On completeness of eigenfunctions of an elliptic operator on a manifold with conical points

Yu. Egorov, V. Kondratiev, and B.–W. Schulze*

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Contents

1 Introduction

This paper is aimed at proving the completeness of the system of eigen- and associated functions (i.e., root functions) of an elliptic operator on a manifold with a finite number of conical points.

If the manifold is smooth, the completeness of the system of root functions for a general elliptic system with boundary conditions of Lopatinsky type was proved by S. Agmon in [1].

Earlier for the Dirichlet boundary value problem for an elliptic operator of order m having a real principal part the completeness of the system of root functions has been established by F. Browder [6]–[8].

Essential progress was achieved in the work [13] by M.V. Keldysh, who considered non–self-adjoint operators. He studied elliptic operators of second order in a smooth domain with Dirichlet boundary conditions. The methods of this work have been used later on by many mathematicians.

In the article [3] by M.S. Agranovich the completeness of the system of root functions for a general elliptic system with boundary conditions of Lopatinsky type has been proved under much weaker conditions on smoothness of the principal coefficients and the domain than in [1].

The case of a non–smooth domain has been studied considerably less. N.M. Krukovsky in [15] proved the completeness of the system of root functions for an elliptic operator of second order in an arbitrary bounded domain with Dirichlet

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boundary conditions and in a Lipschitz domain with Neumann boundary conditions. The case of sectorial operators was investigated by M.S. Agranovich in [3], this class is wider than the class of strongly elliptic operators, but depends on certain properties of the characteristic polynomial.

In the present paper we consider elliptic differential operators on a manifold with a finite number of conical points. B.-W. Schulze in [18] proved the existence and smoothness theorems. The corresponding estimates play an essential role in the present paper.

2 Definitions

Our main definition is as follows. A *manifold B with conical points* is a topological space with a finite subset of points

$$B_0 = \{b_1, \dots, b_M\} \subset B$$

such that the following properties hold:

a) the manifold $B \setminus B_0$ is C^∞-smooth;

b) each point $b \in B_0$ has an open neighborhood U in B such that there exists a diffeomorphism

$$\Phi : U \setminus \{b\} \to X^* = \mathbb{R}^+ \times X,$$

where $X = X(b)$ is a closed compact C^∞-manifold, and Φ is extendible to a homeomorphism

$$\overline{\Phi} : U \to X^\triangle = (\overline{\mathbb{R}}^+ \times X)/(\{0\} \times X).$$

If $\Psi : U \setminus \{b\} \to X^*$ is another diffeomorphism extendible to a corresponding homeomorphism to U, then Φ and Ψ are equivalent if $\Phi\Psi^{-1} : X^* \to X^*$ is the restriction of a diffeomorphism

$$\overline{\mathbb{R}}^+ \times X \to \overline{\mathbb{R}}^+ \times X.$$

B is assumed to be paracompact; for simplicity we consider the case that B_0 contains only one point b_0, $\dim B_0 \setminus b_0 = n, \dim X = n - 1$.

Now we pass to differential operators on a manifold B with a conical point b_0. An operator $A \in \mathrm{Diff}^m(B \setminus b_0)$ is said to be *an operator of Fuchs type*, if there exists a neighborhood U of b_0 such that the operator A in local coordinates $(r, x) \in X^*$ has the form

$$A = r^{-m} \sum_{k=0}^{m} a_k(r) \left(-r \frac{\partial}{\partial r} \right)^k$$

with coefficients $a_k \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{m-k}(X))$. Here, $\mathrm{Diff}^m(X)$ is the space of all differential operators of order m with smooth coefficients defined on X. In local coordinates $x \in X$ we have

$$a_k(r) = \sum_{|\beta| \le m-k} b_\beta(r, x) D_x^\beta,$$

where $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$, $|\beta| = \beta_1 + \cdots + \beta_{n-1}$, $D^\beta = \partial^{|\beta|}/\partial x_1^{\beta_1} \ldots \partial x_{n-1}^{\beta_{n-1}}$.

