GENERALIZED RIEMANN-HILBERT TRANSMISSION AND BOUNDARY VALUE PROBLEMS, FREDHOLM PAIRS AND BORDISMS

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Abstract. We present classical and generalized Riemann-Hilbert problem in several contexts arising from K-theory and bordism theory. The language of Fredholm pairs turns out to be useful and unavoidable. We propose an abstract formulation of a notion of bordism in the context of Hilbert spaces equipped with splittings.

§1. Introduction

The concept of a Fredholm pair $\mathcal{P} = (H^-, H^+)$ of closed subspaces H^-, H^+ of a Hilbert (or Banach) space was introduced in 1966 by T. Kato in his studies of stability properties of closed, mainly unbounded operators, [19].

Recall that the pair $\mathcal{P} = (H^-, H^+)$ is a Fredholm pair if the algebraic sum $H^- + H^+$ is closed and the numbers $\alpha_{\mathcal{P}} = \dim(H^- \cap H^+)$ and $\beta_{\mathcal{P}} = codim(H^- + H^+)$ are both finite. We also assume that H^- and H^+ are of infinite dimensions. The difference $\alpha_{\mathcal{P}} - \beta_{\mathcal{P}}$ was defined in [19] as the index of the pair, $Ind\,\mathcal{P}$, and the crucial observation of T. Kato was that $Ind(H^-, H^+)$ is not changed by „small" deformations of the pair. More precisely, the set $\mathcal{F}Gr^2(H)$ of all Fredholm pairs of a Hilbert space appears then as an open subset of the Cartesian product $Gr(H) \times Gr(H)$ of the Grassmannian of closed subspaces of H supplied with the usual „minimal gap" metric, see [17, 19]. In this context the notation $\mathcal{F}Gr^2(H)$ can be interpreted as the Fredholm bi-Grassmannian of the Hilbert space H which generalizes in a natural way to the Fredholm multi-Grassmannian $\mathcal{F}Gr^n(H)$, when instead of pairs of subspaces we consider n-tuples $(M_1, \ldots, M_n)$ of closed subspaces forming Fredholm fans, [4, 6].

There was no doubt from the outset that the theory of Fredholm pairs and their generalizations should be studied in close relationship with the theory of Fredholm operators. Thus Fredholm pairs in Kato's [19] were considered as a convenient extension of the theory of Fredholm operators. For a Fredholm, possibly unbounded, closed operator $A : H_1 \to H_2$ acting between Hilbert (Banach) spaces, the pair $\mathcal{P}_A = (graph\,A, \widetilde{H}_1)$ was in [19] the basic example of a Fredholm pair. Here

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the „coordinate" subspace $\widetilde{H}_1 = H_1 \oplus 0$ and the graph of A are closed subspaces in the direct sum $H = H_1 \oplus H_2$. Moreover,

$$Ind\,\mathcal{P}_A = ind\,A$$

where $ind\,A$ denotes here and in the sequel the index of the Fredholm operator A. Also in [4] the bi-Grassmannian $\mathcal{F}Gr^2(H)$, understood there also as the space of abstract Riemann-Hilbert transmission problems, was parameterized by a family of Fredholm operators $\mathcal{L}_\mathcal{P}$ associated with projectors (P^-, P^+), not necessarily orthogonal, onto the spaces of the pair $\mathcal{P}$.

The theory of Fredholm operators in Hilbert space turned out to be an important tool for studying topology of manifolds and K-theory, especially the geometrical and topological invariants defined by elliptic differential and pseudodifferential operators in spaces of sections of smooth vector bundles on manifolds. The highlight along that road was the famous solution by M. Atiyah and I. Singer, [2], of the index problem for elliptic operators. In the abstract functional analytic setting the space $\mathcal{F}(H)$ of Fredholm operators in the Hilbert space H, topologized as a subset of the Banach algebra $\mathcal{B}(H)$ of bounded operators in H, turned out to be the classifying space for the functor $K^0(-)$, the 0-th term of the generalized cohomology theory $K^*(-)$, [1]. Later the K-homology $K_*(X)$ of a topological space X (or $K^*(A)$ for a $\mathbb{C}^*$-algebra in the noncommutative case) was introduced, [18]. According to Kasparov the generators of $K_*(X)$ are realized by certain Fredholm operators acting in Hilbert space, which is equipped with an action of the algebra of functions $C(X)$.

The roots of the extremely successful applications of the Fredholm operators in global analysis, geometry of elliptic operators and K-theory, undoubtedly are related with the following basic features of the class $\mathcal{F}(H)$:

(i) The set $\mathcal{F}(H)$ is stable under sufficiently small perturbations in $\mathcal{B}(H)$ i.e.

$$A \in \mathcal{F}(H) \Rightarrow A + B + K \in \mathcal{F}(H)$$

for $B \in \mathcal{B}(H)$, $\|B\| < \varepsilon_A$ (for a sufficiently small ε_A) and $K \in \mathcal{K}(H)$, where $\mathcal{K}(H)$ denotes the ideal of compact operators in H;

(ii) Composition law:

$$A \in \mathcal{F}(H)\,,\; B \in \mathcal{F}(H) \Rightarrow A \circ B \in \mathcal{F}(H)$$

and $ind\,A \circ B = ind\,A + ind\,B$. The index homomorphism

$$ind : \mathcal{F}(H) \to \mathbb{Z}$$

is surjective and describes the set of components $\pi_0(\mathcal{F}(H))$;

(iii) In interesting and important cases, arising in the theory of partial differential equations and boundary value problems, the Hilbert space appears as a function space over a manifold, usually a function space of Sobolev type. Therefore it was natural to consider Hilbert spaces equipped with an action of the algebra of functions $B = C(X)$ over a topological space (usually a manifold) X. More generally, it was assumed in [18] that the considered Hilbert spaces are Hilbert modules with some $\mathbb{C}^*$-algebra B action

$$r : B \to \mathcal{B}(H)\,.$$

The condition

$$\forall b \in B \;:\; [r(b), A] \in \mathcal{K}$$

distinguishes a class of operators $A \in \mathcal{B}$ which is of special interest. In consequence it restricts also the class of Fredholm operators. It is a remarkable fact, that the elliptic pseudodifferential operators belong to the described above class for the standard multiplication representation of the algebra of continuous functions.

