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The smallest Dirac eigenvalue in a spin-conformal class and cmc-immersions

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Abstract

Let us fix a conformal class $[g_0]$ and a spin structure σ on a compact manifold M. For any $g \in [g_0]$, let $\lambda_1^+(g)$ be the smallest positive eigenvalue of the Dirac operator D on (M, g, σ). In a previous paper we have shown that

$$\lambda(M, g, \sigma) := \inf_{g \in [g_0]} \lambda_1^+(g) \mathrm{vol}(M, g)^{1/n} > 0.$$

In the present article, we enlarge the conformal class by certain singular metrics. We will show that if $\lambda(M, g, \sigma) < \lambda(S^n)$, then the infimum is attained on the enlarged conformal class. For proving this, we have to solve a system of semi-linear partial differential equations involving a nonlinearity with critical exponent:

$$D\varphi = \lambda |\varphi|^{2/(n-1)} \varphi.$$

The solution of this problem has many analogies to the solution of the Yamabe problem. However, our reaoning is more involved than in the Yamabe problem as the spectrum of the Dirac operator is not bounded from below. The solution may have a nonempty zero set because a maximum principle is not available.

Using the Weierstraß representation, the solution of this equation in dimension 2 provides a tool for the construction of new constant mean curvature surfaces.

Keywords: Dirac operator, eigenvalues, conformal geometry, critical Sobolev exponents

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1 Introduction

Let M be a compact manifold, on which we fix a conformal class $[g_0]$ and a spin structure σ. For each metric $g \in [g_0]$ let $\lambda_1^+(g)$ be the smallest positive eigenvalue of the (classical) Dirac operator D on (M, g, σ). Note that the dimension of the kernel of the Dirac operator is a conformal invariant.

The main subject of this article is the following infimum:

$$\lambda(M, g, \sigma) := \inf_{g \in [g_0]} \lambda_1^+(g) \mathrm{vol}(M, g)^{1/n}. \tag{1.1}$$

If the dimension is ≥ 3 or if D is invertible, one knows [Amm03] that

$$\lambda(M, g, \sigma) \leq \lambda(S^n), \tag{1.2}$$

where S^n carries the round metric. The main result (Theorem 2.2) of our article states that if strict inequality holds in (1.2), then the nonlinear partial differential equation

$$D\varphi = \lambda |\varphi|^{2/(n-1)} \varphi \tag{1.3}$$

has a solution. The nonlinearity is critical in the sense that the exponents of the Sobolev embeddings involved are critical. The solution of this equation has two major applications. The first application tells us that the infimum in (1.1) is actually attained if we slightly enlarge the conformal class. The second application is that if $\dim M = 2$, then we obtain an existence result for surfaces with constant mean curvature.

We want to emphasize that several equations with similar critical nonlinearities have been the subject of intensive research. We will give two examples:

The first one is the Yamabe problem [LP87]. Yamabe, Trudinger and Aubin have shown that if the Yamabe invariant $\lambda_Y = \lambda_Y(M, [g])$ is smaller than the Yamabe invariant of the round sphere, then the Yamabe equation

$$\left(4 \frac{n-1}{n-2} \Delta + \mathrm{scal} \right) u = \lambda_Y u^{p-1}$$

admits a positive smooth solution u. By deep results, explained e.g. in [LP87], it has been shown that any manifold M which is not conformal to the round sphere actually satisfies $\lambda_Y(M, [g]) < \lambda_Y(S^n)$. As a consequence, there is a metric with constant scalar curvature on M, conformal to g.

The second one is the Yamabe problem on CR-manifolds, studied by Jerison and Lee [JL89]. For any compact, orientable, strictly pseudoconvex $(2n+1)$-dimensional CR-manifold M, they construct an invariant $\lambda^{CR}(M)$ satisfying

$$\lambda^{CR}(M) \leq \lambda^{CR}(S^{2n+1}). \tag{1.4}$$

Jerison and Lee prove that if strict inequality holds in (1.4), then M admits a pseudohermitian structure with constant (Webster or Fefferman) scalar curvature.

2 Statement of results and Overview

Let us summarize what is known about the invariant λ from above.

If D is invertible, then J. Lott has proven [Lot86] that λ is greater than zero. In [Amm03], we have shown that this even holds if D is not invertible. By a blow-up argument [Amm03] one sees that if $\dim M \geq 3$ or if D is invertible, then $\lambda(M, g, \sigma) \leq \lambda(S^n)$. In the remaining case, i.e. if $\dim M = 2$ and D is not invertible, it is still unclear whether $\lambda(M, g, \sigma) \leq \lambda(S^2)$ or not. Hijazi [Hij86] proved for $\dim M \geq 3$ that $\lambda^2(M, g, \sigma) \geq \frac{n}{4(n-1)} \lambda_Y(M, g)$. Christian Bär [Bär92] showed a 2-dimensional analogue, which is only non-trivial if M has genus zero, i.e. $M \cong S^2$. Indeed he proved

$$\lambda^2(S^2, g_{\mathrm{can}}) := \inf_{g \in [g_{\mathrm{can}}]} \left(\lambda_1^+(S^2, g)\right)^2 \mathrm{area}(S^2, g) = 4\pi$$

and the infimum is attained exactly for the metrics of constant Gauss curvature.

In order to formulate our main result, we have to introduce certain types of singularities.

Definition. Let $\mathcal{M}(g_0)$ be the set of metrics in $[g_0]$ with unit volume. We define the set of *generalized conformal metrics* $\overline{\mathcal{M}(g_0)}$ by

$$\begin{aligned}\overline{\mathcal{M}(g_0)} := \{g = f^{2/(n-1)} \cdot g_0 \mid \quad & f \geq 0,\, f \in C^{1,\alpha}(M) \cap C^\infty(M \setminus f^{-1}(0))\ \forall \alpha \in (0,1), \\ & \mathrm{supp}\, f = M,\ \mathrm{vol}(M, g) := \textstyle\int f^{n/(n-1)} = 1\}.\end{aligned}$$

We say that g is *regular*, if $f^{-1}(0) = \emptyset$. Otherwise g is *singular* and $\mathcal{S}_g := f^{-1}(0)$ is called the *singular set.*

Note that singular metrics are not complete.

