Characterization of Coerciveness for single-valued and set-valued mappings versus the Palais-Smale condition

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The paper establishes characterizations of coerciveness property for functionals with single- and set-valued type respectively, satisfying a suitable Palais-Smale condition. These results are obtained by applying the Ekeland variational principle in both case. Comparison with other Palais-Smale condition is provided for single-valued case.

Key words: coerciveness, Palais–Smale condition, variational principle, set-valued mappings

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1 Introduction

Many authors established the basic property that the suitable Palais-Smale condition implies the coerciveness, see [2], [15], [14], [11], [10] and [5].

The main object of this paper is to guarantee the equivalence between the coerciveness of mappings and the suitable Palais-Smale condition. We trait the both case, i.e. the behavior of the single-valued and set-valued mappings via the coercity.

The paper is organized as follows. The Section 2 traits the single-valued case. We recall some definitions and we prove that the main result from the paper of D. Motreanu and V. Motreanu [12] remain true in some weakened condition. Moreover, we prove for a special class of functions that the Palais-Smale condition introduced here is equivalent with the (PSB) condition introduced by J.N. Corvellec in the paper [5], although the relation between the strong slope, see [8], and the Clarke derivative is feeble.

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The Section 3 deals with the set-valued case. We introduce the Palais-Smale condition for set-valued mappings, the notion of critical point and we study some properties of these notions. In the proof of the main result of this section we use a general form of the Ekeland variational principle for set-valued mappings, see Chen, G.Y., Huang, X.X., and Hou, H. [7], i.e. the equivalence between the coerciveness and Palais-Smale condition for this class of mappings.

Throughout in the paper X will be a Banach space with norm $\|\cdot\|$.

2 Single-valued case

First, we recall the definition of the strong slope for lower semicontinuous function, see the paper of M., Degiovanni and M., Marzocchi [8].

Definition 2.1 *Let $f : X \longrightarrow \overline{\mathbb{R}}$ be a lower semicontinuous function and $u \in \mathcal{D}(f)$ a fixed point, where $\mathcal{D}(f) = \{u \in X \ |f(x) < \infty \ \}$. We define*

$$| \bigtriangledown f|(u) = \begin{cases} \limsup\limits_{v \to u} \dfrac{f(u) - f(v)}{d(u,v)}, & \textit{if } u \textit{ is not a local minimum of } f \\ 0 \ , & \textit{if } u \textit{ is a local minimum of } f. \end{cases}$$

The extended real number $| \bigtriangledown f|(u)$ is called the strong slope of f at u.

Definition 2.2 (Corvellec, [5]) *Let $f : X \to (-\infty, +\infty]$ be a lower semicontinuous function. We say that f satisfies condition (PSB), if for every sequence $\{u_n\} \subset X$ such that $\{f(u_n)\}$ is bounded and $|\nabla f|(u_n) \to 0$, the sequence $\{u_n\}$ has a bounded subsequence.*

The following class of functionals was be introduced by D. Motreanu and P.D. Panagiotopoulos [13].

Let $I : X \to (-\infty, +\infty]$ be a functional satisfying the following structure condition:

(H) $I = \phi + \psi$, with $\phi : X \to \mathbb{R}$ locally Lipschitz and $\psi : X \to (-\infty, +\infty]$ proper, convex and lower semicontinuous.

Definition 2.3 *We say that the functional $I : X \to (-\infty, +\infty]$ with property (H) satisfy the weak Palais-Smale condition (in short (w-PS)), if every sequence $\{u_n\} \subset X$ such that $\{I(u_n)\}$ is bounded and*

$$\phi^0(u_n, v - u_n) + \psi(v) - \psi(u_n) \geq -\varepsilon_n \|v - u_n\|, \ (\forall) \ v \in X,$$

for a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+$ *with* $\lim_{n\to\infty} \varepsilon_n = 0$, $\{u_n\}$ *contains a bounded subsequence, where* ϕ^0 *is the Clarke derivative.*

Definition 2.4 *We say that the functional* I *is coercive if for* $\|u\| \to \infty$*, we have* $f(u) \to \infty$.

