FORMALITY OF EQUIVARIANT INTERSECTION COHOMOLOGY OF ALGEBRAIC VARIETIES

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Abstract. We prove that the equivariant intersection cohomology of any complete algebraic variety acted by an algebraic group G is a free module over $H^*(BG)$.

Introduction

Let X be a complex algebraic variety and G be a linear algebraic group acting algebraically on X. Denote by $IH^*_G(X) = H^*(EG \times_G X; \mathcal{IC}^\bullet_G)$ the equivariant intersection cohomology of X with rational coefficients, first introduced in [B]. It is a module over $H^*(BG)$. We say that X is formal if $IH^*_G(X)$ is free. We have a spectral sequence

$$E_2^{pq} = H^p(BG, IH^q(X)) \Rightarrow IH^{p+q}_G(X)\,.$$

Suppose that G is connected. Then BG is 1-connected and the coefficients are not twisted. Then the spectral sequence degenerates if and only if X is formal.

The aim of this note is to show that complete G-varieties are formal. We will prove:

Theorem 1. *Let X be a complete algebraic variety acted by a connected algebraic linear group. Then X is formal.*

In this way we add to the list of [GKM], 14.1 an important class of examples. In fact Theorem 1 is known for specialists. After all, in the most general case, it follows from a simple weight argument, as presented in §3. Nevertheless, we think it is worth to compare several approaches to equivariant intersection cohomology and ways of possible proofs. Our proof of Theorem 1 uses decomposition theorem or reduction to a finite characteristic, which are strong tools. Thus we think that it is worth to note that for projective X the situation is much simpler. We can proceed as follows:

Step 1: We choose an equivariant embedding of X into a projective space. This is possible by a result of Sumihiro [S], Theorem 1.

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Step 2: Let $h \in H^2(X)$ be the class of the hyperplane section. We show that h can be lifted to $H^2_G(X)$. This is so for $X = \mathbb{P}^N$ and $G = Gl(N+1)$ acting naturally. Indeed, take a infinite Grasmanian as a model of BG, then

$$EG \times_G \mathbb{P}^N = \{(V, l) : l \subset V \subset \mathbb{C}^\infty , \dim l = 1 , \dim V = N+1 \} .$$

One has a tautological bundle, its Chern class is a lift of $-h$. If $X \subset \mathbb{P}^N$ and G acts linearly, we construct a lift for $\mathbb{P}^N$ by naturality with respect to G and we restrict it to X.

Step 3: We have a map of spectral sequences

$$L : E_r^{p,q} \to E_r^{p,q+2} ,$$

which is induced by the multiplication by a cycle (or a form) representing h. Let $d = \dim X$. By hard Lefschetz theorem

$$L^k : E_2^{p,d-k} = H^p(BG) \times IH^{d-k}(X) \to E_2^{p,d+k} = H^p(BG) \times IH^{d+k}(X)$$

is an isomorphism for every p and k. From Lefschetz criterion, [D1] it follows that the spectral sequence degenerates, as noticed [Br], 4.2.2.

This argument works for varieties over any field of any characteristic. One should replace the ordinary cohomology and intersection cohomology by their étale version. In the step 3 an isomorphism $IH^*_G(X) \simeq H^*(BG) \otimes IH^*(X)$ can be made canonical by [D3]. It would depend only on the embedding $X \subset \mathbb{P}^N$ and the representation $G \to Gl(k)$.

The step 2 can also be made by methods of symplectic geometry. Let K be a maximal compact subgroup of G. Then $H^*_G(\mathbb{P}^N) \simeq H^*_K(\mathbb{P}^N)$. The class h is represented by a K-equivariant symplectic form ω. One can define a moment map by an explicit formula [K], 2.5. The choice of the moment map is precisely a lift of ω to $H^2_K(\mathbb{P}^N)$ in the Cartan model of $\Omega^*(EK \times_K \mathbb{P}^N)$, see [AB], 6.18. Another proof for smooth projective varieties, can be obtained by means of a Morse function coming from a moment map, see [K], 5.8.

Note that, according to [GKM] 1.6.2(2), if the equivariant intersection cohomology is formal then the (non equivariant) intersection cohomology is trivial as $H_{>0}(G)$-module. This generalizes a result of Deligne [D2] 9.1.7.

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In the rest of the paper we will show how to deduce formality for an arbitrary complete variety, not necessarily complete.

1. Algebraic models of $EG \times_G X$

There are several constructions of BG that can be carried on in the category of algebraic varieties. Of course BG itself cannot be an algebraic variety, since it cannot be of a finite dimension. We can only approximate BG. First recall the construction of B. Totaro, [T].

a. The construction of B. Totaro.

Suppose V is a representation of G and $S \subset V$ is a G-invariant subset such that G acts freely on $V \setminus S$ and the geometric quotient $(V \setminus S)/G$ exists. Then $(V \setminus S)/G$ is a homotopy approximation of BG up to dimension $2\operatorname{codim} S - 1$. This means that there exists a map $(V \setminus S)/G \to BG$, which induces an isomorphism on homotopy groups in the degrees $< 2\operatorname{codim} S - 1$ and surjection in the degree $2\operatorname{codim} S - 1$.

b. The classical model.