For any generalized conformal metric g, we define the *smallest positive Dirac eigenvalue* on (M, g, σ) as

$$\lambda_1^+(g) := \inf \left\{ \frac{(\psi, \psi)_g}{(\psi, D_g^{-1}\psi)_g} \,\middle|\, \psi \in \mathrm{im}_{C^\infty} D_g, \quad \|\psi\|_{L^\infty} < \infty, \quad (\psi, D_g^{-1}\psi)_g > 0 \right\}. \tag{2.1}$$

Here $\mathrm{im}_{C^\infty} D_g$ is the image of the Dirac operator on $(M \setminus \mathcal{S}_g, g)$ acting on smooth spinors. The scalar product $(.,.)_g$ is the L^2-scalar product on spinors on $M \setminus \mathcal{S}_g$.

Obviously, this definition coincides with the smallest positive Dirac eigenvalue in the ordinary sense if g is regular. In Lemma 8.1 we show that

$$\inf_{g\in\overline{\mathcal{M}(g_0)}} \lambda_1^+(g) = \inf_{g\in\mathcal{M}(g_0)} \lambda_1^+(g).$$

THEOREM 2.2. *Let M be a compact manifold of dimension ≥ 2 with a fixed conformal class $[g_0]$ and a spin structure σ. Assume that $\lambda = \lambda(M,g,\sigma) := \inf\{\lambda_1^+(g) \mid g \in \mathcal{M}(g_0)\}$ satisfies*

$$\lambda(M,g,\sigma) < \lambda(S^n) = \frac{n}{2}\,\omega_n^{-1/n}. \tag{2.3}$$

Then

(A) *there is a spinor field $\varphi \in C^{1,\alpha}(\Sigma M) \cap C^\infty(\Sigma(M \setminus \varphi^{-1}(0)))$ on (M,g_0) such that*

$$D_{g_0}\varphi = \lambda\,|\varphi|^{2/(n-1)}\varphi, \qquad \|\varphi\|_{2n/(n-1)} = 1, \tag{2.4}$$

(B) *there is a generalized conformal metric $g \in \overline{\mathcal{M}(g_0)}$ such that*

$$\lambda_1^+(g) = \lambda.$$

It is not difficult to see that statement (B) follows directly from statement (A). If we have a solution as in (A), then we set $g_1 := f^{2/(n-1)}g_0$ with $f = \langle\varphi,\varphi\rangle$. Note that $\mathrm{vol}(M,g_1) = \int |\varphi|^{2n/(n-1)} = 1$, hence $g_1 \in \overline{\mathcal{M}(g_0)}$.

The transformation formula for the Dirac operator under conformal changes (Proposition 4.1) implies that there is a spinor φ_1 on (M,g_1,σ) such that

$$D_{g_1}\varphi_1 = \lambda\varphi_1, \qquad |\varphi_1|_{g_1} \equiv 1$$

on $M \setminus \mathcal{S}_{g_1}$. Then obviously, $\lambda_1^+(g_1) = \lambda$.

Proof of (A)**.** For any $q \geq q_D := 2n/(n+1)$ we define

$$\mu_q := \sup \frac{(\psi, D^{-1}\psi)}{\|\psi\|_{L^q}^2}$$

where the infimum ranges over all smooth non-zero spinors on (M,g_0,σ) in the image of D. For $q > q_D$, the problem is "subcritical" and we will see in Proposition 6.4 that there is a weak solution ψ_q of

$$D^{-1}\psi_q = \mu_q|\psi_q|^{q-2}\psi_q, \qquad \psi_q \in L^q(\mathcal{D}), \qquad \|\psi_q\|_{L^q} = 1. \tag{2.5}$$

Our Regularity Theorem (Theorem 6.3) will imply that the solution is $C^{0,\alpha}$.

By a straightforward calculation we see that the following duality principle holds.

Lemma 2.6 (Duality principle).
Let $p, q > 1$, $\lambda, \mu \in \mathbb{R}^+$ with $p^{-1} + q^{-1} = 1$ and $\lambda\mu = 1$.

(i) *If ψ satisfies (2.5), then $\varphi := \lambda\, D^{-1}\psi$ satisfies*

$$D\varphi = \lambda\, |\varphi|^{p-2}\varphi, \qquad \varphi \in L^p(\mathcal{D}), \qquad \|\varphi\|_{L^p} = 1 \tag{2.7}$$

(ii) *If φ satisfies (2.7), then $\psi := \mu\, D\varphi$ satisfies (2.5).*

We will study the behavior of the solutions $\varphi_q := D^{-1}\psi_q$ for $q \to q_D$. Theorem 6.5 shows that the solutions φ_q are uniformly bounded in L^∞. Applying the Regularity Theorem (Theorem 6.3) once again, we see that ψ_q is even uniformly bounded in $C^{0,\alpha}$, hence φ_q is uniformly bounded in $C^{1,\alpha}$. After taking a subsequence $q_i \to q_D$, the spinor fields φ_{q_i} converge to a $C^{1,\alpha}$-solution of (2.4). □

The structure of the article is as follows. In Section 3 we recall several facts from the standard elliptic theory that will be needed later on. In Section 4 we recall the formula for the change of the Dirac operator under conformal change of the metric. Section 5 introduces a functional $\mathcal{F}_q$ whose critical points are solutions to (2.5). It remains to find conditions under which the supremum $\mathcal{F}_q$ is attained. Section 6 proves the regularity theorem. We obtain $C^{1,\alpha}$-solutions to the subcritical problem and we see that (2.3) implies the existence of a solution to the critical problem. Section 7 deals with the singularities that may appear. The last section sketches how to apply our results to the construction of constant mean curvature surfaces.

