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Non-Commutative Simplicial Complexes and the Baum-Connes Conjecture

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Abstract

We associate a non-commutative C^*-algebra with any locally finite simplicial complex. We determine the K-theory of these algebras and show that they can be used to obtain a conceptual explanation for the Baum-Connes conjecture.

1 Introduction

The Baum-Connes conjecture predicts a formula for the K-theory of group C^*-algebras $C^*_{red}\Gamma$ and, more generally, for the K-theory of reduced crossed product C^*-algebras $A \rtimes_r \Gamma$, [2], [3], [4], [5]. According to the formula, this K-theory should be isomorphic, via a map defined by Baum-Connes, to a topologically defined K-theory, with coefficients in A, of a classifying space $\underline{\mathrm{E}}_\Gamma$ for proper actions of the given group.

The Baum-Connes conjecture contains the Novikov conjecture and the generalized Kadison conjecture and plays an important motivating role in current research on topological K-theory. While counterexamples to the conjecture have recently been constructed (at least for crossed products), by various authors (Higson, Lafforgue-Skandalis, Yu), it is still expected to be true for a very large class of groups.

The conjecture had been elaborated by Baum and Connes over the years with amazing insight, based mainly on the evidence given by a certain (at the time quite limited) number of examples. It is remarkable that it has indeed been verified for

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large and important classes of groups (see e.g. [10], [7], [11]). There was however no compelling explanation for the choice of the left hand side of the conjecture, i.e. the topological K-theory of $\underline{\mathrm{E}}_\Gamma$.
In this article we present such a conceptual explanation. We analyze projections and in particular projections with only finitely many non-zero coefficients in crossed product algebras, or more generally in algebras with a Γ -grading. We then show that the coefficients of such projections lead naturally to representations of a non-commutative classifying space which is exactly the noncommutative analogue of $\underline{\mathrm{E}}_\Gamma$. More precisely, we obtain a projective system of C^*-algebras which are a very straightforward noncommutative generalization of the algebra of functions on certain simplicial complexes. Such a C^*-algebra can be associated naturally with any locally finite simplicial complex.
We develop to some extent the theory of these noncommutative simplicial complexes which, we think, have an interest of their own. They do seem to be very natural objects in connection with noncommutative topology. We mention that similar noncommutative simplicial complexes have been used by W.Winter, [13] to obtain a definition of covering dimension for nuclear C^*-algebras that has good properties. We show in particular that, under certain finiteness conditions, the noncommutative algebras associated with a simplicial complex have the same K-theory as their classical counterpart. This can also be done equivariantly - a fact, which is important in connection with the Baum-Connes conjecture.
In section 3 we investigate another variant of a noncommutative C^*-algebra that can be associated with a simplicial complex. As we show later, these algebras, too, have a natural connection to a version of finite K-theory for algebras with a Γ-grading. Their K-theory however has less good properties. We compute the K-theory for a natural example.
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2 Noncommutative simplicial complexes.

Recall that a simplicial complex Σ can be defined as a set of non-empty finite subsets (the simplexes in Σ) of a set V_Σ (the set of vertices of Σ) satisfying the following axioms:

- if $s \in V_\Sigma$ then $\{s\} \in \Sigma$
- if $F \in \Sigma$ and $\emptyset \neq E \subset F$, then $E \in \Sigma$.

Σ is called locally finite, if every vertex of Σ is contained in only finitely many simplexes of Σ. In the following, we will always assume that Σ is a locally finite simplicial complex. With a simplicial complex Σ one can associate canonically a topological space $|\Sigma|$ - the "geometric realization" of the complex. It can be defined as the space of maps $f : V_\Sigma \to [0,1]$ with finite support such that

$$\sum_{s\in V_\Sigma} f(s) = 1$$

with the topology of pointwise convergence. If Σ is locally finite, then $|\Sigma|$ is locally compact.
With a locally finite complex we associate in this section two canonical C^*-algebras:

(1) $\mathcal{C}_\Sigma$ is the universal C^*-algebra with positive generators $h_s, s \in V_\Sigma$ satisfying the relations

- $h_{a_0}h_{a_1}\dots h_{a_n} = 0$ whenever $\{a_0, \dots, a_n\}$ is not a simplex in Σ (note that here repetitions in the a_i are allowed).
- $\sum_{s\in V_\Sigma} h_s h_t = h_t,\ t \in V_\Sigma$

The sum in the second condition is necessarily finite and this condition may be abbreviated formally to $\sum_{s\in V_\Sigma} h_s = 1$.

(2) $\mathcal{C}^{ab}_\Sigma$ is the universal C^*-algebra with generators and relations as in (1) but satisfying in addition $h_s h_t = h_t h_s$ for all $s, t \in V_\Sigma$.

Homomorphisms from $\mathcal{C}_\Sigma$ (or $\mathcal{C}^{ab}_\Sigma$) are easily defined by specifying the images of the generators. Whenever $h'_s, s \in V_\Sigma$ are positive elements in some C^*-algebra A satisfying the relations in (1) (or in (2)), then there is a unique *-homomorphism from $\mathcal{C}_\Sigma$ (or from $\mathcal{C}^{ab}_\Sigma$) to A mapping each h_s to h'_s.
The algebra $\mathcal{C}^{ab}_\Sigma$ is just the abelianization of $\mathcal{C}_\Sigma$ and we have a canonical surjective map

$$\mathcal{C}_\Sigma \longrightarrow \mathcal{C}^{ab}_\Sigma$$

The natural maps between simplicial complexes Σ and Σ' in our context are the proper simplicial maps. Such a map is given by a map $\varphi : V_\Sigma \to V_{\Sigma'}$ such that $\{\varphi(s_0), \varphi(s_1), \dots, \varphi(s_n)\}$ is a simplex in Σ', whenever $\{s_0, s_1, \dots, s_n\}$ is a simplex in Σ and such that the preimage of any $s \in V_{\Sigma'}$ is finite. Any such map between locally finite simplicial complexes induces a homomorphism $\mathcal{C}_{\Sigma'} \to \mathcal{C}_\Sigma$ mapping a generator h_s to $\sum_{\varphi(t)=s} h_t$ (h_s is mapped to 0 if s is not in the image of φ). Important special cases, which will be used extensively, are the evaluation maps which are induced by the inclusion of a subcomplex. Σ_0 is a subcomplex of Σ if $V_{\Sigma_0} \subset V_\Sigma$ and any simplex in Σ_0 is also a simplex in Σ. There is a natural evaluation homomorphism

$$\pi_{\Sigma_0,\Sigma} : \mathcal{C}_\Sigma \longrightarrow \mathcal{C}_{\Sigma_0}$$

which sends a generator $h_s, s \in V_{\Sigma_0}$ of $\mathcal{C}_\Sigma$ to the corresponding generator h_s of $\mathcal{C}_{\Sigma_0}$ and sends the other generators $h_s, s \notin V_{\Sigma_0}$ to 0.

Proposition 2.1 *The (abelian) C^*-algebra $\mathcal{C}_\Sigma^{ab}$ is isomorphic to the algebra $\mathcal{C}_0(|\Sigma|)$ of continuous functions vanishing at infinity on the geometric realization $|\Sigma|$ of Σ.*

Proof. We know that $\mathcal{C}_\Sigma^{ab} \cong \mathcal{C}_0(X)$ where $X = \mathrm{Spec}\ \mathcal{C}_\Sigma^{ab}$. By definition, the elements of X are the homomorphisms $\mathcal{C}_\Sigma^{ab} \to \mathbb{C}$. Such a homomorphism is uniquely determined by the values it takes on the generators h_s. These values are real numbers t_s between 0 and 1 with sum equal to 1. The set of all s for which t_s is non-zero form necessarily a simplex in Σ, since then the product of the corresponding h_s has to be non-zero. Thus we get a bijection between such homomorphisms and maps $f : V_\Sigma \to [0,1]$ supported on a simplex in Σ satisfying $\sum_{s\in V_\Sigma} f(s) = 1$. This is exactly the description of the geometric realization for Σ given above. □

The C^*-algebras $\mathcal{C}_\Sigma$ and $\mathcal{C}_\Sigma^{ab}$ have a natural filtration by ideals which reduces to the skeleton filtration in the abelian case. For each n, we let I_n denote the ideal in $\mathcal{C}_\Sigma$ generated by all products of generators h_s containing at least $n+1$ pairwise different generators. We also denote by I_n^{ab} the image of I_n in $\mathcal{C}_\Sigma^{ab}$.
The subcomplex Σ^{n-1} of Σ that has the same vertices and whose simplexes are exactly the k-simplexes of Σ with $k \le n-1$ is called the $n-1$-skeleton of Σ.

Proposition 2.2 *The ideal I_n is exactly the kernel of the evaluation map*

$$\pi_{\Sigma^{n-1},\Sigma} : \mathcal{C}_\Sigma \longrightarrow \mathcal{C}_{\Sigma^{n-1}}$$

The abelian C^*-algebra $\mathcal{C}_\Sigma^{ab}/I_n^{ab}$ is isomorphic to $\mathcal{C}_0(|\Sigma^{n-1}|)$.

