GENERALIZED CYLINDERS IN SEMI-RIEMANNIAN AND SPIN GEOMETRY

CHRISTIAN BÄR, PAUL GAUDUCHON, AND ANDREI MOROIANU

ABSTRACT. We use a construction which we call generalized cylinders to give a new proof of the fundamental theorem of hypersurface theory. It has the advantage of being very simple and the result directly extends to semi-Riemannian manifolds and to embeddings into spaces of constant curvature. We also give a new way to identify spinors for different metrics and to derive the variation formula for the Dirac operator. Moreover, we show that generalized Killing spinors for Codazzi tensors are restrictions of parallel spinors. Finally, we study the space of Lorentzian metrics and give a criterion when two Lorentzian metrics on a manifold can be joined in a natural manner by a 1-parameter family of such metrics.

1. INTRODUCTION

In this paper we give various applications of a construction which we call *generalized cylinders*. Let M be a manifold and let g_t be a smooth 1-parameter family of semi-Riemannian metrics on M, $t \in I \subset \mathbb{R}$. Then we call the manifold $\mathcal{Z} = I \times M$ with the metric $dt^2 + g_t$ a generalized cylinder over M. On the one hand, this ansatz is very flexible. Locally, near a semi-Riemannian hypersurface with spacelike normal bundle every semi-Riemannian manifold is of this form. The restriction to spacelike normal bundle, i. e. to the positive sign in front of dt^2 in the metric of $\mathcal{Z}$ is made for convenience only. Changing the signs of the metrics on M as well as on $\mathcal{Z}$ reduces the case of a timelike normal bundle to that of a spacelike normal bundle. On the other hand, this ansatz still allows to closely relate the geometries of M and $\mathcal{Z}$.

In Section 2 we collect basic material on spinors and the Dirac operator on semi-Riemannian manifolds. We do this to fix notation and for the convenience of the reader. Some of the material, such as the spin geometry of submanifolds, is not so easily found in the literature unless one restricts oneself to the Riemannian situation.

In Section 3 we study spinors on a manifold foliated by semi-Riemannian hypersurfaces. In particular, we derive a formula for the commutator of the leafwise Dirac operator and the normal derivative. This formula will be important later.

In Section 4 we collect formulas relating the curvature of a generalized cylinder to geometric data on M.

After these preliminaries we give a first application in Section 5. One technical difficulty when dealing with spinors comes from the fact that the definition of spinors depends on the metric on the manifold. This problem does not arise when one works with tensors. Thus if one wants to compare the Dirac operators for two different metrics, then one first has to identify the spinor bundles in a natural manner. This identification problem can be split into two steps. First, construct an identification for 1-parameter families of metrics and, secondly, given two metrics construct a natural 1-parameter family joining them.

Date: March 7, 2003.

2000 *Mathematics Subject Classification.* 53C27,53A07,53B30.

Key words and phrases. generalized cylinder, identification of spinors, variation formula for Dirac operator, energy-momentum tensor of a spinor, fundamental theorem of hypersurface theory, generalized Killing spinors, space of Lorentzian metrics.

The second step is trivial for Riemannian metrics; just use linear interpolation. For indefinite semi-Riemannian metrics the situation is much more complicated. In fact, two semi-Riemannian metrics on a manifold cannot always be joined by a continuous path of metrics even if they have the same signature. In Section 9 we study this problem in detail for Lorentzian metrics and we give a criterion when two Lorentzian metrics can be joined in a natural manner.

The first step, identifying spinors for 1-parameter families of semi-Riemannian metrics, is carried out in Section 5. The idea is very simple. Given a 1-parameter family of metrics take the corresponding generalized cylinder and use parallel transport on this cylinder. It turns out that this identification is the same as the one constructed differently by Bourguignon and the second author in [3] for Riemannian metrics. The commutator formula from Section 3 directly translates to the variation formula for Dirac operators.

This variation formula is what one needs to compute the energy-momentum tensor for spinors. To make this precise we briefly summarize Lagrangian field theory in Section 6 and we give a general definition of energy-momentum tensors. Then we compute the example of the Lagrangian for spinors given by the Dirac operator.

In Section 7 we give a new and simple proof of the fundamental theorem of hypersurface theory. A hypersurface of $\mathbb{R}^{n+1}$ inherits a Riemannian metric and its Weingarten map must satify the Gauss and Codazzi-Mainardi equations. The fundamental theorem says that, conversely, any Riemannian manifold M with a symmetric endomorphism field of TM satisfying the Gauss and Codazzi-Mainardi equations can, at least locally, be embedded isometrically into $\mathbb{R}^{n+1}$ with Weingarten map given by this endomorphism field. Our proof goes like this: We write down an *explicit* metric on the cylinder $\mathcal{Z} = I \times M$ and we then check that this metric is flat. Since every flat Riemannian manifold is locally isometric to Euclidean space the theorem follows. This approach directly extends to semi-Riemannian manifolds and to embeddings into spaces of constant sectional curvature not necessarily zero. This kind of approach to the fundamental theorem for hypersurfaces was suggested, but not carried out, by Petersen in [9, p. 95].

In Section 8 we study generalized Killing spinors. They are characterized by the overdetermined equation $\nabla^{\Sigma M}_X \psi = \frac{1}{2} A(X) \cdot \psi$ where A is a given symmetric endomorphism field. We show that if A is a Codazzi tensor, then the manifold can be embedded as a hypersurface into a Ricci flat manifold equipped with a parallel spinor which restricts to ψ. This generalizes the case of Killing spinors, $A = \lambda\,\mathrm{id}$. The classification of manifolds admitting Killing spinors in [1] was based on the observation that the cone over such a manifold possesses a parallel spinor. This also generalizes the case that A is parallel which was studied in [7].

Acknowledgements. The authors would like to thank W. Ballmann and H. Karcher for valuable suggestions. The authors have been partially supported by the Research and Training Network HPRN-CT-2000-00101 "EDGE" funded by the European Commission. The first author has also been partially supported by the Research and Training Network HPRN-CT-1999-00118 "Geometric Analysis". The first author would like to thank the Ecole Polytechnique, Palaiseau, and the the Max-Planck-Institut für Mathematik, Bonn, for their hospitality.

2. The Dirac operator on semi-Riemannian manifolds

In this section we collect the basic facts and conventions concerning spinors and Dirac operators on semi-Riemannian manifolds. For a detailed introduction the reader may consult the book [2]. We start with some algebraic preliminaries. Let $r + s = n$ and consider the nondegenerate symmetric bilinear form of signature

(r, s)

$$\langle v, w\rangle := \sum_{i=1}^{r} v^i w^i - \sum_{i=r+1}^{n} v^i w^i$$

on $\mathbb{R}^n$. Define the corresponding *orthogonal group* by

$$\mathrm{O}(r, s) := \{A \in \mathrm{GL}(n, \mathbb{R}) \mid \langle Av, Aw\rangle = \langle v, w\rangle \text{ for all } v, w \in \mathbb{R}^n\}$$

and the *special orthogonal group* by

$$\mathrm{SO}(r, s) := \{A \in \mathrm{O}(r, s) \mid \det(A) = 1\}.$$

If $r = 0$ or $s = 0$, then $\mathrm{SO}(r, s)$ is connected, otherwise it has two connected components.

Now let $\mathrm{Cl}_{r,s}$ be the *Clifford algebra* corresponding to the symmetric bilinear form $\langle\cdot,\cdot\rangle$. This is the unital algebra generated by $\mathbb{R}^n$ subject to the relations

$$v \cdot w + w \cdot v + 2\langle v, w\rangle \cdot 1 = 0 \tag{1}$$

for all $v, w \in \mathbb{R}^n$. There is a decomposition into even and odd elements

$$\mathrm{Cl}_{r,s} = \mathrm{Cl}^0_{r,s} \oplus \mathrm{Cl}^1_{r,s}$$

such that $\mathbb{R}$ injects naturally into $\mathrm{Cl}^0_{r,s}$ and $\mathbb{R}^n$ into $\mathrm{Cl}^1_{r,s}$. The *spin group* is defined by

$$\mathrm{Spin}(r, s) := \{v_1 \cdots v_k \in \mathrm{Cl}^0_{r,s} \mid v_j \in \mathbb{R}^n \text{ such that } \langle v_j, v_j\rangle = \pm 1 \text{ and } k \text{ is even}\}$$

with multiplication inherited from $\mathrm{Cl}_{r,s}$. Given $v \in \mathbb{R}^n$ such that $\langle v, v\rangle \neq 0$ and arbitrary $w \in \mathbb{R}^n$ we see directly from relation (1) that $v^{-1} = -\frac{v}{\langle v,v\rangle}$ and

$$\mathrm{Ad}_v(w) := v^{-1} \cdot w \cdot v = -w + 2\frac{\langle v, w\rangle}{\langle v, v\rangle}v.$$

Hence $-\mathrm{Ad}_v$ is the reflection across the hyperplane $v^\perp$ and, in particular, leaves $\mathbb{R}^n \subset \mathrm{Cl}_{r,s}$ invariant. Thus conjugation gives an action of $\mathrm{Spin}(r, s)$ on $\mathbb{R}^n$ by an even number of reflections across hyperplanes. This yields the exact sequence

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} = \{1, -1\} \longrightarrow \mathrm{Spin}(r, s) \xrightarrow{\mathrm{Ad}} \mathrm{SO}(r, s) \longrightarrow 1.$$

If $n = r + s$ is even the Clifford algebra possesses an irreducible complex module $\Sigma_{r,s}$ of complex dimension dimension $2^{n/2}$, the complex *spinor module*. When restricted to $\mathrm{Cl}^0_{r,s}$ the spinor module decomposes into

$$\Sigma_{r,s} = \Sigma^+_{r,s} \oplus \Sigma^-_{r,s},$$

the submodules of spinors of *positive* resp. *negative chirality*. In particular, the spin group $\mathrm{Spin}(r, s) \subset \mathrm{Cl}^0_{r,s}$ acts on $\Sigma^+_{r,s}$ and on $\Sigma^-_{r,s}$. This action

$$\rho = \rho^+ \oplus \rho^- : \mathrm{Spin}(r, s) \to \mathrm{Aut}(\Sigma^+_{r,s}) \times \mathrm{Aut}(\Sigma^-_{r,s}) \subset \mathrm{Aut}(\Sigma_{r,s})$$

is called the *spinor representation* of $\mathrm{Spin}(r, s)$. Given an orientation on $\mathbb{R}^n$ the $\mathrm{Cl}^0_{r,s}$-submodules $\Sigma^+_{r,s}$ and $\Sigma^-_{r,s}$ can be characterized by the action of the volume element $\mathrm{vol} := e_1 \cdots e_n \in \mathrm{Cl}^0_{r,s}$ which acts on $\Sigma^+_{r,s}$ as $+i^{s+n(n+1)/2}\mathrm{id}$ and on $\Sigma^-_{r,s}$ as $-i^{s+n(n+1)/2}\mathrm{id}$ where $e_1, \dots, e_n$ is a positively oriented orthonormal basis of $\mathbb{R}^n$.

If n is odd, then $\mathrm{Cl}_{r,s}$ has two inequivalent irreducible modules $\Sigma^0_{r,s}$ and $\Sigma^1_{r,s}$, both of complex dimension $2^{(n-1)/2}$. These two modules are again distinguished by the action of the volume element $\mathrm{vol} = e_1 \cdots e_n \in \mathrm{Cl}^1_{r,s}$, namely vol acts as $+i^{s+n(n+1)/2}\mathrm{id}$ on $\Sigma^0_{r,s}$ and as $-i^{s+n(n+1)/2}\mathrm{id}$ on $\Sigma^1_{r,s}$. When restricted to $\mathrm{Cl}^0_{r,s}$ the two modules become equivalent and we simply write $\Sigma_{r,s} := \Sigma^0_{r,s}$. This time the spinor representation

$$\rho : \mathrm{Spin}(r, s) \to \mathrm{Aut}(\Sigma_{r,s})$$

is irreducible. All spinor modules carry nondegenerate symmetric sesquilinear forms $\langle\cdot,\cdot\rangle$ (in general not definite) which are invariant under the action of $\mathrm{Spin}(r, s)$. The action of a vector $v \in \mathbb{R}^n \subset \mathrm{Cl}_{r,s}$ on $\Sigma_{r,s}$ is skewsymmetric with respect to $\langle\cdot,\cdot\rangle$, i. e. $\langle v\cdot\sigma_1,\sigma_2\rangle = -\langle\sigma_1, v\cdot\sigma_2\rangle$.

To prepare for the study of submanifolds later on we now look at an embedding of $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ such that $(\mathbb{R}^n)^\perp$ is spacelike. Let $(\mathbb{R}^n)^\perp$ be spanned by a spacelike unit vector e_0. The map $\mathbb{R}^n \to \mathrm{Cl}_{r+1,s}$, $v \mapsto e_0\cdot v$, induces an algebra isomorphism $\mathrm{Cl}_{r,s} \to \mathrm{Cl}^0_{r+1,s}$ under which the volume element of $\mathrm{Cl}_{r,s}$ is mapped to the volume element of $\mathrm{Cl}_{r+1,s}$ in case n is odd.

If n is even, then $\Sigma_{r+1,s}$ pulls back to $\Sigma_{r,s}$ under this algebra isomorphism. In other words, we can regard $\Sigma_{r+1,s}$ as the spinor representation of $\mathrm{Cl}_{r,s}$ provided we define the action of $\mathrm{Cl}_{r,s}$ on $\Sigma_{r+1,s}$ by

$$v\otimes\sigma \mapsto e_0\cdot v\cdot\sigma$$

where $v\in\mathbb{R}^n$ and $\cdot$ denotes the action of $\mathrm{Cl}_{r+1,s}$.

Similarly, if n is odd, then the action of the volume forms shows that $\Sigma^+_{r+1,s}$ pulls back to $\Sigma^0_{r,s}$ while $\Sigma^-_{r+1,s}$ pulls back to $\Sigma^1_{r,s}$.

Now we turn to geometry. Let X denote an oriented n-dimensional differentiable manifold. The bundle $P_{\mathrm{GL}^+}(X)$ of positively oriented tangent frames forms a $\mathrm{GL}^+(n,\mathbb{R})$-principal bundle over X. Here and henceforth $\mathrm{GL}^+(n,\mathbb{R})$ denotes the group of real $n\times n$-matrices with positive determinante and $A : \widetilde{\mathrm{GL}}^+(n,\mathbb{R}) \to \mathrm{GL}^+(n,\mathbb{R})$ its connected twofold covering group. A *spin structure* of X is a $\widetilde{\mathrm{GL}}^+(n,\mathbb{R})$-principal bundle $P_{\widetilde{\mathrm{GL}}}(X)$ over X together with a twofold covering map $\Theta : P_{\widetilde{\mathrm{GL}}^+}(X) \to P_{\mathrm{GL}^+}(X)$ such that the following diagram commutes

$$\begin{tikzcd}
P_{\widetilde{\mathrm{GL}}^+}(X)\times\widetilde{\mathrm{GL}}^+(n,\mathbb{R}) \arrow[r] \arrow[d, "\Theta\times A"] & P_{\widetilde{\mathrm{GL}}^+}(X) \arrow[d, "\Theta"'] \arrow[dr] & \\
 & & X \\
P_{\mathrm{GL}^+}(X)\times\mathrm{GL}^+(n,\mathbb{R}) \arrow[r] & P_{\mathrm{GL}^+}(X) \arrow[ur] &
\end{tikzcd} \tag{2}$$

where the horizontal arrows denote the group actions on the principal bundles. This definition of a spin structure has the advantage of being independent of the choice of any semi-Riemannian metric on X. An oriented manifold together with a spin structure will be called a *spin manifold*.

Let X now in addition carry a semi-Riemannian metric of signature (r,s), $r+s=n$. The bundle $P_{\mathrm{SO}}(X)\subset P_{\mathrm{GL}^+}(X)$ of positively oriented *orthonormal* tangent frames forms an $\mathrm{SO}(r,s)$-principal bundle over X. Restricting $A : \widetilde{\mathrm{GL}}^+(n,\mathbb{R}) \to \mathrm{GL}^+(n,\mathbb{R})$ to the preimage of $\mathrm{SO}(r,s)\subset\mathrm{GL}^+(n,\mathbb{R})$ we recover $\mathrm{Ad} : \mathrm{Spin}(r,s) \to \mathrm{SO}(r,s)$. Putting $P_{\mathrm{Spin}}(X) := \Theta^{-1}(P_{\mathrm{SO}}(X))$ we get a $\mathrm{Spin}(r,s)$-principal bundle and and the maps in diagram (2) restrict to the following commutative diagram

$$\begin{tikzcd}
P_{\mathrm{Spin}}(X)\times\mathrm{Spin}(r,s) \arrow[r] \arrow[d, "\Theta\times\mathrm{Ad}"] & P_{\mathrm{Spin}}(X) \arrow[d, "\Theta"'] \arrow[dr] & \\
 & & X \\
P_{\mathrm{SO}}(X)\times\mathrm{SO}(r,s) \arrow[r] & P_{\mathrm{SO}}(X) \arrow[ur] &
\end{tikzcd}$$

Very often in the literature $P_{\mathrm{Spin}}(X)$ is called a spin structure of X and we will call X together with $P_{\mathrm{Spin}}(X)$ a *semi-Riemannian spin manifold*.