As a result we obtain

Principle for construction of cmc-surfaces. *For any Riemann spin surface (M, g, σ) with $\lambda^2 < 4\pi$, there is a (branched) periodic conformal immersion of $(\widetilde{M}, g, \sigma)$ into $\mathbb{R}^3$ with constant mean curvature.*

Remark. *The largest negative eigenvalue.* Denote the largest negative eigenvalue of the Dirac operator by $-\lambda_1^-(g)$, i.e. $-\lambda_1^-(g)$ is the negative eigenvalue closest to 0. All statements of the present article hold analogously if we replace the smallest positive eigenvalue $\lambda_1^+(g)$ by $\lambda_1^-(g)$. We omit the proofs as they are the same up to some sign changes.

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3 Elliptic regularity

In this section we want to collect some facts about elliptic regularity for Dirac operators. The proofs of these statements are analogous to proofs of the corresponding statements for the Laplace operators as done e.g. in [GT77] and [Ada75]. Details will be given in [Amm].

Let (M, g) be a manifold with a spin structure σ.

Definition (Sobolev spaces). Let us introduce the function spaces we need. For any smooth spinor ψ, and $q \in (1, \infty)$, $k \in \mathbb{N}$, we define the H_k^q-norm of ψ as

$$\|\psi\|_{H_k^q} = \|\underbrace{\nabla \dots \nabla}_{k-\text{times}} \psi\|_{L^q}.$$

Obviously, the H_q^k-norms for different connections are equivalent.

If M is compact, an alternative way to introduce Sobolev norms on spinors is by setting

$$\|\psi\|_{\widetilde{H}_k^q} = \Big\| \, |D|^k \psi \Big\|_{L^q} + \|\pi\psi\|_1,$$

where π is the L^2-orthogonal projection to the kernel of D, where $\| \cdot \|_1$ is an arbitrary norm on the kernel, and where $|D^k|$ should be understood in the spectral sense, i.e. if φ is an eigenspinor of D to the eigenvalue λ, then $|D|^k \varphi = |\lambda|^k$. Such powers of differential operators are well-understood because M is compact (see [See67, Tay81]). One consequence of the properties of such operators is that the norms $\tilde{H}_k^q$ are equivalent to the H_k^q-norms. The definition of the $\widetilde{H}_k^q$-norms has the advantage that it extends to arbitrary $k \in \mathbb{R}$.

We define the Sobolev space $H_k^q(\Sigma M) = H_k^q$ as the completion of the smooth spinors with respect to this norm.

Definition (Hölder spaces). For $\alpha \in (0, 1]$, the *Hölder-spaces* $C^{0,\alpha}(\Sigma M)$ and $C^{1,\alpha}(\Sigma M)$ are defined to be the completions of $C^\infty(\Sigma M)$ with respect to the *Hölder norms*

$$\begin{aligned}
\|\varphi\|_{C^{0,\alpha}} &:= \text{höl}_\alpha(\varphi) \\
\|\varphi\|_{C^{1,\alpha}} &:= \|\varphi\|_{\sup} + \text{höl}_\alpha(\nabla\varphi) \\
\text{höl}_\alpha(Q) &:= \sup\Big\{ \frac{|Q(x) - P_\gamma Q(y)|}{d(x,y)^\alpha} \mid x, y \in M, x \neq y, P_\gamma \text{ is the} \\
&\qquad \text{parallel transport along a shortest geodesic } \gamma \text{ from } x \text{ to } y. \Big\}
\end{aligned}$$

From elliptic theory, we know the following statements [Amm].

Theorem 3.1 (Sobolev embedding theorem). *Let $k, s \in \mathbb{R}$, $k \geq s$ and $q, r \in (1, \infty)$.*

(a) *If*
$$\frac{1}{r} - \frac{s}{n} \geq \frac{1}{q} - \frac{k}{n}, \tag{3.2}$$
then $H^q_k(\Sigma M)$ is continuously embedded into $H^r_s(\Sigma M)$.

(b) (Rellich-Kondrakov theorem). *If strict inequality holds in (3.2), then the inclusion $H^q_k(\Sigma M) \hookrightarrow H^r_s(\Sigma M)$ is a compact operator.*

(c) *Suppose $0 < \alpha < 1$, $m \in \{0, 1\}$ and*
$$\frac{1}{q} \leq \frac{k - m - \alpha}{n}.$$
Then $H^q_k(\Sigma M)$ is continuously embedded into $C^{m,\alpha}(\Sigma M)$.

Theorem 3.3 (Interior L^p estimates). *Let Ω be open in $\mathbb{R}^n$, equipped with a Riemannian metric and the spin structure σ induced from $\mathbb{R}^n$, and $K \subset \Omega$ compact. We assume $g(v, v) \geq \zeta g_{\mathrm{eucl}}(v, v)$ for all $v \in T\Omega$ and $g \in C^{k+1}$ with $\|g\|_{C^{k+1}} \leq Z$. Let D be the Dirac operator on (Ω, g, σ). Let ψ be in $H^q_k(\Sigma\Omega)$ and let φ be a weak solution of*
$$D\varphi = \psi$$
on Ω. Then $\varphi \in H^q_{k+1}(\Sigma K)$ and
$$\|\varphi\|_{H^q_{k+1}(\Sigma K)} \leq C \cdot \Big(\|\psi\|_{H^q_k(\Sigma\Omega)} + \|\varphi\|_{L^q(\Sigma\Omega)} \Big),$$
where $C = C(\zeta, Z, \Omega, K)$.