Let us give another definition of a manifold with a conical point equivalent to the one before.

Let ω be a set on the unit sphere in $\mathbb{R}^n$. Set

$$K^{\triangle}(\omega) = \{x : x \in \mathbb{R}^n, 0 \leq |x| < \infty, \frac{x}{|x|} \in \omega \ \text{ if } \quad x \neq 0\}.$$

Moreover, let

$$K^*(\omega) = \{x : x \in \mathbb{R}^n, 0 < |x| < \infty, \frac{x}{|x|} \in \omega\}$$

which is a cone in $\mathbb{R}^n$ with base ω.

A *manifold B with a conical point* $b_0 \in B$ is a topological space such that

a) the manifold $B \setminus B_0$ is smooth and has dimension n;

b) there exists a neighborhood U of b_0 in B such that the set $U \setminus b_0$ can be covered by a finite number of open sets $U_i, i = 1, \ldots, L$, such that $U \subset \bigcup U_i$, and there exist diffeomorphisms

$$\Phi_i : U_i \to K^*(\omega_i)$$

that are extendible to homeomorphisms

$$\overline{\Phi}_i : U_i \cup b_0 \to K^{\triangle}(\omega_i).$$

The covering defines a system of local coordinates in a neighborhood of the point b_0. In fact, if $x \in U$ then $x \in U_i$ for some i (may be for several i), $\Phi_i(x) \in K^*(\omega)$ and one can consider (r, y), where $y = \Phi_i(x)/|\Phi_i(x)|$, as local coordinates. If U' is another neighborhood of x and $\overline{U'} \subset U$, then the set $S = \overline{U'} \setminus U' = \partial U'$ is a smooth manifold and $\bigcup_i U_i \cap S$ forms an atlas on S.

Let us introduce *weighted Sobolev spaces* $W^s_\alpha(B)$, where $s \geq 0$ is an integer, $\alpha \in \mathbb{R}$. A distribution u belongs to $W^s_\alpha(B)$, if

(i) $u \in H^s_{loc}(B \setminus b_0)$,

(ii) the function $u(y) = u_i(\Phi_i^{-1} y)$, where $y \in K_i^*$ and i is such that $x \in U_i$, satisfies the condition

$$\int_{K^*(\omega)} \sum_{|\beta| \leq s} r^{2|\beta| - 2\alpha - 2s} |D^\beta u_i|^2 dy < \infty.$$

Note that $W^s_\alpha(B)$ is a Banach space. It is easy to see that an operator A of Fuchs type of order m defines a bounded operator $W^s_\alpha(B) \to W^{s-m}_\alpha(B)$.

An operator A of Fuchs type is said to be *elliptic of weight* α if

(i) it is elliptic on $B \setminus b_0$,

(ii) the operator

$$\sigma(A) = \sum_{k=0}^{m} a_k(0) z^k$$

is an isomorphism $H^{2m}(X) \to H^0(X)$ for all $z \in \Gamma_{n/2+\alpha/2-2m}$, where Γ_γ is the line $\operatorname{Re} z = \gamma$ in the complex plane.

As is shown in [18], an operator $A : W^{2m}_\alpha(B) \to W^0_\alpha(B)$ that is elliptic of weight α is a Fredholm operator.

Let us define unbounded operators $\mathcal{A} : L_2(B) \to L_2(B),\ L_2(B) = W^0_0(B)$, related to an elliptic operator A of Fuchs type. Suppose that the domain $D_\mathcal{A}$ of $\mathcal{A}$ equals $W^{2m}_\alpha(B)$ for some α, $0 \le \alpha < 2m$, and set $\mathcal{A}u = Au$ for $u \in D_\mathcal{A}$.