Proposition 2.1 *Let* $I : X \to (-\infty, \infty]$ *be a bounded below functional satisfying the (H). If* I *satisfies the condition (w-PS), then* I *is coercive.*

Proof. We suppose the contrary, i.e. $c = \liminf_{\|u\|\to\infty} I(u)$, c finite. Then there exists an element $u_n \in X$ such that $I(u_n) \leq c + \dfrac{1}{n}$ and $\|u_n\| \geq 2n$. We apply the classical Ekeland variational principle for I, $\varepsilon := c + \dfrac{1}{n} - \inf_{u\in X} I(u)$ and $\lambda := n$. Then there exists $v_n \in X$ such that we have:

$$I(v_n) \leq I(u_n) \leq c + \frac{1}{n} \tag{2.1}$$

$$I(v) - I(v_n) \geq -\frac{1}{n}\left(c + \frac{1}{n} - \inf_{u\in X} I(u)\right)\|v - v_n\|,\ (\forall)\ v \in X \tag{2.2}$$

$$\|u_n - v_n\| \leq n \tag{2.3}$$

Let $v = v_n + t(u - v_n)$, where $u \in X$ is a fixed element and $t \in (0, 1]$. From (2.2) we get

$$\begin{aligned}&\phi(v_n + t(u - v_n)) - \phi(v_n) + \psi(v_n + t(u - v_n)) - \psi(v_n) \geq \\ &\geq -\frac{1}{n}\left(c + \frac{1}{n} - \inf_{u\in X} I(u)\right)\|v_n + t(u - v_n) - v_n\|.\end{aligned}$$

We introduce the notation

$$\varepsilon_n := \frac{c + \frac{1}{n} - \inf_{u\in X} I(u)}{n}.$$

Since the function ψ is convex, we have:

$$\phi(v_n + t(u - v_n)) - \phi(v_n) + t(\psi(u) - \psi(v_n)) \geq -\varepsilon_n t\|u - v_n\| \tag{2.4}$$

If we divide this relation by $t > 0$ we obtain

$$\frac{\phi(v_n + t(u - v_n)) - \phi(v_n)}{t} + \psi(u) - \psi(v_n) \geq -\varepsilon_n\|u - v_n\|.$$

Since

$$\phi^0(v_n, u - v_n) \geq \limsup_{t\searrow 0} \frac{\phi(v_n + t(u - v_n)) - \phi(v_n)}{t},$$

we obtain

$$\phi^0(v_n, u - v_n) + \psi(u) - \psi(v_n) \geq -\varepsilon_n \|u - v_n\|. \tag{2.5}$$

From the relation (2.3) we get

$$\|v_n\| \geq \|u_n\| - \|u_n - v_n\| \geq 2n - n = n. \tag{2.6}$$

From the relation (2.1) we have $I(v_n) \leq c + \frac{1}{n}$, that is $\{I(v_n)\}$ is bounded. Since $\lim_{n\to\infty} \varepsilon_n = 0$, from the relation (2.5) and condition (w-PS), we obtain that the sequence $\{v_n\}$ contains a bounded subsequence, which is in contradiction with (2.6). In conclusion we have that I is coercive. $\square$

Corollary 2.1 *Let $I : X \to (-\infty, \infty]$ a bounded bellow function, which satisfies the condition (H). Then I is coercive, if and only if I satisfies condition (w-PS).*

Proof. If I satisfies condition (w-PS), then I is coercive, see the Proposition 2.1. Conversely, we suppose that I is coercive and not satisfies condition (w-PS). Then there exists a sequence $\{u_n\} \subset X$ such that the sequence $\{I(u_n)\}$ is bounded and we have:

$$\phi^0(u_n, v - u_n) + \psi(v) - \psi(u_n) \geq -\varepsilon_n \|v - v_n\|,$$

for all $v \in X$ with $\{\varepsilon_n\} \subset \mathbb{R}_+$, $\lim\limits_{n\to\infty} \varepsilon_n = 0$ and every subsequence $\{u_{n_k}\}$ of $\{u_n\}$ is unbounded. Therefore $\{u_n\}$ is unbounded and from the fact that I is coercive we get that $\lim\limits_{n\to\infty} I(u_n) = +\infty$. But this is in contradiction with the fact that $\{I(u_n)\}$ is bounded. $\square$

Therefore we have the following result:

Theorem 2.1 *Let $I : X \to (-\infty, +\infty]$ a bounded below function which satisfies (H) and for all $u \in \mathcal{D}(I)$, there exists $u_h \to u$ with $I(u_h) \to I(u)$. The function I satisfies condition (PSB) if and only if I satisfies condition (w-PS).*

Proof. The equivalence follows from the Corollary 2.1 and [5, Corollary 1]. $\square$