Theorem 3.4 (Interior Schauder estimates). *Let Ω be open in $\mathbb{R}^n$, equipped with a Riemannian metric and the spin structure induced from $\mathbb{R}^n$, and $K \subset \Omega$ compact. Let ψ be a $C^{0,\alpha}$-spinor on Ω. We assume $g(v, v) \geq \zeta g_{\mathrm{eucl}}(v, v)$ for all $v \in T\Omega$ and $\|g\|_{C^{k+1,\alpha}(\Omega)} \leq Z$. Then for any C^1-solution φ of*
$$D\varphi = \psi$$
we have $\varphi \in C^{k+1,\alpha}(K)$ and
$$\|\varphi\|_{C^{k+1,\alpha}(K)} \leq C \cdot (\|\psi\|_{C^{k,\alpha}(\Omega)} + \|\varphi\|_{C^0(\Omega)})$$
where C only depends on n, α, $\mathrm{diam}(\Omega)$, $\mathrm{dist}(K, \partial\Omega)$, ζ and Z.

By gluing together the local versions via charts, we obtain global L^p and Schauder estimates.

Theorem 3.5 (Global L^p estimates). *Let (M, g, σ) be a compact Riemannian spin manifold and $\psi \in H_k^q(\Sigma M)$. Then any weak solution of*

$$D\varphi = \psi$$

satisfies $\varphi \in H_{k+1}^q(\Sigma M)$, and there is a constant $C = C(M, g, \sigma)$ such that

$$\|\varphi\|_{H_{k+1}^q} \leq C\left(\|\psi\|_{H_k^q} + \|\varphi\|_{L^q}\right).$$

Theorem 3.6 (Global Schauder estimates). *Let (M, g, σ) be a compact Riemannian spin manifold and $\psi \in C^{k,\alpha}(\Sigma M)$. Then any solution of*

$$D\varphi = \psi,$$

satisfies $\varphi \in C^{k+1,\alpha}(\Sigma M)$, and there is a constant $C = C(M, g, \sigma)$ such that

$$\|\varphi\|_{C^{k+1,\alpha}} \leq C\left(\|\psi\|_{C^{k,\alpha}} + \|\varphi\|_{C^0}\right).$$

4 Conformal changes and the Dirac operator

In this section we recall the formula for the Dirac operator under conformal changes of the metric.

Proposition 4.1 ([Hit74, Hij86]). *Let $g = e^{2u}g_0$. There is an isomorphism of vector bundles $F : \Sigma(M, g_0, \sigma) \to \Sigma(M, g, \sigma)$ which is a fiberwise isometry such that*

$$D_g(F(\varphi)) = F\left(e^{-\frac{n+1}{2}u} D_{g_0} e^{\frac{n-1}{2}u}\varphi\right).$$

For understanding the idea behind several proofs in this paper, it might be helpful to the reader to identify a spinor field $\psi \in \Gamma(\Sigma(M, g_0, \sigma))$ with $F(e^{-\frac{n+1}{2}u}\psi) \in \Gamma(\Sigma(M, g, \sigma))$. Then the above transformation formula reads as

$$D_g(e^u\psi) = D_{g_0}(\psi).$$

With this identification, the expression $(\psi, D^{-1}\psi)$ is conformally independent. Furthermore, $\|\psi\|_{L^q}$ is conformally independent if and only if q equals to the critical exponent $q_D := 2n/(n+1)$.

All spinor fields for which this identification is helpful will be denoted by ψ, all other spinor fields by φ.

5 The functional

For any $q \in [q_D, \infty)$ we define the functional

$$\mathcal{F}_q : (\mathrm{im}_{C^\infty} D_{g_0}) \smallsetminus \{0\} \to \mathbb{R}, \quad \psi \mapsto \frac{(\psi, D^{-1}\psi)}{\|\psi\|_{L^q}^2}.$$

Here we choose the preimage $D^{-1}\psi$ of ψ orthogonal to the kernel of $\ker D$.

Wet set

$$\mu_q =: \mu_q(M, g_0, \sigma) := \sup_{\psi \in (\mathrm{im}_{C^\infty} D_{g_0}) \smallsetminus \{0\}} \mathcal{F}_q(\psi).$$

Because D_{g_0} has positive eigenvalues, we see that $\mu_q \in (0, \infty]$.

Lemma 5.1. *Let $q \in [q_D, \infty)$. Then*

(1) $\mu_q < \infty$,

(2) $\mathcal{F}_q$ *extends to a differentiable functional on $L^q(\mathrm{im}_{C^\infty} D_{g_0}) \smallsetminus \{0\}$ and the derivation is given by*

$$d\mathcal{F}_q(\psi)(\varphi) = \frac{2}{\|\psi\|_{L^q}^2} \int \langle D^{-1}\psi - \rho_{q,\psi} |\psi|^{q-2}\psi, \varphi \rangle, \tag{5.2}$$

where $\rho_{q,\psi} = \mathcal{F}_q(\psi) \|\psi\|_{L^q}^{2-q}$.

Proof. The Embedding Theorem 3.1 implies that for any q there is a constant C_q such that

$$\|\psi\|_{H^2_{-1/2}} \le C_q \| \psi \|_{L^q} \qquad \forall \psi \in \mathrm{im}_{C^\infty} D_{g_0}.$$

Hence

$$\frac{(\psi, |D|^{-1}\psi)}{\|\psi\|_{L^q}^2} \le C'$$

and hence (1). Similarly one proves that $\mathcal{F}_q$ extends to $L^q(\mathrm{im}_{C^\infty} D_{g_0}) \smallsetminus \{0\}$.