Moreover, assume that A is elliptic of weight α. Then $\mathcal{A}$ is a closed operator $L_2(B) \to L_2(B)$ with dense domain in $L_2(B)$. The kernel of $\mathcal{A}$ has a finite dimension, its image is closed. If its spectrum does not coincide with the whole complex plane, i.e., the resolvent

$$R(\lambda, \mathcal{A}) = (\lambda I - \mathcal{A})^{-1} : L_2(B) \to L_2(B)$$

is defined for some complex λ_0, then $R(\lambda_0, \mathcal{A})$ is compact, and $R(\lambda, \mathcal{A})$ is defined for all λ, except for a discrete set of points $\{\lambda_k\}$, which are the eigenvalues of $\mathcal{A}$.

3 Rays of minimal growth

A ray $\arg\lambda = \theta$ is said to be *a ray of minimal growth* of $R(\lambda, \mathcal{A})$, if the resolvent exists on this ray for all λ of sufficiently large absolute value, and for these λ we have

$$\|R(\lambda, \mathcal{A})\|_{L_2(B)\to L_2(B)} \le C|\lambda|^{\delta-1}$$

for some δ, $0 \le \delta < 1$.

In order to state whether a ray $\arg\lambda = \theta$ is a ray of minimal growth one needs to study Fuchs type operators depending on a parameter in more detail cf. Agmon's work [1] for the case of differential operators in a smooth domain. Concerning a machinery for conical singularities, cf. Gil [12].

Let $B' = B \setminus S^1$, where S^1 is a circle on the unit sphere in $\mathbb{R}^n$. Then B' is a manifold with edge $b_0 \times S^1$. Consider on B' the operator

$$\mathcal{L} = \mathcal{A} - (-1)^m e^{i\theta} \frac{\partial^m}{\partial t^m}.$$

Suppose that $\mathcal{L}$ is a Fredholm operator

$$W^{2m}_\alpha(B') \to W^0_\alpha(B').$$

The space $W^s_\alpha(B')$ consists of the functions u such that

$$\|u\|^2_{s,\alpha} = \int_0^{2\pi} \|u(\cdot,t)\|^2_{W^s_\alpha(B')} dt + \int_0^{2\pi} \left\| \frac{\partial^s u(\cdot,t)}{\partial t^s} \right\|^2_{W^0_\alpha(B')} dt < \infty.$$

The ellipticity implies the estimate

$$\|v\|^2_{2m,\alpha} \le C[\|\mathcal{L}v\|^2_{0,\alpha} + \|v\|^2_{L_2(B')}]. \tag{1}$$

Setting $v = e^{ikt}u$, we get

$$\mathcal{L}v = e^{ikt}(\mathcal{A}u - (-1)^m e^{i\theta} k^m u),$$

and estimate (1) gives us

$$\|ue^{ikt}\|^2_{2m,\alpha} \le C_1\|\mathcal{A}u - (-1)^m e^{i\theta} k^m u\|^2_{2m,\alpha} + C_2\|u\|^2_{L_2(B')}. \tag{2}$$

On the other hand, we have

$$\begin{aligned} C_3\|ue^{ikt}\|^2_{2m,\alpha} &\ge \int\limits_0^{2\pi} \left\| \frac{\partial^{2m} v(\cdot,t)}{\partial t^{2m}} \right\|^2_{W^0_\alpha(B')} dt + \int\limits_0^{2\pi} \|v\|^2_{W^0_\alpha(B')} dt \\ \ge \int\limits_{r<\rho} k^{2m} r^\alpha |u|^2 r^{n-1} dr dx &+ \int\limits_{r<\rho} r^\alpha |u|^2 r^{n-1} dr dx + \int\limits_{B\setminus U} k^{2m} u^2 ds + \int\limits_{B\setminus U} u^2 ds, \end{aligned} \tag{3}$$

where U is a neighborhood of the conical point, (r, x) are local coordinates, and ds is a fixed Riemannian metric on $B \setminus U$.