On a semi-Riemannian spin manifold we define the *spinor bundle* of X as the complex vector bundle associated to the spinor representation, i. e.

$$\Sigma X := P_{\mathrm{Spin}}(X) \times_\rho \Sigma_{r,s}.$$

In other words, for $p \in X$ the fiber of $\Sigma_p X$ of ΣX over p consists of equivalence classes of pairs $[b, \sigma]$ where $b \in P_{\mathrm{Spin}}(X)_p$ and $\sigma \in \Sigma_{r,s}$ subject to the relation

$$[b, \sigma] = [bg^{-1}, g\sigma]$$

for all $g \in \mathrm{Spin}(r, s)$. Unfortunately, the spinor bundle cannot be defined independently of the metric using $P_{\widetilde{\mathrm{GL}}^+}(X)$ instead of $P_{\mathrm{Spin}}(X)$ because the spinor representation ρ of $\mathrm{Spin}(r, s)$ on $\Sigma_{r,s}$ does not extend to a representation of $\widetilde{\mathrm{GL}}^+(n, \mathbb{R})$ on $\Sigma_{r,s}$. We will come back to this problem in Section 5.

Note that the tangent bundle can also be written in a similar manner, $TX = P_{\mathrm{SO}}(X) \times_\tau \mathbb{R}^n$ where τ is the standard representation of $\mathrm{SO}(r, s)$ on $\mathbb{R}^n$. One defines *Clifford multiplication* $T_pX \otimes \Sigma_p X \to \Sigma_p X$ by

$$[\Theta(b), v] \cdot [b, \sigma] := [b, v \cdot \sigma]$$

where $b \in P_{\mathrm{Spin}}(X)_p$, $v \in \mathbb{R}^n$, and $\sigma \in \Sigma_{r,s}$. For $g \in \mathrm{Spin}(r, s)$ we see from

$$\begin{aligned} [\Theta(bg), v] \cdot [bg, \sigma] &= [\Theta(b)\mathrm{Ad}_g, v] \cdot [bg, \sigma] = [\Theta(b), \mathrm{Ad}_g v] \cdot [b, g\sigma] \\ &= [b, gvg^{-1}g\sigma] = [b, gv\sigma] = [bg, v\sigma] \end{aligned}$$

that this is well-defined. It is this point that goes wrong when one tries to work with nonoriented manifolds and pin structures. Had we defined $\Sigma_{r,s} = \Sigma^1_{r,s}$ instead of $\Sigma_{r,s} = \Sigma^0_{r,s}$ in odd dimensions, then we would have obtained the Clifford multiplication with the opposite sign.

Clifford multiplication inherits the relations of the Clifford algebra, i. e. for $X, Y \in T_pX$ and $\varphi \in \Sigma_p X$ we have

$$X \cdot Y \cdot \varphi + Y \cdot X \cdot \varphi + 2 \langle X, Y \rangle \varphi = 0.$$

In even dimensions the spinor bundle splits into the positive and the negative *half-spinor bundles*,

$$\Sigma X = \Sigma^+ X \oplus \Sigma^- X \tag{3}$$

where $\Sigma^\pm X = P_{\mathrm{Spin}}(X) \times_{\rho^\pm} \Sigma^\pm_{r,s}$. Clifford multiplication by a tangent vector interchanges $\Sigma^+ X$ and $\Sigma^- X$.

The $\mathrm{Spin}(r, s)$-invariant nondegenerate symmetric sesquilinear forms on $\Sigma_{r,s}$ and $\Sigma^\pm_{r,s}$ induce (in general indefinite) inner products on ΣX and $\Sigma^\pm X$ which we again denote by $\langle \cdot, \cdot \rangle$.

The connection 1-form ω^X on $P_{\mathrm{SO}}(X)$ for the Levi-Civita connection ∇^X can be lifted via Θ to $P_{\mathrm{Spin}}(X)$, i. e. $\omega^{\Sigma X} := \mathrm{Ad}_*^{-1} \circ \Theta^*(\omega^X)$. Composing with Ad_*^{-1} is necessary because the connection 1-form on $P_{\mathrm{Spin}}(X)$ must take values in the Lie algebra of $\mathrm{Spin}(r, s)$ rather than in that of $\mathrm{SO}(r, s)$. Now $\omega^{\Sigma X}$ induces a covariant derivative $\nabla^{\Sigma X}$ on ΣX.

An equivalent, but less invariant, way of describing $\nabla^{\Sigma X}$ is as follows: If b is a local section in $P_{\mathrm{Spin}}(X)$, then $\Theta(b) = (e_1, \ldots, e_n)$ is a local oriented orthonormal tangent frame, $\langle e_i, e_j \rangle \equiv \varepsilon_i \delta_{ij}$ where $\varepsilon_i = \pm 1$. The Christoffel symbols of ∇^X with respect to this frame are given by

$$\nabla^X_{e_i} e_j = \sum_{k=1}^n \Gamma^k_{ij}\, e_k.$$

Now the covariant derivative of a locally defined spinor field $\varphi = [b, \sigma]$, σ a function with values in $\Sigma_{r,s}$, is given by

$$\nabla^{\Sigma X}_{e_i} \varphi = \left[b, d_{e_i}\sigma + \frac{1}{2} \sum_{j<k} \Gamma^k_{ij}\, \varepsilon_j\, e_j \cdot e_k \cdot \sigma \right]. \tag{4}$$

One checks that $\nabla^{\Sigma X}$ is a metric connection and that it leaves the splitting (3) in even dimensions invariant. Moreover, it satisfies the following Leibniz rule:

$$\nabla^{\Sigma X}_Z (Y \cdot \varphi) = (\nabla^X_Z Y) \cdot \varphi + Y \cdot \nabla^{\Sigma X}_Z \varphi$$

for all vector fields Z and Y and all spinor fields φ.

The curvature tensor $R^{\Sigma X}$ of $\nabla^{\Sigma X}$ can be computed in terms of the curvature tensor R^X of the Levi-Civita connection,

$$R^{\Sigma X}(Y, Z)\varphi = \frac{1}{2} \sum_{i<j} \varepsilon_i \varepsilon_j \left\langle R^X(Y, Z)e_i, e_j \right\rangle e_i \cdot e_j \cdot \varphi.$$

Using the first Bianchi identity one easily computes

$$\sum_{i=1}^{n} \varepsilon_i\, e_i \cdot R^{\Sigma X}(e_i, Y)\varphi = \frac{1}{2} \mathrm{Ric}^X(Y) \cdot \varphi. \tag{5}$$

Here Ric^X denotes the *Ricci curvature* considered as an endomorphism field on TM. The Ricci curvature considered as a symmetric bilinear form will be written $\mathrm{ric}^X(Y, Z) = \left\langle \mathrm{Ric}^X(Y), Z \right\rangle$.

The *Dirac operator* maps spinor fields to spinor fields and is defined by

$$D^X \varphi = \sum_{i=1}^{n} \varepsilon_i e_i \cdot \nabla^{\Sigma X}_{e_i} \varphi.$$

Given two spinor fields φ and ψ one can define a vector field Y by the requirement $\langle Y, Z\rangle = \langle Z \cdot \varphi, \psi\rangle$ for all vector fields Z and one easily computes

$$\mathrm{div}(Y) = \left\langle D^X \varphi, \psi \right\rangle - \left\langle \varphi, D^X \psi \right\rangle.$$

Hence the Dirac operator is formally selfadjoint, i. e. if the intersection of the supports of φ and ψ is compact, then

$$(D^X \varphi, \psi) = (\varphi, D^X \psi)$$

where $(\varphi, \psi) = \int_M \langle \varphi, \psi \rangle\, dV$.

3. The Dirac operator on manifolds foliated by hypersurfaces

Let $\mathcal{Z}$ be an oriented $(n+1)$-dimensional semi-Riemannian spin manifold. Let $\Theta : P_{\mathrm{Spin}}(\mathcal{Z}) \to P_{\mathrm{SO}}(\mathcal{Z})$ be a spin structure on $\mathcal{Z}$. Let $M \subset \mathcal{Z}$ be a semi-Riemannian hypersurface with trivial spacelike normal bundle. This means there is a vector field ν on $\mathcal{Z}$ along M satisfying $\langle \nu, \nu \rangle = +1$ and $\langle \nu, TM \rangle = 0$. If the signature of M is (r, s), then the signature of $\mathcal{Z}$ is $(r+1, s)$.

In this situation M inherits a spin structure as follows: The bundle of oriented orthonormal frames of M, $P_{\mathrm{SO}}(M)$, can be embedded into the bundle of oriented orthonormal frames of $\mathcal{Z}$ restricted to M, $P_{\mathrm{SO}}(\mathcal{Z})|_M$, by the map $\iota : (e_1, \ldots, e_n) \mapsto (\nu, e_1, \ldots, e_n)$. Then $P_{\mathrm{Spin}}(M) := \Theta^{-1}(\iota(P_{\mathrm{SO}}(M)))$ defines a spin structure on M. We will always implicitly assume that this spin structure be taken on M. The same discussion is possible on the level of $\widetilde{\mathrm{GL}}^+(n, \mathbb{R})$-bundles.

The algebraic remarks in the previous section show that if n is even, then

$$\Sigma\mathcal{Z}|_M = \Sigma M$$

where the Clifford multiplication with respect to M is given by $X \otimes \varphi \mapsto \nu \cdot X \cdot \varphi$ and "$\cdot$" always denotes the Clifford multiplication with respect to $\mathcal{Z}$. If n is odd, then

$$\Sigma^+\mathcal{Z}|_M = \Sigma M$$

and again Clifford multiplication with respect to M is given by $X \otimes \varphi \mapsto \nu \cdot X \cdot \varphi$ while

$$\Sigma^-\mathcal{Z}|_M = \Sigma M$$

with Clifford multiplication with respect to M given by $X \otimes \varphi \mapsto -\nu \cdot X \cdot \varphi$. The minus sign comes from the fact that in odd dimensions we defined $\Sigma_{r,s} = \Sigma^0_{r,s}$ while $\Sigma^1_{r,s}$ leads to the opposite sign for the Clifford multiplication. The identifications preserve the natural inner products $\langle\cdot,\cdot\rangle$.

Let W denote the *Weingarten map* with respect to ν, i. e.

$$\nabla^{\mathcal{Z}}_X Y = \nabla^M_X Y + \langle W(X), Y\rangle\, \nu \tag{6}$$

for all vector fields X and Y on M. The Weingarten map is symmetric with respect to the semi-Riemannian metric, $\langle W(X), Y\rangle = \langle X, W(Y)\rangle$ and is also given by $W(X) = -\nabla^{\mathcal{Z}}_X \nu$. If we denote the Christoffel symbols of M with respect to a local orthogonal tangent frame $(e_1, \ldots, e_n)$ by $\Gamma^{M,k}_{ij}$ and the Christoffel symbols of $\mathcal{Z}$ with respect to $(e_0, e_1, \ldots, e_n)$, $e_0 = \nu$, by $\Gamma^{\mathcal{Z},k}_{ij}$, then (6) implies for $1 \le i,j,k \le n$

$$\Gamma^{\mathcal{Z},k}_{ij} = \Gamma^{M,k}_{ij}, \tag{7}$$

$$\Gamma^{\mathcal{Z},0}_{ij} = \langle W(e_i), e_j\rangle\,, \tag{8}$$

$$\Gamma^{\mathcal{Z},k}_{i0} = -\varepsilon_0\varepsilon_k\Gamma^{\mathcal{Z},0}_{ik} = -\varepsilon_k \langle W(e_i), e_k\rangle\,. \tag{9}$$

Plugging this into (4) we get for a section $\varphi = [b, \sigma]$ of $\Sigma\mathcal{Z}|_M$ and $1 \le i \le n$

$$\begin{aligned}
\nabla^{\Sigma\mathcal{Z}}_{e_i}\varphi &= \left[b, d_{e_i}\sigma + \frac{1}{2}\left(-\sum_{k=1}^n \varepsilon_k \langle W(e_i), e_k\rangle\, \varepsilon_0 e_0 \cdot e_k + \sum_{1\le j<k\le n} \Gamma^{M,k}_{ij}\varepsilon_j e_j \cdot e_k\right)\cdot\sigma\right] \\
&= \left[b, d_{e_i}\sigma + \frac{1}{2}\left(-e_0 \cdot W(e_i) + \sum_{1\le j<k\le n} \Gamma^{M,k}_{ij}\varepsilon_j e_0 \cdot e_j \cdot e_0 \cdot e_k\right)\cdot\sigma\right] \\
&= \nabla^{\Sigma M}_{e_i}\varphi - \frac{1}{2}\nu \cdot W(e_i)\cdot\varphi.
\end{aligned}$$

Hence for each $X \in TM$ and each section φ of $\Sigma\mathcal{Z}|_M$ we have

$$\nabla^{\Sigma\mathcal{Z}}_X\varphi = \nabla^{\Sigma M}_X\varphi - \frac{1}{2}\nu\cdot W(X)\cdot\varphi. \tag{10}$$

Now let φ be a section of $\Sigma\mathcal{Z}$ defined in a neighborhood of M. On the one hand,

$$D^{\mathcal{Z}}\varphi = \sum_{i=1}^n \varepsilon_i e_i \cdot \nabla^{\Sigma\mathcal{Z}}_{e_i}\varphi + \nu\cdot\nabla^{\Sigma\mathcal{Z}}_\nu\varphi.$$

On the other hand by (10),

$$\begin{aligned}
\sum_{i=1}^n \varepsilon_i e_i \cdot \nabla^{\Sigma\mathcal{Z}}_{e_i}\varphi &= \sum_{i=1}^n \varepsilon_i\, e_i \cdot \nabla^{\Sigma M}_{e_i}\varphi - \frac{1}{2}\sum_{i=1}^n \varepsilon_i\, e_i\cdot\nu\cdot W(e_i)\cdot\varphi \\
&= -\nu\cdot\sum_{i=1}^n \varepsilon_i\,\nu\cdot e_i\cdot\nabla^{\Sigma M}_{e_i}\varphi + \frac{1}{2}\sum_{i=1}^n \varepsilon_i\,\nu\cdot e_i\cdot W(e_i)\cdot\varphi \\
&= -\nu\cdot\tilde{D}^M - \frac{1}{2}\,\mathrm{tr}(W)\nu\cdot\varphi
\end{aligned}$$

where $\tilde{D}^M = D^M$ if n is even and $\tilde{D}^M = \begin{pmatrix} D^M & 0 \\ 0 & -D^M \end{pmatrix}$ if n is odd. Thus the Dirac operators on M and on $\mathcal{Z}$ are related by

$$\nu \cdot D^{\mathcal{Z}} = \tilde{D}^M + \frac{n}{2}H - \nabla^{\Sigma\mathcal{Z}}_\nu \tag{11}$$

where $H = \frac{1}{n}\operatorname{tr}(W)$ denotes the mean curvature.

Next we consider the situation that $\mathcal{Z}$ carries a semi-Riemannian foliation by hypersurfaces. The commutator of the leafwise Dirac operator and the normal derivative will be of central importance later.

Proposition 3.1. *Let $\mathcal{Z}$ be an $(n+1)$-dimensional semi-Riemannian spin manifold. Let $\mathcal{Z}$ carry a semi-Riemannian foliation by hypersurfaces with trivial spacelike normal bundle, i. e. the leaves M are semi-Riemannian hypersurfaces and there exists a vector field ν on $\mathcal{Z}$ perpendicular to the leaves such that $\langle \nu, \nu\rangle = 1$ and $\nabla^{\mathcal{Z}}_\nu \nu = 0$. Let W denote the Weingarten map of the leaves with respect to ν and let $H = \frac{1}{n}\operatorname{tr}(W)$ be the mean curvature.*

Then the commutator of the leafwise Dirac operator and the normal derivative is given by

$$[\nabla^{\Sigma\mathcal{Z}}_\nu, \tilde{D}^M]\,\varphi = \mathfrak{D}^W\varphi - \frac{n}{2}\,\nu\cdot \operatorname{grad}^M(H)\cdot\varphi + \frac{1}{2}\,\nu\cdot\operatorname{div}^M(W)\cdot\varphi.$$

Here grad^M denotes the leafwise gradient, $\operatorname{div}^M(W) = \sum_{i=1}^n \varepsilon_i\,(\nabla^M_{e_i}W)(e_i)$ denotes the leafwise divergence of the endomorphism field W, $\mathfrak{D}^W\varphi = \sum_{i=1}^n \varepsilon_i\,\nu\cdot e_i\cdot\nabla^{\Sigma M}_{W(e_i)}\varphi$, and "$\cdot$" denotes Clifford multiplication on $\mathcal{Z}$.