Totaro's models include the classical one. Suppose $G \subset Gl(k)$. Let $V = Hom(\mathbb{C}^k, \mathbb{C}^{n+k})$. Let $V \setminus S$ be the set of the maps of maximal rank. This is the Stiefel variety. The group G acts on it via the representation in $Gl(k)$. The Stiefel variety can be presented as:

$$V(k,n) = Gl(n+k)/H\,, \qquad H = \begin{pmatrix} 1 & * \\ 0 & Gl(n) \end{pmatrix}.$$

Then

$$B_nG = Gl(n+k)/K\,, \qquad K = \begin{pmatrix} G & * \\ 0 & Gl(n) \end{pmatrix}.$$

It is an approximation of BG up to degree $2n+1$.

c. Induced $Gl(n+k)$-space.

Another approximation, which might be perhaps more useful in applications, is

$$E'_nG = Gl(n+k)/H' \qquad H' = \begin{pmatrix} I & 0 \\ 0 & Gl(n) \end{pmatrix} \simeq Gl(n)\,.$$

It fibres over E_nG with the fibre A^{kn}. Then

$$B'_nG = Gl(n+k)/K'\,, \qquad K' = \begin{pmatrix} G & 0 \\ 0 & Gl(n) \end{pmatrix} \simeq G \times Gl(n)\,.$$

Now we construct the associated bundle. For example, using the model 'c.' we have $E'_nG \times_G X = Gl(n+k) \times_{K'} X$ (where $Gl(n)$ acts trivially on X). But even here, with such a simple construction the resulting space does not have to be a variety. The spaces $E'_nG \times_G X$ (approximations of the Borel construction) are quotients of $E'_nG \times X$ by G and it can be shown, that they are algebraic spaces. Let us quote [B-B]:

Theorem 2. *Let G be an algebraic group. Let E and X be algebraic spaces and let E be a principal locally isotrivial G-fibration. Assume that E is normal. Then $E \times_G X$ exists in the category of algebraic spaces. If, moreover E is an algebraic variety, X normal and can be covered by G-invariant open quasi-projective subsets, then $E \times_G X$ is an algebraic variety.*

Note that, by [B-B] Theorem 3, if H is a connected affine algebraic group, $G \subset H$ and X normal, then $H \times_G X$ is an algebraic variety if and only if the assumption about covers is fulfilled. One would like to apply decomposition theorem (or other methods based on Weil conjectures). It is clear that either we have to make some assumptions about X, or to generalize the results of [BBD] for algebraic spaces.

d. Simplicial model.

In order to avoid algebraic spaces it is possible to apply a simplicial construction of BG, see [D2] 6.1. Then $(BG)_\bullet$ is a simplicial variety: $(BG)_n = G^n$. The associated Borel construction is also a simplicial variety: $(EG \times_G X)_n = G^n \times X$.

Remark 3. Applying any of the construction 'a.–d.'we obtain a Hodge structure on $H^*(BG)$ which is pure, see [D2] 9.1. If we perform the same constructions for varieties over a finite field, we also obtain a pure structure with respect to the action of the Frobenius authomorphism.

2. Degeneration of the spectral sequence without assumption of projectivity

Suppose X is a complete algebraic variety. We replace X by its normalization. It does not change the intersection cohomology. Assume that X can be covered by G-invariant open quasi-projective subsets. Then the space $E_nG \times_G X$ (or $E'_nG \times_G X$) is an algebraic variety by Theorem 2. The projection

$$E_nG \times_G X \to B_nG$$

is a proper algebraic morphism.

Corollary 4. *If the normalization of X can be covered by G-invariant quasiprojective open subsets, then the spectral sequence converging to $IH^*(E_nG \times_G X)$ degenerates.*

Proof. The Corollary follows from decomposition theorem of [BBD]. □

Remark 5. We would be able to remove the assumption about covers, provided that we proved the decomposition theorem for separable algebraic spaces. I was assured by Prof. Deligne that it is possible; the statement of [BBD] 6.2.5 remains true. To prove it one would have to develop the formalism of mixed sheaves for algebraic spaces over finite fields.

3. The end of the proof

If X can be covered by G-invariant quasiprojective open subsets. Then $EG \times_G X$ can be approximated by algebraic varieties. The formality follows from Corollary 4. For arbitrary complete X we would rather apply the simplicial construction. We reduce the base ring to a finite field. Now we follow Deligne's suggestion: then the Galois group acts on the spectral sequence

$$E_2^{pq} = H^p(BG) \otimes IH^q(X) \Rightarrow IH_G^{p+q}(X)\,.$$

Since $H^*(BG)$ and $IH^*(X)$ are pure, E_2^{pq} is pure of weight $p+q$. Thus all the differentials in the spectral sequence vanish because they mix weights. □

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