Inequalities (2) and (3) imply that

$$C_4\|\mathcal{A}u\|^2_{m,\alpha} + C_5\|u\|^2_{L_2(B)} \ge k^{2m-\alpha}\|u\|^2_{L_2(B)}.$$

Since

$$\lambda = (-1)^m e^{i\theta} k^m, \quad |\lambda| = |k|^m, \quad \arg\lambda = \theta - m\pi,$$

the latter inequality means that

$$\|R(\lambda, \mathcal{A})\| \le C_5|\lambda|^{\frac{\alpha}{2m}-1}. \tag{4}$$

In the works [18], [19] by B.-W. Schulze there are the conditions for the operator $\mathcal{L}$ to be Fredholm. They include condition (i)

$$\arg \sum_{|\beta|=2m} a_\beta(x)\xi^\beta \ne 0$$

and condition (ii) of ellipticity on the edge. It can be stated as follows. Consider the operator

$$\mathcal{A}_0 = r^{-m} \sum_{k=0}^m a_k(0) \left(-r\frac{\partial}{\partial r} \right)^k - e^{i\theta}$$

on X^*. It is easy to see that this is a Fredholm operator from $W^{2m}_\alpha(X^*) \cap W^0_\alpha(X^*)$ to $W^0_\alpha(X^*)$.

The definition of the spaces $W^s_\alpha(X^*)$ is a natural generalization of the definition of $W^s_\alpha(B)$ given above. The set X^* can be covered by a finite set of neighborhoods each of which is homeomorphic to an unbounded cone in the Euclidian space, and we have $u \in W^s_\alpha(X^*)$ if condition (ii) from the definition of $W^s_\alpha(B)$ is fulfilled.

The ellipticity condition for the operator $\mathcal{L}$ means that $\mathcal{A}_0$ is an isomorphism between $W^{2m}_\alpha(X^*) \cap W^0_\alpha(X^*)$ and $W^0_\alpha(X^*)$. This condition can be considered as the Lopatinsky condition on the edge. Such a condition for elliptic boundary value problems was introduced in the article [16] by V.G. Mazya and B.A. Plamenevsky.

Estimate (4) implies, in particular, that the map $u \mapsto (\mathcal{A} - \lambda)u$ is bijective for all λ with sufficiently large absolute value and argument equal to $\theta - \pi m$. In order to show that the ray $\arg \lambda = \theta - \pi m$ is a ray of minimal growth, it suffices to prove that the image of $\mathcal{A} - \lambda I$ coincides with $L_2(B)$. This rather long proof is based on the methods of the proof of invertibility of elliptic differential operators on a manifold with conical singularities, cf. Remark 2.4.50 in [19].

4 Completeness of root functions

Now we can state our main result, the theorem on the completeness of the system of root functions of an elliptic operator on a manifold with a conical point.

Theorem 1 *Let $\mathcal{A}$ be an operator of Fuchs type, elliptic of weight α, $0 \le \alpha < 2m$. Suppose that there exist rays $\arg \lambda = \theta_i$, $i = 1, \dots, N$ in the complex plane that are rays of minimal growth for the resolvent of $\mathcal{A}$, where the angles between pairs of neighbouring rays are $\le \pi(2m - \alpha)/2n$. Then the spectrum of the operator is discrete, and the root functions form a complete system in $L_2(B)$.*

Proof. Let z_0 be a point not belonging to the spectrum of $\mathcal{A}$. Set $T = (\mathcal{A} - z_0 I)^{-1}$. We can suppose that $z_0 = 0$. Let $R(\lambda, T) = (T - \lambda I)^{-1}$. Theorem 1 then easily follows from the following lemma.

Lemma 1 *If $\mathcal{A}$ is an operator of Fuchs type, elliptic of weight α, there exists a sequence of circles $|\lambda| = \rho_i$ such that $\rho_i \to 0$ and*

$$\|R(\lambda, T)\|_{L_2(B) \to L_2(B)} \le \exp(|\lambda|^{-n/\gamma - \varepsilon}), \ |\lambda| = \rho_i,$$

where $\gamma = (\alpha - 2m)/2$.