Now take p with $p^{-1} + q^{-1} = 1$. If ψ is in L^q, $q \ge \frac{2n}{n+1}$, then $D^{-1}\psi$ is in H_1^q and hence in L^p. Also $|\psi|^{q-2}\psi \in L^p$. Hence the right hand side of (5.2) defines a continuous functional on L^q which we denote by $\varphi \mapsto \mathrm{RHS}_\psi(\varphi)$. Similarly one sees that

$$\mathcal{F}_q(\psi + \varphi) - \mathcal{F}_q(\psi) - \mathrm{RHS}_\psi(\varphi) \le o(\|\varphi\|_{L^q}),$$

hence $\mathcal{F}_q$ is Fréchet differential with derivative RHS_ψ. $\square$

PROPOSITION 5.3 (Properties of μ_q). *The function* $[q_D, \infty) \to (0, \infty), \quad q \mapsto \mu_q$ *is*

(1) *nonincreasing in* q*,*

(2) *is continuous from the right,*

(3) $\mu_2 = (\lambda_1^+(g_0))^{-1}$.

(4) μ_{q_D} *is conformally invariant, i. e. for* $g_1 \in [g_0]$

$$\mu_{q_D}(M, g_0, \sigma) = \mu_{q_D}(M, g_1, \sigma).$$

(5) $\mu_{q_D}(M, g, \sigma) \geq \mu_{q_D}(S^n, g_{\mathrm{can}})$ *if* D *is invertible or if* $n \geq 3$.

Here g_{can} is the metric on S^n of constant sectional curvature and volume 1.

Proof.
(1) This is evident as $q \mapsto \|\psi\|_{L^q}$ is nondecreasing,
(2) For a given $q \geq q_D$, we take a smooth spinor field ψ such that $\mathcal{F}_q(\psi) \geq \mu_q - \varepsilon$. Observe that for $q' \geq q$

$$\mathcal{F}_{q'}(\psi) = \frac{\|\psi\|_{L^q}}{\|\psi\|_{L^{q'}}} \mathcal{F}_q(\psi).$$

The function $q' \mapsto \|\psi\|_{L^{q'}}$ is continuous, hence if q' is sufficiently close to q, then

$$\mu_{q'} \geq \mathcal{F}_{q'}(\psi) \geq \mathcal{F}_q(\psi) - \varepsilon \geq \mu_q - 2\varepsilon.$$

Because $q \mapsto \mu_q$ is nonincreasing, the statement follows.
(3) follows directly, by decomposing $L^2(\mathrm{im}_{C^\infty} D_{g_0})$ into eigenspaces for D.
(4) This follows from section 4.
(5) This has been proven in [Amm03]. The idea of the proof is as follows. In a small neighborhood $B_\varepsilon(p)$ of a given point p, we conformally blow up a sphere on which we construct a suitable test spinor. The corresponding Rayleigh quotient for D^2 is close to the smallest eigenvalue of D^2 on the sphere. Hence, the statement follows for the case that D is invertible. If D is not invertible, we have to ensure that the test spinor is almost orthogonal to the kernel of D. If $n \geq 3$, this can be done by choosing $B_\varepsilon(p)$ sufficiently small. $\square$

In Proposition 6.4 we will see that the supremum defining μ_q is attained for $q > q_D$ by a $C^{0,\alpha}$-function which in turn implies that the function

$$[q_D, \infty) \to (0, \infty], \quad q \mapsto \mu_q$$

is also continuous from the left.

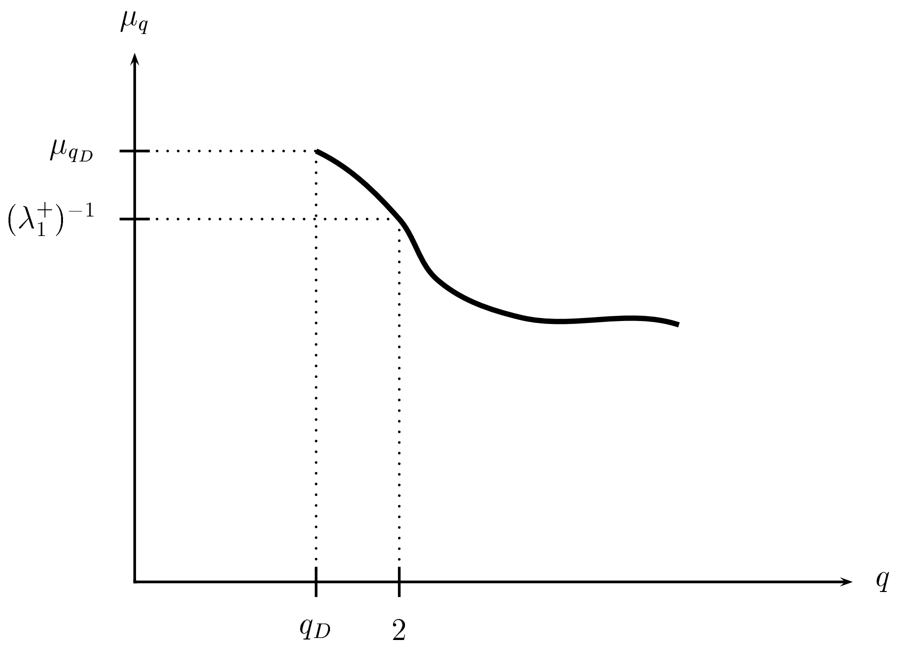

Figure 1: μ_q as a function of q

COROLLARY 5.4. *The smallest positive eigenvalue of the Dirac operator is bounded from below by $\mu_{q_D}^{-1}$:*

$$\lambda_1^+ \geq \mu_{q_D}^{-1}.$$

6 Solution of the equation

If the supremum of $\mathcal{F}_q$ is actually attained by a function ψ_q with $\|\psi_q\|_{L^q} = 1$, then because of (5.2) ψ_q is a solution of

$$D^{-1}\psi = \|\psi\|_{L^q}^{2-q}\mu_q \quad |\psi|^{q-2}\psi. \tag{6.1}$$

Obviously, $\mathcal{F}_q(r\psi) = \mathcal{F}_q(\psi)$ for all $r \in \mathbb{R}^+$. As a consequence, any solution can be rescaled to one with $\|\psi\|_{L^q} = 1$. Hence, we study

$$D^{-1}\psi = \mu_q|\psi|^{q-2}\psi, \qquad \|\psi\|_{L^q} = 1, \qquad \psi \in L^q(\mathcal{D}). \tag{6.2}$$