The calculus of commutators and their traces was the starting point for A. Connes for introducing cyclic cohomology and proclaiming the program of noncommutative geometry, [12].

The natural and intimate connection of the classical Riemann-Hilbert transmission problems and the theory of Fredholm pairs in a Hilbert space H was first described in middle seventies by the first author, [4]. In particular the mentioned above basic properties (i), (ii) and (iii) appear in a decisive way in [4]. In the linear transmission problems for Cauchy-Riemann systems, generalized Cauchy-Riemann systems, Dirac operators as well as higher dimensional transmission problems related with the Cauchy data spaces for higher order elliptic operators in vector bundles on manifolds, the Fredholm pair approach is more direct then the usual reduction process to systems of elliptic ΨDO's on the boundary or the splitting submanifold. In [4] a variety of concepts have been introduced. Besides the named above Fredholm bi-Grassmannian and the abstract Riemann-Hilbert transmission problem let's mention here the discussion of the role of Calderón projectors on the Cauchy data spaces in general vector bundle setting, Green formulas and pairing between Cauchy data spaces for D and the formally adjoint operator D^*, splitting index formulas. In the case when $H = H_1 \oplus H_2$ Fredholm pairs were discussed as pairs of correspondences (relations), which may be composed, leading to a generalization of composition rules for Fredholm operators. This is crucial for the case of bordisms, [8] and §5 below.

Applications to topology of Fredholm pairs are not enough investigated so far. Except for the articles [Bo1-4], there are very few papers exploring this subject. One should mention [9-10].

The purpose of this note and its expanded version [8] is to give an introduction to a systematic treatment of the Fredholm pairs theory applied to geometry and topology. In terms of boundary values of solutions the Riemann-Hilbert problem translates directly to the language of Fredholm pairs. One can recover the index of the original problem as well as the kernel and the cokernel. Developing this idea we study an application of Fredholm pairs to bordisms. We consider a bordism of smooth manifolds

$$M_1 \sim_X M_2$$

equipped with an elliptic differential operator D acting on the sections of a vector bundle ξ over X. Let $H_i = L^2(M_i;\xi)$. The generalized boundary values of the solutions of $Du = 0$ form a subspace L contained in the direct sum $H_1 \oplus H_2$.

This space cannot be represented as a graph of an operator $H_1 \to H_2$, but it may be treated as a morphism from H_1 to H_2. It transports certain family of linear subspaces of H_1 to H_2. The correspondence L allows to couple spaces $H_1^- \subset H_1$ with spaces $H_2^+ \subset H_2$. In general the index is defined when $(L, H_1^- \oplus H_2^+)$ is a Fredholm pair.

As in the case of Fredholm operators, the properties (i)–(iii) play the decisive role:

(i) The index of Fredholm pairs is stable under deformations;

(ii) Although the index is not additive under the composition of correspondences, but the defect is well understood;

(iii) The constructions are motivated and illustrated by examples coming from the boundary value problems of elliptic operators.

It appears that the concept of Fredholm pairs and correspondences creates a natural analytical setting for an abstract theory of bordisms, expressed in terms of linear functional analysis.

The concept of Fredholm pairs and its generalizations provide a convenient approach to a variety of problems in partial differential equations: both local, as classical boundary value problems, or non-local, when in the boundary conditions a global operator (e.g. spectral restriction or additional pointwise translation) is present. In the setting of Fredholm pairs the given differential operators and their parametrices exist on the same footing. Families of Fredholm pairs and the bi-Grassmannian $\mathcal{F}Gr^2$ appear as a classifying space for K-functor. The algebraic construction of K-homology $K_*(X)$ suggested by Atiyah and realized by Kasparov, based on the theory of elliptic or Fredholm operators in $C(X)$ (or $C^\infty(X)$) modules, have direct analogies in the Fredholm pairs setting. Some construction e.g. description of the differential

$$\delta : K_0(X) \to K_1(M)$$

in the K-homology for the Mayer-Vietoris exact sequence for a splitting $X = X_- \cup_M X_+$ is easier then in the Fredholm operator setting, [8]. In some situations, like the Cauchy data spaces for elliptic operators or the bordism category, the language of Fredholm pairs and correspondences seems unavoidable. The concept of abstract Fredholm pairs and bordisms admits natural and well motivated generalizations: Fredholm fans, [6], also treated in [8].

§2. Classical and abstract Riemann-Hilbert problem

The classical Riemann-Hilbert problem is understood as follows. Let $\mathbb{CP}^1 = \mathbb{D}_- \cup_{S^1} \mathbb{D}_+$ be the usual decomposition of the Riemann sphere (i.e. the complex projective line). Here $\mathbb{D}_+$ is the unit disk and $\mathbb{D}_-$ is the complementary disk containing infinity. Given a function (a loop) $\phi : S^1 \to GL(\mathbb{C}^n)$, describe the totality of holomorphic vector-valued functions $s_\pm : \mathbb{D}_\pm \to \mathbb{C}^n$, such that $s_+(z) = \phi(z)s_-(z)$ for $z \in S^1$. Due to Birkhoff decomposition, [22, 26], if ϕ is differentiable then the Riemann-Hilbert problem is the same as looking for a section of the holomorphic bundle defined by ϕ, [21, 5, 7, 9, 10]. If ϕ is piecewise constant, then this is the 21$^{\text{st}}$ problem, as stated by Hilbert, see [11]. It's a question about existence of a system of singular differential equations with prescribed monodromy.