3 Set-valued case

Throughout in this section $F : X \longrightarrow \mathbb{R}$ is a set-valued mapping with nonempty values and $F(x) \subset \mathbb{R}$ is compact set for every $x \in X$. We define the graph of the function F by

$$Graph(F) = \{\ (u, c) \in X \times \mathbb{R} \mid c \in F(u)\ \}.$$

Definition 3.1 *We say that the set-valued function $F : X \to \mathbb{R}$ is upper semicontinuous at x if and only if for any neighbourhood U of $F(x)$, $\exists\, \eta > 0$ such that for every $x' \in B_X(x, \eta)$ we have $F(x') \subset U$.*

Definition 3.2. *We say that the function $F : X \to \mathbb{R}$ is Lipschitz around $x \in X$ if there exists a positive constant l and a neighbourhood U of x such that*

$$\forall\ x_1, x_2 \in U, \quad F(x_1) \subset F(x_2) + l\|x_1 - x_2\|[-1, 1].$$

The set-valued function F is upper semicontinuous if F is upper semicontinuous in every point $x \in X$. If F is Lipschitz around any $x \in X$, we say that F is locally Lipschitz.

Proposition 3.1. *If $F : X \to \mathbb{R}$ is Lipschitz around $x \in X$, then F is upper semicontinuous at x.*

Proof. Let U be a fixed neighbourhood of $F(x)$. Then there exists an $\delta_1 > 0$ such that $B(F(x), \delta_1) \subset U$, because $F(x)$ is a compact subset of $\mathbb{R}$. Since F is Lipschitz around x, we have a neighbourhood of x, say U_1, and $l > 0$ such that

$$\forall\ x_1, x_2 \in U_1 : \quad F(x_1) \subset F(x_2) + l\|x_1 - x_2\|[-1, 1].$$

In particular for every $x_1 \in U_1$ we have $x_1 \in U_1 : \quad F(x_1) \subset F(x) + l\|x_1 - x\|[-1, 1]$.
Let $\eta > 0$ such that $\eta < \min\left\{\frac{\delta_1}{l}, \gamma_1\right\}$, where $B_X(x, \gamma_1) \subset U_1$.
For every $x_1 \in B_X(x, \eta)$ we have: $F(x_1) \subset U$. □

We introduce the following notation, see [1]

$$\mathrm{Limsup}_{x' \to x} F(x') := \left\{ y \in \mathbb{R} \ \mid\ \liminf_{x' \to x} d(y, F(x')) = 0 \right\}.$$

Now, we recall the notion of contingent derivative, see [1, p.186]. If F is Lipschitz around x, then for $(x, y) \in Graph(F)$ and $u \in X$

$$v \in DF(x, y)(u) \ \Leftrightarrow\ \liminf_{h \to 0^+} d\left(v, \frac{F(x + hu) - y}{h}\right) = 0,$$

see [1, Proposition 5.1.4], thus

$$DF(x, y)(u) = \operatorname*{Limsup}_{h \to 0^+} \frac{F(x + hu) - y}{h},$$

using the above notation. The $DF(x, y)(u)$ is the contingent derivative at $(x, y) \in Graph(F)$ in the direction u.

In the sequel, we will use a particular form of the Corollary 4.1 from the paper [7], i.e. the Ekeland variational principle for set-valued mapping.

Proposition 3.2 *Let $\lambda_1 > 0$ and $\varepsilon > 0$. Suppose that F is upper semicontinuous and there exists $(v_1, y_1) \in Graph(F)$ such that:*

(a) $y_1 = \min F(v_1)$

(b) $(F(v) - y_1 + \varepsilon) \cap -\mathbb{R}_+ = \emptyset, \ \forall \ v \in X \setminus \{v_1\}$ *and*

(c) $F(X) \cap (y_1 - \mathbb{R}_+)$ *is* $\mathbb{R}_+$ *lower bounded, i.e. there exists* $r \in \mathbb{R}$, *such that*

$$F(X) \cap (y_1 - \mathbb{R}_+) - r \subset \mathbb{R}_+.$$

Then, there exists $(v_\varepsilon, y_\varepsilon) \subset Graph(F)$ such that:

(i) $y_\varepsilon = \min F(v_\varepsilon)$

(ii) $y_\varepsilon - y_1 + \dfrac{\varepsilon}{\lambda_1}\|v_1 - v_\varepsilon\| \in -\mathbb{R}_+$

(iii) $\left(F(v) - y_\varepsilon + \dfrac{\varepsilon}{\lambda_1}\|v - v_\varepsilon\|\right) \cap -\mathbb{R}_+ = \emptyset, \ \forall \ v \in X \setminus \{v_\varepsilon\}$

(iv) $\|v_1 - v_\varepsilon\| < \lambda_1$.