THEOREM 6.3 (Regularity theorem). *Suppose that $\psi \in L^q$, $q \geq q_D$ is a solution of equations (2.5). Suppose that there is an $r > q_D$ such that $\|\psi\|_{L^r} < \infty$. We choose $k, K > 0$ such that $\|\psi\|_{L^r} < k$ and $\mu_q \geq K$. Then for any $\alpha \in (0,1)$ there is* *a constant C depending only on (M, g, σ), q, r, K, k and α with*

$$\|\psi\|_{C^{0,\alpha}} \leq C \qquad \textit{and} \qquad \|D^{-1}\psi\|_{C^{1,\alpha}} \leq C$$

Proof. Without loss of generality we can assume $r < n$. We apply the Global L^p-estimates 3.5 to $\psi \in L^r$, and then the Sobolev embedding 3.1 and obtain

$$D^{-1}(\psi) \in H_1^r \hookrightarrow L^s$$

with $1 - (n/r) = -n/s$, or equivalently $s = rn/(n-r)$. Hence, by equation 2.5, we know $\psi \in L^{s(q-1)} \hookrightarrow L^{s(q_D-1)}$. We set

$$r' := s(q_D - 1) = \frac{n-1}{n+1}\,\frac{rn}{(n-r)}.$$

The inequality $r > q_D$ implies $r' > r$. By iterating this bootstrap argument we obtain higher and higher regularity for ψ. The function

$$r \mapsto \frac{n-1}{n+1}\,\frac{rn}{(n-r)}$$

tends to ∞ if r converges to n from below. This shows that $\psi \in L^r$ for all $r \in (0, \infty)$. Thus $\psi \in H_1^r$ for all $r \in (0, \infty)$ and by the Sobolev embedding theorem 3.1 (c) one obtains $\psi \in C^{0,\alpha}$ for any $\alpha > 0$. And finally by the Schauder estimates 3.6 $D^{-1}\psi \in C^{1,\alpha}$. The uniform upper bound C of the norms is now clear from the construction. $\square$

Remark. Outside the zero locus $\psi^{-1}(0)$ we can continue the bootstrap argument and apply inductively the interior Schauder estimates Theorem 3.4. We conclude that ψ is smooth on $M \setminus \psi^{-1}(0)$. Similarly, if $p := (q-1)^{-1} + 1$ is an even integer, then $\varphi \mapsto |\varphi|^{p-2}\varphi$ is also smooth in 0, and hence ψ is smooth on M.

PROPOSITION 6.4. *For any $q > q_D$ the supremum μ_q is attained by a spinor field $\psi_q \in C^{0,\alpha}$ which is a solution of* (2.5).

Proof. Let ψ_i be a maximizing sequence for $\mathcal{F}_q$, i. e. $\mathcal{F}_q(\psi_i) \to \mu_q$. We may assume $\|\psi_i\|_{L^q} = 1$. After taking a subsequence there is a $\psi_\infty \in L^q$ such that ψ_i converges weakly to ψ_∞ in L^q. Because the embedding $L^q \hookrightarrow H^2_{-1/2}$ is compact, we can again choose a subsequence, and we obtain, in addition to the weak convergence in L^q, strong convergence to ψ_∞ in $H^2_{-1/2}$. Hence,

$$\mu_q \leq \limsup \mathcal{F}_q(\psi_i) \leq \limsup \frac{\|\psi_i\|^2_{H^2_{-1/2}}}{\|\psi_i\|^2_{L^q}} \leq \mathcal{F}_q(\psi_\infty) \leq \mu_q.$$

As a consequence, we have equality in all inequalities, in particular $\|\psi_\infty\|_{L^q} = 1$. By the variational formula ψ_∞ is a solution of

$$D^{-1}\psi_\infty = \mu_q|\psi_\infty|^{q-2}\psi_\infty.$$

Now the regularity theorem tells us that ψ_∞ is actually $C^{0,\alpha}$. $\square$

THEOREM 6.5. *Let ψ be a solution of (2.5) with $q \in (q_D, 2]$ and $\mu_q \geq \mu_{q_D}^{S^n} + \varepsilon$, $\varepsilon > 0$. Then there is a constant $C = C(M, g, \sigma, \varepsilon)$ such that*

$$\|\psi\|_{C^0} < C.$$

Proof. Assume that such a constant does not exist. Then we find a sequence of solutions ψ_k to (2.5) with $q = q_k$, $\mu_q = \mu_k \geq \mu_{q_D}^{S^n} + \varepsilon$ and

$$\|\psi_k\|_{C^0} \to \infty. \tag{6.6}$$

Let us assume for a moment that $q_\infty := \limsup q_k > q_D$. In this case, we can choose a subsequence with $q_k \to q_\infty$. Then the regularity theorem (Theorem 6.3) applied to a real number $r \in (q_D, q_\infty)$, says that $\|\psi_k\|_{C^0}$ is bounded, in contradiction to (6.6). Hence, $\lim q_k = q_D$.

We now study

$$\varphi_k := \frac{1}{\mu_k} D^{-1} \psi_k,$$

which are solutions of the dual equation (2.7).

There is a sequence of points $s_k \in M$ with

$$m_k := |\varphi_k(s_k)| = \max\{\varphi_k(x) \mid x \in M\} \to \infty.$$

Since M is compact, we can assume, after passing to a further subsequence, that s_k converges to $p \in M$.

