THE MINIMAL NUMBER OF CRITICAL POINTS OF MAPS BETWEEN MANIFOLDS OF THE SAME DIMENSION

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ABSTRACT. One computes the minimal number of critical points of a smooth map between two manifolds of the same dimension. Some non-trivial examples for higher codimension spheres are presented.

1. Introduction

Set $\varphi(M, N)$ for the minimal number of critical points of a smooth map $f : M \to N$ between two manifolds. One can consult [3] which surveys various features of this invariant (see also [12]). Generally one derived topological conditions ensuring that $\varphi(M, N)$ is infinite.

In the present paper one computes $\varphi(M, N)$ when M and N have the same dimension. In particular one computes it in the case of surfaces, and one shows that in higher dimensions φ is either 0, 1 or ∞. The problem is closely related to the existence of ramified coverings between two given manifolds (see for instance [7, 9]).

One seeks then for a refined invariant measuring how far the manifold M is from a fibration over N. Further one gives some examples of nontrivial φ in non-zero codimension.

We will consider henceforth that all manifolds are closed and connected unless the opposite is stated.

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2. Surfaces

Necessary and sufficient conditions were given by Patterson in [11] for the existence of a covering of some surface with prescribed degree and ramification orders. Specifically his result states as follows:

Proposition 2.1. *Let X be a Riemann surface of genus $g \geq 1$. Let $p_1, ..., p_k$ be distinct points of X and $m_1, ..., m_k$ strictly positive integers so that*

$$\sum_{i=1}^{k}(m_i - 1) = 0(mod\ 2).$$

Let d be an integer such that $d \geq max_{i=1,...,k} m_i$. Then there exists a Riemann surface Y and a holomorphic covering map $f : Y \to X$ of degree d such that there

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exist k points $q_1, ..., q_k$ in Y so that $f(q_j) = p_j$, f is ramified to order m_j at q_j and is unramified outside the set $\{q_1, ..., q_k\}$.

One observes now that a smooth map $f : Y \to X$ between surfaces has finitely many critical points if and only if it is a ramified covering. Furthermore $\varphi(Y, X)$ is the minimal number of ramification points of a covering $Y \to X$. Estimations could be obtained from the previous result. Denote by Σ_g the genus g surface. Our principal result in this section is the following:

Proposition 2.2.
1. *If $G < g$ then $\varphi(\Sigma_G, \Sigma_g) = \infty$.*
2. *For $G \geq 1$ one has $\varphi(\Sigma_G, S^2) = 3$.*
3. *If $G \geq 2$ then $\varphi(\Sigma_G, \Sigma_1) = 1$.*
4. *If $2 \leq g \leq G \leq 2g - 2$ then $\varphi(\Sigma_G, \Sigma_g) = \infty$.*
5. *If $2 \leq g < 2g - 1 \leq G$ let us write $G - 1 = a(g - 1) + b$, where a is the integer part $\left[\frac{G-1}{g-1}\right]$ and $b \in \{0, 1, ..., g - 2\}$. Then*

$$\varphi(\Sigma_G, \Sigma_g) = \left[\left[\frac{2b}{a-1}\right]\right],$$

where the brackets $[[r]]$ state for the smallest integer larger than r. In particular if $G \geq 2(g-1)^2$ then

$$\varphi(\Sigma_G, \Sigma_g) = \begin{cases} 0 & \text{if} \quad \frac{G-1}{g-1} \in \mathbf{Z}_+ \\ 1 & \text{otherwise} \end{cases}$$

Proof. The first claim is obvious.

Further $\varphi(\Sigma_G, S^2) \leq 3$ because any surface is a branched cover of the 2-sphere in 3 points (from [2]). A deeper result is that the same inequality holds in the holomorphic framework. In fact Belyi's theorem states that any Riemann surface defined over a number field admits a meromorphic function on it with only three critical points (see e.g. [14]). On the other hand if the punctured Σ_G covers a punctured sphere then the map between the respective fundamental groups must be injective. This rules out the case of twice punctured sphere, whose fundamental group is cyclic abelian. Therefore the second claim holds.

Next the nonramified covers of a torus are tori, hence our covering should be ramified. The existence of a degree $d = 2G - 1$ covering with a single ramification point of multiplicity $2G - 1$ yields the claim.

Lemma 2.3. *$\varphi(\Sigma_G, \Sigma_g)$ is the smallest integer k for which one has*

$$2(g-1)\left[\left[\frac{2(G-1)+k}{2(g-1)+k}\right]\right] \leq 2(G-1) - k$$

Here the brackets $[[a]]$ state for the smallest integer larger than a.

Proof. Suppose that Σ_G is a degree d covering of Σ_g ramified in k points with the multiplicities $m_i = d - \lambda_i$, where $0 \leq \lambda_i \leq d - 2$. Set $\lambda = \sum_i \lambda_i$. The Hurwitz formula states that:

$$d(2(g-1) + k) = 2(G-1) + k + \lambda.$$

Conversely, if there are solutions d and λ of the Hurwitz diophantine equation above, such that $d \geq 1$ and $\lambda \leq k(d-2)$ then one can find multiplicities m_i in the given range and conclude from Patterson's theorem.

For given k the smallest positive λ is

$$(2(g-1)+k)\left[\left[\frac{2(G-1)+k}{2(g-1)+k}\right]\right]-(2(G-1)+k),$$

and this proves the claim. □

Assume now that $g < G \leq 2g-2$. The Hurwitz relation and the constraint on λ imply that

$$d(2(g-1)+k) \leq 2(G-1)+k+k(d-2).$$

Then $G \leq 2g-2$ leads to $d < 2$, hence $d = 1$ which is impossible, so $\varphi(\Sigma_G, \Sigma_g) = \infty$ holds.