Definition 3.3 *We say that $u_0 \in X$ is critical point of F if there exists $v_0 \in F(u_0)$ such that for every $u \in X$ we have $DF(u_0, v_0)(u - u_0) \subset \mathbb{R}_+$. The v_0 is a critical value of F.*

Definition 3.4 *We say that $x_0 \in X$ is local minimum of F if there exists a neighborhood U of x_0 such that $F(x) - \min F(x_0) \subset \mathbb{R}_+, \ \forall \ u \in U$. $x_0 \in X$ is global minimum of F if the above holds for $U := X$.*

Proposition 3.3 *Each local minimum of F is necessarily a critical point of F, with critical value equal to the minimum.*

Proof. Let u_0 be the local minimum point, i.e.

$$F((1-t)u_0 + tu) - v_0 \subset \mathbb{R}_+$$

for t small, $v_0 = \min F(u_0)$ and for all $u \in X$. Dividing the above relation by $t > 0$, we get

$$\frac{F(u_0 + t(u - u_0)) - v_0}{t} \subset \mathbb{R}_+,$$

since $\mathbb{R}_+$ is a cone. From this we have

$$\text{Limsup}_{t \to 0^+} \frac{F(u_0 + t(u - u_0)) - v_0}{t} \subset \mathbb{R}_+, \ \forall \ u \in X.$$

Therefore

$$DF(u_0, v_0)(u - u_0) \subset \mathbb{R}_+, \ \forall \ u \in X.$$

i.e. u_0 is a critical point. $\square$

Definition 3.5 *$F : X \to \mathbb{R}$ is pseudo-convex at $(x, y) \in Graph(F)$ if $\forall x' \in X$, $F(x') \subset DF(x, y)(x' - x) + y$.*

Proposition 3.4 *Let $x_0 \in X$ be a critical point of F. If F is pseudo-convex at (x_0, y_0), where y_0 is the critical value of F, then x_0 is global minimum point of F and $y_0 = \min F(x_0)$.*

Proof. Since $DF(x_0, y_0)(u - x_0) \subset \mathbb{R}_+, \ \forall \ u \in X$, from the Definition 3.5 we get

$$F(u) - y_0 \subset \mathbb{R}_+, \ \forall u \in X \tag{3.1}$$

Put in the above relation $u := x_0$, we get $F(x_0) - y_0 \subset \mathbb{R}_+$. From this, we have necessarily $y_0 = \min F(x_0)$. From (3.1) we get that x_0 is global minimum point of F. $\square$

Next we define the notion of the weak Palais-Smale condition for set-valued functionals.

Definition 3.6 (w-PS-condition) *We say that the set-valued function $F : X \longrightarrow \mathbb{R}$ satisfies the weak Palais-Smale condition (shortly (w-PS)), if for every sequence $\{u_n\} \subset X$, such that*

$$DF(u_n, v_n)(u - u_n) + \varepsilon_n \|u - u_n\| \subset \mathbb{R}_+, \ \forall \ u \in X$$

$v_n \in F(u_n)$ where $\varepsilon_n \to 0$, and $\{v_n\}$ is bounded, the sequence $\{u_n\}$ possesses a bounded subsequence.

Lemma 3.1 *Let X be a Banach space, $F : X \to \mathbb{R}$ be a locally Lipschitz set-valued mapping with compact and non-empty values, such that $\inf F(X) > -\infty$. Define $c = \liminf\limits_{\|u\| \to \infty} \min F(u)$. Then, if $c \in \mathbb{R}$, there exists a sequence $\{v_n\} \subset X$ such that:*

(i) $\|v_n\| \to +\infty$;

(ii) $z_n = \min F(v_n) \to c$;

(iii) $DF(v_n, z_n)(u - v_n) + \varepsilon_n \|u - v_n\| \subset \mathbb{R}_+$, *where* $\varepsilon_n \to 0, \ \forall \ u \in X$.

Proof. Since F is Lipschitz around every point $x \in X$, from Proposition 3.1 is results that F is upper semicontinuous. From the assumption, there exists for any $n \in \mathbb{N}^*$ an element $u_n \in X$ such that

$$y_n = \min F(u_n) \leq c + \frac{1}{n} \tag{3.2}$$

and

$$\|u_n\| \geq 2n. \tag{3.3}$$

By Proposition 3.2, with $\lambda_1 = n$ and $\varepsilon = c+\frac{2}{n}-\inf F(X)$, we observe that the assumptions (a), (b) and (c) holds. In fact, (a) is obvious from (3.2). For (b), let $v \in X$ and $y \in F(v)$ an arbitrary element. We have that:

$$y - y_n + \varepsilon = y - y_n + c + \frac{2}{n} - \inf F(X) \geq y + c + \frac{2}{n} - \inf F(X) - c - \frac{1}{n} =$$

$$= y - \inf F(X) + \frac{1}{n} > 0.$$

Therefore, $(F(v) - y_n) + \varepsilon) \cap -\mathbb{R}_+ = \emptyset, \ \forall \ v \in X.$