Denote by $H^{\pm}$ the space of boundary values of holomorphic vector-functions on $\mathbb{D}_{\pm}$. This is a Fredholm pair in $H = L^2(S^1;\mathbb{C}^n)$. The pair $(\phi(H^-), H^+)$ is also Fredholm, where $\phi(H^-)$ is the image of H^- with respect to the obvious multiplication representation of the loop group $\Omega GL(\mathbb{C}^n)$ on H.

If we normalize s_- by the condition $s_-(\infty) = 0$, then we obtain a subspace $H^\flat \subset H^-$. Set $H^\sharp = H^+$. Then $H = H^\flat \oplus H^\sharp$. According to [4] the question about the pair $(\phi(H^-), H^+)$ being Fredholm is reduced to the abstract problem of studying the operator

$$\mathcal{L}_\phi = \phi P^\flat + P^\sharp : H \to H$$

or the Toeplitz operator

$$P^\flat \phi : H^\flat \to H^\flat .$$

The projectors $P^\flat$ and $P^\sharp$ are the projectors in the direct sum $H = H^\flat \oplus H^\sharp$, but they can be substituted by the projectors of Sohotski-Plemelj, [25, 15], which are singular integral operators.

In order to explain the deep meaning of the operator $\mathcal{L}_\phi$ let us summarize some facts about Fredholm pairs. We will follow [4] and [6]. Suppose that H is decomposed into a direct sum

$$H = H^\flat \oplus H^\sharp ,$$

both summands closed of infinite dimension. We can say that this decomposition is given by a symmetry S: a ,,sign" or ,,signature" operator with $S^2 = 1$. Let $P^\flat$ and $P^\sharp$ be the corresponding projectors and $S = P^\sharp - P^\flat$. This is the basic one-dimensional singular integral operator. It is well known that for any continuous loop ϕ the commutator $[\phi, S] = \phi S - S\phi$ is a compact operator (Mikhlin lemma, [26]).

Let $\mathcal{I} \subset \mathcal{B}(H)$ be an ideal containing the ideal of finite rank operators and contained in the ideal of compact operators. Define $GL(S,\mathcal{I}) \subset GL(H)$ to be the set of all invertible authomorphisms of H commuting with S up to the ideal $\mathcal{I}$:

$$GL(S,\mathcal{I}) = \{\phi \in GL(H) \,:\, [\phi, S] \in \mathcal{I}\} .$$

We have the following classification result.

Theorem 2.1. [4] *Let $H^{\pm}$ be a Fredholm pair with $H^+ = H^\sharp$. Suppose, that H^- is given by a projector P^- satisfying $P^\flat - P^- \in \mathcal{I}$. Then there exists $\phi \in GL(S,\mathcal{I})$, such that $H^- = \phi(H^\flat)$. Moreover, the operator $\mathcal{L}_\phi = \phi P^\flat + P^\sharp$ is Fredholm and*

$$ind(\mathcal{L}_\phi) = Ind(H^-, H^+) .$$

The map

$$\widetilde{\kappa} : GL(S,\mathcal{I}) \to \mathbb{Z}$$
$$\widetilde{\kappa}(\phi) = ind(\mathcal{L}_\phi)$$

is a group homomorphism

$$\widetilde{\kappa}(\phi \circ \psi) = \widetilde{\kappa}(\phi) + \widetilde{\kappa}(\psi) .$$

We remark that the correspondence $\phi \mapsto \mathcal{L}_\phi$ between the group $GL(S,\mathcal{I})$ and the Fredholm operators is an ,,almost" homomorphism, i.e.

$$\mathcal{L}_{\phi\circ\psi} = \mathcal{L}_\phi \circ \mathcal{L}_\psi + T(\phi, \psi)$$

with $T(\phi,\psi) = \frac{1}{4}(1-\phi)[S,\psi](1-S) \in \mathcal{I}$. If we write $\phi \in GL(S,\mathcal{I})$ in a matrix form with respect to the splitting $H = H^\flat \oplus H^\sharp$:

$$\phi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

then α, δ are Fredholm operators, β, γ are in $\mathcal{I}$ and $\widetilde{\kappa}(\phi) = ind(\alpha) = -ind(\delta)$. It follows that

$$Ind(H^-, H^+) = ind(P^\flat\phi : H^\flat \to H^\flat) = ind(P^\sharp\phi^{-1} : H^\sharp \to H^\sharp)\,.$$

Note, that if ϕ is arbitrary, possibly not invertible, then $\phi P^\flat + P^\sharp : H \to H$ and $P^\flat\phi : H^\flat \to H^\flat$ are Fredholm operators of equal indices, provided that α is Fredholm. These indices are not necessarily equal to $Ind(\phi(H^\flat), H^\sharp)$. The equality holds if and only if $\phi_{|H^\flat}$ is injective. It is better to distinguish between the domain of ϕ (we write H_1) and its target (H_2). There is another expression for $\widetilde{\kappa}(\phi)$, which will be useful later:

Proposition 2.2. *The pair $(graph\,\phi, H_1^\flat \oplus H_2^\sharp)$ in $H_1 \oplus H_2$ is Fredholm and its index is equal to $\widetilde{\kappa}(\phi)$.*

We have introduced a splitting of the Hilbert space $H = L^2(S^1;\mathbb{C}^n) = H^\flat \oplus H^\sharp$. It's a good moment now to expose its fundamental role. The splitting comes from the division of the Fourier exponents into subsets

$$\mathbb{Z} = \mathbb{Z}_{<0} \cup \mathbb{Z}_{\geq 0}\,.$$

A finite perturbation of this set is also an admissible decomposition. The need of introducing a splitting was clear already in [4]:

- It was used to the study of Fredholm pairs with application to Riemann-Hilbert problem in [4].
- Splitting also came into light in the paper of Kasparov, [18] who introduced homological K-theory built from Hilbert modules. The program of noncommutative geometry of A.Connes develops this idea.
- Splitting plays an important role in the theory of loop groups in [26].
- There is also a number of papers in which surgery of the Dirac operator is studied. Splitting serves as a boundary condition, see e.g. [14, 27]. These papers originate from [3].