Proof. Obvious from the proof of 2.1. □

Let Δ be an n- simplex with vertices $\{t_0,\dots,t_n\}$ and $\Delta^i, i=0,\dots,n$ the $(n-1)$-face of Δ with vertices $\{t_0,\dots,t_n\}\setminus\{t_i\}$. By definition, $\mathcal{C}_\Delta$ is the unital C^*-algebra with generators h_s, $s \in \{t_0,\dots,t_n\}$ such that $h_s \ge 0$ and $\sum h_s = 1$. There are natural evaluation maps $\pi_{\Delta^i,\Delta} : \mathcal{C}_\Delta \to \mathcal{C}_{\Delta^i}$ mapping the generator h_{t_i} to 0. We denote by J_Δ the ideal in $\mathcal{C}_\Delta$ generated by products of generators containing all the $h_{t_i}, i = 0,\dots,n$. Note that J_Δ is exactly the intersection of the kernels of all the $\pi_{\Delta^i,\Delta}$.
This is nothing but the ideal I_n in $\mathcal{C}_\Delta$. The abelian version J_Δ^{ab} is obviously isomorphic to the algebra of continuous functions on the Euclidean n-simplex

$$|\Delta| \cong \{(x_0,\dots,x_n)\in\mathbb{R}^{n+1} \mid 0\le x_i\le 1, \sum_{i=0}^n x_i = 1\}$$

that vanish on the boundary.

Lemma 2.3 *The ideal J_Δ is essential in $\mathcal{C}_\Delta$.*

Proof. We proceed by induction on n and assume that the assertion has been proved already for $n-1$ (it is obviously true for $n = 0, 1$). Given t_i in the vertex set $\{t_0,\dots,t_n\}$ for Δ, and $t \in [0,1]$, we define a homomorphism $\varphi_t : \mathcal{C}_\Delta \to \mathcal{C}_{\Delta^i}$

by $\varphi_t(h_s) = (1-t)h_s, s \neq t_i$, and $\varphi_t(h_{t_i}) = t1$. Let φ denote the corresponding homomorphism from $\mathcal{C}_\Delta$ into the algebra $\mathcal{C}_{\Delta^i}[0,1]$ of continuous functions on the unit interval with values in $\mathcal{C}_{\Delta^i}$. The image of J_Δ under φ equals $J_{\Delta^i}(0,1)$, i.e. the functions with values in J_{Δ^i} vanishing in 0 and 1. If $xJ_\Delta = \{0\}$ for $x \in \mathcal{C}_\Delta$, then $\varphi(x)J_{\Delta^i}(0,1) = \{0\}$. Since J_{Δ^i} is essential in $\mathcal{C}_{\Delta^i}$ by hypothesis, this implies that $\varphi_t(x) = 0$ for $0 < t < 1$ and by continuity also that $\varphi_0(x) = \pi_{\Delta^i,\Delta}(x) = 0$. It follows that $x \in J_\Delta = \bigcap_i \operatorname{Ker} \pi_{\Delta^i,\Delta}$ whence $x = 0$. $\square$

As a consequence of the previous lemma, the natural map from $\mathcal{C}_\Delta$ into the multiplier algebra $\mathcal{M}(J_\Delta)$ is injective.

Lemma 2.4 *Let Σ be a locally finite simplicial complex and I_n be the ideal in $\mathcal{C}_\Sigma$ defined above. Then*

$$I_n/I_{n+1} \cong \bigoplus_\Delta J_\Delta$$

where $J_\Delta \subset \mathcal{C}_\Delta$ is the ideal defined above and the direct sum is taken over all n-simplexes Δ in Σ. The natural map $\mathcal{C}_\Sigma/I_{n+1} \to \mathcal{M}(J_\Delta)$, for each Δ, factors as

$$\mathcal{C}_\Sigma/I_{n+1} \stackrel{\pi_{\Delta,\Sigma^n}}{\longrightarrow} \mathcal{C}_\Delta \underset{\cong}{\subseteq} \mathcal{M}(J_\Delta)$$

Proof. The first part of the assertion follows from the fact that, mod I_{n+1}, any product containing $n+1$ different generators is orthogonal to any further generator. The second part follows from Lemma 2.3 and from the fact that the natural map into the multiplier algebra of J_Δ annihilates all h_s for $s \notin \Delta$. $\square$

Call a simplex of Σ maximal if it is not strictly contained in any other simplex of Σ. Obviously, in a locally finite complex, any simplex is contained in a maximal simplex.

Lemma 2.5 *Let Σ be a locally finite simplicial complex and $x \in \mathcal{C}_\Sigma$. If $\pi_{\Delta,\Sigma}(x) = 0$ for all simplexes Δ in Σ, then $x = 0$.*
The same holds if $\pi_{\Delta,\Sigma}(x) = 0$ for all maximal simplexes Δ in Σ.

Proof. It follows from 2.4 by induction on n that x has to lie in I_n for each n.
On the other hand, for a locally finite complex, one has for each generator h_s, that $h_sI_k = 0$ for sufficiently large k. Thus, if z is any polynomial in the generators of $\mathcal{C}_\Sigma$ such that $\|x - z\| < \varepsilon$, then $\overline{\mathcal{C}_\Sigma z \mathcal{C}_\Sigma} \cap I_k = 0$ for sufficiently large k. This implies that the image of z in the quotient by I_k has the same norm as z, so that $\|x\| < \varepsilon$ for any ε.
The second statement follows from the fact that, if F is a face of Δ, then $\pi_{F,\Sigma} = \pi_{F,\Delta} \circ \pi_{\Delta,\Sigma}$. $\square$

Remark 2.6 *As a consequence of the preceding lemma, we obtain, for any x in $\mathcal{C}_\Sigma$*

$$\|x\| = \sup\{\, \|\pi_{\Delta,\Sigma}(x)\| \mid \Delta \textit{ is a maximal simplex in } \Sigma\}$$

It follows that, for any locally finite simplicial complex Σ, the algebra $\mathcal{C}_\Sigma$ is a fibered product, over all simplexes Δ in Σ, of algebras $\mathcal{C}_\Delta$.

Theorem 2.7 *The algebra $\mathcal{C}_\Sigma$ is isomorphic to the subalgebra A_Σ of the direct sum $\bigoplus \mathcal{C}_\Delta$ over all simplexes in Σ, consisting of families $(x_\Delta)_{\Delta\in\Sigma}$ such that*

$$\pi_{F,\Delta}(x_\Delta) = \pi_{F,\Delta'}(x_{\Delta'})$$

whenever F is a common face of Δ and Δ'.
In the direct sum it suffices to take the sum over all maximal simplexes.

Proof. It follows from Lemma 2.5 that the map $\sigma = \prod \pi_{\Delta,\Sigma} : \mathcal{C}_\Sigma \to \bigoplus \mathcal{C}_\Delta$ is injective. Its image is obviously in A_Σ.
On the other hand, if $x = (x_\Delta)$ is an element of A_Σ and Δ is a maximal simplex in Σ, we can find y_Δ in the subalgebra of $\mathcal{C}_\Sigma$ generated by the h_s, $s \in \Delta$, such that $x_\Delta = \pi_{\Delta,\Sigma}(y_\Delta)$. Similarly, an element of $\mathcal{C}_\Delta$, that vanishes under the evaluation on a face F, can be lifted to an element in the ideal generated by the h_s, $s \notin F$ in the subalgebra of $\mathcal{C}_\Sigma$ generated by the h_s, $s \in \Delta$. By iterating this, we can find y in $\mathcal{C}_\Sigma$ such that $x_\Delta = \pi_{\Delta,\Sigma}(y)$ for any finite collection of maximal simplexes Δ.
It follows that, for any $\varepsilon > 0$, there is $y \in \mathcal{C}_\Sigma$ such that $\|x - \sigma(y)\| < \varepsilon$. □

Lemma 2.8 *Let Σ be a locally finite simplicial complex and Δ a simplex in Σ.*
The canonical evaluation map $\pi_\Delta : \mathcal{C}_\Sigma \to \mathcal{C}_\Delta$ admits a canonical completely positive lift $\varphi_\Delta : \mathcal{C}_\Delta \to \mathcal{C}_\Sigma$. This lift is equivariant for the action of any group on Σ mapping Δ into itself.