Proof. We choose a local oriented orthonormal tangent frame $(e_1,\dots,e_n)$ for the leaves and we may assume for simplicity that $\nabla^{\mathcal{Z}}_\nu e_i = 0$. We compute

$$\begin{aligned}
[\nabla^{\Sigma\mathcal{Z}}_\nu, \tilde{D}^M]\,\varphi &= \sum_{i=1}^n \varepsilon_i\left(\nabla^{\Sigma\mathcal{Z}}_\nu(\nu\cdot e_i\cdot\nabla^{\Sigma M}_{e_i}\varphi) - \nu\cdot e_i\cdot\nabla^{\Sigma M}_{e_i}\nabla^{\Sigma\mathcal{Z}}_\nu\varphi\right)\\
&= \sum_{i=1}^n \varepsilon_i\,\nu\cdot e_i\cdot\left(\nabla^{\Sigma\mathcal{Z}}_\nu\nabla^{\Sigma M}_{e_i}\varphi - \nabla^{\Sigma M}_{e_i}\nabla^{\Sigma\mathcal{Z}}_\nu\varphi\right)\\
&\overset{(10)}{=} \sum_{i=1}^n \varepsilon_i\,\nu\cdot e_i\cdot\Big(\nabla^{\Sigma\mathcal{Z}}_\nu(\nabla^{\Sigma\mathcal{Z}}_{e_i} + \frac{1}{2}\nu\cdot W(e_i))\\
&\qquad -(\nabla^{\Sigma\mathcal{Z}}_{e_i} + \frac{1}{2}\nu\cdot W(e_i))\nabla^{\Sigma\mathcal{Z}}_\nu\Big)\varphi\\
&= \sum_{i=1}^n \varepsilon_i\,\nu\cdot e_i\cdot\Big(R^{\Sigma\mathcal{Z}}(\nu,e_i) + \nabla^{\Sigma\mathcal{Z}}_{[\nu,e_i]} + \frac{1}{2}\nu\cdot(\nabla^{\mathcal{Z}}_\nu W)(e_i)\Big)\varphi\\
&\overset{(5)}{=} -\frac{1}{2}\nu\cdot\operatorname{Ric}^{\mathcal{Z}}(\nu)\cdot\varphi + \sum_{i=1}^n \varepsilon_i\,\nu\cdot e_i\cdot\Big(\nabla^{\Sigma\mathcal{Z}}_{W(e_i)} + \frac{1}{2}\nu\cdot(\nabla^{\mathcal{Z}}_\nu W)(e_i)\Big)\varphi\\
&\overset{(10)}{=} -\frac{1}{2}\nu\cdot\operatorname{Ric}^{\mathcal{Z}}(\nu)\cdot\varphi\\
&\qquad + \sum_{i=1}^n \varepsilon_i\,\nu\cdot e_i\cdot\Big(\nabla^{\Sigma M}_{W(e_i)} - \frac{1}{2}\nu\cdot W^2(e_i) + \frac{1}{2}\nu\cdot(\nabla^{\mathcal{Z}}_\nu W)(e_i)\Big)\varphi\\
&= -\frac{1}{2}\nu\cdot\operatorname{Ric}^{\mathcal{Z}}(\nu)\cdot\varphi + \mathfrak{D}^W\varphi\\
&\qquad + \frac{1}{2}\sum_{i=1}^n \varepsilon_i\,e_i\cdot\Big(-W^2(e_i) + (\nabla^{\mathcal{Z}}_\nu W)(e_i)\Big)\varphi. \qquad (12)
\end{aligned}$$

The Riccati equation for the Weingarten map $(\nabla^{\mathcal{Z}}_\nu W)(X) = R^{\mathcal{Z}}(X,\nu)\nu + W^2(X)$ yields

$$
\begin{aligned}
[\nabla^{\Sigma\mathcal{Z}}_\nu, \tilde{D}^M]\,\varphi &= -\frac{1}{2}\nu\cdot \mathrm{Ric}^{\mathcal{Z}}(\nu)\cdot\varphi + \mathfrak{D}^W\varphi + \frac{1}{2}\sum_{i=1}^n \varepsilon_i\, e_i\cdot(R^{\mathcal{Z}}(e_i,\nu)\nu)\cdot\varphi \\
&= -\frac{1}{2}\nu\cdot \mathrm{Ric}^{\mathcal{Z}}(\nu)\cdot\varphi + \mathfrak{D}^W\varphi + \frac{1}{2}\mathrm{ric}^{\mathcal{Z}}(\nu,\nu)\varphi \\
&= \mathfrak{D}^W\varphi - \frac{1}{2}\sum_{i=1}^n \varepsilon_i\,\mathrm{ric}^{\mathcal{Z}}(\nu,e_i)\,\nu\cdot e_i\cdot\varphi. \qquad (13)
\end{aligned}
$$

The Codazzi-Mainardi equation [8, p. 115] gives for $X, Y, V \in T_pM$

$$\left\langle R^{\mathcal{Z}}(X,Y)V,\nu\right\rangle = \left\langle(\nabla^M_X W)(Y),V\right\rangle - \left\langle(\nabla^M_Y W)(X),V\right\rangle.$$

Thus

$$
\begin{aligned}
\mathrm{ric}^{\mathcal{Z}}(\nu,X) &= \sum_{i=1}^n \varepsilon_i\left\langle R^{\mathcal{Z}}(X,e_i)e_i,\nu\right\rangle \\
&= \sum_{i=1}^n \varepsilon_i\left(\left\langle(\nabla^M_X W)(e_i),e_i\right\rangle - \left\langle(\nabla^M_{e_i} W)(X),e_i\right\rangle\right) \\
&= \mathrm{tr}(\nabla^M_X W) - \left\langle \mathrm{div}^M(W),X\right\rangle.
\end{aligned}
$$

Plugging this into (13) we get

$$
\begin{aligned}
[\nabla^{\Sigma\mathcal{Z}}_\nu, \tilde{D}^M]\,\varphi &= \mathfrak{D}^W\varphi - \frac{1}{2}\sum_{i=1}^n \varepsilon_i\left(\mathrm{tr}(\nabla^M_{e_i}W) - \left\langle\mathrm{div}^M(W),e_i\right\rangle\right)\,\nu\cdot e_i\cdot\varphi \\
&= \mathfrak{D}^W\varphi - \frac{1}{2}\sum_{i=1}^n \varepsilon_i\, d_{e_i}\,\mathrm{tr}(W)\nu\cdot e_i\cdot\varphi + \frac{1}{2}\nu\cdot\mathrm{div}^M(W)\cdot\varphi \\
&= \mathfrak{D}^W\varphi - \frac{n}{2}\nu\cdot\mathrm{grad}^M(H)\cdot\varphi + \frac{1}{2}\nu\cdot\mathrm{div}^M(W)\cdot\varphi.
\end{aligned}
$$

□

4. THE GENERALIZED CYLINDER

Let M be an n-dimensional differentiable manifold, let g_t be a smooth 1-parameter family of semi-Riemannian metrics on M, $t \in I$ where $I \subset \mathbb{R}$ is an interval. We define the *generalized cylinder* by

$$\mathcal{Z} := I \times M$$

with semi-Riemannian metric

$$g_{\mathcal{Z}} := dt^2 + g_t.$$

The generalized cylinder is an $(n+1)$-dimensional semi-Riemannian manifold (with boundary if I has boundary) of signature $(r+1,s)$ if the signature of g_t is (r,s). The vector field $\nu := \frac{\partial}{\partial t}$ is spacelike of unit length and orthogonal to the hypersurfaces $M_t := \{t\}\times M$. Let W denote the Weingarten map of M_t with respect to ν and let H be the mean curvature.

If X is a local coordinate field on M, then $\langle X,\nu\rangle = 0$ and $[X,\nu] = 0$. Thus

$$
\begin{aligned}
0 &= d_\nu\langle X,\nu\rangle = \left\langle\nabla^{\mathcal{Z}}_\nu X,\nu\right\rangle + \left\langle X,\nabla^{\mathcal{Z}}_\nu\nu\right\rangle = \left\langle\nabla^{\mathcal{Z}}_X\nu,\nu\right\rangle + \left\langle X,\nabla^{\mathcal{Z}}_\nu\nu\right\rangle \\
&= -\langle W(X),\nu\rangle + \left\langle X,\nabla^{\mathcal{Z}}_\nu\nu\right\rangle = \left\langle X,\nabla^{\mathcal{Z}}_\nu\nu\right\rangle
\end{aligned}
$$

and differentiating $\langle\nu,\nu\rangle = 1$ yields $\left\langle\nu,\nabla^{\mathcal{Z}}_\nu\nu\right\rangle = 0$. Hence

$$\nabla^{\mathcal{Z}}_\nu\nu = 0,$$

i. e. for $p \in M$ the curves $t \mapsto (t,p)$ are geodesics parametrized by arclength. So the assumptions of Proposition 3.1 are satisfied for the foliation $(M_t)_{t\in I}$.

Now fix $p \in M$ and $X, Y \in T_pM$. We define the first and second derivative of g_t by

$$\begin{aligned} \dot g_t(X,Y) &:= \frac{d}{dt}(g_t(X,Y)), \\ \ddot g_t(X,Y) &:= \frac{d^2}{dt^2}(g_t(X,Y)). \end{aligned}$$

Then $\dot g_t$ and $\ddot g_t$ are smooth 1-parameter families of symmetric $(2,0)$-tensors on M.

Proposition 4.1. *On a generalized cylinder $\mathcal{Z} = I \times M$ with semi-Riemannian metric $g^{\mathcal{Z}} = \langle\cdot,\cdot\rangle = dt^2 + g_t$ the following formulas hold:*

$$\langle W(X), Y\rangle = -\frac{1}{2}\dot g_t(X,Y), \tag{14}$$

$$\begin{aligned} \left\langle R^{\mathcal{Z}}(U,V)X,Y\right\rangle &= \left\langle R^{M_t}(U,V)X,Y\right\rangle \\ &\quad +\frac{1}{4}\left(\dot g_t(U,X)\dot g_t(V,Y) - \dot g_t(U,Y)\dot g_t(V,X)\right), \end{aligned} \tag{15}$$

$$\left\langle R^{\mathcal{Z}}(X,Y)U,\nu\right\rangle = \frac{1}{2}\Big((\nabla^{M_t}_Y \dot g_t)(X,U) - (\nabla^{M_t}_X \dot g_t)(Y,U)\Big), \tag{16}$$

$$\left\langle R^{\mathcal{Z}}(X,\nu)\nu,Y\right\rangle = -\frac{1}{2}\left(\ddot g_t(X,Y) + \dot g_t(W(X),Y)\right), \tag{17}$$

$$\mathrm{ric}^{\mathcal{Z}}(\nu,\nu) = \mathrm{tr}(W^2) - \frac{1}{2}\,\mathrm{tr}_{g_t}(\ddot g_t), \tag{18}$$

$$\mathrm{ric}^{\mathcal{Z}}(X,\nu) = d_X\,\mathrm{tr}(W) - \Big\langle \mathrm{div}^M(W), X\Big\rangle, \tag{19}$$

$$\begin{aligned} \mathrm{ric}^{\mathcal{Z}}(X,Y) &= \mathrm{ric}^{M_t}(X,Y) + 2\langle W(X), W(Y)\rangle \\ &\quad - \mathrm{tr}(W)\langle W(X),Y\rangle - \frac{1}{2}\ddot g_t(X,Y), \end{aligned} \tag{20}$$

$$\mathrm{Scal}^{\mathcal{Z}} = \mathrm{Scal}^{M_t} + 3\,\mathrm{tr}(W^2) - \mathrm{tr}(W)^2 - \mathrm{tr}_{g_t}(\ddot g_t), \tag{21}$$

where $X,Y,U,V \in T_pM$, $p \in M$.

Proof. To show (14) we extend X and Y to local coordinate fields on M so that all Lie brackets vanish. Then the Koszul formula [8, p. 61] for the Levi-Civita connection of $\mathcal{Z}$ yields

$$\begin{aligned} \langle W(X), Y\rangle &= -\left\langle \nabla^{\mathcal{Z}}_X \nu, Y\right\rangle = -\frac{1}{2}\left(d_X\langle \nu,Y\rangle + d_\nu\langle Y,X\rangle - d_Y\langle X,\nu\rangle\right) \\ &= -\frac{1}{2}d_\nu\langle Y,X\rangle = -\frac{1}{2}\frac{\partial}{\partial t}g_t(X,Y) = -\frac{1}{2}\dot g_t(X,Y). \end{aligned}$$

Equation (15) follows directly from (14) and the Gauss equation [8, p. 100]

$$\begin{aligned} \left\langle R^{\mathcal{Z}}(U,V)X,Y\right\rangle &= \left\langle R^{M_t}(U,V)X,Y\right\rangle + \langle W(U),X\rangle\langle W(V),Y\rangle \\ &\quad - \langle W(U),Y\rangle\langle W(V),X\rangle. \end{aligned}$$

Equation (16) follows directly from (14) and the Codazzi-Mainardi equation [8, p. 115]

$$\left\langle R^{\mathcal{Z}}(X,Y)U,\nu\right\rangle = \Big\langle (\nabla^{M_t}_X W)(Y), U\Big\rangle - \Big\langle (\nabla^{M_t}_Y W)(X), U\Big\rangle.$$

The Riccati equation for W

$$(\nabla^{\mathcal{Z}}_\nu W)(X) = R^{\mathcal{Z}}(X,\nu)\nu + W^2(X)$$

gives

$$
\begin{aligned}
\left\langle R^{\mathcal{Z}}(X,\nu)\nu, Y\right\rangle &= \left\langle (\nabla^{\mathcal{Z}}_{\nu} W)(X), Y\right\rangle - \left\langle W^2(X), Y\right\rangle \\
&= \frac{\partial}{\partial t}\left\langle W(X), Y\right\rangle - \left\langle W(\nabla^{\mathcal{Z}}_{\nu} X), Y\right\rangle - \left\langle W(X), \nabla^{\mathcal{Z}}_{\nu} Y\right\rangle \\
&\quad + \frac{1}{2}\dot{g}_t(W(X), Y) \\
&= -\frac{1}{2}\frac{\partial}{\partial t}\dot{g}_t(X,Y) - \left\langle W(\nabla^{\mathcal{Z}}_{X}\nu), Y\right\rangle - \left\langle W(X), \nabla^{\mathcal{Z}}_{Y}\nu\right\rangle \\
&\quad + \frac{1}{2}\dot{g}_t(W(X), Y) \\
&= -\frac{1}{2}\ddot{g}_t(X,Y) + \langle W(W(X)), Y\rangle + \langle W(X), W(Y)\rangle \\
&\quad + \frac{1}{2}\dot{g}_t(W(X), Y) \\
&= -\frac{1}{2}\ddot{g}_t(X,Y) - \frac{1}{2}\dot{g}_t(W(X), Y)
\end{aligned}
$$

which is (17). The Ricci curvature is now easily computed.