Lemma 1 will be verified below. Now we show how Theorem 1 follows from it.

In fact, let Lemma 1 be true and assume that there exists a function $f^* \in L_2(B)$, orthogonal to all eigen- and associated functions of the operator $\mathcal{A}$. We shall show that $f^* = 0$. This will imply the completeness of the system of root functions. Consider the function

$$F(\lambda) = (f^*, R(\frac{1}{\lambda}, T)f),$$

where $f \in L_2(B)$, $(\cdot, \cdot)$ is the scalar product in $L_2(B)$.

Since the resolvent of $\mathcal{A}$ is a meromorphic function with poles at the points of the spectrum of $\mathcal{A}$, the function F is analytic for λ outside the eigenvalues of $\mathcal{A}$. This follows from the expansion

$$R(\lambda, T)f = \frac{\Phi_1}{(\lambda - \lambda_k)^j} + \frac{\Phi_2}{(\lambda - \lambda_k)^{j-1}} + \cdots + \frac{\Phi_j}{\lambda - \lambda_k} + \sum_{i=0}^{\infty} g_i (\lambda - \lambda_k)^i$$

in a neighborhood of the point $\lambda = \lambda_k$, λ_k is a pole of R. Here $j \ge 1, \Phi_i \ne 0, \Phi_i \in L_2(B), g_i \in L_2(B), \Phi_1, \Phi_2, \dots, \Phi_j$ is a chain of the associated functions. This expansion implies that λ_k is a regular point of $F(\lambda)$, since f^* is orthogonal to all Φ_i. Therefore, $F(\lambda)$ is an entire function.

Lemma 1 implies that

$$|F(\lambda)| \le \exp(|\lambda|^{n/\gamma+\varepsilon}), \quad \gamma = (\alpha - 2m)/2$$

for $|\lambda| = r_i,\ r_i \to \infty$. Consider $F(\lambda)$ for λ in the closure of an angle between the rays $\arg\lambda = \theta_j$ and $\arg\lambda = \theta_{j+1}$. Its size is less than $\pi\gamma/n$. Since

$$R(\lambda, T) = \lambda I - \lambda^2 R(\lambda, \mathcal{A})$$

and the ray $\arg\lambda = \theta_i$ is a ray of minimal growth, we have

$$|F(\lambda)| = O(|\lambda|^{2-\kappa}),\ \kappa > 0,\ \lambda \to \infty,\ \arg\lambda = \theta_i.$$

Applying the Phragmen-Lindelöf theorem we obtain that $|F(\lambda)| = O(|\lambda|^{2-\kappa})$ as $|\lambda| \to \infty$ in the whole complex plane. Therefore, $F(\lambda)$ is a linear function, i.e.

$$F(\lambda) = c_0 + c_1\lambda.$$

On the other hand, we have

$$R(1/\lambda, T) = \lambda I + \lambda^2 + \dots,$$

and therefore,

$$F(\lambda) = \lambda(f^*, f) + \lambda^2(f^*, Tf) + \dots.$$

Since F is linear, we have $(f^*, Tf) = 0$ for all $f \in L_2(B)$. Since the range of the operator T is dense in $L_2(B)$, we have $f^* = 0$. Thus, the system of root functions of the operator $\mathcal{A}$ is complete in $L_2(B)$.

Definition 1 *An operator T belongs to the class C_p, $0 < p < \infty$, if*

$$\sum_i |\mu_i(T)|^p < \infty,$$

where $\mu_i(T)$ are the eigenvalues of the operator $(T^*T)^{1/2}$.

Lemma 1 follows from the following result:

Lemma 2 (see [10]) *If T is a compact operator in a Hilbert space belonging to the class C_p, $0 < p < \infty$, then there exists a sequence ρ_i, $\rho_i \to 0$ such that*

$$\|R(\lambda, T)\| \le \exp(c|\lambda|^{-p})$$

for $|\lambda| = \rho_i$.

It remains to prove that $T \in C_p$ with $p > n/\gamma$. To this end we use the following result of Agmon.