Let us come back to the decomposition of $L^2(S^1;\mathbb{C}^n)$ originating from the classical Riemann-Hilbert problem. It is given by a pair of pseudodifferential projectors. Suppose that the authomorphism ϕ is the multiplication by a matrix with entries being continuous functions. Then $\phi \in GL(S,\mathcal{K})$ and the Theorem 2.1. applies.

§3. Fredholm bi-Grassmannian

We will describe the homotopy types of the spaces involved in our constructions.

3.1. *The Grassmannian of the closed linear subspaces $M \subset H$ with* $\dim(M) = codim(M) = \infty$. We denote this set by $Gr_\infty(H)$. The linear group $GL(H)$ acts on it transitively. Let $S \in GL(H)$ be a symmetry decomposing H into direct sum $H^\flat \oplus H^\sharp$ of closed subspaces of infinite dimensions. The stabilizer of $H^\sharp \in Gr_\infty(H)$ consists of linear isomorphisms, which can be written in the block form $\begin{pmatrix} \alpha & 0 \\ \gamma & \delta \end{pmatrix}$ with α, δ being isomorphisms and γ arbitrary linear map. We can write $Gr_\infty(H) = GL(H)/Stab(H^\sharp)$. We endow this set with the quotient topology. By a result of Kuiper, [20], the topological spaces $GL(H)$ and $Stab(H^\sharp) = GL(H^\flat) \times Hom(H^\flat, H^\sharp) \times GL(H^\sharp)$ are contractible. Hence $Gr_\infty(H)$ is contractible as well.

3.2. *The set of Fredholm pairs (H^-, H^+) in H.* We denote this set by $\mathcal{F}Gr^2(H)$. This is a subset of $Gr_\infty(H) \times Gr_\infty(H)$. The projection on the second factor (forgetting about H^-) is a fibration. Denote the fiber over $H^\sharp$ by $Gr_{H^\sharp}(H)$

$$Gr_{H^\sharp}(H) \hookrightarrow \mathcal{F}Gr^2(H) \twoheadrightarrow Gr_\infty(H)\,.$$

Since the base of the fibration is contractible by §3.1, the inclusion of the fiber is a homotopy equivalence.

3.3. The fiber $Gr_{H^\sharp}(H)$ is identified with the subset of $Gr_\infty(H)$ consisting of *the closed linear subspaces $H^- \subset H$, such that the pair $(H^-, H^\sharp)$ is Fredholm.* It is the orbit of $H^\flat$ with respect to the action of the ,,parabolic up to $\mathcal{K}$" subgroup $P(S,\mathcal{K}) \subset GL(H)$

$$Gr_{H^\flat}(H) = P(S,\mathcal{K}) \cdot H^\flat \subset Gr_\infty(H)\,.$$

The group $P(S,\mathcal{K})$ consists of isomorphism of the form $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with α, δ being Fredholm operators and β compact operator. Consider the projection $P(S,\mathcal{K}) \to \mathcal{F}(H^\flat)$ sending an element of $\phi \in P(S,\mathcal{K})$ to the operator $\alpha = P^\flat \phi_{|H^\flat}$. Arguing as in [26] 6.2.4 we prove that this map has the contractible fibers. Therefore it is a homotopy equivalence.

Let us sum up the results and compare it with [26], where the restricted Grassmannian was studied.

Theorem 3.4. *The following maps are homotopy equivalences:*

$$\begin{array}{ccccc} GL(S,\mathcal{K}) & \hookrightarrow & P(S,\mathcal{K}) & \twoheadrightarrow & \mathcal{F}(H^\flat) \\ \downarrow & & \downarrow & & \\ Gr_{res}(S,\mathcal{K}) & \hookrightarrow & Gr_{H^\sharp}(H) & \hookrightarrow & \mathcal{F}Gr^2(H)\,, \end{array}$$

where $Gr_{res}(S,\mathcal{K})$ consists of the subspaces $H^- \subset H$ of the form $H^- = \phi(H^\flat)$ for $\phi \in GL(S,\mathcal{K})$.

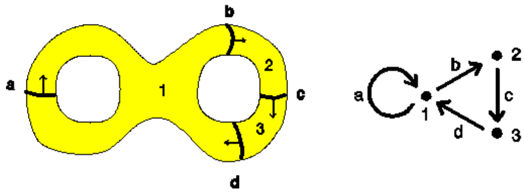
In our constructions the ideal of compact operators $\mathcal{K}$ can be substituted by any smaller ideal containing the ideal of the finite rank operators. In fact in [26] the ideal of Hilbert-Schmidt operators appears.

§4. Some general remarks and comments

We recapitulate: there are several aspects of the general Riemann-Hilbert problem.

4.1. *The underlying geometric and combinatorial setting.* The simplest example of the Riemann-Hilbert problem deals with the operator $\overline{\partial}$ on the Riemannian sphere $S^2 = \mathbb{CP}^1$ (or a Riemann surface) divided into two complementary domains by a curve Γ, say $\Gamma = S^1$, as described. In the literature much more general situations have been studied, [25, 15, 22].