Proof. Let V be the set of vertices of Σ, $h_s, s \in V$ the canonical generators of $\mathcal{C}_\Sigma$ and $\Delta = \{s_0, s_1, \ldots, s_n\}$.
Let D be the C^*-subalgebra of $\mathcal{C}_\Sigma$ generated by $h_{s_0}, \ldots, h_{s_n}$ and denote by $h'_0, h'_1, \ldots, h'_n$ the generators of $\mathcal{C}_\Delta$. We define a homomorphism $\alpha : \mathcal{C}_\Delta \to \tilde{D}$ by putting

$$\alpha(h'_i) = h_{s_i} + \tfrac{1}{n+1}(1 - \sum_{i=0}^{n} h_{s_i})$$

The desired map φ_Δ can now be obtained as

$$\varphi_\Delta(x) = (h_{s_0} + \ldots + h_{s_n})\alpha(x)(h_{s_o} + \ldots + h_{s_n}) \in D \subset \mathcal{C}_\Delta.$$

□

Lemma 2.9 *Let $\Delta = \{s_0, s_1, \ldots, s_n\}$ be a simplex and $F^i = \{s_0, s_1, \ldots, s_n\} \setminus \{s_i\}$, $i = 0, 1, \ldots, n$ its faces. Let K_i be the kernel of the evaluation map $\mathcal{C}_\Delta \to \mathcal{C}_{F^i}$, i.e. the ideal generated by h_i.*
Then there are multipliers M_i, $i = 0, 1, \ldots, n$, of J_Δ such that $0 \le M_i \le 1$, $\sum M_i = 1$, M_i commute with $\mathcal{C}_\Delta$ modulo J_Δ and $M_i K_j \subset J_\Delta$ for $j \neq i$.
Moreover, the M_i can be chosen equivariant for the action of any group on Δ.

Proof. This is exactly Kasparov's technical theorem, [8]. □

Proposition 2.10 *Let Γ be a group acting on the locally finite complex Σ by simplicial maps. The quotient map $\mathcal{C}_\Sigma/I_{n+1} \to \mathcal{C}_\Sigma/I_n$ admits an equivariant completely positive splitting.*

Proof. The map $\mathcal{C}_\Sigma/I_{n+1} \to \mathcal{C}_\Sigma/I_n$ corresponds to the inclusion map $\Sigma^{n-1} \to \Sigma^n$ between the skeletons.
By Theorem 2.7, $\mathcal{C}_{\Sigma^{n-1}}$ is isomorphic to a subalgebra $A_{\Sigma^{n-1}}$ of the direct sum $\bigoplus_{\Delta'\in E'} \mathcal{C}_{\Delta'}$, where E' is the set of all maximal simplexes Δ' in Σ^{n-1}. Similarly, $\mathcal{C}_{\Sigma^n}$ is contained in the direct sum $\bigoplus_{\Delta\in E} \mathcal{C}_\Delta$, where E is the set of all maximal simplexes Δ in Σ^n. Note that some of the simplex algebras appear in both direct sums.
We define a completely positive map

$$\psi : \bigoplus \mathcal{C}_{\Delta'} \longrightarrow \bigoplus \mathcal{C}_\Delta$$

as a sum over maps $\psi_{\Delta,\Delta'} : \mathcal{C}_{\Delta'} \to \mathcal{C}_\Delta$, where, given Δ' in E', Δ is in the set $M_{\Delta'}$ of all simplexes Δ in E such that $\Delta \supset \Delta'$, and where

$$\psi_{\Delta',\Delta'} = \mathrm{id} \quad \text{if } \Delta' \text{ is not a face of an } n\text{-simplex in } \Sigma^n \text{ i.e. if } M_{\Delta'} = \{\Delta'\}$$

$$\psi_{\Delta,\Delta'}(x) = M_F^{\frac{1}{2}} \varphi_F(x) M_F^{\frac{1}{2}} \quad \text{if } \Delta' = F \text{ is a face of the } n\text{-simplex } \Delta \text{ in } \Sigma^n$$

Here, φ_F is as in 2.8 and M_F is as in 2.9. It is easily checked that the sum ψ over the $\psi_{\Delta,\Delta'}$ makes sense and that ψ is a lift for the quotient map $\mathcal{C}_{\Sigma^n} \to \mathcal{C}_{\Sigma^{n-1}}$. □

An equivariant homomorphism $A_1 \to A_2$ between two C^*-algebras equipped with an action of the group G induces a KK^G-equivalence, if, for any G-algebra B the induced maps $KK^G(A_2, B) \to KK^G(A_1, B)$ and $KK^G(B, A_1) \to KK^G(B, A_2)$ are isomorphisms.

Theorem 2.11 *Let G be a group acting by simplicial maps on a simplex Δ and on a finite simplicial complex Σ .*

(a) There is an equivariant map $\alpha : \mathcal{C}_\Delta \to \mathbb{C}$ which is an equivariant homotopy inverse to the natural inclusion $\mathbb{C} \to \mathcal{C}_\Delta$ and thus induces a KK^G-equivalence.

(b) The canonical map $J_\Delta \to J_\Delta^{ab}$ induces a KK^G-equivalence.

(c) The canonical map $\mathcal{C}_\Sigma \to \mathcal{C}_\Sigma^{ab}$ induces a KK^G-equivalence.

Proof. (a) If $h_0, h_1, \dots, h_n$ denote the generators of $\mathcal{C}_\Delta$, we put $\alpha(h_i) = \frac{1}{n+1} 1$. If we compose α with the inclusion $\mathbb{C} \to \mathcal{C}_\Delta$ it becomes homotopic to the identity map via the equivariant homotopy defined by

$$\alpha_t(h_i) = th_i + \tfrac{1-t}{n+1} 1 \,, \qquad t \in [0,1]$$

We prove (b) and (c) simultaneously by induction on the dimension n of Δ and Σ. Both assertions are trivially true for $n = 0$.
Let Δ be an n-dimensional simplex and S its boundary, i.e. the standard simplicial $n-1$-sphere. There is an extension

$$0 \longrightarrow J_\Delta \longrightarrow \mathcal{C}_\Delta \longrightarrow \mathcal{C}_S \longrightarrow 0$$

and the analogous extension for the abelianized algebras. By (a) the map $\mathcal{C}_\Delta \to \mathcal{C}_\Delta^{ab}$ is a KK^G-equivalence and by induction hypothesis, the map $\mathcal{C}_S \to \mathcal{C}_S^{ab}$ is so, too. Therefore, by the five lemma and using 2.10, the third map $J_\Delta \to J_\Delta^{ab}$ is a KK^G-equivalence.
Let now (I_k) be the skeleton filtration for the n-dimensional finite complex $\mathcal{C}_\Sigma$. We have $I_{n+1} = 0$ and $I_n \cong \bigoplus J_\Delta$ where the sum is over all n-dimensional simplexes in Σ. There is an extension

$$0 \longrightarrow I_n \longrightarrow \mathcal{C}_\Sigma \longrightarrow \mathcal{C}_{\Sigma^{n-1}} \longrightarrow 0$$

where Σ^{n-1} is the $n-1$-skeleton of Σ. Since the assertion is true for $\mathcal{C}_{\Sigma^{n-1}}$, by induction hypothesis, and for I_n, by (b), which we just proved, we can again use the five lemma, to conclude that the map is a KK^G-equivalence. □

Proposition 2.12 *Let Σ be a locally finite simplicial complex and (I_k) the skeleton filtration for $\mathcal{C}_\Sigma$. Assume that G is a group acting on Σ (by simplicial maps). Then the map $I_n/I_{n+1} \to I_n^{ab}/I_{n+1}^{ab}$ is a KK^G-equivalence for each n.*

Proof. We know that

$$I_n/I_{n+1} \cong \bigoplus_\Delta J_\Delta \qquad I_n^{ab}/I_{n+1}^{ab} \cong \bigoplus_\Delta J_\Delta^{ab}$$

where the sums are taken over all n-simplexes in Σ. The map in question is a KK-equivalence on each summand J_Δ, equivariant for the action of the stabilizer group for Δ by 2.11. To obtain the inverse in $KK^G(I_n^{ab}/I_{n+1}^{ab}, I_n/I_{n+1})$ for the KK^G-element defined by the canonical map $I_n/I_{n+1} \to I_n^{ab}/I_{n+1}^{ab}$ we can take the direct sum (with the obvious G-action) of the Kasparov bimodules representing the inverse in $KK(J_\Delta^{ab}, J_\Delta)$ for the element defined by the quotient map $J_\Delta \to J_\Delta^{ab}$. □

Theorem 2.13 *Let Σ be a finite-dimensional locally finite simplicial complex and G a group acting simplicially on Σ. The canonical quotient map $\mathcal{C}_\Sigma \to \mathcal{C}_\Sigma^{ab}$ is a KK^G-equivalence.*

Proof. The skeleton filtration (I_k) for $\mathcal{C}_\Sigma$ is finite. Since all quotient maps with respect to these ideals admit equivariant completely positive splittings by 2.10, we can use the exactness properties of KK^G to deduce the assertion from repeated application of the five lemma (the spectral sequence associated with the filtration) from 2.12. □

3 Noncommutative flag complexes.

In our applications to the Baum-Connes conjecture below we will be concerned with simplicial complexes that have a special property. We say, that a simplicial complex Σ is full, if it is determined by its 1-simplexes through the following condition:

a finite subset $F \neq \emptyset$ of V_Σ belongs to Σ if $\{s,t\} \in \Sigma$ for all $\{s,t\} \subset F$.