For (c), we use the fact that $\inf F(X) > -\infty$. Therefore, there exists $(v_n, z_n) \in Graph(F)$ such that

$$z_n = \min F(v_n) \tag{3.4}$$

$$z_n - y_n + \frac{\varepsilon}{n}\|u_n - v_n\| \in -\mathbb{R}_+ \tag{3.5}$$

$$\Big(F(v) - z_n + \frac{\varepsilon}{n}\|v - v_n\|\Big) \cap -\mathbb{R}_+ = \emptyset, \ \forall \ v \in X \setminus \{v_n\} \tag{3.6}$$

$$\|v_n - u_n\| < n. \tag{3.7}$$

From (3.7) and (3.3), we have

$$\|v_n\| \geq \|u_n\| - \|v_n - u_n\| > 2n - n = n,$$

i.e. $\|v_n\| \to \infty$, as $n \to \infty$, which represent exactly (i).

From (3.5) and (3.2), we have

$$z_n \leq y_n - \frac{\varepsilon}{n}\|u_n - v_n\| \leq y_n \leq c + \frac{1}{n}.$$

From the definition of c, we have

$$z_n = \min F(v_n) \to c,$$

as $n \to \infty$, exactly the (ii).

From (3.6), we have that

$$F(v) - z_n + \varepsilon_n \|v - v_n\| \subset \mathbb{R}_+, \ \forall \ v \in X,$$

with $\varepsilon_n = \dfrac{\varepsilon}{n}$. Evidently $\varepsilon_n \to 0$, as $n \to \infty$. Let $v := v_n + h(u - v_n)$, with $h \in (0, 1]$, then we get

$$F(v_n + h(u - v_n)) - z_n + h\varepsilon_n \|u - v_n\| \subset \mathbb{R}_+, \ \forall \ u \in X.$$

Dividing by $h > 0$, we get

$$\frac{F(v_n + h(u - v_n)) - z_n}{h} + \varepsilon_n \|u - v_n\| \subset \mathbb{R}_+, \ \forall \ u \in X.$$

Letting $h \to 0^+$, we get

$$DF(v_n, z_n)(u - v_n) + \varepsilon_n \|u - u_n\| \subset \mathbb{R}_+, \ \forall \ u \in X,$$

which is exactly the (iii). Thus the proof of lemma is complete. □

Definition 3.7 *We say that the set-valued function* $F : X \to \mathbb{R}$ *is coercive, if for* $\|u\| \to \infty$ *we have* $\min F(u) \to \infty$.

Theorem 3.1 *Let* X *be a Banach space,* $F : X \to \mathbb{R}$ *be a locally Lipschitz set-valued mapping with compact and non-empty values, such that* $\inf F(X) > -\infty$. *F satisfies the (w-PS) condition if and only if* F *is coercive.*

Proof. If F is not coercive, let $c = \liminf_{\|u\| \to \infty} \min F(u)$, with $c \in \mathbb{R}$. Then by Lemma 3.1, there exists a sequence $\{v_n\}$ such that

(i) $z_n = \min F(v_n) \to c$

(ii) $\|v_n\| \to \infty$

(iii) $DF(v_n, z_n)(u - v_n) + \varepsilon_n \|u - v_n\| \subset \mathbb{R}_+$, where $\varepsilon_n \to 0, \ \forall \ u \in X$.

From (i) and (iii), using the (w-PS) condition, we can obtain a bounded subsequence of $\{v_n\}$. But from (ii), $\|v_n\| \to \infty$, which is a contradiction.

Conversely, let $\{x_n\} \subset X$ such that $DF(x_n, y_n)(u - x_n) + \varepsilon_n \|u - x_n\| \subset \mathbb{R}_+$, $\forall u \in X, \ \ \varepsilon_n \to 0, \ \ y_n \in F(x_n)$ and $\{y_n\}$ is bounded. Suppose that we can not subtract a bounded subsequence from $\{x_n\}$. From this, we have that $\|x_n\| \to \infty$. Using the coercivity of F, we have $\min F(x_n) \to \infty$, i.e. $\{y_n\}$ is unbounded, which is a contradiction. Therefore the (w-PS) holds true. □

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