Let Q be a cube in $\mathbb{R}^n$,

$$Q = \{x \in \mathbb{R}^n : |x_i| < \pi, i = 1, \dots, n\}.$$

Represent $u \in L_2(Q)$ in the form

$$u(x) = \sum_{k_1,\dots,k_n} a_{k_1,\dots,k_n} e^{i(k_1x_1+\dots+k_nx_n)}.$$

Let H_r be the space of functions u with a finite norm

$$\|u\|_r^2 = |a_0|^2 + \sum_{k\neq 0} |k|^{2r}|a_k|^2, \quad r > 0.$$

Let H_r^N be the space of N-dimensional vectors, each component of which belongs to H_r.

Lemma 3 *Let T be a compact operator in H_r^N such that $TH_r^N \subset H_{r+s}^N$. Then there exists a sequence ρ_i, $\rho_i \to 0$ such that*

$$\|R(\lambda, T)\| \leq \exp(c|\lambda|^{-n/s-\varepsilon})$$

for $|\lambda| = \rho_i$.

This follows from Theorem A1.1', p. 137 in [1].

We shall reduce the spectral problem on a conical manifold to a spectral problem for an operator on H_r^N. Let

$$B = \Big(\bigcup_{i=1}^{l} \Omega_i\Big) \cup \Big(\bigcup_{i=1}^{L} \Omega_i'\Big),$$

where $b_0 \notin \Omega_i$, each Ω_i is homeomorphic to a cube, i.e. $\{\Omega_i\}$ form an atlas on $B \setminus U_0$, where U_0 is a neighborhood of b_0, which is covered by L sets Ω_i', each of them being homeomorphic to a cone K_i in $\mathbb{R}^n$.

Set

$$P_i u = u(\Phi_i^{-1} y),$$

where Φ_i is a diffeomorphism of Ω_i on the cube Q_i. We can extend $P_i(u)$ to the whole space as a periodic function. Let us suppose for simplicity that the period of this function is 2π in each variable y_i.

Consider the function

$$P_i' u = u((\Phi_i')^{-1} y),$$

where Φ_i' is a diffeomorphism of Ω_i' on the cone K_i. Let us take a partition of unity in K_i such that

$$1 \equiv \sum \theta_j(t),\ \theta_j \in C_0^\infty(\mathbb{R}_+^n),\ \operatorname{supp}\theta_j \subset K_{i,j},\ |D^\alpha \theta_j| \leq C_\alpha 2^{j|\alpha|},$$

where

$$K_{i,j} = \{y : y \in K_i,\ 2^{-j-2} < |y| < 2^{-j-1}\}.$$

Set

$$u_{i,j} = \theta_j P_i' u$$

in $K_{i,j}$. It is easy to see that for $|\beta| \leq 2m$

$$\int_{K_{i,j}} \Big|\frac{\partial^\beta u_{i,j}}{\partial y^\beta}\Big|^2 r^{2|\beta|-4m+\alpha} dx \leq C\|u_{i,j}\|^2_{W_\alpha^{2m}(K_{i,j})}.$$

Without loss of generality we can assume that the base of the cone is a domain with smooth boundary and $u_{i,j}$ can be extended to the domain

$$S_j = \{y : 2^{-j-2} < |y| < 2^{-j-1}\}$$

so that it vanishes for $|y| \geq 2^{-j-1}$ and for $|y| \leq 2^{-j-2}$. The interpolation inequality implies that

$$\varepsilon^{2s}\|u_{i,j}\|^2_{H^s(\mathbb{R}^n)} \leq C_1 \varepsilon^{4m} \int_{K_{i,j}} \sum_{|\beta|=2m} |D^\beta u_{i,j}|^2 dy + C_2 \int_{K_{i,j}} |u_{i,j}|^2 dy$$

or

$$\varepsilon^{2s-4m+\alpha}\|u_{i,j}\|^2_{H^s(\mathbb{R}^n)} \le C_3\varepsilon^\alpha \int\limits_{S_j} \sum_{|\beta|=2m} |D^\beta u_{i,j}|^2 dy + C_4\varepsilon^{\alpha-4m}\int\limits_{S_j} |u_{i,j}|^2 dy.$$