Such complexes are sometimes called flag complexes in the literature. We will use both terminologies in the sequel. Note that the barycentric subdivision of any simplicial complex is full.
For a full locally finite complex we introduce a new noncommutative algebra $\mathcal{C}_\Sigma^{flag}$ as the universal C^*-algebra with generators $h_s, s \in V_\Sigma$ satisfying the relations $h_s \geq 0$ for all s, $h_s h_t = 0$ for $\{s,t\} \notin \Sigma$ and $\sum_{s \in V_\Sigma} h_s h_t = h_t, t \in V_\Sigma$. As before, the sum in the last condition is necessarily finite and this condition may be abbreviated formally to $\sum_{s \in V_\Sigma} h_s = 1$. The motivation for this definition is the fact that C^*-algebras associated in this way with flag complexes appear completely canonically in connection with the "finite" K-theory introduced in section 4.
Note that this C^*-algebra can also be defined for a simplicial complex that is not full. It will then coincide with the C^*-algebra $\mathcal{C}_{\Sigma^{flag}}^{flag}$ associated with the saturation Σ^{flag} of Σ. (We define Σ^{flag} to be the unique full simplicial complex with the same vertex set and the same edges as Σ. A subset F of vertices belongs to Σ^{flag} if $\{s,t\} \in \Sigma$ for all $s,t \in F$.)
For a flag complex, there are canonical surjective maps

$$\mathcal{C}_\Sigma^{flag} \longrightarrow \mathcal{C}_\Sigma \longrightarrow \mathcal{C}_\Sigma^{ab}$$

Unfortunately, without the skeleton filtration, the C^*-algebra $\mathcal{C}_\Sigma^{flag}$ does not, in general, retain the topological information of Σ. We are now going to determine the K-theory for a typical example of a noncommutative flag complex. This example is given by a complex that models a simplicial sphere.
Consider the universal C^*-algebra $C^*(x_0, x_1, \ldots, x_n)$ with self-adjoint generators x_i satisfying the relation that $\sum_i x_i^2 = 1$. The abelianization of this C^*-algebra is obviously isomorphic to the algebra of continuous functions on the n-sphere S^n.
Consider the positive and negative parts $(x_i)_+$ and $(x_i)_-$ if x_i. We have $x_i^2 = (x_i)_+^2 + (x_i)_-^2$ and $(x_i)_+^2 (x_i)_-^2 = 0$. If we put $h_{i^+} = (x_i)_+^2$ and $h_{i^-} = (x_i)_-^2$ we get $2n+2$ new generators for $C^*(x_0, x_1, \ldots, x_n)$ that are positive and satisfy

$$\sum h_{i^+} + \sum h_{i^-} = 1, \qquad h_{i^+} h_{i^-} = 0 \quad \forall i$$

Thus $C^*(x_0, x_1, \ldots, x_n)$ is exactly isomorphic to the algebra $\mathcal{C}_\Sigma^{flag}$ for the flag complex Σ_{S^n} with vertices $\{0^+, 1^+, \ldots, n^+, 0^-, \ldots, n^-\}$ and the condition that exactly the edges $\{i^+, i^-\}$ do not belong to Σ_{S^n}. This complex manifestly is a simplicial model for the n-sphere.

We will denote the C^*-algebra $\mathcal{C}_{\Sigma_{Sn}}^{flag}$ by S_n^{nc}.
For any vertex s in a flag complex there is a natural evaluation map $\mathcal{C}_{\Sigma}^{flag} \to \mathbb{C}$ mapping the generator h_s to 1 and all the other generators to 0. We will now consider various homomorphisms from S_n^{nc} to S_n^{nc} and from S_n^{nc} to $M_2(S_n^{nc})$.
Denote by κ_i the homomorphism from S_n^{nc} to S_n^{nc} that maps the generator h_{i^+} to 1 and all the other generators to 0. This is simply the composition of the evaluation map for the vertex i^+ with the natural inclusion map $\mathbb{C} \to S_n^{nc}$.
We will further consider two homomorphisms α and β from S_n^{nc} to $M_2(S_n^{nc})$ of the form

$$\alpha = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & \alpha_1 \end{pmatrix} \qquad \beta = \begin{pmatrix} \beta_0 & 0 \\ 0 & \beta_1 \end{pmatrix}$$

where id is the identity homomorphism of S_n^{nc} and α_1, β_0, β_1 are the homomorphisms $S_n^{nc} \to S_n^{nc}$ determined by

$$\alpha_1(h_{i^+}) = h_{i^+} + h_{i^-},\ \alpha_1(h_{i^-}) = 0 \text{ for } i = 0, 1,$$
$$\alpha_1(h_s) = h_s \text{ for all other generators } h_s$$

$$\beta_0(h_{0^+}) = h_{0^+} + h_{0^-},\ \beta_0(h_{0^-}) = 0, \quad \beta_0(h_s)) = h_s \text{ for all other generators } h_s$$

$$\beta_1(h_{1^+}) = h_{1^+} + h_{1^-},\ \beta_1(h_{1^-}) = 0, \quad \beta_1(h_s)) = h_s \text{ for all other generators } h_s$$

Lemma 3.1 *We have the following homotopies:*

(a) α is homotopic to β.

(b) α_1 is homotopic to κ_0.

(c) β_0 is homotopic to κ_0 and β_1 is homotopic to κ_1.

Proof. (a) is proved by rotating $\beta(h_{0^+})$ and $\beta(h_{0^-})$ to $\alpha(h_{0^+})$ and $\alpha(h_{0^-})$, i.e. we use the homomorphisms φ_t from S_n^{nc} to $M_2(S_n^{nc})$ mapping h_0^+, h_0^- to $R_t\beta(h_0^+)R_t^*$, $R_t\beta(h_0^-)R_t^*$ and all the other generators h_s to $\beta(h_s)$, where R_t are rotation matrices, $t \in [0, \pi/2]$. We have $\varphi_0 = \beta$ and $\varphi_{\pi/2} = \alpha$.
(b) A homotopy between α_1 and κ_0 is obtained by considering the homomorphisms mapping h_{0^+} to $t(h_{0^+} + h_{0^-}) + (1-t)1$, h_{0^-} to 0, h_{1^+} to $t(h_{1^+} + h_{1^-})$, h_{1^-} to 0 and all the other generators h_s to th_s, $t \in [0, 1]$.
(c) A homotopy for β_0 is given by the homomorphisms mapping h_{0^+} to $t(h_{0^+} + h_{0^-}) + (1-t)1$, h_{0^-} to 0 and all the other generators h_s to th_s. The homotopy for β_1 is then obtained by exchanging the role of 0 and 1. □

From this we can easily deduce the K-theory of S_n^{nc}.

Proposition 3.2 *The evaluation map $S_n^{nc} \to \mathbb{C}$ at the vertex 1^+ (and thus at any vertex) induces an isomorphism $K_*(S_n^{nc}) \longrightarrow K_*(\mathbb{C})$.*

Proof. We have to show that $K_*(\kappa_1)$ is an isomorphism. This follows from the fact that

$$K_*(\mathrm{id}) + K_*(\kappa_0) = K_*(\kappa_0) + K_*(\kappa_1)$$

which is a consequence of the preceding lemma. $\square$

In order to recover the topological information in S_n^{nc} we have to use its natural filtration. Let (I_k^{nc}) be the natural skeleton filtration, i.e. I_k^{nc} is the ideal in S_n^{nc} generated by products containing at least $k+1$ different generators.
One can show that, for $n = 1$, the natural maps $S_1^{nc}/I_2^{nc} \to \mathcal{C}(S^1)$ and $S_1^{nc}/I_3^{nc} \to \mathcal{C}(S^1)$ induce isomorphisms in K-theory. Moreover one has $K_0(I_2^{nc}) = K_0(I_3^{nc}) = \mathbb{Z}$ and $K_1(I_2^{nc}) = K_1(I_3^{nc}) = 0$.
For $n = 2$, one can show that the natural map $S_2^{nc}/I_3^{nc} \to \mathcal{C}(S^2)$ induces an isomorphism in K-theory. However already here we have somewhat unexpectedly, that $K_0(I_3^{nc}/I_4^{nc}) \cong \mathbb{Z}^3$.
For higher n, the situation becomes much more complicated.
To end this section, we want to emphasize that the arguments used to determine the K-theory of S_n^{nc} in this section are far from being equivariant for natural group actions on S_n^{nc}.