Suppose that there is given an oriented contour M on a Riemann surface X. It may consist of a finite number of parameterized curves admitting transversal intersections and self-intersections. We have on each component of M positive and negative side (i.e. an orientation of the normal bundle).



Configuration of curves and its combinatorial model.

If

$$\phi : M \to GL(\mathbb{C}^n)$$

is a continuous map, then for a vector-function $s(z)$ holomorphic on each component U of $X \setminus M$, admitting in some natural sense boundary extension to $\overline{U}$, the Riemann-Hilbert transmission condition

$$s_+(z) = \phi(z) s_-(z)\,, \quad \text{for} \quad z \in M$$

is meaningful. Here $s_\pm(z)$ are boundary values of $s(z)$ on positive and negative side of M.

If $M = \bigcup_{i=0}^m \Gamma_i$ is the sum of boundaries of disjoint discs $\Gamma_i = \partial \mathbb{D}_i$, $i = 0, \ldots m$, then we have the classical Riemann-Hilbert problem in a non-simplyconnected domain. Splitting M into two disjoint parts $M = M_1 \sqcup M_2$ provides an example of a bordism. Such bordisms admit compositions, if the boundaries are matching. One can formulate local and global index formulas in the realm of conformal field theory. For details see §6 and [8].

In higher dimensions one considers manifolds with a configurations of hypersurfaces (of codimension one) non-intersecting or with transversal intersections. The most relevant here is the case of submanifolds realizing a decomposition of X into bordisms

$$X_0 \cup_{M_1} X_1 \cup_{M_2} \cdots \cup_{M_m} X_m\,.$$

Recall that, as beautifully described in [24], the bordisms form a category with oriented $n-1$-dimensional manifolds as objects and bordisms as morphisms. In our abstract bordism model each splitting manifold M_i has an associated Hilbert space H_i supplied with an involution S_i also called signature operator. These define splittings $H_i = H_i^\flat \oplus H_i^\sharp$ into incoming and outgoing components and should be considered as a part of the structure. The bordism $M_{i-1} \sim_{X_1} M_i$ together with an elliptic first order operator D on X_i gives rise to a closed linear subspace $L \subset H_{i-1} \oplus H_i$, the space of Cauchy data on $\partial X_i = M_{i-1} \sqcup M_i$ of solutions of homogeneous equation $D = 0$ on X_i. The space L is a correspondence from H_{i-1} to H_i. We will illustrate our point of view by a simple but instructive Example 5.1.

4.2. *The generalized Riemann-Hilbert problem.* In the simplest case $m = 1$

$$X_- \cup_M X_+$$

Fredholm pair arises from consideration of the Cauchy data on X_- and X_+. More precisely let

$$D : C^\infty(X;\xi) \to C^\infty(X;\eta)$$

be an elliptic operator of the first order. The Dirac operator is of special interest. One defines the spaces $H^{\pm}(D) \subset H = L^2(M;\xi)$, which are the spaces of boundary values of solutions of homogeneous equations $Ds = 0$ on the manifolds $X_{\pm}$. The pair $H^{\pm}(D)$ is Fredholm. In order to study $\mathit{ker}\, D$ and $\mathit{coker}\, D$ it is convenient to assume that D and D^* have the unique extension property, i.e. the solutions of D and D^* are determined by the boundary values on M. Then

$$\mathit{ker}\, D = H^+(D) \cap H^-(D)\,, \quad \mathit{coker}\, D = H/(H^+(D) + H^-(D))\,.$$

As in the case described in §2 corresponding Cauchy data spaces admit projectors. There are associated Calderón projectors $P^{\pm}(D)$ onto $H^{\pm}(D)$. They are complementary up to a compact operator: $P^-(D) + P^+(D) - 1 \in \mathcal{K}$. The operation S given by

$$S = P^+ - P^-$$

is the fundamental singular operator. The group $GL(S, \mathcal{K})$ is naturally involved.

4.3. *The Riemann-Hilbert problem in an abstract Hilbert space H:* suppose we have an involution $S \in \mathcal{B}(H)$ defining a splitting $H^\flat \oplus H^\sharp$. We consider the following objects:

- the group of $GL(S, \mathcal{K})$ of the linear isomorphisms of H commuting with S up to $\mathcal{K}$,
- the bi-Grassmannian $\mathcal{F}Gr^2_{res}$ consisting of pairs of the form $(\phi(H^\flat), H^\sharp)$, with $\phi \in GL(S, \mathcal{K})$, called restricted Grassmannian in [26].
- the bi-Grassmannian $\mathcal{F}Gr^2(H)$, the set of all Fredholm pairs in H.

These spaces are homotopy equivalent, they are classifying spaces of K-theory $BU \times \mathbb{Z}$. A family of Fredholm pairs (say over T) defines an element of $K^0(T)$. Moreover, the classical Riemann-Hilbert problem gives us a way of constructing a Fredholm pair in $H^n = L^2(S^1;\mathbb{C}^n)$ out of a given loop in $U_n \subset GL(\mathbb{C}^n)$. The assignment

$$\Omega U_n \to \mathcal{F}Gr^2(H^n) \simeq BU \times \mathbb{Z}$$

passes to a map

$$\Omega U_\infty \to \mathcal{F}Gr^2(H^\infty) \simeq BU \times \mathbb{Z}\,,$$

which can be interpreted as the Bott periodicity map.