Now, we define rescaled geodesic normal coordinates $(\sigma_k)^{-1}$ via the formula

$$\sigma_k(x) = \exp_p \Big(\delta_k x + \exp_p^{-1}(z_k) \Big),$$

where $\delta_k = m_k^{2-p_k} \to 0$.

Then a straightforward calculation shows that

$$\widetilde{\varphi}_k(x) := m_k^{-1} \varphi_k \circ \sigma_k(x)$$

is a solution of

$$D_k \widetilde{\varphi}_k = \frac{1}{\mu_k} |\widetilde{\varphi}_k|^{p_k - 2} \widetilde{\varphi}_k,$$

where D_k is the Dirac operator associated to the metric $g_k := \delta_k^{-2} \sigma_k^*(g)$. Writing the metric g on M in geodesic normal coordinates centered in p one immediately sees that on any closed ball the sequence g_k converges to the euclidean metric in the C^∞-topology.

Note that with respect to g_k

$$\|\widetilde{\varphi}_k\|_{C^0} \leq |\widetilde{\varphi}_k(0)| = 1.$$

Hence, we may apply the interior L^p- and Schauder-estimates 3.3 and 3.4 to conclude that

$$\|\widetilde{\varphi}_k\|_{C^{1,\alpha}} \leq C(R), \qquad \forall k > k(R),$$

with constants $C(R)$ and $k(R)$.

Compare D with the Dirac operator D^{flat} on flat $\mathbb{R}^n$ (see e.g. [Pfä02] or [AHM03] for explicit computations). We have

$$\|(D^{\mathrm{flat}} - D_k)\widetilde{\varphi}_k\|_{C^{0,\alpha}(B_R(0))} \leq \tau_k \|\widetilde{\varphi}_k\|_{C^{1,\alpha}(B_R(0))},$$

with $\tau_k \to 0$.

We choose a sequence of radii $R_m \to \infty$. After passing to a diagonal sequence, we see that there is a spinor $\widetilde{\varphi}_\infty$ on $\mathbb{R}^n$, such that $\widetilde{\varphi}_k|_{B_R(0)}$ converges to $\widetilde{\varphi}_\infty|_{B_R(0)} \in C^{1,\alpha}(B_R(0))$.

For any $\varepsilon > 0$ and $R > 0$ there is a $k_0(R, \varepsilon)$ such that

$$\|\widetilde{\varphi}_k\|_{L^{p_k}(B_R(0))} \leq 1 + \varepsilon$$

for all $k \geq k_0$. Because of the C^1-convergence $\tilde{\varphi}_k \to \tilde{\varphi}_\infty$, Fatou's lemma yields

$$\|\widetilde{\varphi}_\infty\|_{L^{q_D}(B_R(0))} \leq 1$$

for any R, and finally for $R = \infty$. Then $\widetilde{\varphi}_\infty$ is a solution of

$$D^{\mathrm{flat}}\widetilde{\varphi}_\infty = \mu_{q_D} \, |\widetilde{\varphi}_\infty|^{p_D - 2}\widetilde{\varphi}_\infty,$$

and then $\widetilde{\psi}_\infty := \mu_{q_D} \, D^{\mathrm{flat}}\widetilde{\varphi}_\infty$ is a solution of

$$(D^{\mathrm{flat}})^{-1}\widetilde{\psi}_\infty = \mu_{q_D}|\widetilde{\psi}_\infty|^{q_D - 2}\widetilde{\psi}_\infty.$$

We identify $\widetilde{\psi}_\infty$ via stereographic projection with an L^{q_D}-spinor $\widehat{\psi}_\infty$ on S^n with the identification provided in section 4. Note that

$$\begin{aligned}
\|\widehat{\psi}_\infty\|_{L^{q_D}(S^n)} &= \|\widetilde{\psi}_\infty\|_{L^{q_D}(\mathbb{R}^n)} \leq 1 \\
(D^{-1}\widehat{\psi}_\infty, \widehat{\psi}_\infty)_{S^n} &= (D^{-1}\widetilde{\psi}_\infty, \widetilde{\psi}_\infty)_{\mathbb{R}^n} = \mu_{q_D}\|\widetilde{\psi}_\infty\|^{q_D}_{L^{q_D}(\mathbb{R}^n)}, \\
\mu^{S^n}_{q_D} \geq \mathcal{F}^{S^n}_{q_D}(\widehat{\psi}_\infty) &= \mu_{q_D}\|\widetilde{\psi}_\infty\|^{q_D - 2}_{L^{q_D}(\mathbb{R}^n)} \geq \mu_{q_D}
\end{aligned}$$

which is apparently a contradiction to our assumption $\mu_{q_D} \geq \mu^{S^n}_{q_D} + \varepsilon$. $\square$

PROPOSITION 6.7. *If there is a $q_0 > q_D$ and an $r > q_D$ such that for all $t \in (q_D, q_0)$ there is a solution ψ_t of equation* (2.5) *with $q = t$ such that $\|\psi_q\|_{L^r}$ is bounded by a constant C independent from t, then there is a sequence $t_i \to q_D$ such that ψ_{t_i} converges in the C^1-topology to a solution of equation* (2.5) *with $q = q_D$.*

Proof. For q sufficiently close to q_D, we know because of Proposition 5.3 that μ_q is bounded from below by a positive constant. Thus, we can apply the regularity theorem (Theorem 6.3) which tells us that (ψ_t) is uniformly bounded in $C^{0,\alpha}$. By elliptic regularity, $D^{-1}(\psi_t)$ is uniformly bounded in $C^{1,\alpha}$. Hence, for a sequence (t_i) with $t_i > q_D$, converging to q_D, the spinor fields $D^{-1}(\psi_{t_i})$ converge in the C^1-topology to a C^1-spinor field φ_{q_D}. Then $D\varphi_{q_D}$ is a solution of equation (2.5) with $q = q_D$. □

7 The size of the singular set

In this section we recall the weak Unique Continuation Property. We obtain directly that the set on which a solution of (6.1) vanishes does not contain any non-empty open set.