4 Applications to the Baum-Connes conjecture

Let Γ be a discrete group. We denote by λ_s, $s \in \Gamma$ the unitary operators in $\mathcal{L}(\ell^2\Gamma)$ given by left translation $\lambda_s\xi_t = \xi_{st}$ on the canonical orthormal basis ξ_s in $\ell^2\Gamma$. As usual, $C^*_{\mathrm{red}}\Gamma$ then denotes the C^*-algebra generated by the λ_s, $s \in \Gamma$. There is a natural coproduct $\delta_\Gamma : C^*_{\mathrm{red}}\Gamma \to C^*_{\mathrm{red}}\Gamma \otimes C^*_{\mathrm{red}}\Gamma$ given by $\delta_\Gamma(\lambda_s) = \lambda_s \otimes \lambda_s$ (all C^*-algebra tensor products in this paper will be minimal tensor products).
We are going to examine the K-theory of the reduced group C^*-algebra $C^*_{\mathrm{red}}\Gamma$ or, more generally, the K-theory of a reduced crossed product $A \rtimes_r \Gamma$ for any C^*-algebra A carrying an action of Γ.
Now, such crossed products carry, as an additional structure, the dual action of $\hat{\Gamma}$, i.e. an action of the Hopf algebra $C^*_{\mathrm{red}}\Gamma$ (or a coaction of Γ).
An action of $\hat{\Gamma}$ on a C^*-algebra B is, by definition, a nondegenerate (i.e. the hereditary subalgebra generated by the image is everything) homomorphism

$$\delta_B : B \to B \otimes C^*_{\mathrm{red}}\Gamma$$

satisfying the coassociativity condition $(\delta_B \otimes 1)\delta_B(x) = (1 \otimes \delta_\Gamma)\delta_B(x)$. Following [1] we say that B is a $\hat{\Gamma}$-algebra if, in addition, δ_B is injective.
Any $\hat{\Gamma}$-algebra carries a natural Γ-grading, see [1]. A C^*-algebra B is Γ-graded if for each $s \in \Gamma$, there is a closed linear subspace B_s of B such that

- $B = \overline{\bigoplus_{s\in\Gamma} B_s}$
- $B_s B_t \subset B_{st}$ for all $s, t \in \Gamma$

- $B_s^* = B_{s^{-1}}$

For a $\hat{\Gamma}$-algebra B the projection onto the subspace B_s is obtained by taking the coefficient at s of $\delta_B(x)$. For $B = A \rtimes_r \Gamma$ we obviously can take $B_s = Au_s$, u_s the unitary implementing $s \in \Gamma$.
Let B be a $\hat{\Gamma}$-algebra. A $\hat{\Gamma}$-equivariant Hilbert-B-module is a Hilbert-B-module E with a (coassociative nondegenerate) coaction $\delta_E : E \to E \otimes B$ satisfying

$$\delta_E(\xi)\delta_B(a) = \delta_E(\xi b) \qquad \delta_B((\xi \,|\, \eta)) = (\delta_E(\xi) \,|\, \delta(\eta))$$

for $\xi, \eta \in E$, $b \in B$ and the B-valued scalar product $(\cdot\,|\,\cdot)$, see [1], Definition 2.2 (this definition also makes sense if the coaction on B is not injective). Using the coaction we can define subspaces E_s of degree s in E.
We now use the Γ-grading to define groups that approximate the K-theory of algebras with an action of $\hat{\Gamma}$. To define the finite and strongly finite K-theory of B we consider triples $(p, \bar{p}, E)$ where E is a $\hat{\Gamma}$-equivariant Hilbert B-module and $p, \bar{p}$ are projections in $\mathcal{L}(E)$ such that $p - \bar{p} \in \mathcal{K}(E)$. Here, as usual, $\mathcal{L}(E)$ and $\mathcal{K}(E)$ denote the algebras of bounded and compact operators on E, respectively. We denote by $\langle p, \bar{p}, E\rangle$ the equivalence class, for unitary $\hat{\Gamma}$-equivariant equivalence of such a triple.
For any $\hat{\Gamma}$-equivariant Hilbert B-module E we say that an element $x \in \mathcal{L}(E)$ has degree s if $xE_t \subset E_{st}$ for all $t \in \Gamma$. This definition makes $\mathcal{K}(E)$ into a Γ-graded C^*-algebra. The algebra $\mathcal{L}(E)$ is not Γ-graded, since linear combinations of homogeneous elements with a fixed degree are not necessarily dense. However, for any $x \in \mathcal{L}(E)$, we can define its component x_s of degree s by $x_s\xi = P_{E_{st}}x\xi$ for $\xi \in E_t$ and $P_{E_{st}}$ the projection onto E_{st}. Let x be an element and A be a subalgebra of $\mathcal{L}(E)$. We define their Γ-support as

$$\mathrm{supp}_\Gamma x = \{s \in \Gamma \mid x_s \neq 0\}$$

$$\mathrm{supp}_\Gamma A = \{s \in \Gamma \mid A_x \neq 0\}$$

Let $F \subset \Gamma$ be a finite subset. Set

$$C_0^F(B) = \{\langle p, \bar{p}, E\rangle \mid \mathrm{supp}_\Gamma p,\ \mathrm{supp}_\Gamma \bar{p} \subset F\}$$

$$C_0^{fin}(B) = \{\langle p, \bar{p}, E\rangle \mid \mathrm{supp}_\Gamma p,\ \mathrm{supp}_\Gamma\ \bar{p} \text{ are finite}\}$$

$$C_0^{sF}(B) = \{\langle p, \bar{p}, E\rangle \mid \mathrm{supp}_\Gamma C^*(\{p_s | s \in \Gamma\}),\ \mathrm{supp}_\Gamma(C^*(\{\bar{p}_s | s \in \Gamma\})) \subset F\}$$

$$C_0^{sfin}(B) = \{\langle p, \bar{p}, E\rangle \mid C^*(\{p_s\}), C^*(\{\bar{p}_s\}) \text{ have finite } \Gamma-\text{support}\}$$

A homotopy between classes $\langle p_0, \bar{p}_0, E_0\rangle$ and $\langle p_1, \bar{p}_1, E_1\rangle$ in $C_0^F(B)$, $C_0^{\mathrm{fin}}(B)$, $C_0^{sF}(B)$, $C_0^{\mathrm{sfin}}(B)$, respectively, is as usually a class $\langle p, \bar{p}, E[0,1]\rangle$ in $C_0^F(B[0,1])$, $C_0^{\mathrm{fin}}(B[0,1])$, $C_0^{sF}(B[0,1])$, $C_0^{\mathrm{sfin}}(B[0,1])$, respectively, whose restrictions to 0 and $1 \in [0,1]$ are $\langle p_0, \bar{p}_0, E_0\rangle$ and $\langle p_1, \bar{p}_1, E_1\rangle$ (here, as usually, $B[0,1]$ denotes the algebra of B-valued continuous functions on the unit interval $[0,1]$). We denote by $[p_0, \bar{p}_0, E_0]$ the homotopy class of $\langle p_0, \bar{p}_0, E_0\rangle$.
We can now define for any $\hat{\Gamma}$-algebra B

Definition 4.1

$$K_0^F(B) = \{[p,\bar{p},E] \mid \langle p,\bar{p},E\rangle \in C_0^F(B)\}$$
$$K_0^{\mathrm{fin}}(B) = \{[p,\bar{p},E] \mid \langle p,\bar{p},E\rangle \in C_0^{\mathrm{fin}}(B)\}$$
$$K_0^{sF}(B) = \{[p,\bar{p},E] \mid \langle p,\bar{p},E\rangle \in C_0^{sF}(B)\}$$
$$K_0^{\mathrm{sfin}}(B) = \{[p,\bar{p},E] \mid \langle p,\bar{p},E\rangle \in C_0^{\mathrm{sfin}}(B)\}$$

With direct sum as addition $K_0^F(B), K_0^{\mathrm{fin}}(B), K_0^{sF}(B)$ and $K_0^{\mathrm{sfin}}(B)$ are abelian groups. Moreover, we obviously have

$$K_0^{\mathrm{fin}}(B) = \varinjlim_F K_0^F(B)$$

$$K_0^{\mathrm{sfin}}(B) = \varinjlim_F K_0^{sF}(B)$$

where the inductive limits are taken over all finite subsets F of Γ.
There are natural maps $K_0^{\mathrm{sfin}}B \to K_0^{\mathrm{fin}}B \to K_0B$. We may similarly define K_1^{sfin} and K_1^{fin} starting from triples (p,T,E) where E is a $\hat{\Gamma}$-equivariant Hilbert C^*-module, $p \in \mathcal{L}(E)$ is a projection with support condition using F and $T \in \mathcal{L}(E)$ is an operator such that $T = T^*, T^2 = 1, T = T_1$ (i.e $\mathrm{supp}_\Gamma T \subset \{1\}$) and $Tp - pT \in \mathcal{K}(E)$. Equivalently, we could define K_1^{sfin} and K_1^{fin} by taking suspensions.
Let now E be a $\hat{\Gamma}$-equivariant Hilbert B-module and $p \in \mathcal{L}(E)$ a projection with finite Γ-support. The relations $p = p^*$ and $p^2 = p$, combined with the identity $p = \sum_{s\in\Gamma} p_s$ give, respectively:

$$(P_1) \quad p_s^* = p_s^{-1}$$
$$(P_2) \quad p_t = \textstyle\sum_{s\in\Gamma} p_s p_{s^{-1}t}$$

Note that, in particular, $p_1^2 = \sum_{s\in\Gamma} p_s p_s^*$ ($1 \in \Gamma$ the neutral element). This shows that p_1 is a strictly positive element in the C^*-algebra generated by the $p_s, s \in \Gamma$.