4.4. *Quantum Riemann-Hilbert problem:* There is another structure which one cannot forget keeping in mind geometric applications. The Hilbert space H comes with an action of the algebra $C(M)$ of functions on $M = \partial X_{\pm}$. The pseudodifferential operator $P^+(D) - P^-(D)$ almost commutes with the algebra action. Hence it defines an element in the odd K-homology $K_1(M)$. From the point of view of

Kasparov theory we can replace $P^+(D) - P^-(D)$ by the almost equal operator $S = P^\sharp - P^\flat$, $S^2 = 1$. But now as in [13], pp 287-289, it is easier to express the pairing with $K^1(M)$. If $\phi \in GL(S^{\oplus n}, \mathcal{K})$ is defined by a matrix of functions $\widetilde{\phi} : M \to GL(\mathbb{C}^n)$ (i.e. $\widetilde{\phi}$ is a generator of $K^1(M)$), then the effect of the pairing of $[\widetilde{\phi}] \in K^1(M)$ with $[S] \in K_1(M)$ is equal to $\widetilde{\kappa}(\phi)$. One can ask what element of $K_1(M)$ is defined by S. It's not hard to guess that:

Theorem 4.5. *The Fredholm module $(H = L^2(M;\xi), S)$ is the image of $[D]$ with respect to the differential $\delta : K_0(X) \to K_1(M)$ in homological Mayer-Vietoris sequence of the triple (X, X_-, X_+).*

Since our proof in [8] is obtained by means of duality, the result holds modulo torsion in $K_1(M)$.

It is clear that the algebra of functions $C(S^1)$ (or $C(M)$) may be replaced by an arbitrary $\mathbb{C}^*$-algebra, possibly noncommutative. The framework of noncommutative geometry, [12, 13], is another possible setup for studying Riemann-Hilbert problem and corresponding Fredholm pairs.

§5. Bordisms

Now we would like to describe more general objects than the operators $\phi \in GL(S, \mathcal{K})$ considered so far. We study relations in H or correspondences from H_1 to H_2. Our approach is motivated by the geometric theory of bordisms, [24]. First, let us present an example:

Example 5.1. For $0 < r < R$ consider the ring

$$X = \{z \in \mathbb{C} : r \le |z| \le R\}.$$

Then

$$\partial X = M_1 \sqcup M_2 = S^1_R \sqcup S^1_r.$$

The functions $e_i = z^i : M_1 \to \mathbb{C}$ and $\epsilon_i = z^i : M_2 \to \mathbb{C}$ for $i \in \mathbb{Z}$ form a basis of the Hilbert spaces

$$H_1 = L^2(M_1; \mathbb{C}) = \left\{\textstyle\sum_{i\in\mathbb{Z}} a_i e_i : \sum_{i\in\mathbb{Z}} a_i^2 R^{2i} < \infty\right\}$$

and

$$H_2 = L^2(M_2; \mathbb{C}) = \left\{\textstyle\sum_{i\in\mathbb{Z}} b_i \epsilon_i : \sum_{i\in\mathbb{Z}} b_i^2 r^{2i} < \infty\right\}.$$

Consider the Cauchy-Riemann operator acting on the complex-valued functions on X. The space of the boundary values of solutions L is the graph of the unbounded operator

$$\Phi : H_1 \dashrightarrow H_2, \qquad \Phi\left(\textstyle\sum_{i\in\mathbb{Z}} a_i e_i\right) = \sum_{i\in\mathbb{Z}} a_i \epsilon_i.$$

The maximal domain of Φ is

$$\left\{\textstyle\sum_{i\in\mathbb{Z}} a_i e_i : \sum_{i\in\mathbb{Z}} a_i^2 (R^{2i} + r^{2i}) < \infty\right\}.$$

The above condition can be substituted by $\sum_{i<0} a_i^2 r^{2i} + \sum_{i\ge 0} a_i^2 R^{2i} < \infty$. Set

$$H_1^\flat = span\{e_i : i < 0\}, \quad H_1^\sharp = span\{e_i : i \ge 0\},$$

$$H_2^\flat = span\{\epsilon_i : i < 0\}, \quad H_2^\sharp = span\{\epsilon_i : i \ge 0\}.$$

Now Φ restricted to $H_1^\sharp$ is bounded and moreover, it is *compact*. Indeed, $\Phi_{|H_1^\sharp}$ is given by Cauchy integral

$$\Phi(f)(\zeta) = \frac{1}{2\pi i}\int_{|z|=R} \frac{f(\zeta)d\zeta}{z-\zeta}\,.$$

Similarly Φ^{-1} restricted to $H_2^\flat$ is a compact operator. The space of the boundary values of the solutions is the direct sum of the graphs

$$L = graph(\phi_1) \oplus graph(\phi_2)\,,$$

where

$$\phi_1 = \Phi_{|H_1^\sharp} : H_1^\sharp \to H_2^\sharp\,, \quad \phi_2 = \Phi^{-1}_{|H_2^\flat} : H_2^\flat \to H_1^\flat\,.$$

Note, that

- The pair $(L, H_1^\flat \oplus H_2^\sharp)$ spans $H_1 \oplus H_2$ as a direct sum,
- the projection of L onto $H_1^\sharp \oplus H_2^\flat$ along $H_1^\flat \oplus H_2^\sharp$ is an isomorphism,
- L is a direct sum of graphs of the compact operators $\phi_1 = \Phi_{|H_1^\sharp}$ and $\phi_2 = \Phi^{-1}_{|H_2^\flat}$.

Now consider the Cauchy-Riemann operator D on the projective line $\mathbb{CP}^1$ decomposed into the subsets

$$\begin{aligned} X_1 =& \{z \in \mathbb{C} \,:\, |z| \geq R\} \cup \{\infty\}, \\ X =& \{z \in \mathbb{C} \,:\, r \leq |z| \leq R\}, \\ X_2 =& \{z \in \mathbb{C} \,:\, |z| \leq r\}\,. \end{aligned}$$

We have the spaces of boundary values of holomorphic functions on X_1 and X_2

$$\begin{aligned} H_1^- =& span\{e_i \,:\, i \leq 0\} = H_1^\flat \oplus \langle e_0\rangle, \\ H_2^+ =& span\{\epsilon_i \,:\, i \geq 0\} = H_2^\sharp. \end{aligned}$$

The space $L \subset H_1 \oplus H_2$ is not a graph of a bounded operator, but as in the case of Riemann-Hilbert transmission problem, we can write

$$ind(D) = Ind(L(H_1^-), H_2^+)\,,$$

where $L(H_1^-) = \{y \in H_2 \,:\, \exists x \in H_1\,,\, (x,y) \in L\}$.