$$
\begin{aligned}
\mathrm{ric}^{\mathcal{Z}}(\nu,\nu) &= \sum_{i=1}^{n} \varepsilon_i \left\langle R^{\mathcal{Z}}(e_i,\nu)\nu, e_i\right\rangle \\
&\overset{(17)}{=} -\frac{1}{2}\sum_{i=1}^{n}\varepsilon_i \left(\ddot{g}_t(e_i,e_i) + \dot{g}_t(W(e_i), e_i)\right) \\
&\overset{(14)}{=} -\frac{1}{2}\,\mathrm{tr}_{g_t}(\ddot{g}_t) + \mathrm{tr}(W^2)
\end{aligned}
$$

which is (18). Moreover,

$$
\begin{aligned}
\mathrm{ric}^{\mathcal{Z}}(X,\nu) &= \sum_{i=1}^{n} \varepsilon_i \left\langle R^{\mathcal{Z}}(X,e_i)e_i, \nu\right\rangle \\
&\overset{(16)}{=} \frac{1}{2}\sum_{i=1}^{n}\varepsilon_i \left((\nabla^{M_t}_{e_i}\dot{g}_t)(X,e_i) - (\nabla^{M_t}_{X}\dot{g}_t)(e_i,e_i)\right) \\
&\overset{(14)}{=} -\sum_{i=1}^{n}\varepsilon_i \left(\left\langle (\nabla^{M_t}_{e_i} W)(X), e_i\right\rangle - \left\langle (\nabla^{M_t}_{X} W)(e_i), e_i\right\rangle\right) \\
&= -\left\langle \mathrm{div}^{M_t}, X\right\rangle + \mathrm{tr}(\nabla^{M_t}_{X} W) \\
&= -\left\langle \mathrm{div}^{M_t}, X\right\rangle + d_X\,\mathrm{tr}(W)
\end{aligned}
$$

thus showing (19). Furthermore,

$$
\begin{aligned}
\mathrm{ric}^{\mathcal{Z}}(X,Y) &= \sum_{i=1}^{n} \varepsilon_i \left\langle R^{\mathcal{Z}}(e_i,X)Y,e_i\right\rangle + \left\langle R^{\mathcal{Z}}(\nu,X)Y,\nu\right\rangle \\
&\overset{(15),(17)}{=} \sum_{i=1}^{n} \varepsilon_i \Big(\left\langle R^{M_t}(e_i,X)Y,e_i\right\rangle + \frac{1}{4}\dot{g}_t(e_i,Y)\dot{g}_t(X,e_i) \\
&\quad -\frac{1}{4}\dot{g}_t(e_i,e_i)\dot{g}_t(X,Y)\Big) - \frac{1}{2}\left(\ddot{g}_t(X,Y) + \dot{g}_t(W(X),Y)\right) \\
&= \mathrm{ric}^{M_t}(X,Y) + \sum_{i=1}^{n} \varepsilon_i (\langle W(e_i),Y\rangle \langle W(X),e_i\rangle \\
&\quad - \langle W(e_i),e_i\rangle \langle W(X),Y\rangle) - \frac{1}{2}\ddot{g}_t(X,Y) + \left\langle W^2(X),Y\right\rangle \\
&= \mathrm{ric}^{M_t}(X,Y) + 2\langle W(X),W(Y)\rangle - \mathrm{tr}(W)\langle W(X),Y\rangle \\
&\quad -\frac{1}{2}\ddot{g}_t(X,Y)
\end{aligned}
$$

shows (20). Formula (21) for the scalar curvature follows from (18) and (20). □

Example 4.2. A simple special case of a generalized cylinder is that of a *warped product*, i. e. $g_t = f(t)^2 g$ where $f : I \to \mathbb{R}$ is a smooth positive function. Then $\dot{g}_t = 2\,f\,\dot{f}\,g = \frac{2\dot{f}}{f} g_t$ and $\ddot{g}_t = 2(\dot{f}^2 + f\ddot{f})g = 2\frac{\dot{f}^2+f\ddot{f}}{f^2} g_t$ and the formulas in Proposition 4.1 reduce to

$$
\begin{aligned}
W &= -\frac{\dot{f}}{f}\mathrm{id}, \\
R^{\mathcal{Z}}(X,Y)U &= R^{M_t}(X,Y)U + \frac{\dot{f}^2}{f^2}\left(\langle X,U\rangle Y - \langle Y,U\rangle X\right), \\
R^{\mathcal{Z}}(X,\nu)\nu &= -\frac{\ddot{f}}{f}X, \\
\mathrm{ric}^{\mathcal{Z}}(X,Y) &= \mathrm{ric}^{M_t}(X,Y) - \frac{(n-1)\dot{f}^2 + f\ddot{f}}{f^2}\langle X,Y\rangle, \\
\mathrm{ric}^{\mathcal{Z}}(X,\nu) &= 0, \\
\mathrm{ric}^{\mathcal{Z}}(\nu,\nu) &= -n\frac{\ddot{f}}{f}, \\
\mathrm{Scal}^{\mathcal{Z}} &= \mathrm{Scal}^{M_t} - n\,\frac{(n-1)\dot{f}^2 + 2f\ddot{f}}{f^2},
\end{aligned}
$$

compare [8, Ch. 7].

5. Identifying spinors and the variation formula for the Dirac operator

It is an annoying problem that the definition of spinors, in contrast to that of differential forms and tensors, depends on the semi-Riemannian metric of the manifold. Hence if one wants to compare the Dirac operators for two different metrics one first has to identify the underlying spinor bundles.

The problem of constructing such identifications can be split into two steps: First construct identifications for any two metrics in a 1-parameter family of metrics. The identification of spinors for two metrics will in general depend on the 1-parameter family of metrics joining them. Secondly, given two metrics construct a natural curve of metrics joining them.

Both steps have been carried out very satisfactorily for the case of Riemannian metrics in [3]. In the present section we will deal only with the first step. The second step cannot always be carried out. In Section 9 we will discuss this problem for the case of Lorentz metrics in great detail.

Now let $g_t, t \in I$, be a smooth 1-parameter family of semi-Riemannian metrics of signature (r, s) on a manifold M. We form the generalized cylinder $\mathcal{Z} := I \times M$ with metric $g = dt^2 + g_t$. For $t \in I$ we abbreviate the semi-Riemannian manifold (M, g_t) by M_t.

Spin structures on M and on $\mathcal{Z}$ are in 1-1-correspondence. As explained in Section 3 spin structures on $\mathcal{Z}$ can be restricted to spin structures on $M_t = M$. Conversely, given a spin structure on M it can be pulled back to $I \times M$ yielding a $\widetilde{\mathrm{GL}}^+(n, \mathbb{R})$-principal bundle on $\mathcal{Z}$. Enlarging the structure group via the embedding $\widetilde{\mathrm{GL}}^+(n, \mathbb{R}) \hookrightarrow \widetilde{\mathrm{GL}}^+(n+1, \mathbb{R})$ covering the standard embedding $\mathrm{GL}^+(n, \mathbb{R}) \hookrightarrow \mathrm{GL}^+(n+1, \mathbb{R})$, $a \mapsto \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$, yields the spin structure on $\mathcal{Z}$ which restricts to the given spin structure on M.

Let us write "$\cdot$" for the Clifford multiplication on $\mathcal{Z}$ and "$\bullet_t$" for the Clifford multiplication on M_t. Recall from Section 3 that $\Sigma\mathcal{Z}|_{M_t} = \Sigma M_t$ as Hermitian vector bundles if $n = r + s$ is even and $\Sigma^+\mathcal{Z}|_{M_t} = \Sigma M_t$ if n is odd. In both cases the Clifford multiplications are related by $X \bullet_t \varphi = \nu \cdot X \cdot \varphi$. For given $x \in M$ and $t_0, t_1 \in I$ parallel translation on $\mathcal{Z}$ along the curve $t \mapsto (t, x)$ is a linear isometry $\tau_{t_0}^{t_1} : \Sigma_x M_{t_0} \to \Sigma_x M_{t_1}$. Since "$\cdot$" and ν are parallel along the curve $t \mapsto (t, x)$ so is the family of Clifford multiplications "$\bullet_t$" and $\tau_{t_0}^{t_1}$ preserves Clifford multiplication in the following sense:

$$\tau_{t_0}^{t_1}(X \bullet_{t_0} \varphi) = (\tau_{t_0}^{t_1} X) \bullet_{t_1} (\tau_{t_0}^{t_1} \varphi).$$

In general, the covariant derivative and hence parallel transport depends on the semi-Riemannian metric and its first derivatives. We note here that for fixed $x \in M$ the parallel transport $\tau_{t_0}^{t_1} : T_x M_{t_0} \to T_x M_{t_1}$ or $\tau_{t_0}^{t_1} : \Sigma_x M_{t_0} \to \Sigma_x M_{t_1}$ is determined by $g_t(x)$ and $\dot{g}_t(x)$, no x-derivatives of g_t enter. Namely, if $x^1, \ldots, x^n$ are local coordinates on M and $X(t, x) = \sum_{j=1}^n \xi^j(x, t) \frac{\partial}{\partial x^j}$ is a parallel vector field along $t \mapsto (t, x)$, then this means by (9) and (14)

$$\begin{aligned} 0 &= \frac{\nabla}{dt} X = \sum_{j=1}^n \left(\dot{\xi}^j + \sum_{k=1}^n \Gamma_{k,0}^{\mathcal{Z},j} \xi^k \right) \frac{\partial}{\partial x^j} \\ &= \sum_{j=1}^n \left(\dot{\xi}^j + \frac{1}{2} \sum_{k,\ell=1}^n g_t^{j\ell} \dot{g}_{t,k\ell} \xi^k \right) \frac{\partial}{\partial x^j}. \end{aligned}$$

Thus $\tau_{t_0}^{t_1} : T_x M_{t_0} \to T_x M_{t_1}$ is given by solving the system of ordinary differential equations

$$\dot{\xi}^j(t, x) = -\frac{1}{2} \sum_{k,\ell=1}^n g_t^{j\ell}(x) \dot{g}_{t,k\ell}(x) \xi^k(t, x).$$

For spinors the situation is similar. By [3, Prop. 2] this shows that our identification $\tau_{t_0}^{t_1}$ of spinors for different metrics coincides with the one in [3].

Now we rewrite the commutator formula of Proposition 3.1. For a section φ of $\Sigma\mathcal{Z}$ (or $\Sigma^+\mathcal{Z}$ if n is odd) we have

$$[\nabla_\nu^{\Sigma\mathcal{Z}}, D^{M_t}]\varphi = \mathfrak{D}^{W_t}\varphi - \frac{n}{2} \operatorname{grad}^{M_t}(H_t) \bullet_t \varphi + \frac{1}{2} \operatorname{div}^{M_t}(W_t) \bullet_t \varphi \tag{22}$$

where D^{M_t} is the Dirac operator of M_t, $\operatorname{grad}^{M_t}$ is the gradient and div^{M_t} the divergence (of endomorphisms) on M_t, W_t is the Weingarten map of M_t in $\mathcal{Z}$ and $H_t = \frac{1}{n} \operatorname{tr}(W_t)$ the mean curvature and finally $\mathfrak{D}^{W_t}\varphi = \sum_{i=1}^n \varepsilon_i e_i \bullet_t \nabla_{W_t(e_i)}^{\Sigma M_t} \varphi$ for any orthonormal basis $e_1, \ldots, e_n$. From (14) we have $\operatorname{div}^{M_t}(W_t) = -\frac{1}{2} \operatorname{div}^{M_t}(\dot{g}_t)$, $H_t = -\frac{1}{2n} \operatorname{tr}_{g_t}(\dot{g}_t)$ and $\mathfrak{D}^{W_t} = -\frac{1}{2}\mathfrak{D}^{\dot{g}_t}$ where $\mathfrak{D}^{\dot{g}_t}\varphi = \sum_{i,j=1}^n \varepsilon_i \varepsilon_j \dot{g}_t(e_i, e_j) e_i \bullet_t \nabla_{e_j}^{\Sigma M_t} \varphi$. Thus (22) can be rewritten as

$$[\nabla^{\Sigma\mathcal{Z}}_\nu, D^{M_t}]\,\varphi = -\frac{1}{2}\mathfrak{D}^{\dot g_t}\varphi + \frac{1}{4}\,\mathrm{grad}^{M_t}(\mathrm{tr}_{g_t}(\dot g_t)) \bullet_t \varphi - \frac{1}{4}\,\mathrm{div}^{M_t}(\dot g_t) \bullet_t \varphi. \tag{23}$$

Now if φ is parallel along the curves $t \mapsto (t,x)$, i. e. it is of the form $\varphi(t,x) = \tau^t_{t_0}\psi(x)$ for some spinor field ψ on M_{t_0}, then the left hand side of (23) is at $t = t_0$

$$\begin{aligned}[\nabla^{\Sigma\mathcal{Z}}_\nu, D^{M_t}]\,\varphi &= \nabla^{\Sigma\mathcal{Z}}_\nu D^{M_t}\,\varphi = \frac{d}{dt}\Big|_{t=t_0} \tau^{t_0}_t D^{M_t}\,\varphi \\ &= \frac{d}{dt}\Big|_{t=t_0} \tau^{t_0}_t D^{M_t}\tau^t_{t_0}\psi.\end{aligned}$$

We have shown the variation formula for the Dirac operator:

Theorem 5.1. *Let g_t be a smooth 1-parameter family of semi-Riemannian metrics on a spin manifold M. We write briefly M_t for the semi-Riemannian spin manifold (M, g_t). Let $\tau^{t_1}_{t_0}$ be the identification of spinor spaces for M_{t_0} and M_{t_1} constructed above, let D^{M_t} be the Dirac operator of M_t, let "$\bullet_t$" be Clifford multiplication on M_t and let $\mathfrak{D}^{\dot g_t}\varphi = \sum_{i,j=1}^n \varepsilon_i\varepsilon_j \dot g_t(e_i,e_j)e_i \bullet_t \nabla^{\Sigma M_t}_{e_j}\varphi$.*

Then for any smooth spinor field ψ on M_{t_0} we have

$$\frac{d}{dt}\Big|_{t=t_0} \tau^{t_0}_t D^{M_t}\tau^t_{t_0}\psi = -\frac{1}{2}\mathfrak{D}^{\dot g_{t_0}}\psi + \frac{1}{4}\,\mathrm{grad}^{M_{t_0}}(\mathrm{tr}_{g_{t_0}}(\dot g_{t_0})) \bullet_{t_0} \psi - \frac{1}{4}\,\mathrm{div}^{M_{t_0}}(\dot g_{t_0}) \bullet_{t_0} \psi.$$

This is exactly the formula given in [3, Thm. 21] for Riemannian manifolds.

6. Energy-momentum tensors

Theorem 5.1 can be used to compute the energy-momentum tensor for spinors. In order to explain what this means we briefly sketch Lagrangian field theory, see [4, p. 153 ff] for a more detailed introduction. Let M denote a differentiable manifold and let $\mathcal{G}$ be a set of (smooth) semi-Riemannian metrics on M, open in the C^∞-topology. Let $\pi : E \to \mathcal{G} \times M$ be a fiber bundle with finite dimensional fibers. For example, if M carries a spin structure the fiber over $(g,x) \in \mathcal{G} \times M$ could be the spinor space at x with respect to the metric g, $E_{(g,x)} = \Sigma^g_x M$. For each fixed $g \in \mathcal{G}$ the restriction $\pi^{-1}(\{g\} \times M) \to M$ is a fiber bundle over M and we can form the space of smooth sections $\mathcal{S}_g$ of this bundle. These Fréchet manifolds $\mathcal{S}_g$ give rise to a Fréchet fiber bundle $\mathcal{S} := \bigcup_{g\in\mathcal{G}} \mathcal{S}_g \to \mathcal{G}$. Let $\mathcal{F} \subset \mathcal{S}$ be a Fréchet submanifold such that the restriction $\pi : \mathcal{F} \to \mathcal{G}$ is again a Fréchet fiber bundle.

Now let $L : \mathcal{F} \to \Omega^{|n|}(M)$ be a smooth map where $\Omega^{|n|}(M)$ denotes the space of smooth densities on M, i. e. smooth sections of $\Lambda^n T^*M \otimes \mathfrak{o}_M$ where $\mathfrak{o}_M$ is the orientation line bundle. We assume that L is local in the sense that for $\varphi \in \mathcal{F}$ the density $L(\varphi)$ evaluated at $x \in M$ depends only on $\varphi(x)$ and the M-derivatives of φ at x. In other words, $L(\varphi)(x)$ is a function of the jet $j^\infty_M\varphi(x)$. We call L the *Lagrangian density*. In physics it is customary to integrate over M and call $\int_M L(\varphi)$ the *Lagrangian* or the *action*. We avoid this integration since in general the integral $\int_M L(\varphi)$ need not exist.

We call a smooth 1-parameter family $\varphi_t \in \mathcal{F}_g$ with $\varphi_0 = \varphi$ *compactly supported* if it is constant outside a compact subset $K \subset M$, i. e. $\varphi_t(x) = \varphi(x)$ for all $x \in M \setminus K$ and all t. Since L is local $L(\varphi_t)$ is constant outside K as well so that $\int_M (L(\varphi_t) - L(\varphi))$ exists and

$$\frac{d}{dt}\Big|_{t=0} \int_M (L(\varphi_t) - L(\varphi)) = \int_M \frac{d}{dt}\Big|_{t=0} L(\varphi_t).$$

The section $\varphi \in \mathcal{F}_g$ is called *critical* for L if for each compactly supported deformation φ_t

$$\int_M \frac{d}{dt}\bigg|_{t=0} L(\varphi_t) = 0.$$

To explain the concept of energy-momentum tensors we need one more piece of structure. Let $H \subset T\mathcal{F}$ be a connection. This means that for any $\varphi \in \mathcal{F}$ we have $T_\varphi \mathcal{F} = T_\varphi(\mathcal{F}_{\pi(\varphi)}) \oplus H_\varphi$ and the restriction $d\pi|_{H_\varphi} : H_\varphi \to T_{\pi(\varphi)}\mathcal{G}$ is an isomorphism. For fixed $\varphi \in \mathcal{F}$ and $g := \pi(\varphi)$ we have the linear map $dL \circ (d\pi|_{H_\varphi})^{-1} : T_g\mathcal{G} \to \Omega^{|n|}(M)$. Recall that $T_g\mathcal{G}$ is nothing but the space of smooth $(2,0)$-tensors. A smooth symmetric $(2,0)$-tensor Q_φ will be called the *energy-momentum tensor* for φ with respect to the Lagrangian L if

$$dL \circ (d\pi|_{H_\varphi})^{-1}(k) = \langle Q_\varphi, k\rangle_g \; dV_g$$

for all $k \in T_g\mathcal{G}$. Here $\langle \cdot, \cdot \rangle_g$ denotes the (pointwise) metric on symmetric $(2,0)$-tensors induced by g and dV_g is the Riemannian volume measure for g. If it exists Q_φ is obviously unique. By its definition the energy-momentum tensor describes the behavior of the Lagrangian under variations of the metric.