Definition 4.2 *Let F be a finite subset in Γ. We denote by P_F the universal C^*-algebra generated by elements $p_s, s \in F$, satisfying the relations (P_1) and (P_2) (where p_s is understood to be zero for $s \notin F$). We denote by P_F^s the universal C^*-algebra with generators $p_s, s \in F$ satisfying the relations (P_1) and (P_2) and the additional relation $p_{s_1}p_{s_2}\dots p_{s_n} = 0$ whenever $s_1s_2\dots s_n \notin F$.*

Note that for any projection p as above, the relations (P_1) and (P_2) imply that $\mathrm{supp}_\Gamma p = (\mathrm{supp}_\Gamma p)^{-1}$ and $\mathrm{supp}_\Gamma p \neq \emptyset \Rightarrow 1 \in \mathrm{supp}_\Gamma p$. Therefore, in the definition of P_F and P_F^s , we might just as well assume that $F = F^{-1}$ and $1 \in F$.
There is a homomorphism $\delta : P_F \to P_F \otimes C^*_{\mathrm{red}}\Gamma$ defined by $\delta(p_s) = p_s \otimes \lambda_s$ ($\lambda_s \in C^*_{\mathrm{red}}\Gamma$ the unitary defined by the left regular representation λ). Therefore, on P_F we can define a natural Γ-grading (obtained by projecting $\delta(x)$ onto the coefficient of λ_s) such that $\deg p_s = s$. Also, P_F contains a canonical projection, namely $p = \sum p_s$.

The same comments apply to P_F^s.
The algebra P_F^s is a sum of finitely many closed subspaces V_t, $t \in F$, where V_t is the closure of linear combinations of products $p_{s_1} p_{s_2} \dots p_{s_k}$ with $s_1 s_2 \dots s_k = t$. Each V_t is mapped isometrically under the coaction $\delta : P_F^s \to P_F^s \otimes C^*_{red}\Gamma$ to $V_t \otimes \lambda_t$. Thus the coaction is injective and P_F^s is a $\hat{\Gamma}$-algebra. This is not necessarily the case for P_F. We therefore define

Definition 4.3 *We denote by P_F^{red} the C^*-algebra isomorphic to the image of P_F under the map $\delta : P_F \to P_F \otimes C^*_{\mathrm{red}}\Gamma$, i.e. the C^*-algebra generated by the elements $p_s \otimes \lambda_s$, $s \in F$.*

P_F^{red} is a $\hat{\Gamma}$-algebra. If B is a $\hat{\Gamma}$-algebra, i.e. if the coaction $\delta : B \to B \otimes C^*_{\mathrm{red}}\Gamma$ is injective, then any homomorphism $P_F \to B$ that respects the Γ-grading factors through P_F^{red}.

Let E again be a $\hat{\Gamma}$-equivariant Hilbert module over the $\hat{\Gamma}$-algebra B. By the very definition of P_F there is a bijection between:

- projections e in $\mathcal{L}(E)$ with $\mathrm{supp}_\Gamma e \subset F$
- homomorphisms $\varphi : P_F \to \mathcal{L}(E)$ respecting the Γ-grading

given by $\varphi \mapsto e_\varphi = \sum \varphi(p_s)$ and $e \mapsto \varphi_e$ where φ_e is defined by $\varphi_e(p_s) = e_s$. If the coaction on B (and thus on E) is injective, then these objects are moreover in bijection with

- homomorphisms $\varphi : P_F^{\mathrm{red}} \to \mathcal{L}(E)$ respecting the Γ-grading

By definition of P_F^s there is, similarly, a canonical bijection between:

- projections e in $\mathcal{L}(E)$ with $\mathrm{supp}_\Gamma(C^*(\{e_s | s \in \Gamma\})) \subset F$
- homomorphisms $\varphi : P_F^s \to \mathcal{L}(E)$ respecting the Γ-grading

Note also that a homomorphism from P_F or P_F^s into a $\hat{\Gamma}$-algebra B is $\hat{\Gamma}$-equivariant if and only if it respects the $\hat{\Gamma}$-grading. We obtain, basically from the definition of the equivariant KK-groups, [1]

Proposition 4.4 *Let B be a $\hat{\Gamma}$-algebra and F a finite subset of Γ. We have natural isomorphisms*

$$K_*^F(B) \cong KK_*^{\hat{\Gamma}}(P_F, B)$$
$$K_*^{fin}(B) \cong \varinjlim_F KK_*^{\hat{\Gamma}}(P_F, B)$$
$$K_*^{sF}(B) \cong KK_*^{\hat{\Gamma}}(P_F^s, B)$$
$$K_*^{sfin}(B) \cong \varinjlim_F KK_*^{\hat{\Gamma}}(P_F^s, B)$$

The algebra P_F can be replaced by P_F^{red} in the first two isomorphisms.

Proof. The proof follows simply from the fact that $KK_0^{\hat\Gamma}(B_1,B_2)$ can be defined as the set of homotopy classes of triples $(\varphi,\bar\varphi,E)$ where E is a $\hat\Gamma$-equivariant Hilbert B_2-module and $\varphi,\bar\varphi : B_1 \to \mathcal{L}(E)$ are $\hat\Gamma$-equivariant homomorphisms, such that $\varphi(x)-\bar\varphi(x)\in\mathcal{K}(E)$ for all $x\in B_1$ (the operator F appearing in the definition of $KK^{\hat\Gamma}$ in [1] can here be assumed to be $\hat\Gamma$-invariant). In the odd case, $KK_1^{\hat\Gamma}(B_1,B_2)$ can similarly be defined from triples (φ,T,E) where E,φ are as above and $T\in\mathcal{L}(E)$ satisfies $T=T^*, T^2=1, \mathrm{supp}_\Gamma T\subset\{1\}$ and $T\varphi(x)-\varphi(x)T\in\mathcal{K}(E)$ for all $x\in B_1$. These descriptions of KK_*^Γ show that the claimed isomorphisms are simply a translation of the definitions of $K_*^F, K_*^{fin}, K_*^{sF}$ and K_*^{sfin}. □

We are now in a position to apply the Baaj-Skandalis duality. Recall that, for any $\hat\Gamma$-algebra A, one can define a reduced crossed product $A\rtimes_r\hat\Gamma$. The precise definition is irrelevant for our purposes. We only need the following two basic results, [1],[12]:

$$KK_*^{\hat\Gamma}(B_1,B_2)\cong KK_*^\Gamma(B_1\rtimes_r\hat\Gamma, B_2\rtimes_r\hat\Gamma)$$

$$A\rtimes_r\Gamma\rtimes_r\hat\Gamma\cong A\otimes\mathcal{K}(\ell^2\Gamma)$$

which hold for all $\hat\Gamma$-algebras B_1, B_2 and for each Γ-algebra A. We obtain for $B=A\rtimes_r\Gamma$

Proposition 4.5 *Let A be a Γ-algebra. Then*

$$K_*^{fin}(A\rtimes_r\Gamma)\cong\varinjlim_F KK_*^\Gamma(P_F^{\mathrm{red}}\rtimes_r\hat\Gamma, A)$$

$$K_*^{sfin}(A\rtimes_r\Gamma)\cong\varinjlim_F KK_*^\Gamma(P_F^{s}\rtimes_r\hat\Gamma, A)$$

Proof. Combining the Baaj-Skandalis results with 4.5 we obtain isomorphisms

$$K_*^F(A\rtimes_r\Gamma)\cong KK_*^{\hat\Gamma}(P_F^{\mathrm{red}},A\rtimes_r\Gamma)\cong KK_*^\Gamma(P_F^{\mathrm{red}}\rtimes_r\hat\Gamma, A\rtimes_r\Gamma\rtimes_r\hat\Gamma)\cong KK_*^\Gamma(P_F^{\mathrm{red}}\rtimes_r\hat\Gamma, A)$$

and analogous isomorphisms for K_*^{sF}. □

We are now going to determine $P_F^{\mathrm{red}}\rtimes_r\hat\Gamma$ and $P_F^s\rtimes_r\hat\Gamma$ (up to Morita equivalence). For this we define

- $\mathcal{E}_F$ is the universal C^*-algebra with generators $h_s, s\in\Gamma$ satisfying the relations $h_s\geq 0, h_sh_t=0$ if $s^{-1}t\notin F$ and $\sum_s h_sh_t=h_t$
- $\mathcal{E}_F^s$ is the quotient of $\mathcal{E}_F$ by the additional relation

  $$h_sh_{a_1}h_{a_2}\ldots h_{a_n}h_t=0$$

  for all $a_1,\ldots,a_n\in\Gamma$, if $s^{-1}t\notin F$

These algebras are exactly noncommutative simplicial complex algebras, associated with a natural full simplicial complex, in the sense of sections 2 and 3.