The situation described in the example is quite general. Consider a manifold X with boundary, which is the sum of two components $\partial X = M_1 \sqcup M_2$. Let $D : C^\infty(X;\xi) \to C^\infty(X;\eta)$ be an elliptic operator of the first order. Set $H_i = L^2(M_i;\xi)$ for $i = 1,2$ and let L be the closure in $L^2(\partial X;\xi) = H_1 \oplus H_2$ of the space $\{u_{|\partial X} \,:\, Du = 0\,,\, u \in C^\infty(X;\xi)\}$. Let P_L be the Calderón projector

$$P_L : H_1 \oplus H_2 \twoheadrightarrow L\,.$$

Let $\widetilde{\xi_i}$ be the pull back of ξ to $T^*M_i\backslash\{0\}$. The symbol $\sigma(P_L)_{|M_i}$ is an endomorphism of the bundle $\widetilde{\xi_i}$. Let us choose pseudodifferential projectors P_i acting on H_i with $\sigma(P_i) = \sigma(P_L)_{|M_i}$. Then

$$P_1 \oplus P_2 \sim P_L : H_1 \oplus H_2 \to H_1 \oplus H_2\,.$$

Set $P_1^\sharp = P_1$, $P_2^\flat = P_2$. These operators define split Hilbert spaces $H_i = H_i^\flat \oplus H_i^\sharp$ $(i = 1, 2)$. It follows that

- the pair $(L, H_1^\flat \oplus H_2^\sharp)$ is a Fredholm pair,
- the projection $P_1^\sharp \oplus P_2^\flat$ from L onto $H_1^\sharp \oplus H_2^\flat$ along $H_1^\flat \oplus H_2^\sharp$ is a Fredholm operator.

It can be shown that, as in Example 5.1, there are compact operators:

- ϕ_1 transforming the restrictions on M_1 of some solutions $Du = 0$ to the restrictions on M_2
- and ϕ_2 acting in the opposite direction,

such that up to finite dimensional perturbation the space L is equal to the sum of their graphs.

Note, that Calderón projectors are well defined if we restrict our considerations to the space of smooth sections. That means that we work on a pre-Hilbert level. To obtain the abstract Hilbert space model the completion operation should be applied. In the completion process different Sobolev type metrics have to be used according to the Sobolev trace type imbedding theorems. It requires caution and involves some additional technicalities, which have been skipped here. For details see [8]. The remarks above can be clearly seen in our basic cobordism example of the Cauchy-Riemann operator in the ring domain.

We want to define an index of D, regardless of all the possible choices of manifolds closing X. It will be defined with respect to the splittings. The index $Ind(L(H_1^\flat), H_2^\sharp)$ is not stable under a compact perturbation. If we twist L with an authomorphism of the form $1 + K$, $K \in \mathcal{K}$ the index may change. Instead it is wiser to consider the pair $(L, H_1^\flat \oplus H_2^\sharp)$. Its index is stable under such twists. It is worth to say when the considered indices are equal:

Proposition 5.2. *$Ind(L, H_1^\flat \oplus H_2^\sharp) = Ind(L(H_1^\flat), H_2^\sharp)$ provided that both following conditions hold*

- *L is injective on $H_1^\flat$, i.e. if $(x, y) \in L$ and $(x', y) \in L$, $x, x' \in H_1^\flat$, then $x = x'$,*
- *$H_1^\flat + dom\, L = H_1$, where $dom\, L = \{x \in H_1 : \exists y \in H_2, (x, y) \in L\}$.*

Consider again the case of a bordism X, this time closed from both sides by manifolds X_1 and X_2. That is: there is a closed manifold Y with a first order elliptic operator D and Y is decomposed

$$Y = X_1 \cup_{M_1} X \cup_{M_2} X_2 .$$

By Theorem 2.1 there exist authomorphisms ϕ_i of H_i almost commuting with Calderón projectors, such that

$$H_1^\flat = \phi_1 H_1^-(D) \quad \text{and} \quad H_2^+(D) = \phi_2(H_2^\sharp) .$$

Then (provided that D and D^* have the unique extension property)

$$Ind(D) = Ind(L, H_1^- \oplus H_2^+) = \widetilde{\kappa}(\phi_1) + Ind(L, H_1^\flat \oplus H_2^\sharp) + \widetilde{\kappa}(\phi_2) .$$

We see that L plays a role of the twist $\phi_i : H_i \to H_i$, but here L allows us to couple „the lower half" of H_1 with „the upper half" of H_2. We can treat it as a morphism[1] from H_1 to H_2. Note, that H_1 and H_2 are not canonically identified. Indeed the manifolds M_1 and M_2 joined by the bordism X can be quite different. There are also two different algebras $C(M_1)$ and $C(M_2)$ acting. The actions commute with the splittings up to compact operators. The object described here is an abstract substitute of a geometric bordism.