Example 6.1. Let M carry a spin structure, let $\mathcal{G}$ be the set of all semi-Riemannian metrics on M and let E be the universal spinor bundle, $E_{(g,x)} = \Sigma^g_x M$. Then $\mathcal{S}$ is the universal bundle of spinor fields and we put $\mathcal{F} := \mathcal{S}$. We fix $\lambda \in \mathbb{R}$ and we define the Lagrangian L by

$$L(\varphi) := \operatorname{Re}\langle \varphi, (D^g - \lambda)\varphi\rangle_g \; dV_g$$

where D^g is the Dirac operator with respect to the metric $g = \pi(\varphi)$. If φ_t is a compactly supported deformation of φ we write $\frac{d}{dt}|_{t=0}\varphi_t = \dot\varphi$ and we compute

$$\begin{aligned}\int_M \frac{d}{dt}\bigg|_{t=0} L(\varphi_t) &= \int_M \operatorname{Re}(\langle \dot\varphi, (D^g - \lambda)\varphi\rangle_g + \langle \varphi, (D^g-\lambda)\dot\varphi\rangle_g)\, dV_g \\ &= 2\operatorname{Re}\int_M \langle \dot\varphi, (D^g - \lambda)\varphi\rangle_g \; dV_g.\end{aligned}$$

Thus φ is critical if and only if $(D^g - \lambda)\varphi = 0$, i. e. if φ is a Dirac-eigenspinor for the eigenvalue λ.

The connection H is determined by the parallel translation $\tau_{t_0}^{t_1}$ used in the previous section to identify spinors for different metrics. More precisely, H_φ is the set of all $\frac{d}{dt}\big|_{t=0} \tau_0^t\varphi$ for all smooth curves g_t of metrics with $g_0 = \pi(\varphi)$.

Now let g_t be such a 1-parameter family of metrics and write $k := \dot g_0$. We compute

$$\begin{aligned}&dL \circ (d\pi|_{H_\varphi})^{-1}(k) \\ &= \frac{d}{dt}\bigg|_{t=0} L(\tau_0^t\varphi) \\ &= \frac{d}{dt}\bigg|_{t=0} \operatorname{Re}\left\langle \tau_0^t\varphi, (D^{g_t} - \lambda)(\tau_0^t\varphi)\right\rangle_{g_t} \; dV_{g_t} \\ &= \frac{d}{dt}\bigg|_{t=0} \operatorname{Re}\langle \varphi, (\tau_t^0 D^{g_t}\tau_0^t - \lambda)\varphi\rangle_{g_0} \, \frac{dV_{g_t}}{dV_{g_0}}\, dV_{g_0} \\ &= \operatorname{Re}\left(\left\langle \varphi, \frac{d}{dt}\bigg|_{t=0} (\tau_t^0 D^{g_t}\tau_0^t\varphi)\right\rangle_{g_0} + \langle \varphi, (D^{g_0} - \lambda)\varphi\rangle_{g_0} \frac{d}{dt}\bigg|_{t=0}\frac{dV_{g_t}}{dV_{g_0}}\right) dV_{g_0}.\end{aligned}$$

The first term is given by the variation formula for the Dirac operator. Since Clifford multiplication with tangent vectors is skewadjoint all terms of the form $\operatorname{Re}\langle \varphi, X \bullet_{g_0} \varphi\rangle$ vanishes. Thus Theorem 5.1 yields

$$\operatorname{Re}\left\langle \varphi, \frac{d}{dt}\bigg|_{t=0} (\tau_t^0 D^{g_t}\tau_0^t\varphi)\right\rangle_{g_0} = -\frac{1}{2}\operatorname{Re}\langle \varphi, \mathfrak{D}^k\varphi\rangle_{g_0}.$$

For the second term we use

$$\frac{d}{dt}\bigg|_{t=0} \frac{dV_{g_t}}{dV_{g_0}} = \frac{1}{2}\,\mathrm{tr}_{g_0}(k).$$

Thus

$$\begin{aligned} dL \circ (d\pi|_{H_\varphi})^{-1}(k) &= \frac{1}{2}\,\mathrm{Re}\left(- \langle \varphi, \mathfrak{D}^k \varphi \rangle_{g_0} + \langle \varphi, (D^{g_0} - \lambda)\varphi \rangle_{g_0} \mathrm{tr}_{g_0}(k) \right) dV_{g_0} \\ &= \langle Q_\varphi, k \rangle_{g_0} \, dV_{g_0} \end{aligned}$$

for the symmetric $(2,0)$-tensor

$$\begin{aligned} Q_\varphi(X,Y) &= -\frac{1}{4}\,\mathrm{Re}\left(\langle \varphi, X \bullet_{g_0} \nabla^{\Sigma M}_Y \varphi \rangle + \langle \varphi, Y \bullet_{g_0} \nabla^{\Sigma M}_X \varphi \rangle \right) \\ &\quad + \frac{1}{2}\,\mathrm{Re}\,\langle \varphi, (D^{g_0} - \lambda)\varphi \rangle \; g_0(X,Y). \end{aligned}$$

If φ is critical, i. e. if $D^{g_0}\varphi = \lambda\varphi$, then the energy-momentum tensor simplifies to

$$Q_\varphi(X,Y) = -\frac{1}{4}\,\mathrm{Re}\left(\langle \varphi, X \bullet_{g_0} \nabla^{\Sigma M}_Y \varphi \rangle + \langle \varphi, Y \bullet_{g_0} \nabla^{\Sigma M}_X \varphi \rangle \right). \tag{24}$$

Example 6.2. Again, let M carry a spin structure, let $\mathcal{G}$ be the set of all semi-Riemannian metrics on M and let E be the universal spinor bundle, $E_{(g,x)} = \Sigma^g_x M$. Then again $\mathcal{S}$ is the universal bundle of spinor fields and we this time we put $\mathcal{F}_g := \{\varphi \in \mathcal{S}_g \mid \int_M \langle \varphi, \varphi \rangle_g \, dV_g = \pm 1\}$. We define the Lagrangian L by

$$L(\varphi) := \mathrm{Re}\,\langle \varphi, D^g \varphi \rangle_g \; dV_g .$$

Now φ is critical if and only if

$$\int_M \frac{d}{dt}\bigg|_{t=0} L(\varphi_t) = 2\,\mathrm{Re} \int_M \langle \dot\varphi, D^g \varphi \rangle_g \; dV_g = 0$$

for all $\dot\varphi$ perpendicular to φ, i. e. if and only if $D^g\varphi$ is a multiple of φ. This way we obtain all nonnull eigenspinors for all eigenvalues simultaneously as critical φ's.

This time the connection has to be chosen differently because $\tau^{t_1}_{t_0}$ is a pointwise isometry but the volume element dV_g also depends on the semi-Riemannian metric. Therefore $\tau^{t_1}_{t_0}$ does not give an isometry for the L^2-product used to define $\mathcal{F}$. This can be corrected by defining the connection $\bar H$ as the set of all $\frac{d}{dt}\big|_{t=0} \sqrt{\frac{dV_{g_t}}{dV_{g_0}}}\tau^t_0\varphi$ for all smooth curves g_t of metrics with $g_0 = \pi(\varphi)$.

Then we have for such a 1-parameter family of metrics g_t with $k := \dot g_0$

$$dL \circ (d\pi|_{\bar H_\varphi})^{-1}(k) = \mathrm{Re} \left\langle \varphi, \frac{d}{dt}\bigg|_{t=0} (\tau^0_t D^{g_t} \tau^t_0 \varphi) \right\rangle_{g_0} dV_{g_0}$$

and therefore

$$Q_\varphi(X,Y) = -\frac{1}{4}\,\mathrm{Re}\left(\langle \varphi, X \bullet_{g_0} \nabla^{\Sigma M}_Y \varphi \rangle + \langle \varphi, Y \bullet_{g_0} \nabla^{\Sigma M}_X \varphi \rangle \right)$$

for all φ, critical or not.

These two examples show that for noncritical φ the energy-momentum tensor also depends on the choice of connection H. In contrast, for critical φ the differential dL descends to a map $dL : T_\varphi\mathcal{F}/T_\varphi(\mathcal{F}_{\pi(\varphi)}) \to \Omega^{|n|}(M)$. Thus the map $dL \circ d\pi^{-1} : T_{\pi(\varphi)}\mathcal{G} \to \Omega^{|n|}(M)$ is well defined without any reference to H.

7. EMBEDDINGS OF HYPERSURFACES

We will now apply the cylinder construction described in Section 4 to study the question whether a given manifold can be isometrically immersed as a hypersurface into a manifold of constant curvature. The classical example for such a result is the fundamental theorem for hypersurfaces which can be stated as follows:

Theorem 7.1. *Let (M^n, g) be a Riemannian manifold and let A be a field of symmetric endomorphisms of TM satisfying the equations of Gauss and Codazzi-Mainardi:*

$$(\nabla^M_X A)Y = (\nabla^M_Y A)X, \tag{25}$$

$$R^M(X,Y)Z = \langle A(Y), Z\rangle\, A(X) - \langle A(X), Z\rangle\, A(Y) \tag{26}$$

for all $X, Y, Z \in T_pM$, $p \in M$.

Then every point of M has a neighborhood which can be isometrically embedded into Euclidean $(n+1)$-space $\mathbb{R}^{n+1}$, with Weingarten map A. If M is simply connected, then there exists a global isometric immersion of M into $\mathbb{R}^{n+1}$ with the above property.

A proof can be found in [6, Ch. VII.7], but here we will give a more geometrical argument based on the cylinder construction. This will allow us to extend the result without effort to the semi-Riemannian case and to embeddings into model spaces of constant sectional curvature not necessarily zero. We will construct an *explicit* metric of constant curvature on the cylinder $I \times M$, whose restriction to the leaf $\{0\} \times M$ is g.

For a constant $\kappa \in \mathbb{R}$ define the *generalized sine* and *cosine functions*

$$\mathfrak{s}_\kappa(t) := \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}\cdot t) & , \ \kappa > 0 \\ t & , \ \kappa = 0 \\ \frac{1}{\sqrt{|\kappa|}} \sinh(\sqrt{|\kappa|}\cdot t), & \kappa < 0 \end{cases} \quad \text{and} \quad \mathfrak{c}_\kappa(t) := \begin{cases} \cos(\sqrt{\kappa}\cdot t) & , \ \kappa > 0 \\ 1 & , \ \kappa = 0 \\ \cosh(\sqrt{|\kappa|}\cdot t), & \kappa < 0 \end{cases}$$

One easily checks $\mathfrak{s}_\kappa(0) = 0$, $\mathfrak{c}_\kappa(0) = 1$, $\kappa\mathfrak{s}_\kappa^2 + \mathfrak{c}_\kappa^2 = 1$, $\mathfrak{s}_\kappa' = \mathfrak{c}_\kappa$, and $\mathfrak{c}_\kappa' = -\kappa\mathfrak{s}_\kappa$.

Theorem 7.2. *Let (M^n, g) be a semi-Riemannian manifold and let $\kappa \in \mathbb{R}$. Let A be a field of symmetric endomorphisms of TM satisfying*

$$(\nabla^M_X A)Y = (\nabla^M_Y A)X, \tag{27}$$

$$\begin{aligned} R^M(X,Y)Z &= \langle A(Y), Z\rangle\, A(X) - \langle A(X), Z\rangle\, A(Y) \\ &\quad + \kappa(\langle Y, Z\rangle\, X - \langle X, Z\rangle\, Y) \end{aligned} \tag{28}$$

for all $X, Y, Z \in T_pM$, $p \in M$. Define a family of metrics on M by

$$g_t(X,Y) := g((\mathfrak{c}_\kappa(t)\,\mathrm{id} - \mathfrak{s}_\kappa(t)A)^2 X, Y).$$

Then the metric $dt^2 + g_t$ on $\mathcal{Z} = I \times M$ has constant sectional curvature κ on its domain of definition (i. e. for $|t|$ sufficiently small).

Proof. Put $R^{\mathcal{Z}}_\kappa(X,Y)Z := R^{\mathcal{Z}}(X,Y)Z - \kappa(\langle Y, Z\rangle\, X - \langle X, Z\rangle\, Y)$. Having constant sectional curvature κ is equivalent to $R^{\mathcal{Z}}_\kappa \equiv 0$. The proof is based on the following lemma:

Lemma 7.3. *Let $\mathcal{Z} = I \times M$ be a generalized cylinder and let $\kappa \in \mathbb{R}$. Assume that $g(R^{\mathcal{Z}}_\kappa(X,\nu)\nu, Y) = 0$ for all vector fields X and Y on $\mathcal{Z}$, where ν denotes the vector $\frac{\partial}{\partial t}$.*

(i) *If the Weingarten map A of the hypersurface $\{0\} \times M$ of $\mathcal{Z}$ satisfies (27), then $g(R^{\mathcal{Z}}_\kappa(X,Y)Z, \nu) = 0$ for all vector fields X, Y and Z on $\mathcal{Z}$.*

(ii) *If, moreover, A also satisfies (28), then $R^{\mathcal{Z}}_\kappa \equiv 0$, i. e. $\mathcal{Z}$ has constant sectional curvature κ.*

Assume this lemma for a moment. We will check that the metric $dt^2 + g_t$ satisfies the hypothesis of the lemma for $g_t(X,Y) = g((\mathfrak{c}_\kappa(t)\,\mathrm{id} - \mathfrak{s}_\kappa(t)A)^2X,Y)$. Let W_t denote the Weingarten tensor of the hypersurface $\{t\}\times M$ of $\mathcal{Z}$. This gives rise to a tensor field W on $\mathcal{Z}$, vanishing in the direction of ν. From the definition of g_t we compute

$$\begin{aligned}\dot g_t(X,Y) &= -2g((\mathfrak{c}_\kappa(t)\,\mathrm{id} - \mathfrak{s}_\kappa(t)A))(\kappa\mathfrak{s}_\kappa(t)\,\mathrm{id} + \mathfrak{c}_\kappa(t)A)X,Y)\\ &= -2g_t((\mathfrak{c}_\kappa(t)\,\mathrm{id} - \mathfrak{s}_\kappa(t)A))^{-1}(\kappa\mathfrak{s}_\kappa(t)\,\mathrm{id} + \mathfrak{c}_\kappa(t)A)X,Y)\end{aligned}$$

hence by (14)

$$W = (\mathfrak{c}_\kappa(t)\,\mathrm{id} - \mathfrak{s}_\kappa(t)A))^{-1}(\kappa\mathfrak{s}_\kappa(t)\,\mathrm{id} + \mathfrak{c}_\kappa(t)A).$$

Moreover,

$$\ddot g_t(X,Y) = -2g\left([\kappa(\mathfrak{c}_\kappa(t)\,\mathrm{id} - \mathfrak{s}_\kappa(t)A)^2 - (\kappa\mathfrak{s}_\kappa(t)\,\mathrm{id} + \mathfrak{c}_\kappa(t)A)^2]X,Y\right).$$

Equation (17) yields

$$\begin{aligned}g_t(R^{\mathcal{Z}}(X,\nu)\nu,Y) &= -\frac{1}{2}\ddot g_t(X,Y) - \frac{1}{2}\dot g_t(W(X),Y)\\ &= g(\kappa(\mathfrak{c}_\kappa(t)\,\mathrm{id} - \mathfrak{s}_\kappa(t)A)^2X,Y)\\ &= \kappa\, g_t(X,Y),\end{aligned}$$

thus $R^{\mathcal{Z}}(X,\nu)\nu = \kappa X$ and hence $R^{\mathcal{Z}}_\kappa(X,\nu)\nu = 0$. All conditions of the lemma are satisfied and the theorem follows. ☐

Proof of the lemma. The modified curvature tensor $R^{\mathcal{Z}}_\kappa$ has all the symmetries of a curvature tensor including the Bianchi identities.