THEOREM 7.1 (Weak Unique Continuation Property [BBMW, Theorem 2.1]). *Let P be a locally bounded function on a connected Riemannian manifold M and let φ be a solution of*

$$D\varphi = P \cdot \varphi$$

that vanishes on a nonempty open set. Then φ is identically 0.

COROLLARY 7.2. *The singular set of C^1-solutions to equation* (2.7) *does not contain any nonempty open set.*

In case that the function P is smooth, there is a stronger result by Christian Bär.

THEOREM 7.3 (Nodal sets for Dirac Operators [Bär97]). *Let M^n be compact and connected and let φ be a solution of*

$$D\varphi = P\varphi$$

where P is a smooth endomorphism. Then the zero set of φ has at most Hausdorff dimension $n-2$. If $n = 2$, then the zero set is discrete.

In the case $n = 2$ this theorem implies that the zero set of a solution to (2.4) is discrete.

8 The smallest positive Dirac eigenvalue on singular spaces

As before we fix a spin structure σ and a conformal structure $[g_0]$ on a compact manifold M. Let $g = f^{4/(n-1)} \cdot g_0$ be a generalized conformal metric. We define the smallest positive Dirac eigenvalue on (M, g, σ) by (2.1). It is evident, that if g is regular (i. e. $f > 0$), then $\lambda_1^+(g)$ is the smallest positive eigenvalue of the Dirac operator.

Lemma 8.1. *The following identities hold*

$$\inf_{g\in\mathcal{M}(g_0)} \lambda_1^+(g) = \inf_{g\in\overline{\mathcal{M}(g_0)}} \lambda_1^+(g) = \mu_{q_D}^{-1}.$$

Proof. Obviously,

$$\inf_{g\in\mathcal{M}(g_0)} \lambda_1^+(g) \geq \inf_{g\in\overline{\mathcal{M}(g_0)}} \lambda_1^+(g).$$

From the previous section we already know that $\lambda_1^+(g) \geq \mu_{q_D}^{-1}$ for all $\mathcal{M}(g_0)$. Let us show that the argument even holds on $\overline{\mathcal{M}(g_0)}$. For this we write $g = e^{2u} g_0$ on $M \setminus \mathcal{S}_g$, $u \in C^\infty(M \setminus \mathcal{S}_g)$. Using the conformal change formula we see that

$$\lambda_1^+(g) = \inf \left\{ \frac{(h^{-1}\psi, \psi)_{g_0}}{(\psi, D_{g_0}^{-1}\psi)_{g_0}} \,\middle|\, \psi \in \mathrm{im}_{C^\infty} D_{g_0}, \quad \|h^{-(n+1)/2}\psi\|_{L^\infty} < \infty, \quad (\psi, D_{g_0}^{-1}\psi)_{g_0} > 0 \right\}.$$

In particular all test spinors have finite $\|\psi\|_{L^\infty}$. Applying Hölder's inequality and using $\int h^n = 1$, we see that $(h^{-1}\psi, \psi)_{g_0} \geq \|\psi\|_{L^{q_D}}^2$. Hence,

$$\lambda_1^+(g) \geq \mu_{q_D}^{-1} \qquad \forall g \in \overline{\mathcal{M}(g_0)}.$$

On the other hand, let ψ be a spinor field such that $\mathcal{F}_{q_D}(\psi) \geq \mu_{q_D} - \varepsilon$ with arbitrary small $\varepsilon > 0$. We can assume that ψ is a smooth spinor field without zeros. Then for $g := |\psi|^{4/(n+1)} g_0$ we obtain $\lambda_1^+(g) \leq \mathcal{F}_{q_D}(\psi)^{-1}$. Hence

$$\inf_{g\in\mathcal{M}(g_0)} \lambda_1^+(g) \leq \mu_{q_D}^{-1}.$$

□

Remark. Much recent research deals with Laplacians acting on functions on singular spaces. The function Laplacian can be defined on a much larger category of spaces, i. e. on metric spaces carying a measure with certain compatibility conditions. (See e.g. [Gro99],[KMS01], [KS01] and the references therein.)

9 Applications to cmc surfaces

The solution of equation (2.5) provides a strong tool for the construction of constant mean curvature surfaces. Because most of these surfaces have not been known before, we will dedicate a separate article to this subject [**?**].

In this section we want to sketch the underlying ideas.

Definition. Let (M, g) be a compact Riemann surface with universal covering $\widetilde{M}$ with Deck transformation group Γ. A *periodic conformal immersion* F is a branched conformal immersion of $\widetilde{M}$ into $\mathbb{R}^3$ such that there is a group homomorphism $h : \Gamma \to \mathbb{R}^3$ satisfying

$$F(\gamma \cdot p) = F(p) + h(\gamma).$$

Using the Weierstraß representation [KS96, Bär98, Fri98, Amm98], one sees that any parallel spinor on $\mathbb{R}^3$ and any such immersion F induces a spinor φ on M which is a solution of

$$D\varphi = H|\varphi|^2\varphi, \tag{9.1}$$

H being the mean curvature of the immersion. Branching points are in a 1-to-1 correspondence to the zeros of φ.

Conversely, any solution of (9.1) yields a periodic conformal immersion F.

An immediate consequence of Theorem 2.2 is the following principle.

Principle for construction of cmc-surfaces. *For any Riemann spin surface (M, g, σ) with $\lambda^2 < 4\pi$, there is a periodic conformal immersion of $(\widetilde{M}, g, \sigma)$ into $\mathbb{R}^3$ with constant mean curvature.*

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