BS], who considered only the subspaces which are the images of pseudodifferential projectors.

DEFINITION 5.1 We say that a pair of subspaces $H_\pm$ is an *admissible Fredholm pair* if it is Fredholm and there exist projectors $P_\pm$, such that $H_\pm = im\, P_\pm$ and $P_\pm$ commute with the action of $\mathcal{B}$ up to compact operators. Moreover we assume, that

$$P_+ + P_- \sim 1$$

up to compact operators.

PROPOSITION 5.2 *Each admissible Fredholm pair is a good Fredholm pair.*

PROOF. Set $K = P_+ + P_- - 1$. If $v \in H_+ \cap H_-$, then $K(v) = v$. Since K is a compact operator, $\dim(H_+ \cap H_-) < \infty$. To prove that $H_+ + H_-$ is closed and of finite codimension, note that $im(P_+ + P_-) \subset H_+ + H_-$. Since $P_+ + P_-$ is Fredholm its image is closed and of finite codimension. This way we have shown that $H_\pm$ is a Fredholm pair. Now, if we conjugate $P_+^{\oplus n}$ by $\widetilde{A}$ we obtain again an almost complementary pair of projectors. Thus $A \bowtie H_\pm$ is a Fredholm pair as well.$\square$

As in the case of good Fredholm pairs we construct an associated Grothendieck group. We divide it by homotopies and we consider as trivial the pairs coming from direct sums. We denote the resulting group by $AFP(M)$. The generators are Fredholm pairs with fixed projectors.

Let us recall the construction of the map δ in Mayer-Vietoris sequence. It is a composition of the natural maps:

$$K_0(X) \longrightarrow K_0(X, X_-) \stackrel{\simeq}{\longleftarrow} K_0(X_+, M) \stackrel{\delta}{\longrightarrow} K_1(M)\,.$$

Thus we have a commutative diagram:

$$\begin{array}{ccccc} \{Elliptic\ operators\} & \longrightarrow & K_0(X) & \longrightarrow & K_0(X_+, M) \\ \downarrow & & & & \downarrow\delta \\ AFP(M) & \longrightarrow & GFP(M) & \stackrel{\alpha}{\longrightarrow} & K_1(M) \otimes \mathbf{Q}\,. \end{array}$$

We see that the image of an operator D in the grouop $K_1(M)$ depends only on the behaviour on X_+. In fact we have the following natural transformation:

$$\begin{array}{rccc} \beta_+ : & AFP(M) & \to & K_1(M) \\ & (H, P_\pm) & \mapsto & (H, 2P_+ - 1) \end{array}$$

(we remind that the objects generating $K_1(M)$ are odd Fredholm modules, see 2.2). This procedure is simply forgetting about P_-, and $2P_+ - 1$ is s just the symmetry defined by P_+.

THEOREM 5.3 *The image of an elliptic operator in $K_1(M)$ depends only on $P_+(D)$. Precisely*

$$\langle \delta[D], [A] \rangle = \langle \beta_+(P_\pm(D)), [A] \rangle$$

for $[A] \in K_1(M)$.

PROOF. Let $P = P_+(D)$ be the Calderón projector on $X = H_+(D)$. Set $Q = 1 - P$ and $Y = Q(H)$. By 3.2 there exists $\phi \in Gl(P, \mathcal{K})$ such, that $H_-(D) = \phi(Y)$. Then by [Bo1]

$$\begin{aligned} ind(D \otimes \Theta_A) &= Ind(A \bowtie H_\pm(D)) \\ &= Ind(\widetilde{A}(H_+(D)^n), H_-(D)^n) \end{aligned}$$

$$
\begin{aligned}
&= Ind(\widetilde{A}(X^n), (\phi(Y))^n) \\
&= Ind((\phi^{\oplus n})^{-1} \circ \widetilde{A}(X^n), Y^n) \\
&= \widetilde{ind}((\phi^{\oplus n})^{-1} \circ \widetilde{A}) \\
&= -\widetilde{ind}(\phi^{\oplus n}) + \widetilde{ind}(\widetilde{A})\,.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\langle \delta[D], [A] \rangle &= \langle [D], d[A] \rangle \\
&= ind(D \otimes \Theta_A^X) - ind(D \otimes \Theta_{I_n}^X) \\
&= \widetilde{ind}(\widetilde{A}) - \widetilde{ind}(\widetilde{I}_n) \\
&= \widetilde{ind}(\widetilde{A}) \\
&= ind(P^{\oplus n} \circ \widetilde{A} : X^n \to X^n)\,.
\end{aligned}
$$