i) Consider the family of tensors on M defined by $K_t(X,Y,Z)_x := \left\langle R^{\mathcal{Z}}_\kappa(X,Y)Z,\nu\right\rangle_{(t,x)}$. Using the second Bianchi identity on $\mathcal{Z}$, together with the fact that ν commutes with vectors on M and the formula $W(X) = -\nabla^{\mathcal{Z}}_X\nu = -\nabla^{\mathcal{Z}}_\nu X + [\nu,X] = -\nabla^{\mathcal{Z}}_\nu X$ we see

$$\begin{aligned}\dot K_t(X,Y,Z) &= d_\nu\left\langle R^{\mathcal{Z}}_\kappa(X,Y)Z,\nu\right\rangle\\ &= \left\langle(\nabla^{\mathcal{Z}}_\nu R^{\mathcal{Z}}_\kappa)(X,Y)Z,\nu\right\rangle\\ &\quad - \left\langle R^{\mathcal{Z}}_\kappa(W(X),Y)Z + R^{\mathcal{Z}}_\kappa(X,W(Y))Z + R^{\mathcal{Z}}_\kappa(X,Y)W(Z),\nu\right\rangle\\ &= \left\langle(\nabla^{\mathcal{Z}}_X R^{\mathcal{Z}}_\kappa)(\nu,Y)Z,\nu\right\rangle + \left\langle(\nabla^{\mathcal{Z}}_Y R^{\mathcal{Z}}_\kappa)(X,\nu)Z,\nu\right\rangle\\ &\quad +(W^*K_t)(X,Y,Z)\end{aligned}\tag{29}$$

where W^* denotes the induced action of W as a derivation on tensors. From the assumption in the lemma we conclude

$$\begin{aligned}0 &= d_X\left\langle R^{\mathcal{Z}}_\kappa(\nu,Y)Z,\nu\right\rangle\\ &= \left\langle(\nabla^{\mathcal{Z}}_X R^{\mathcal{Z}}_\kappa)(\nu,Y)Z,\nu\right\rangle + \left\langle R^{\mathcal{Z}}_\kappa(\nabla^{\mathcal{Z}}_X\nu,Y)Z,\nu\right\rangle + \left\langle R^{\mathcal{Z}}_\kappa(\nu,\nabla^{\mathcal{Z}}_X Y)Z,\nu\right\rangle\\ &\quad + \left\langle R^{\mathcal{Z}}_\kappa(\nu,Y)\nabla^{\mathcal{Z}}_X Z,\nu\right\rangle + \left\langle R^{\mathcal{Z}}_\kappa(\nu,Y)Z,\nabla^{\mathcal{Z}}_X\nu\right\rangle\\ &= \left\langle(\nabla^{\mathcal{Z}}_X R^{\mathcal{Z}}_\kappa)(\nu,Y)Z,\nu\right\rangle - \left\langle R^{\mathcal{Z}}_\kappa(W(X),Y)Z,\nu\right\rangle + 0\\ &\quad + 0 - \left\langle R^{\mathcal{Z}}_\kappa(\nu,Y)Z,W(X)\right\rangle\end{aligned}$$

thus

$$\left\langle(\nabla^{\mathcal{Z}}_X R^{\mathcal{Z}}_\kappa)(\nu,Y)Z,\nu\right\rangle = \left\langle R^{\mathcal{Z}}_\kappa(W(X),Y)Z,\nu\right\rangle + \left\langle R^{\mathcal{Z}}_\kappa(\nu,Y)Z,W(X)\right\rangle$$

and similarly

$$\left\langle(\nabla^{\mathcal{Z}}_Y R^{\mathcal{Z}}_\kappa)(X,\nu)Z,\nu\right\rangle = \left\langle R^{\mathcal{Z}}_\kappa(X,W(Y))Z,\nu\right\rangle + \left\langle R^{\mathcal{Z}}_\kappa(X,\nu)Z,W(Y)\right\rangle.$$

Plugging this into (29) yields

$$\begin{aligned}\dot K_t(X,Y,Z) &= \left\langle R^{\mathcal{Z}}_\kappa(W(X),Y)Z,\nu\right\rangle + \left\langle R^{\mathcal{Z}}_\kappa(\nu,Y)Z,W(X)\right\rangle\\ &\quad + \left\langle R^{\mathcal{Z}}_\kappa(X,W(Y))Z,\nu\right\rangle + \left\langle R^{\mathcal{Z}}_\kappa(X,\nu)Z,W(Y)\right\rangle\\ &\quad +(W^*K_t)(X,Y,Z).\end{aligned}$$

Hence $\dot{K}_t = F(t)(K_t)$ for some linear endomorphism F of the space of 3-tensors. This is a linear first order ODE for K_t. The initial condition $K_0 = 0$ follows from (16) because $W_0 = A$ is a Codazzi tensor. This shows that $K_t \equiv 0$.

ii) Similarly, using the identity $\left\langle R_\kappa^{\mathcal{Z}}(X,Y)Z,\nu\right\rangle \equiv 0$ that we just obtained, we see that the family of tensors on M defined by $R_t(X,Y,Z,V)_x := \left\langle R_\kappa^{\mathcal{Z}}(X,Y)Z,V\right\rangle_{(t,x)}$ satisfies a linear ODE. Moreover, (15) implies $R_0 \equiv 0$ because $W_0 = A$ satisfies the Gauss equation. Thus $R_t \equiv 0$ for all t. This proves the lemma. $\square$

Now recall that any semi-Riemannian manifold of constant sectional curvature κ is locally isometric to $\mathbb{M}_\kappa^{r,s}$. Here $\mathbb{M}_\kappa^{r,s}$ is the model space of constant sectional curvature κ and signature (r,s). If $\kappa = 0$, then $\mathbb{M}_0^{r,s}$ is semi-Euclidean space $\mathbb{R}^n$ with the metric $g_{r,s} = (dx^1)^2 + \cdots + (dx^r)^2 - (dx^{r+1})^2 - \cdots - (dx^n)^2$. If $\kappa > 0$, then $\mathbb{M}_\kappa^{r,s}$ is a pseudosphere, more precisely, it is the semi-Riemannian hypersurface of $(\mathbb{R}^{n+1}, g_{r+1,s})$ defined by $\langle x,x\rangle_{r+1,s} = 1/\kappa$ and $x^1 > 0$ if $r = 0$. If $\kappa < 0$, then $\mathbb{M}_\kappa^{r,s}$ is a pseudohyperbolic space, more precisely, it is the semi-Riemannian hypersurface of $(\mathbb{R}^{n+1}, g_{r,s+1})$ defined by $\langle x,x\rangle_{r,s+1} = 1/\kappa$ and $x^{n+1} > 0$ if $r = 0$. In all cases $\mathbb{M}_\kappa^{r,s}$ is connected and homogeneous. Moreover, $\mathbb{M}_\kappa^{r,s}$ is simpy connected except for $\mathbb{M}_\kappa^{1,n-1}$ if $\kappa > 0$ and $\mathbb{M}_\kappa^{n-1,1}$ if $\kappa < 0$, compare [8, p. 108 ff].

The local isometry is essentially given by the Riemannian exponential map, see [10, Cor. 2.3.8], and it is uniquely determined by its differential at a point. Applying this to the cylinder constructed in Theorem 7.2 yields the local statement in the fundamental theorem for hypersurfaces for semi-Riemannian manifolds.

Corollary 7.4. *Let (M^n, g) be a semi-Riemannian manifold of signature (r,s) and let $\kappa \in \mathbb{R}$. Let A be a field of symmetric endomorphisms of TM satisfying the equations of Gauss and Codazzi-Mainardi:*

$$
\begin{aligned}
(\nabla_X^M A)Y &= (\nabla_Y^M A)X, \\
R^M(X,Y)Z &= \langle A(Y),Z\rangle\, A(X) - \langle A(X),Z\rangle\, A(Y) \\
&\quad + \kappa(\langle Y,Z\rangle\, X - \langle X,Z\rangle\, Y)
\end{aligned}
$$

for all $X,Y,Z \in T_pM$, $p \in M$.

Then for every point $p \in M$, for every $q \in \mathbb{M}_\kappa^{r+1,s}$, and for every linear isometric embedding $F : T_pM \to T_q\mathbb{M}_\kappa^{r+1,s}$ there exists a neighborhood U of p in M and an isometric embedding $f : U \to \mathbb{M}_\kappa^{r+1,s}$ as a semi-Riemannian hypersurface with Weingarten map A, such that $f(p) = q$ and $df(p) = F$.

Moreover, any two such local embeddings f_1 and f_2 must agree in a neighborhood of p if $f_1(p) = f_2(p) =: q$ and $df_1(p) = df_2(p) : T_pM \to T_q\mathbb{M}_\kappa^{r+1,s}$.

Now that this local result is established exactly the same proof as in [6, Ch. VII, Thm. 7.2] can be used to show the corresponding global immersion statement in the simply connected case.

Corollary 7.5. *Let (M^n, g) be a simply connected semi-Riemannian manifold of signature (r,s), let $\kappa \in \mathbb{R}$ and let A be a field of symmetric endomorphisms of TM satisfying the two equations (27) and (28) above.*

Then M can be isometrically immersed as a semi-Riemannian hypersurface into the model space $\mathbb{M}_\kappa^{r+1,s}$ with Weingarten map A. Any two such immersions differ by an isometry of $\mathbb{M}_\kappa^{r+1,s}$.

8. Generalized Killing spinors

We now turn our attention to restrictions of spinors to hypersurfaces. Let $M^n \subset \mathcal{Z}^{n+1}$ be a hypersurface of a spin manifold $\mathcal{Z}$ admitting a parallel spinor Ψ. If $n+1$ is even, we will assume that Ψ lies in $\Sigma^+\mathcal{Z}$. From the discussion in Section 3 we see that the restriction ψ of Ψ to M is actually a spinor on M and (10) reads

$$0 = \nabla_X^{\Sigma\mathcal{Z}}\Psi = \nabla_X^{\Sigma M}\psi - \frac{1}{2}A(X)\bullet\psi \tag{30}$$

for all $X \in TM$ where A is the Weingarten tensor of the submanifold M and "$\bullet$" denotes Clifford multiplication on M. If ψ is an eigenspinor of the Dirac operator, then A is closely related to the energy-momentum tensor of ψ. More precisely, using (24) one computes

$$Q_\psi(X,Y) = \frac{1}{4}\langle X, A(Y)\rangle\langle\psi,\psi\rangle$$

where $\langle\psi,\psi\rangle$ is constant since ψ is parallel on $\mathcal{Z}$. Spinors satisfying (30) will be called *generalized Killing spinors*. They are closely related to the so–called T–Killing spinors studied by Friedrich and Kim in [5].

Conversely, given a generalized Killing spinor ψ on a manifold M^n with $\nabla_X^{\Sigma M}\psi - \frac{1}{2}A(X)\bullet\psi$, it is natural to ask whether the tensor A can be realized as the Weingarten tensor of some isometric embedding of M in a manifold $\mathcal{Z}^{n+1}$ carrying parallel spinors. Morel studied this problem in the case where the tensor A is parallel, see [7].

The next result provides an affirmative answer to the above question, for the case where the energy-momentum tensor of ψ is a Codazzi tensor.

Theorem 8.1. *Let (M^n, g) be a semi-Riemannian spin manifold and let A be a field of symmetric endomorphisms of TM satisfying equation (25) on M. Let ψ be a spinor on (M^n, g) satisfying for all $X \in TM$*

$$\nabla_X^{\Sigma M}\psi = \frac{1}{2}A(X)\bullet\psi. \tag{31}$$

Then the generalized cylinder $\mathcal{Z} = I \times M$ with the metric $dt^2 + g_t$, where $g_t(X,Y) = g((\mathrm{id} - tA)^2X, Y)$, and with the spin structure inducing the given one on $\{0\}\times M$ by restriction has a parallel spinor, whose restriction to the leaf $\{0\}\times M$ is just ψ.

Proof. The spinor ψ defines a spinor Ψ on $\mathcal{Z}$ by parallel transport along the geodesics $\mathbb{R}\times\{x\}$. More precisely, we define $\Psi_{(0,x)} := \psi_x$ via the identification $\Sigma_x M \cong \Sigma_{(0,x)}\mathcal{Z}$ (resp. $\Sigma^+_{(0,x)}\mathcal{Z}$ for n odd) and $\Psi_{(t,x)} = \tau_0^t\Psi_{(0,x)}$. By construction we have

$$\nabla_\nu^{\Sigma\mathcal{Z}}\Psi \equiv 0 \ \text{ and } \ \nabla_X^{\Sigma\mathcal{Z}}\Psi|_{\{0\}\times M} = 0 \tag{32}$$

for all $X \in TM$.

The explicit form of the metrics g_t yields $\left\langle R^{\mathcal{Z}}(X,\nu)\nu, Y\right\rangle = 0$ on $\mathcal{Z}$ for all X and Y tangent to M as in the proof of Theorem 7.2. Since the Codazzi equation (25) holds Lemma 7.3 (i) yields $\left\langle R^{\mathcal{Z}}(\nu,X)Y, Z\right\rangle = 0$ on all of $\mathcal{Z}$. Hence $R^{\mathcal{Z}}(\nu,X) = 0$ for all $X \in TM$.

Let X be a fixed arbitrary vector field on M, identified as usual with the vector field $(0,X)$ on $\mathcal{Z}$. Using (32) we get $0 = \frac{1}{2}R^{\mathcal{Z}}(\nu,X)\cdot\Psi = \nabla_\nu^{\Sigma\mathcal{Z}}\nabla_X^{\Sigma\mathcal{Z}}\Psi$, thus showing that the spinor field $\nabla_X^{\Sigma\mathcal{Z}}\Psi$ is parallel along the geodesics $\mathbb{R}\times\{x\}$. Now (32) shows that this spinor vanishes for $t = 0$, hence it is zero everywhere on $\mathcal{Z}$. Since X was arbitrary, this shows that Ψ is parallel on $\mathcal{Z}$. $\square$

This theorem generalizes the result from [1] where the case $A = \lambda \cdot \mathrm{id}$ is treated, $\lambda \in \mathbb{R}$, and it is shown that the cone over a manifold with Killing spinors admits parallel spinors, as well as a more recent result by Morel [7] for the case when A is parallel. Nevertheless, the question whether a manifold with a spinor satisfying (31) can be isometrically embedded in a manifold with parallel spinors such that A becomes the Weingarten tensor of the embedding without assuming that A is a Codazzi tensor is left open in the present article.

9. THE SPACE OF LORENTZIAN METRICS

In the final section we address the problem of connecting any two semi-Riemannian metrics of signature (r, s) on some manifold M of dimension $n = r + s$, by a curve g_t of semi-Riemannian metrics of the same signature in a unique and universal manner. The latter requirement reduces this problem to the purely algebraic issue of finding a universal way of relating any two inner products of signature (r, s) on some real vector space $E \cong \mathbb{R}^n$ in the manifold $\mathcal{M}_{r,s}$ of all inner products of signature (r, s) on E.

In the positive or negative definite case an obvious candidate is the linear interpolation $g_t = tg_1 + (1 - t)g_0$ which, however, cannot be used for other signatures. An alternative solution, which has been considered in the definite case, see e.g. [3], but holds in a formally identical way for all signatures, relies on the geometry of $\mathcal{M}_{r,s}$, as a (semi-Riemannian) symmetric space that we now recall briefly.

For any signature (r, s) the identity component of the general linear group $\mathrm{GL}^+(E) \cong \mathrm{GL}^+(n, \mathbb{R})$ acts transitively on $\mathcal{M}_{r,s}$ by

$$(\gamma \cdot g)(u, v) = g(\gamma^{-1}u, \gamma^{-1}v)$$

for $\gamma \in \mathrm{GL}^+(E)$, $g \in \mathcal{M}_{r,s}$, and $u, v \in E$. For any chosen g_0 in $\mathcal{M}_{r,s}$, the isotropy group of g_0 in $\mathrm{GL}^+(E)$ is the special orthogonal group $\mathrm{SO}(g_0)$ relative to g_0. Recall that, except in the definite case where $\mathrm{SO}(g_0)$ is connected, $\mathrm{SO}(g_0)$ has *two* connected components. We thus get the identification $\mathcal{M}_{r,s} = \mathrm{GL}^+(E)/\mathrm{SO}(g_0)$ or, equivalently, $\mathcal{M}_{r,s} = \mathbb{R}^+ \times \mathrm{SL}(E)/\mathrm{SO}(g_0)$, where $\mathbb{R}^+$ acts by homotheties, and $\mathrm{SL}(E) \cong \mathrm{SL}(n, \mathbb{R})$ denotes the special linear group of elements of determinant 1 in $\mathrm{GL}^+(E)$. Hence $\mathcal{M}^0_{r,s} := \mathrm{SL}(E)/\mathrm{SO}(g_0)$ can be regarded as the space of inner products on E of signature (r, s) and with a fixed volume element. Concerning the problem addressed in this section, it is clearly sufficient to restrict our attention to $\mathcal{M}^0_{r,s}$.

The homogeneous geometry of $\mathcal{M}^0_{r,s} = \mathrm{SL}(E)/\mathrm{SO}(g_0)$ can be described as follows. For simplicity, write $G := \mathrm{SL}(E)$, $H := \mathrm{SO}(g_0)$, let $\mathfrak{g}$ be the Lie algebra of G, identified with the Lie algebra of trace-free endomorphisms of E, and let $\mathfrak{h}$ be the Lie algebra of H, identified with the Lie algebra of g_0-skewsymmetric endomorphisms. Denote by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to the Killing form of $\mathfrak{g}$, so that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Recall that the Killing form of $\mathfrak{g}$ equals the bilinear form $a, b \mapsto \mathrm{tr}(ab)$, up to a positive universal constant, so that $\mathfrak{m}$ is the space of g_0-symmetric elements of $\mathfrak{g}$. Since the Killing form is G-invariant, $\mathfrak{m}$ is stable under the adjoint action of H, making $\mathcal{M}^0_{r,s}$ a reductive homogeneous space. Moreover, we clearly have the Lie bracket relations $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ showing that $\mathcal{M}^0_{r,s}$ is actually a symmetric homogeneous space.