Proposition 4.6 *Let Σ_F be the full locally finite simplicial complex with vertex set Γ and given by*

$$\Sigma_F = \{\{s_0, \ldots, s_n\} \subset \Gamma \mid s_i^{-1}s_j \in F, \ \forall i, j = 0, \ldots, n\}$$

Then $\mathcal{E}_F^s \cong \mathcal{C}_{\Sigma_F}$ and $\mathcal{E}_F \cong \mathcal{C}_{\Sigma_F}^{flag}$.

Proof. Obvious from the definitions. □

Let $\mathcal{E}_F^{ab}$ denote the abelianization of $\mathcal{E}_F$, i.e. the quotient of $\mathcal{E}_F$ by the additional relation $h_s h_t = h_t h_s$, $s, t \in \Gamma$. We have natural maps

$$\mathcal{E}_F \longrightarrow \mathcal{E}_F^s \longrightarrow \mathcal{E}_F^{ab}$$

We say that two C^*-algebras A and B with an action of $\hat{\Gamma}$ are $\hat{\Gamma}$-equivariantly Morita equivalent, if there is a C^*-algebra E carrying an action of $\hat{\Gamma}$ with a direct sum decomposition into closed $\hat{\Gamma}$-invariant subspaces A,X,Y and B that multiply like in a 2×2-matrix decomposition

$$\begin{pmatrix} A & X \\ Y & B \end{pmatrix}$$

i.e. A and B are subalgebras of E such that $AB = BA = 0$, $AX = X$, $BY = Y$, $XY = A$, $YX = B$.
Any $\hat{\Gamma}$-equivariant Morita equivalence between two algebras A and B with an action of $\hat{\Gamma}$ induces an invertible element in $KK^{\hat{\Gamma}}(A, B)$, [1] 5.9.

Theorem 4.7 *Let F be a finite subset of Γ (with $F = F^{-1}$ and $1 \in F$).*
The C^*-algebra P_F is $\hat{\Gamma}$-equivariantly Morita equivalent to the full crossed product $\mathcal{E}_F \rtimes \Gamma$ while P_F^{red} is isomorphic to the reduced crossed product $\mathcal{E}_F \rtimes_r \Gamma$. The C^*-algebra P_F^s is $\hat{\Gamma}$-equivariantly Morita equivalent to $\mathcal{E}_F^s \rtimes \Gamma \cong \mathcal{E}_F^s \rtimes_r \Gamma$.

Proof. There is a natural Γ-grading respecting homomorphism α from P_F onto a $\hat{\Gamma}$-invariant C^*-subalgebra of $\mathcal{E}_F \rtimes \Gamma$ mapping $p_s \in P_F$ to $h_1^{\frac{1}{2}} u_s h_1^{\frac{1}{2}}$ (u_s the unitary implementing the automorphism of $\mathcal{E}_F$ given by s). The identity

$$h_1 h_{s_1} h_{s_2} \ldots h_{s_n} u_t h_1 = \alpha(p_1^{\frac{1}{2}} p_{s_1} p_{s_1^{-1}s_2} \ldots p_{s_{n-1}^{-1}s_n} p_{s_n^{-1}t} p_1^{\frac{1}{2}})$$

shows that $\alpha(P_F) = \overline{h_1(\mathcal{E} \rtimes \Gamma)h_1}$. It follows that $\alpha(P_F)$ is a full hereditary subalgebra of $\mathcal{E}_F \rtimes \Gamma$.
To prove the first assertion, it will therefore suffice to show that α is injective (in the definition of Morita equivalence we can then take $X = \overline{h_1(\mathcal{E}_F \rtimes \Gamma)}$ and $Y = \overline{(\mathcal{E}_F \rtimes \Gamma)h_1}$).
Let P_F be faithfully represented as a subalgebra of $\mathcal{L}(H)$ and let $\varrho_s, \lambda_s, s \in \Gamma$ denote the unitary operators in $\mathcal{L}(l^2\Gamma)$ defined by the right and left regular representation

of Γ, respectively.
Consider the operator

$$\hat{p} = \sum_{s\in F} p_s \otimes \varrho_s$$

in $\mathcal{L}(H \otimes l^2\Gamma)$. The relations satisfied by the p_s immediately imply that $\hat{p}$ is a projection (up to replacing ρ_s by λ_s, we have $\hat{p} = \delta_\Gamma(p)$).
Moreover, $\hat{p}$ is invariant under the action of Γ by conjugation by $1 \otimes \lambda_t$, $t \in \Gamma$.
Let e_{st} for $s, t \in \Gamma$ denote the matrix units in $\mathcal{K}(l^2\Gamma)$ (mapping the t-th element in the canonical orthonormal basis for $l^2\Gamma$ to the s-th element) and put

$$f_s = \hat{p}(1 \otimes e_{ss})\hat{p}$$

Then $f_s f_t = \hat{p}(1 \otimes e_{ss})\hat{p}(1 \otimes e_{tt})\hat{p} = \hat{p}(p_{s^{-1}t} \otimes e_{st})\hat{p}$.
Thus $f_s f_t = f_t f_s = 0$ for $s^{-1}t \notin F$ and moreover $\sum_{s\in\Gamma} f_s f_t = f_t$ (since $\sum_s (1 \otimes e_{ss})\hat{p}(1 \otimes e_{tt}) = \hat{p}(1 \otimes e_{tt})$).
Also, obviously $(1 \otimes \lambda_s) f_t (1 \otimes \lambda_s^{-1}) = f_{st}$.
Mapping the generators h_s of $\mathcal{E}_F$ to f_s, we therefore obtain a Γ-covariant representation of $\mathcal{E}_F$ in $H \otimes l^2\Gamma$, whence a representation of $\mathcal{E}_F \rtimes \Gamma$.
In order to show that the homomorphism α, defined at the beginning of the proof, is injective, it is enough to show that the homomorphism $\beta : P_F \to \mathcal{L}(H \otimes \ell^2\Gamma)$ mapping p_s to $f_1^{\frac{1}{2}}(1 \otimes \lambda_s) f_1^{\frac{1}{2}}$ is injective (note that β factors through α).
This however follows from the identity

$$(1 \otimes e_{11})(f_1 \beta(p_{s_1} p_{s_2} \dots p_{s_n}) f_1)(1 \otimes e_{11}) = (p_1 p_{s_1} p_{s_2} \dots p_{s_n} p_1) \otimes e_{11}$$

for all $s_1, s_2, \dots, s_n \in F$, which implies more generally that

$$(1 \otimes e_{11}) f_1 \ \beta(x) \ f_1 \ (1 \otimes e_{11}) = (p_1 x p_1) \otimes e_{11} \quad \text{for all } x \in P_F$$

and the fact that the positive map $P_F \ni x \mapsto p_1 x p_1 \in P_F$ is injective (p_1 is a strictly positive element in P_F).
To treat the case of P_F^{red} one simply has to replace p_s by $p_s \otimes \lambda_s$ everywhere.
The argument for P_F^s is also a verbatim repetition of the preceding discussion. One checks that, in that case

$$f_s f_{a_1} \dots f_{a_n} f_t = \hat{p}\,(p_{s^{-1}a_1} p_{a_1^{-1}a_2} \dots p_{a_n^{-1}t} \otimes e_{st})\,\hat{p}$$

so that the f_s satisfy the relations of $\mathcal{E}_F^s$. $\square$

Corollary 4.8 *Let F be a finite subset of Γ (with $F = F^{-1}$ and $1 \in F$). The C^*-algebra $\mathcal{E}_F$ is Γ-equivariantly Morita equivalent to $P_F^{\mathrm{red}} \rtimes_r \hat{\Gamma}$. The C^*-algebra $\mathcal{E}_F^s$ is Γ-equivariantly Morita equivalent to $P_F^s \rtimes_r \hat{\Gamma}$. We have*

$$K_*^{fin}(A \rtimes_r \Gamma) \quad \cong \quad \varinjlim_F KK_*^\Gamma(\mathcal{E}_F, A)$$

$$K_*^{sfin}(A \rtimes_r \Gamma) \quad \cong \quad \varinjlim_F KK_*^\Gamma(\mathcal{E}_F^s, A)$$

Proof. This follows from 4.7 by Baaj-Skandalis duality. □

Recall now the definition of the Baum-Connes map for a C^*-algebra A with an action of Γ. The topological K-theory of Γ with coefficients in A is defined as $KK_*^\Gamma(\underline{E}_\Gamma, A)$ where $\underline{E}_\Gamma$ is the universal Γ-space with a proper Γ-action. A natural realization of $\underline{E}_\Gamma$ is

$$\underline{E}_\Gamma = \{\, \mu \mid \mu \text{ is a unit measure on } \Gamma \text{ with finite support}\}$$

The technical definition of $KK_*^\Gamma(\underline{E}_\Gamma, A)$ now is

$$KK_*^\Gamma(\underline{E}_\Gamma, A) = \lim_{\overrightarrow{F}} KK_*^\Gamma(\mathcal{C}_0(\underline{E}_F), A)$$

where

$$\underline{E}_F = \{\, \mu \mid \mu \text{ is a unit measure on } \Gamma \text{ with } \operatorname{supp} \mu \subset s_o F \text{ for some } s_0 \in \Gamma\}$$

and where the inductive limit is taken over all finite subsets $F \subset \Gamma$, see e.g.[6]. Note now that $\underline{E}_F$ is exactly the spectrum of $\mathcal{E}_F^{ab}$.