Setting $\varepsilon = 2^{-j}$ we get

$$2^{js_1}\|u_{i,j}\|^2_{H^s(\mathbb{R}^n)} \le C_3 \int\limits_{S_j} \sum_{|\beta|=2m} |y|^\alpha |D^\beta u_{i,j}|^2 dy + C_4 \int\limits_{S_j} |y|^{\alpha-4m}|u_{i,j}|^2 dy,$$

where $s_1 = -\alpha + 4m - 2s$. We have

$$\begin{aligned} \|P_i' u\|_{H^s(\mathbb{R}^n)} &\le \sum_{j=1}^\infty \|u_{i,j}\|_{H^s(\mathbb{R}^n)} \le \Big(\sum_{j=1}^\infty \|u_{i,j}\|^2_{H^s(\mathbb{R}^n)} 2^{js_1}\Big)^{1/2} \Big(\sum_{j=1}^\infty 2^{-js_1}\Big)^{1/2} \\ &\le C_5\Big[\sum_{j=1}^\infty \int\limits_{S_j} \sum_{|\beta|=2m} |y|^\alpha |D^\beta u_{i,j}|^2 dy + \int\limits_{S_j} |y|^{\alpha-4m}|u_{i,j}|^2 dy\Big]^{1/2} \\ &\le C_6\|u\|^2_{W^{2m}_\alpha}. \end{aligned}$$

The function $P_i' u$ can also be extended to the whole space as a periodic function, since it vanishes outside the unit sphere, and we obtain a map $u \mapsto \{\bigcup_i P_i u, \bigcup_i P_i' u\}$. It defines the operator $J : W^{2m}_\alpha(B) \to H^N_s$, $N = l + L$.

Let T be a bounded operator $L_2(B) \to W^{2m}_\alpha(B)$. Using the above constructed operator J we can define an operator T^+ in H^N_0 as follows. Let $f = (f_1, \dots, f_N) \in H^N_0$. To each function f_i there corresponds the function $\tau f_i \in L_2(B)$, $\operatorname{supp}(\tau f_i) \subset U_i$ or U_i'. We have $T\tau f_i \in W^m_\alpha(B)$ and $JT\tau f_i \in H^N_s$. Set $T^+ = JT\tau$. It is clear that T^+ is a compact operator from H^N_0 to H^N_0. It is evident also that if $\lambda \ne 0$ is a point of the spectrum of T, then it belongs to the spectrum of T^+. We have $T\tau = \tau T^+$.

It is easy to see that

$$\|R(\lambda, T)\|_{L_2(B)\to L_2(B)} \le C\|R(\lambda, T^+)\|_{H^N_0 \to H^N_0},$$

if λ is a regular point of the operator T^+. Indeed, let $\lambda \ne 0$ be such a point. If $u \in H^N_0$, then

$$(T - \lambda I)\tau(T^+ - \lambda I)^{-1} u = \tau(T^+ - \lambda I)(T^+ - \lambda I)^{-1} u = \tau u,$$

whence

$$\tau R(\lambda, T^+)u = R(\lambda, T)\tau u$$

or

$$R(\lambda, T)u = \tau R(\lambda, T^+)Ju.$$

Therefore,

$$\|R(\lambda, T)u\| \le \|R(\lambda, T^+)\|\|u\|.$$

Lemma 3 implies that there exists a sequence $\rho_i > 0$ such that $\rho_i \to 0$ and

$$\|R(\lambda, T)\| \le \exp(|\lambda|^{-n/\gamma-\varepsilon}) \quad \text{if} \quad |\lambda| = \rho_i.$$

This completes the proof of Lemma 1 and Theorem 1 follows. $\square$

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