In the positive definite case, $\mathcal{M}^0_{n,0}$ is a Riemannian symmetric space of noncompact type, hence a Hadamard space. It follows that any two points of $\mathcal{M}^0_{n,0}$ can be joined by a unique geodesic. If g and g_0 are any two points of $\mathcal{M}_{n,0}$, then $g = g_0(A\cdot, \cdot)$, for a uniquely defined automorphism A of E, where A is symmetric and positive definite for both g_0 and g. Then $A = \exp(a)$ for a uniquely defined symmetric endomorphism a of E and the unique geodesic connecting g_0 to g is the curve $g_t := g_0(\exp(ta)\cdot,\cdot) = g_0(A^t\cdot,\cdot)$, for $t \in [0,1]$ where $\exp : \mathfrak{g} \to G$ denotes the exponential mapping.

In the general case, the restriction of the Killing form to $\mathfrak{m}$ is an H-invariant inner product of signature $\left(\frac{r(r+1)}{2} + \frac{s(s+1)}{2} - 1, rs\right)$, making $\mathcal{M}^0_{r,s}$ a *semi-Riemannian* symmetric space of this signature.

The fact that $\mathcal{M}^0_{r,s}$ is symmetric, as a semi-Riemannian homogeneous space, implies that the Levi-Civita connection of the semi-Riemannian metric coincides with the canonical homogeneous connection. In particular, all (semi-Riemannian) geodesics emanating from g_0 are of the form

$$\exp(tX) \cdot g_0$$

for $X \in \mathfrak{m} = T_{g_0}\mathcal{M}_{r,s}$.

As a homogeneous semi-Riemannian manifold $\mathcal{M}^0_{r,s}$ is certainly geodesically complete in the sense that geodesics are defined on all of $\mathbb{R}$, but for $(r,s) \neq (n,0),(0,n)$, it is not longer true that any two points can be joined by a geodesic and, if so, there is no guarantee that the geodesic be unique. This will be illustrated firstly in the case that $(r,s) = (1,1)$, then in the general Lorentzian case when $(r,s) = (n-1,1)$.

9.1. **The space of Lorentzian inner products in dimension** 2. Let E denote an oriented real vector space of dimension 2. We fix a positive generator ω of the real line $\Lambda^2 E^*$, which can be viewed as a symplectic form on E. Now $G \cong \mathrm{SL}(2,\mathbb{R})$, $\mathfrak{g} \cong \mathfrak{sl}(2,\mathbb{R})$ is the Lie algebra of trace-free endomorphisms of E, and $\mathcal{M}^0_{1,1}$ is the space of all Lorentzian inner products on E, whose volume form with respect to the given orientation is ω. For any chosen point $g_0 \in \mathcal{M}^0_{1,1}$ we then have $\mathcal{M}^0_{1,1} = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)$. Note that $\mathrm{SO}(1,1)$ has two connected components. The connected component of the identity $\mathrm{SO}_0(1,1)$ is isomorphic the the additive group $\mathbb{R}$ of real numbers via the isomorphism $t \mapsto \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$. The other connected component equals $-\mathrm{SO}_0(1,1)$. Differentiation with respect to t shows that the corresponding isotropy Lie algebra $\mathfrak{h}$ is the Lie algebra of 2×2-matrices of the form $\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$, for $b \in \mathbb{R}$.

An endomorphism α of E is tracefree if and only if it is "antisymmetric" with respect to ω, i. e. if and only if it satisfies: $\omega(\alpha\cdot,\cdot) + \omega(\cdot,\alpha\cdot) = 0$.

For any $g \in \mathcal{M}^0_{1,1}$ there is one and only one automorphism I_g of E such that

$$g = \omega(\cdot, I_g\cdot). \tag{33}$$

Since g is symmetric I_g is trace-free. Its determinant equals -1 because g is Lorentzian, with volume form equal to ω. In particular, $I_g^2 = 1$. The light cone of g is the union of the two eigenspaces of I_g, for the eigenvalues ± 1. The latter are generated by $v \pm I_g v$ respectively, for any nonzero $v \in E$.

Conversely, for any automorphism I of E of trace equal to 0 and of determinant equal to -1, the bilinear form g defined by $g = \omega(\cdot, I\cdot)$ is a Lorentzian inner product, with volume form equal to ω and $I = I_g$.

The automorphism I_g belongs to the Lie algebra $\mathfrak{g}$, on which G acts by the adjoint representation, and the map $g \mapsto I_g$ is G-equivariant. Indeed, by definition of G, we have that $\omega(\gamma\cdot,\gamma\cdot) = \omega(\cdot,\cdot)$ for each $\gamma \in G$, so that

$$\gamma \cdot g = g(\gamma^{-1}\cdot,\gamma^{-1}\cdot) = \omega(\gamma^{-1}\cdot, I_g\,\gamma^{-1}\cdot) = \omega(\cdot, \gamma\, I_g \gamma^{-1}\cdot).$$

The map $g \mapsto I_g$ is then a G-equivariant identification of $\mathcal{M}^0_{1,1}$ with the adjoint orbit of all elements of $\mathfrak{g}$ of determinant equal to -1.

As a function defined on $\mathfrak{g} \cong \mathbb{R}^3$, the opposite of the determinant is a nondegenerate quadratic form of signature (2, 1), equal to the (suitably normalized) Killing form. We denote the symmetric bilinear form corresponding to $-\det$ by $\langle\cdot,\cdot\rangle$, i. e. $\langle u,u\rangle = -\det(u) = \frac{1}{2}\operatorname{tr}(u^2)$. The adjoint orbit is then the pseudosphere $\mathbb{M}_1^{1,1}$ of elements u such that $\langle u,u\rangle = 1$ in the 3-dimensional Minkowski space $(\mathfrak{g},\langle\cdot,\cdot\rangle)$. The restriction of $\langle\cdot,\cdot\rangle$ to $\mathbb{M}_1^{1,1}$ makes the latter a G-homogeneous Lorentzian manifold, known as the 2-dimensional *de Sitter universe*.

The map $\mathcal{M}^0_{1,1} \to \mathbb{M}_1^{1,1}$, $g \mapsto I_g$, is a G-equivariant isometry.

Reflection with respect to $\langle\cdot,\cdot\rangle$ about a vector subspace is an isometry of $(\mathfrak{g},\langle\cdot,\cdot\rangle)$ and it preserves $\mathbb{M}_1^{1,1}$. Since the fixed point set of an isometry is a totally geodesic submanifold the geodesics of $\mathbb{M}_1^{1,1}$ are precisely the intersections of $\mathbb{M}_1^{1,1}$ with 2-dimensional vector subspaces $E \subset \mathfrak{g}$. There are three types of geodesics: timelike geodesics (hyperbolas) corresponding to Minkowski planes, spacelike geodesics (ellipses) corresponding to spacelike planes, and null geodesics (straight lines) corresponding to degenerate planes (tangent to the light cone).

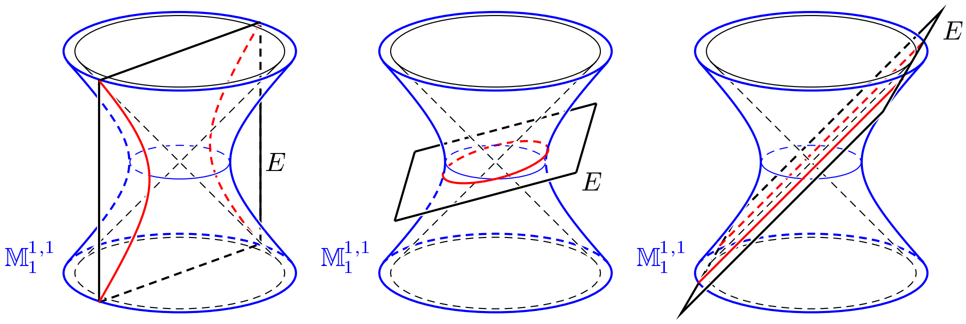

Fig. 1

Now let I, I' be two different points in $\mathbb{M}_1^{1,1}$. If $I' = -I$, then each plane E containing I also contains I'. In the timelike or in the null case I' lies on the other connected component of $E \cap \mathbb{M}_1^{1,1}$. Thus all spacelike geodesics emanating from I hit $I' = -I$, but the timelike and null geodesics emanating from I miss $I' = -I$.

If $I' \neq -I$, then I and I' are linearly independent, so the plane E containing I and I' is uniquely determined. Thus I' is hit by the geodesic emanating from I if and only if it does not lie on the "wrong" connected component of $E \cap \mathbb{M}_1^{1,1}$ (in the timelike or null case). In other words, the points on $\mathbb{M}_1^{1,1}$ which cannot be reached by a geodesic emanating from I are precisely the ones lying on timelike or null geodesics emanating from $-I$.

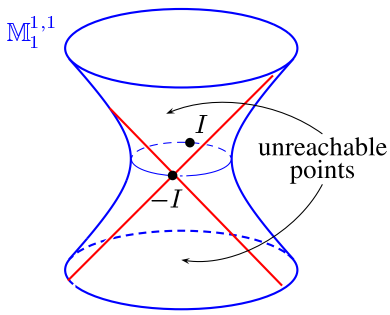

Fig. 2

The two null geodesics emanating from $-I$ are cut out of $\mathbb{M}_1^{1,1}$ by the affine plane $\{\langle I, I'\rangle = -1\}$. Thus the points $I' \in \mathbb{M}_1^{1,1}$ with $\langle I, I'\rangle < -1$ cannot be attained by a geodesic from I.

Similarly, by looking at the affine plane $\{\langle I, I'\rangle = +1\}$ we see that the points I' with $\langle I, I'\rangle > 1$ are the ones that lie on timelike geodesics emanating from I, the ones with $\langle I, I'\rangle = 1$ are the ones that lie on null geodesics emanating from I, and the ones with $-1 < \langle I, I'\rangle < 1$ lie on spacelike geodesics emanating from I.

We now retranslate this information back to $\mathcal{M}^0_{1,1}$. If $g, g' \in \mathcal{M}^0_{1,1}$, then

$$g' = g(A\cdot, \cdot),$$

with

$$A = I_g^{-1} I_{g'} = I_g I_{g'}.$$

We then have

$$\langle I_g, I_{g'}\rangle = \frac{1}{2}\mathrm{tr}\, A.$$

Note that A is g- and g'-symmetric and of determinant equal to $+1$.

By choosing g as a base-point, we conclude that $\mathcal{M}^0_{1,1}$ can also be identified with the space of all g-symmetric automorphisms of determinant 1 of E. We summarize:

Proposition 9.1. *The space $\mathcal{M}^0_{1,1}$ of Lorentzian inner products on a 2-dimensional real vector space that have a fixed volume element carries a natural Lorentzian metric making it $\mathrm{SL}(2, \mathbb{R})$-equivariantly isometric to the 2-dimensional de Sitter universe. For $g, g' \in \mathcal{M}^0_{1,1}$ there is a unique endomorphism A such that $g' = g(A\cdot, \cdot)$. Moreover, the following holds:*

- *If $\mathrm{tr}(A) > 2$, then there is a unique geodesic in $\mathcal{M}^0_{1,1}$ joining g and g'. This geodesic is timelike.*
- *If $\mathrm{tr}(A) = 2$, then there is a unique geodesic in $\mathcal{M}^0_{1,1}$ joining g and g'. This geodesic is null.*
- *If $-2 < \mathrm{tr}(A) < 2$, then there is a unique geodesic in $\mathcal{M}^0_{1,1}$ joining g and g'. This geodesic is spacelike.*
- *If $\mathrm{tr}(A) < -2$, then there is no geodesic in $\mathcal{M}^0_{1,1}$ joining g and g'.*
- *If $\mathrm{tr}(A) = -2$ and $g \neq -g'$, then there is no geodesic in $\mathcal{M}^0_{1,1}$ joining g and g'.*
- *If $\mathrm{tr}(A) = -2$ and $g = -g'$, then all spacelike geodesics in $\mathcal{M}^0_{1,1}$ emanating from g pass through g' while the timelike and null geodesics in $\mathcal{M}^0_{1,1}$ emanating from g miss g'.*

This proposition shows that given two Lorentzian metrics on a 2-dimensional manifold we can construct a canonical 1-parameter family of Lorentzian metrics joining them only if the endomorphism field A relating the two metrics satisfies $\mathrm{tr}(A) > -2$. A restriction like this does not come as a surprise because there are pairs of Lorentzian metrics e. g. on the 2-torus which cannot even be joined by any continuous curve of Lorentzian metrics.

9.2. **The space of Lorentzian inner products in higher dimensions.** We now consider the manifold $\mathcal{M}_{n-1,1} = \mathbb{R}^+ \times \mathcal{M}^0_{n-1,1}$ of all Lorentzian inner products of signature $(n-1, 1)$ on some n-dimensional real vector space E.

As observed before the manifold $\mathcal{M}^0_{n-1,1}$ is a symmetric semi-Riemannian space of signature $\left(\frac{n(n-1)}{2}, n-1\right)$ and the geodesics emanating from any chosen base-point g_0 are of the form $\exp(tX) \cdot g_0$, where X belongs to the space $\mathfrak{m}$ of trace-free g_0-symmetric endomorphisms of E, $\mathfrak{m}$ being naturally identified with the tangent space $T_{g_0}\mathcal{M}^0_{n-1,1}$.

The goal of this section is to determine the set of elements $g \in \mathcal{M}_{n-1,1}$ which can be joined from g_0 by a geodesic in $\mathcal{M}_{n-1,1}$, and whether or not this geodesic is unique. This has just been done in detail in the case that $n = 2$ and, as we shall see, the general case can essentially be reduced to the 2-dimensional case. More precisely, we have

Proposition 9.2. *Let g_0 and g be two distinct points in $\mathcal{M}_{n-1,1}$. Then there is the following alternative: Either*

(i) *E splits as*

$$E = E_{1,1} \oplus E_{n-2,0},$$

where the sum is orthogonal, $E_{1,1}$ is of signature $(1,1)$, $E_{n-2,0}$ is of signature $(n-2,0)$ for g_0 and g. Both g_0 and g belong to the corresponding totally geodesic submanifold $\mathcal{M}_{1,1} \times \mathcal{M}_{n-2,0} \subset \mathcal{M}_{n-1,1}$. Thus the issue of the existence and uniqueness of geodesics connecting g_0 to g is reduced to the same issue for the 2-dimensional Lorentzian metrics $g_{0|E_{1,1}}$ and $g_{|E_{1,1}}$ in $\mathcal{M}_{1,1}$ as described in Proposition 9.1, or

(ii) *E splits as*

$$E = E_{2,1} \oplus E_{n-3,0},$$

where the sum is orthogonal, $E_{2,1}$ is of signature $(2,1)$, $E_{n-3,0}$ is of signature $(n-3,0)$ for g_0 and g. Both g_0 and g belong to the corresponding totally geodesic submanifold $\mathcal{M}_{2,1} \times \mathcal{M}_{n-3,0} \subset \mathcal{M}_{n-1,1}$. The 3-dimensional Lorentzian metrics $g_{0|E_{2,1}}$ and $g_{|E_{2,1}}$ are related by $g_{|E_{2,1}} = g_{0|E_{2,1}}(B\cdot,\cdot)$, where B is an automorphism of $E_{2,1}$ of the form $k(\mathrm{id} + x)$, where k is a positive real number and x is an endomorphism of $E_{2,1}$ satisfying $x^3 = 0$ but $x^2 \neq 0$. Thus g_0 and g are connected by a unique geodesic whose $E_{2,1}$-part is of the form

$$g_{t|E_{2,1}} = g_{0|E_{2,1}}(B_t\cdot,\cdot),$$

with $B_t = k^t \exp(t(x - \frac{1}{2}x^2)) = k^t \Big(1 + tx + \frac{t(t-1)}{2}x^2\Big)$.

The rest of the paper is devoted to the proof of Proposition 9.2.

Recall that for any g and g_0 in $\mathcal{M}_{n-1,1}$, there exists a uniquely defined automorphism A of E — with $\det A > 0$ — such that $g = g_0(A\cdot,\cdot)$: $A = (\gamma^{-1})^*\gamma^{-1}$, for any $\gamma \in \mathrm{GL}(E)$ such that $g = \gamma \cdot g_0$ and A is symmetric relative to both g and g_0. Then g_0 can be joined with g by a geodesic in $\mathcal{M}_{n-1,1}$ if and only if A is of the form $A = \exp(a)$, for some g_0-symmetric endomorphism a of E, and the corresponding geodesic is then the curve $g_t := g_0(\exp(ta)\cdot,\cdot)$ for $t \in [0,1]$.