Definition 5.3. *A* **restricted bordism** $H_1 \overset{L}{\rightsquigarrow} H_2$ *between split Hilbert spaces* $H_i = H_i^\flat \oplus H_i^\sharp$ *(for* $i = 1, 2$*) is a closed linear subspace* $L \subset H_1 \oplus H_2$*, which is the image of a projector* $P_L \sim P_1^\sharp \oplus P_2^\flat$*.*

The widest class of linear correspondences, which allows us to define the index is the following:

Definition 5.4. *A* **Fredholm bordism** $H_1 \overset{L}{\rightsquigarrow} H_2$ *between split Hilbert spaces is a closed linear subspace* $L \subset H_1 \oplus H_2$*, such that the pair* $(L, H_1^\flat \oplus H_2^\sharp)$ *in* $H_1 \oplus H_2$ *is Fredholm. The index of* L *is the index of this pair. It is denoted by* $\kappa(L)$ *or* $\kappa(H_1^\flat | L | H_2^\sharp)$ *to expose the role of splittings.*

Note, that by Proposition 2.2 the graph of an isomorphism $\phi \in GL(S, \mathcal{K})$ is a Fredholm bordism and $\kappa(graph\,\phi) = \widetilde{\kappa}(\phi)$.

We can say that the class of Hilbert spaces with involutions (splittings) and Fredholm bordisms $H_1 \sim_L H_2$ form a category, which may be considered as an abstract functional theoretic counterpart of of the category of geometric bordisms. Each elliptic differential operator on any geometric bordism, the Calderón projectors and the corresponding involutions gives rise to a Fredholm bordism.

§6. Riemann surfaces with boundary

Let us consider another example which is classical, now also studied under the name of conformal field theory. We consider the Hilbert space of complex functions on the circle: $H = L^2(S^1; \mathbb{C})$. Let Y_g be a Riemann surface of genus g. Suppose we have $k + l$ disjoined holomorphic disks $\mathbb{D}_i$ $(i = 1, \ldots, k)$, $\mathbb{D}'_j$ $(j = 1, \ldots, l)$ contained in Y_g. Let X be the complement of the disks. We think of X as of a bordism between k circles and l circles. Let $L \subset H^k \oplus H^l$ be the space of boundary values of the holomorphic functions on X. Denote by $H(\lambda)$ (for $\lambda \in \mathbb{Z}$) the space H equipped with the splitting

$$H^\flat(\lambda) = z^\lambda H^\flat = span\langle z^i \,:\, i < \lambda\rangle\,,$$

$$H^\sharp(\lambda) = z^\lambda H^\sharp = span\langle z^i \,:\, i \geq \lambda\rangle\,.$$

For sequences of integers $\lambda_\bullet = (\lambda_1, \ldots, \lambda_k)$ and $\mu_\bullet = (\mu_1, \ldots, \mu_l)$ we have splittings

$$H_1 = H^k(\lambda_\bullet) = \Big(H^\flat(\lambda_1) \oplus \cdots \oplus H^\flat(\lambda_k)\Big) \oplus \Big(H^\sharp(\lambda_1) \oplus \cdots \oplus H^\sharp(\lambda_k)\Big)\,,$$

$$H_2 = H^l(\mu_\bullet) = \Big(H^\flat(\mu_1) \oplus \cdots \oplus H^\flat(\mu_k)\Big) \oplus \Big(H^\sharp(\mu_1) \oplus \cdots \oplus H^\sharp(\mu_k)\Big)\,.$$

We will compute the index of L with respect to these splittings. An element of the intersection $L \cap (H_1^\flat \oplus H_2^\sharp)$ defines a meromorphic function on Y_g with zeros

[1] A different approach to bordisms, based on quantum field theory point of view, is presented in [28], Lecture 2. Dirac operators are of special interest.

(resp. poles) at the centers of $\mathbb{D}_i$'s (resp. $\mathbb{D}'_j$'s) of the order at least λ_i (resp. smaller then μ_j). This is a section of a sheaf

$$\mathcal{O}\left(-\sum_{i=1}^k \lambda_i d_i + \sum_{j=1}^l (\mu_j - 1)d'_j\right) .$$

Here d_i and d'_j are the centers of the disks. The index is equal to the Euler characteristic of Y with coefficients in this sheaf, that is

$$1 - g - \sum_{i=1}^k \lambda_i + \sum_{j=1}^l \mu_j - l .$$

In particular, if we want to compute the index of the Cauchy-Riemann operator on Y, then we set $\lambda_i = 0$, $\mu_j = 1$. These splittings agree with the spaces of boundary values of solutions on the disks: the index is $1 - g$. Again we can write $Ind(L, H_1^\flat \oplus H_2^\sharp) = Ind(L(H_1^\flat), H_2^\sharp)$. This number is denoted by $\kappa(H_1^\flat|L|H_2^\sharp)$ according to Definition 5.4.

Note, that the bordisms, as well as correspondences can be composed. We consider the composition with the splittings coinciding. If we deal only with connected surfaces then

$$\kappa(L_1 \circ L_2) = \kappa(L_1) + \kappa(L_2) .$$

If we admit disconnected bordism, then it may happen, that a closed component is created while sewing the bordisms. The defect $\Delta = \kappa(L_1) + \kappa(L_2) - \kappa(L_1 \circ L_2)$ equals to the index on this component. This remark generalizes to an arbitrary elliptic differential operator D of the first order. In consequence a decomposition of a closed manifold

$$X = \emptyset \sim_{X_0} M_1 \sim_{X_1} \cdots \sim_{X_{n-1}} M_n \sim_{X_n} \emptyset$$

gives rise to a sequence of restricted bordisms

$$0 \overset{L_0}{\rightsquigarrow} H_1 \overset{L_1}{\rightsquigarrow} \dots \overset{L_{n-1}}{\rightsquigarrow} H_n \overset{L_n}{\rightsquigarrow} 0 .$$

Theorem 6.1. [8] *Suppose D and D^* have the unique extension property. Fix splittings S_i of H_i. Then the global index of D is equal to the sum of partial indices:*

$$ind\, D = \sum_{i=0}^n \kappa(H_i^\flat|L_i|H_{i+1}^\sharp) .$$

We refer to [8] for further discussion of 'local to global' formulas.

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