Proposition 4.9 *The C^*-algebra $\mathcal{E}_F^{ab}$ is isomorphic to $\mathcal{C}_0(\underline{E}_F)$. We have*

$$KK_*^\Gamma(\underline{E}_\Gamma, A) \cong \lim_{\overrightarrow{F}} KK_*^\Gamma(\mathcal{E}_F^{ab}, A)$$

The Baum-Connes map $\mu : KK_*^\Gamma(\underline{E}_\Gamma, A) \to K_*(A \rtimes_r \Gamma)$ is exactly the composition of the maps

$$KK_*^\Gamma(\underline{E}_\Gamma, A) = \lim_{\overrightarrow{F}} KK_*^\Gamma(\mathcal{E}_F^{ab}, A) \to \lim_{\overrightarrow{F}} KK_*^\Gamma(\mathcal{E}_F, A) \cong K_*^{fin}(A \rtimes_r \Gamma) \to K_*(A \rtimes_r \Gamma)$$

where the first arrow is induced by the natural map $\mathcal{E}_F \to \mathcal{E}_F^{ab}$.

Proof. The Baum-Connes map is constructed by choosing a so called cut-off function f on $\underline{E}_F$. A cut-off function is a positive function with compact support such that the sum over all translates $s(f)$, $s \in \Gamma$ is the constant function 1 and such that $f s(f) \neq 0$ only for finitely many s. One then considers the projection $e = \sum_{s \in \Gamma} f^{\frac{1}{2}} u_s f^{\frac{1}{2}} \in \mathcal{C}_0(\underline{E}_F) \rtimes_r \Gamma$. The map μ then maps an element $\alpha \in KK_*^\Gamma(\underline{E}_F, A)$ to the Kasparov product $[e] \cdot [\alpha \rtimes_r \Gamma] \in K_*(A \rtimes_r \Gamma)$ where $[\alpha \rtimes_r \Gamma]$ is the element of $KK_*(\mathcal{C}_0(\underline{E}_F) \rtimes_r \Gamma, A \rtimes_r \Gamma)$ obtained from α by "descent".
Now, the generator $h_1 \in \mathcal{E}_F^{ab} \cong \mathcal{C}_0(\underline{E}_F)$ is precisely a possible cut-off function. Under the natural map $P_F^{\text{red}} \subset \mathcal{E}_F \rtimes_r \Gamma \to \mathcal{C}_0(\underline{E}_F) \rtimes_r \Gamma$ the universal projection p is mapped to e. If u and v denote the natural maps $K_*(P_F^{\text{red}}) \to K_*(\mathcal{E}_F^{ab} \rtimes_r \Gamma)$ and $KK_*(\mathcal{E}_F^{ab} \rtimes_r \Gamma, A \rtimes_r \Gamma) \to KK_*(P_F^{\text{red}}, A \rtimes_r \Gamma)$, then $u([p]) \cdot [\alpha \rtimes_r \Gamma] = [p] \cdot v([\alpha \rtimes_r \Gamma])$. However, $[p] \cdot v([\alpha \rtimes_r \Gamma])$ is exactly the image of $v([\alpha \rtimes_r \Gamma])$ under the natural map $KK_*(P_F^{\text{red}}, A \rtimes_r \Gamma) \to K_*(A \rtimes_r \Gamma)$. □

The commutative diagram

$$\begin{array}{ccc} KK_*^\Gamma(\mathcal{E}_F^s, A) & \longrightarrow & K_*^{sfin}(A \rtimes_r \Gamma) \\ \downarrow & & \downarrow \\ KK_*^\Gamma(\mathcal{E}_F, A) & \longrightarrow & K_*^{fin}(A \rtimes_r \Gamma) \end{array}$$

shows that the Baum-Connes map is also equal to the composition of the maps

$$KK_*^\Gamma(\underline{E}_\Gamma, A) \xrightarrow{\varphi} \varinjlim_F KK_*^\Gamma(\mathcal{E}_F^s, A) \cong K_*^{sfin}(A \rtimes_r \Gamma) \xrightarrow{\psi} K_*(A \rtimes_r \Gamma).$$

We will now show that φ is always an isomorphism. Therefore the Baum-Connes map μ is an isomorphism if and only if the natural map

$$K_*^{sfin}(A \rtimes_r \Gamma) \xrightarrow{\psi} K_*(A \rtimes_r \Gamma)$$

is an isomorphism.

Proposition 4.10 *Let F be a finite subset of Γ. The canonical map $\mathcal{E}_F^s \to \mathcal{E}_F^{ab}$ is a KK^Γ-equivalence.*

Proof. This follows from 2.13, since the dimension of Σ_F is bounded by $|F|-1$. □

We summarize the preceding discussion in the following theorem, which is our main result .

Theorem 4.11 *Let A be a C^*-algebra on which Γ acts by automorphisms. The left hand side $KK_*^\Gamma(\underline{E}_\Gamma, A)$ of the Baum-Connes conjecture is canonically isomorphic to $K_*^{sfin}(A \rtimes_r \Gamma)$. The Baum-Connes map $\mu : KK_*^\Gamma(\underline{E}_\Gamma, A) \to K_*(A \rtimes_r \Gamma)$ corresponds, under this isomorphism, to the natural map $K_*^{sfin}(A \rtimes_r \Gamma) \to K_*(A \rtimes_r \Gamma)$.*

We thus have the following factorizations for the Baum-Connes map:

$$KK_*^\Gamma(\underline{E}_\Gamma, A) \cong K_*^{sfin}(A \rtimes_r \Gamma) \longrightarrow K_*(A \rtimes_r \Gamma).$$

and

$$KK_*^\Gamma(\underline{E}_\Gamma, A) \longrightarrow K_*^{sfin}(A \rtimes_r \Gamma) \longrightarrow K_*^{fin}(A \rtimes_r \Gamma) \longrightarrow K_*(A \rtimes_r \Gamma)$$

In view of the discussion in section 3, it seems quite uncertain if the map

$$K_*^{sfin}(A \rtimes_r \Gamma) \longrightarrow K_*^{fin}(A \rtimes_r \Gamma)$$

could be an isomorphism in general. We mention however that, if A is proper in the sense of [6], then it is nearly trivial that the map $K_*^{fin}(A \rtimes_r \Gamma) \to K_*(A \rtimes_r \Gamma)$ is an isomorphism. Thus, by the result in [6] combined with Theorem 4.11, the map $K_*^{sfin}(A \rtimes_r \Gamma) \to K_*^{fin}(A \rtimes_r \Gamma)$ has to be an isomorphism.
The connection between $K_*^{fin}(A \rtimes_r \Gamma)$ and $K_*^{sfin}(A \rtimes_r \Gamma)$ becomes however much closer if we take the skeleton filtration into account. Given a finite subset F of Γ, let (I_k) and (I_k^s) be the skeleton filtrations in $\mathcal{E}_F$ and $\mathcal{E}_F^s$, respectively. We denote by J_F the kernel of the natural map $\mathcal{E}_F \to \mathcal{E}_F^s$.

Lemma 4.12 *Let F be a finite subset of Γ with $F = F^{-1}$ and $1 \in F$, and F^n the set of all products of n elements in F. Let moreover denote h_s, $s \in \Gamma$, the generators of $\mathcal{E}_F$ and $\mathcal{E}^s_F$.*

(a) If $s^{-1}t \notin F^n$ for s, t in Γ, then for all $t_1, \dots, t_{n-1}$ in Γ, the product

$$h_s h_{t_1} \dots h_{t_{n-1}} h_t$$

is zero in $\mathcal{E}_F$.

(b) The natural map $\mathcal{E}_{F^n} \longrightarrow \mathcal{E}_F/I_{n+1}$ maps J_{F^n} to zero.

(c) The natural map $\mathcal{E}_{F^n}/I_{n+1} \longrightarrow \mathcal{E}_F/I_{n+1}$ factors as

$$\mathcal{E}_{F^n}/I_{n+1} \longrightarrow \mathcal{E}^s_{F^n}/I^s_{n+1} \longrightarrow \mathcal{E}_F/I_{n+1}$$

Proof. Obvious. □

Proposition 4.13 *Let A be a C^*-algebra with an action of Γ. For each fixed n, we have an isomorphism*

$$\lim_{\overrightarrow{F}} KK^\Gamma_*(\mathcal{E}_F/I_n, A) \;\cong\; \lim_{\overrightarrow{F}} KK^\Gamma_*(\mathcal{E}^s_F/I^s_n, A) \tag{1}$$

Proof. This follows from 4.12 (c). □

If we take the inductive limit over n in the right hand side of equation 1 we obtain $K^{sfin}_*(A \rtimes_r \Gamma)$.

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