The proof of Proposition 9.2 requires the spectral analysis of A. For this purpose it is convenient to introduce a positive definite *Euclidean* inner product $(\cdot,\cdot)$ on E such that $g_0 = (I\cdot,\cdot)$ where I is of the form

$$I = \mathrm{id} - 2(u,\cdot)u, \tag{34}$$

for some element $u \in E$ such that $|u|^2 = 1$. Here, and henceforth, $|\cdot|$ denotes the norm with respect to $(\cdot,\cdot)$. For g_0 the vector u is timelike with $g_0(u,u) = -1$. Conversely, any such u determines a Euclidean inner product as above.

Then $g = g_0(A\cdot,\cdot)$ can be written as $g = (S\cdot,\cdot)$ for a uniquely defined $(\cdot,\cdot)$-symmetric automorphism S of E with exactly $n-1$ positive and 1 negative eigenvalues.

Conversely, for any such automorphism S, the inner product $g = (S\cdot,\cdot)$ belongs to $\mathcal{M}_{n-1,1}$ with

$$A = I^{-1}S = IS.$$

The spectral decomposition of S reads

$$S = \lambda_0\Pi_0 + \bigoplus_{r=1}^{\ell} \lambda_j\Pi_r,$$

with $\lambda_0 < 0 < \lambda_1 < \ldots \lambda_\ell$, where Π_j denotes the $(\cdot,\cdot)$-orthogonal projection onto the d_j-dimensional eigenspace E_j of S corresponding to the eigenvalue λ_j, $j = 0, 1, \ldots, \ell$. Note that $d_0 = 1$.

Via the decomposition $E = E_0 \oplus \bigoplus_{r=1}^{\ell} E_r$ the unit vector u appearing in (34) splits as

$$u = u_0 + u_1 + \ldots + u_\ell.$$

We denote by Δ the subset of $j \in \{0, 1, \ldots, \ell\}$ such that $u_j \neq 0$, and by m the cardinality of Δ. For each $j \in \Delta$ such that $d_j > 1$ we denote by $\tilde{E}_j$ the $(\cdot,\cdot)$-orthogonal complement of u_j in E_j. Let $\tilde{E}$ be the subspace of E defined by

$$\tilde{E} := \bigoplus_{j\in\Delta, d_j>1} \tilde{E}_j \oplus \bigoplus_{j\notin\Delta} E_j, \tag{35}$$

and W the m-dimensional subspace of E defined by

$$W = \bigoplus_{j\in\Delta} \mathbb{R}u_j \tag{36}$$

so that

$$E = \tilde{E} \oplus W.$$

Both $\tilde{E}$ and W are left invariant by A, I, and S. The sum is orthogonal with respect to $(\cdot,\cdot)$, g_0, and g.

Note that if $0 \notin \Delta$, i. e. if $u_0 = 0$, then $\tilde{E}$ is of signature $(n - m - 1, 1)$ and W is of signature $(m, 0)$, whereas, if $0 \in \Delta$, i. e. if $u_0 \neq 0$, W is of signature $(m - 1, 1)$ and $\tilde{E}$ is of signature $(n - m, 0)$ for g (but W is always of signature $(m - 1, 1)$ for g_0, as $\tilde{E}$ is orthogonal to u).

Since $\tilde{E}$ is orthogonal to u, $I_{|\tilde{E}} = \mathrm{id}$ and $A_{|\tilde{E}} = S_{|\tilde{E}}$. In particular, $A_{|\tilde{E}}$ is symmetric for g_0, g and $(\cdot,\cdot)$ and its spectral decomposition coincides with the one of $S_{|\tilde{E}}$, given by (35), with eigenvalues λ_j for each $j \notin \Delta$ and each $j \in \Delta$ with $d_j > 1$.

The spectral study of A is then reduced to the spectral study of $A_{|W}$ and the latter is summarized by the following lemma.

Lemma 9.3. (i) *The characteristic polynomial P of $A_{|W}$ defined by $P(t) = \det(t\,\mathrm{id} - A_{|W})$ is given by*

$$P(t) = \prod_{j\in\Delta}(t - \lambda_j) + 2\sum_{j\in\Delta} \lambda_j |u_j|^2 \prod_{k\in\Delta\setminus\{j\}} (t - \lambda_k). \tag{37}$$

In particular, the roots of P are all distinct from the λ_j, $j \in \Delta$.

(ii) *For each real root μ of P the corresponding eigenspace is the one-dimensional vector space generated by the element $v_\mu \in W$ defined by*

$$v_\mu = \sum_{j\in\Delta} \frac{u_j}{\mu - \lambda_j}. \tag{38}$$

Moreover,

$$g(v_\mu, v_\mu) = \mu\, g_0(v_\mu, v_\mu) = -\frac{1}{2}\frac{P'(\mu)}{Q(\mu)} \tag{39}$$

where Q denotes the polynomial defined by $Q(t) = \prod_{j\in\Delta}(t - \lambda_j)$. In particular, v_μ is a null-vector — for both g and g_0 — if and only if μ is a multiple root of P.

Proof. By definition, any $v \in W$ is of the form $v = \sum_{j\in\Delta} y_j u_j$, for real numbers $y_1, \ldots, y_m$, so that

$$Av = ISv = \sum_{j\in\Delta} (\lambda_j y_j - 2(Su, v))\, u_j.$$

Note that v is an eigenvector of $A_{|W}$ for some eigenvalue μ if and only if

$$(\mu - \lambda_j)\, y_j = -2(Su, v), \tag{40}$$

for each $j \in \Delta$. It is easily checked that (Su, v) cannot be equal to 0 if $v \neq 0$. Indeed, suppose for a contradiction that v satisfies (40) with $(Su, v) = 0$ and $v \neq 0$. Since $v \neq 0$, one of the y_j, say y_1, is nonzero, so that $\mu = \lambda_1$. This implies $\mu \neq \lambda_j$, for $j \neq 1$, as the λ_j are pairwise distinct. It follows that $y_j = 0$ for all $j \neq 1$, so that $v = y_1 u_1$. Then $(Su, v) = \lambda_1 y_1 |u_1|^2 \neq 0$ as $y_1 \neq 0$, a contradiction.

In particular, this shows $\mu \neq \lambda_j$ for each $j \in \Delta$ so that we can write

$$v = -2(Su, v) \sum_{j \in \Delta} \frac{u_j}{\mu - \lambda_j}. \tag{41}$$

Moreover, by computing $(Su, v) = (Sv, u)$ from (41), we get

$$\sum_{j \in \Delta} \frac{\lambda_j |u_j|^2}{\mu - \lambda_j} = -\frac{1}{2}. \tag{42}$$

It follows that each eigenvalue of $A_{|W}$ is a root of the polynomial P defined by (37). Since P is monic and of degree m, it must coincide with the characteristic polynomial of $A_{|W}$. We readily see from (37) that the roots of P are distinct from the λ_j (recall that the latter are pairwise distinct). From (41) we immediately see that the eigenspace corresponding to μ is generated by the vector v_μ defined by (38).

Conversely, for each root μ of P the vector v_μ defined by (38) is certainly an eigenvector of $A_{|W}$ for the eigenvalue μ.

Since the roots of P are distinct from the λ_j, P can also be expressed by

$$\frac{P(t)}{Q(t)} = 1 + 2 \sum_{j \in \Delta} \frac{\lambda_j |u_j|^2}{t - \lambda_j}, \tag{43}$$

where we put $Q(t) := \prod_{j \in \Delta} (t - \lambda_j)$. Differentiating (43) at $t = \mu$, we get (39). It follows that v_μ is a null vector if and only if $P'(\mu) = 0$, meaning that μ is a multiple root. □

For further use, we need more information about the sign of the characteristic polynomial P at $t = \lambda_j$, $j \in \Delta$, and at $t = 0$. In the sequel, we use the notation $P(t_0) \equiv (-1)^r$, for some integer r, to mean that P has the sign of $(-1)^r$ — in particular is not zero — at $t = t_0$.

Lemma 9.4. (i) *If* $0 \notin \Delta$, *we re-label the* λ_j *so that* $\Delta = \{1, \ldots, m\}$, *and* $0 < \lambda_1 < \ldots < \lambda_m$. *We then have:*

$$\begin{aligned} &P(-\infty) \equiv P(\lambda_0) \equiv (-1)^m, \\ &P(0) \equiv (-1)^{m-1}, \\ &P(\lambda_j) \equiv (-1)^{m-j}, \quad j = 1, \ldots, m. \end{aligned} \tag{44}$$

In particular, P *has then exactly* m *distinct real roots* $\mu_0 < 0 < \mu_1 < \ldots < \mu_{m-1}$, *with* $\mu_0 \in (\lambda_0, 0)$ *and* $\mu_i \in (\lambda_i, \lambda_{i+1})$, *for* $i = 1, \ldots, m-1$.

(ii) *If* $0 \in \Delta$, *we re-label the* λ_j *so that* $\Delta = \{0, 1, \ldots, m-1\}$ *and* $\lambda_0 < 0 < \lambda_1 < \ldots < \lambda_{m-1}$. *We then have*

$$\begin{aligned} &P(-\infty) \equiv P(\lambda_0) \equiv P(0) \equiv (-1)^m, \\ &P(\lambda_j) \equiv (-1)^{m-j-1}, \quad j = 1, \ldots, m-1. \end{aligned} \tag{45}$$

In particular, P *has then at least* $(m-2)$ *distinct real roots* $0 < \mu_1 < \ldots < \mu_{m-2}$, *with* $\mu_i \in (\lambda_i, \lambda_{i+1})$, *for* $i = 1, \ldots, m-2$.

Proof. Easy consequence of (37). □

We now consider the two cases when 0 does or does not belong to Δ.

Case 1: $0 \notin \Delta$.

According to Lemma 9.4 (i), $A_{|W}$ is diagonalizable (over $\mathbb{R}$) with one negative eigenvalue μ_0 and $m-1$ distinct positive eigenvalues. Moreover, we easily see from (39) that the m corresponding eigenvectors v_μ, defined by (38), are all spacelike. On the other hand, $A_{|\tilde{E}}$ is also diagonalizable with one negative eigenvalue, namely λ_0 — whose eigenspace is E_0 — and $n-m-1$ positive eigenvalues. Denote by $E_{1,1}$ the direct sum of E_0 and the (one-dimensional) eigenspace of μ_0, and by $E_{n-2,0}$ the orthogonal complement of $E_{1,1}$ for g or g_0. Then, both g and g_0 are of signature $(1,1)$ on $E_{1,1}$ and positive definite on $E_{n-2,0}$. Accordingly, A splits as the sum of two operators $A = A_{1,1} \oplus A_{n-2,0}$, where $A_{1,1}$ acts trivially on $E_{n-2,0}$ and is diagonalizable, with negative eigenvalues on $E_{1,1}$, whereas $A_{n-2,0}$ acts trivially on $E_{1,1}$ and is positive definite, as well as g_0- and g-symmetric on $E_{n-2,0}$. This can be interpreted as follows. Denote by $\mathcal{M}_{1,1}$ the space of Lorentzian inner products of $E_{1,1}$, by $\mathcal{M}_{n-2,0}$ the space of positive definite inner products of $E_{n-2,0}$. Then the product $\mathcal{M}_{1,1} \times \mathcal{M}_{n-2,0}$ is naturally embedded as a totally geodesic submanifold of $\mathcal{M}_{n-1,1}$ and both $g = g_{|E_{1,1}} \oplus g_{|E_{n-2,0}}$ and $g_0 = g_{0|E_{1,1}} \oplus g_{0|E_{n-2,0}}$ belong to it. In $\mathcal{M}_{n-2,0}$ any two elements, in particular $g_{|E_{n-2,0}}$ and $g_{0|E_{n-2,0}}$, are joined by a unique geodesic. The situation concerning $\mathcal{M}_{1,1}$ has been explored in detail in the first part of this section. In the present case, $g_{|E_{1,1}}$ and $g_{0|E_{1,1}}$ are related by the automorphism $A_{|E_{1,1}}$ which is diagonalizable with distinct negative eigenvalues, so that $g_{|E_{1,1}}$ and $g_{0|E_{1,1}}$ cannot be linked by a geodesic.

Case 2: $0 \in \Delta$.

According to Lemma 9.4 (ii), there exist at least $m-2$ distinct positive eigenvalues of $A_{|W}$, namely $0 < \mu_1 < \ldots < \mu_{m-2}$. Then, either these eigenvalues are all simple roots of P, or one of them — and only one — is a triple root. The case that two of them are double roots is impossible since, according to Lemma 9.3 (ii), the corresponding eigenvectors defined by (38) would then form an orthogonal pair of nonzero null vectors in the Lorentzian space (E, g).

In the case when all μ_i are simple roots, we easily check by using (39) that the corresponding eigenvectors are all spacelike. Denote by $E_{n-2,0}$ the direct sum of the corresponding eigenspaces and $\tilde{E}$, and by $E_{1,1} \subset W$ the orthogonal complement of $E_{n-2,0}$ for g or g_0. Then, both g and g_0 are positive definite on $E_{n-2,0}$ and of signature $(1,1)$ on $E_{1,1}$. The situation is then quite similar to the previous one, except that all cases considered in Section 9.1 for $\mathcal{M}_{1,1}$ may now happen, depending on whether the missing two roots of P are complex conjugate, both positive (equal or distinct) or both negative (equal or distinct).

It remains to consider the case that one of the μ_i, say $\mu_j := k > 0$, is a triple root of P. Then, according to Lemma 9.3 (iii), the corresponding eigenvector v_{μ_j} is a null vector. Again, it is easily checked that the v_{μ_i}, for $i \neq j$, are all spacelike. Denote by $E_{n-3,0}$ the direct sum of the eigenspaces corresponding to the μ_i, $i \neq j$, and E^0, and by $E_{2,1} \subset W$ the orthogonal complement of $E_{n-3,0}$ for g or g_0. Then, both g and g_0 are positive definite on $E_{n-3,0}$ and of signature $(2,1)$ on $E_{2,1}$. It follows that g and g_0 both belong to a same totally geodesic subspace $\mathcal{M}_{2,1} \times \mathcal{M}_{n-3,0}$. Moreover, the restriction of A to $E_{2,1}$, which relates $g|_{E_{2,1}}$ and $g_{0|E_{2,1}}$, is of the form $k(\mathrm{id} + x)$, where x is nilpotent and regular (this is because μ_j has no other eigenvector than v_{μ_j}). Now, $\mathrm{id} + x$ is the exponential of $\mathrm{id} + x - \frac{x^2}{2}$, which is certainly symmetric for both g_0 and g (since $x = (\mathrm{id} + x) - \mathrm{id}$ is symmetric) and is the only symmetric "logarithm" of $\mathrm{id} + x$. We thus get a unique (null) geodesic between $g_{0|E_{2,1}}$ and $g_{|E_{2,1}}$ in $\mathcal{M}_{2,1}$, hence also between g_0 and g in $\mathcal{M}_{n,1}$.

This completes the proof of Proposition 9.2. $\square$

REFERENCES

[1] C. Bär, *Real Killing spinors and holonomy*, Commun. Math. Phys. **154** (1993), 509–521.
[2] H. Baum, *Spin-Strukturen und Dirac-Operatoren über pseudoriemannschen Mannigfaltigkeiten*, B. G. Teubner Verlagsgesellschaft, Leipzig, 1981.
[3] J.-P. Bourguignon and P. Gauduchon, *Spineurs, opérateurs de Dirac et variations de métriques*, Commun. Math. Phys. **144** (1992), 581–599.
[4] P. Deligne et al (Eds.), *Quantum fields and strings: A course for mathematicians, Vol. 1*, AMS, 1999.
[5] T. Friedrich and E. C. Kim, *Some remarks on the Hijazi inequality and generalizations of the Killing equation for spinors*, J. Geom. Phys. **37** (2001), 1–14.
[6] S. Kobayashi and K. Nomizu, *Foundations of differential geometry, Vol. 2*, Interscience Publishers, John Wiley & Sons, New York, Chichester, Brisbane, Toronto, 1969.
[7] B. Morel, *The energy-momentum tensor as a second fundamental form*, Preprint, 2003, `math.DG/0302205`.
[8] B. O'Neill, *Semi-Riemannian geometry*, Academic Press, New York, London, 1983.
[9] P. Petersen, *Riemannian geometry*, Springer Verlag, New York, 1998.
[10] J. Wolf, *Spaces of constant curvature*, Publish or Perish, Wilmington, Delaware, 1984.

UNIVERSITÄT HAMBURG, FB MATHEMATIK, BUNDESSTR. 55, 20146 HAMBURG, GERMANY

ÉCOLE POLYTECHNIQUE, CENTRE DE MATHÉMATIQUES, 91128 PALAISEAU CEDEX, FRANCE

ÉCOLE POLYTECHNIQUE, CENTRE DE MATHÉMATIQUES, 91128 PALAISEAU CEDEX, FRANCE

E-mail address: `baer@math.uni-hamburg.de`

E-mail address: `pg@math.polytechnique.fr`

E-mail address: `am@math.polytechnique.fr`