This is nothing but the value of the pairing of $[A]$ with the odd Fredholm module

$$
(L^2(M, \xi_0), 2P_+(D) - 1) = \beta_+(P_\pm(D))\,,
$$

as described in 2.4.□

For Dirac type operators the proof of 5.3 can be done more geometricaly. One can perform a surgery as in [BW], §9. We replace X by the double of X_+. Then we use a localization principle [T], 12.1:

$$
\begin{aligned}
ind(D \otimes \Theta_A^X) - ind(D \otimes \Theta_{I_n}^X) &= ind(D \otimes \Theta_A^{2X_+}) - ind(D \otimes \Theta_{I_n}^{2X_+}) \\
&= ind(D \otimes \Theta_A^{2X_+})\,,
\end{aligned}
$$

since the operator $D \otimes \Theta_{I_n}^{2X_+}$ is invertible. The last term is equal to

$$
ind(P^{\oplus n} \circ \widetilde{A} : X^n \to X^n)\,.
$$

Our construction is the same when the roles of P_+ and P_- are exchanged. We obtain:

Corollary 5.4 *There is an equality of indices*

$$
ind(P_+(D)^{\oplus n} \circ \widetilde{A} : H_+ \to H_+) = ind(P_-(D) \circ \widetilde{A}^{-1} : H_- \to H_-)
$$

It is worth to see what kind of information is lost when we forget about $P_-(D)$. Consider the space of admissible Fredholm pairs in $L^2(M,\xi_0)$. This is the set of pairs of the pseudodifferential projectors (P_+,P_-) with symbols (p_+,p_-) such that $p_+ + p_- = 1$. Forgetting about P_- is a fibration with the fiber being the Grassmannian Gr_{p_-}, which is the set of pseudodifferential projectors with symbol $p_- = 1 - p_+$. In [BW], 15.9 there are computed homotopy groups:

$$\pi_{2i}(Gr_p) = \mathbf{Z}\,, \qquad \pi_{2i+1}(Gr_p) = 0\,, \qquad \text{for} \quad i \geq 0\,.$$

It is clear that (counting at the connected components) the only information wich we loose while forgetting P_- is the index of the original Fredholm pair.

The following demands a rigorous proof, but we state it here. One can literally repeat the computation of homotopy groups of Gr_p in the abstract setting. There is a fiber bundle

$$Gr_p \hookrightarrow \{Admissible\ Fredholm\ Pairs\} \xrightarrow{\beta_+} \{Admissible\ Projectors\}\,.$$

It has a cross section:

$$P_+ \mapsto (P_+, 1 - P_-)\,.$$

Thus we obtain:

Corollary 5.5 *There is an isomorphism*

$$AFP(M) \simeq K_1(M) \oplus \mathbf{Z}\,.$$

It is reasonable to study the higher homotopy groups of the set of admissible Fredholm pairs. This task we will undertake in future.

6 Operators without the unique estension property

It may happen that D does not have the unique estension property. This is so for example when X is not connected. Then the Cauchy data $H_\pm(D)$ does not say anything about the index of the operator on the components not intersecting M. There are also known elliptic operators without the unique estension property on connected manifolds. In this case the Bojarski formula for index is modified.

Set

$$a_{\pm} = \dim\{f \in C^{\infty}(X_{\pm}, \xi_{|X_{\pm}}) \ : \ Df = 0\}\,,$$
$$b_{\pm} = \dim\{f \in C^{\infty}(X_{\pm}, (\eta^*)_{|X_{\pm}}) \ : \ D^* f = 0\}\,.$$

Then

$$\begin{aligned} \dim(ker\, D) &= \dim(H_+(D) \cap H_-(D)) + a_+ + a_-\,, \\ \dim(coker\, D) &= \dim(ker\, D^*) \\ &= \dim(H_+(D^*) \cap H_-(D^*)) + b_+ + b_- \\ &= \operatorname{codim}(H_+(D) + H_-(D)) + b_+ + b_-\,. \end{aligned}$$

Thus

$$ind(D) = Ind(H_{\pm}) + a_+ + a_- - b_+ - b_-\,.$$

The twisted operator $D \otimes \Theta^X_A$ for $A \in Gl_n(C(M))$ has index

$$ind(D \otimes \Theta^X_A) = Ind(A \bowtie H_{\pm}(D)) + n(a_+ + a_- - b_+ - b_-)\,.$$

The conclusion of 4.6 remains true:

$$Ind(A \bowtie H_{\pm}(D)) - Ind(I_n \bowtie H_{\pm}(D)) = ind(D \otimes \Theta^X_A) - ind(D \otimes \Theta^X_{I_n})\,.$$

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