Set $G' = G-1$, $g' = g-1$ for simplicity. One has to compute the minimal k fulfilling

$$\left[\left[\frac{2ag'+2b+k}{2g'+k}\right]\right] \leq \frac{2ag'+2b-k}{2g'},$$

or equivalently,

$$\left[\left[\frac{2b+(1-a)k}{2g'+k}\right]\right] \leq \frac{2b-k}{2g'}.$$

The smallest k for which the quantity in the brackets is non-positive is $k = \left[\left[\frac{2b}{a-1}\right]\right]$, for which

$$\left[\left[\frac{2b+(1-a)k}{2g'+k}\right]\right] \leq 0 \leq \frac{2b-k}{2g'}.$$

For smaller k one has a strictly positive integer on the left hand side, which is therefore at least 1. But the right hand side is strictly smaller than 1, hence the inequality could not hold. This proves the claim. □

3. Higher dimensions

The situation changes completely in dimension $n \geq 3$. The first who noticed that a smooth map $\mathbf{R}^n \to \mathbf{R}^n$ $(n \geq 3)$ which has only an isolated critical point p is actually a local homeomorphism at p was H.Hopf according to ([6], II, p.535). Our result below is an easy application of this fact. We outline the proof for the sake of completeness.

Proposition 3.1. *If $\varphi(M^n, N^n)$ is finite and $n \geq 3$ then $\varphi(M^n, N^n) \in \{0, 1\}$. Further $\varphi(M^n, N^n) = 1$ if and only if M^n is the connected sum of a finite covering $\widehat{N^n}$ of N^n with an exotic sphere, hence being a manifold homeomorphic but not diffeomorphic to $\widehat{N^n}$.*

Proof. There exists a smooth map $f : M^n \to N^n$ which is a local diffeomorphism on the preimage of the complement of a finite subset of points. Notice that f is a proper map.

Fix a small ball B around a singular value q. The restriction of f to $f^{-1}(B-\{q\})$ is a proper submersion hence a covering of $B-\{q\}$. But all such coverings are trivial for $n \geq 3$, since $\pi_1(B-\{q\}) = 0$. Hence $f^{-1}(B-\{q\})$ is a disjoint union of punctured balls and the map f is then a diffeomorphism on each component of $f^{-1}(B-\{q\})$. Choose one component $B'-\{q'\}$ of $f^{-1}(B-\{q\})$. Using Alexander's trick one sees that the diffeomorphism $f : \partial B' \to \partial B$ is isotopic to identity, among homeomorphisms. However we cannot assure that this is smoothly isotopic to identity. This shows that the restriction of f to $\partial B'$ extends to a homeomorphism $B' \to B$. Thus one can replace f by a proper map which is a local homeomorphism (using a recurrence on the number of critical values) and therefore a a topological covering map. This proves that M^n is homeomorphic to a covering of N^n.

Let us remove the disk B' from M^n and glue it back using another diffeomorphism, say ψ of $\partial B'$. One can choose this diffeomorphism action in order to cancel that of f. Then the restriction of f to $M - B'$ extends over the disk B' to $M - B' \cup_{\psi} B' \to N - B \cup B$ by a local diffeomorphism on B'. Thus the induced map from the modified M^n to N^n has one critical point less.

Such modifications on M amount to taking the connect sum with an exotic sphere. Once we do that for all components and critical values we obtain a covering $\widehat{N^n}$ (since no more critical points left). But this covering is obtained from M^n by taking the connect sum with one exotic sphere, namely the sum of all exotic spheres used in the process. In particular $\varphi(M^n, N^n) \leq 1$. The equality holds if and only if M^n is not diffeomorphic to $\widehat{N^n}$ (hence the latter must admit exotic structures). Notice however that not all exotic structures can be obtained from a given structure by connect sum with an exotic sphere. □

Corollary 3.1. In dimension $n \leq 6$ either φ is 0 or else it is infinite.

Proof. In fact two 3-manifolds which are homeomorphic are diffeomorphic. On the other hand Cerf's result (see [5]) implies that the gluing two 4-disks along the 3-sphere does not produce an exotic 4-sphere. In the remaining dimensions there are no exotic spheres (Kervaire and Milnor). □

Remark 3.2. A careful analysis of open maps between manifolds of the same dimensions was carried out in [6]. In particular one proved that an open map of finite degree has a branch locus of codimension 2 (see [6],II) in the PL or TOP framework.

4. A substitute for φ in dimension 3

One saws that $\varphi(M^n, N^n)$ is not an interesting invariant for $n \geq 3$. One would like to have an invariant of the pair (M^n, N^n) measuring the defect of M^n being an unramified covering of N^n. First one has to know whether there is a branched covering $M^n \to N^n$ and next if the branch locus could be empty.

Notice for instance that an old theorem of Alexander ([2]) states that any n-manifold is a branched covering of the sphere S^n. Moreover the ramification locus can be taken as the $(n-2)$-skeleton of the standard n-simplex. There exists an obvious obstruction to the existence of a ramified covering $M^n \to N^n$, namely the existence of a map of non-zero degree $M^n \to N^n$. In particular a necessary condition is

$$\| M \| \geq \| N \|,$$

where $\| M \|$ denotes the simplicial volume of M (see [8, 10]).

Remark 4.1. Notice that this condition is far from being sufficient. In fact there is no ramified covering $S^n \to X^m$, for $m, n \geq 2$, where X^m denotes an aspherical compact m-manifold. In fact any open map $S^n \to X^m$ should be esential (see e.g.

[6], III), which would contradict the asphericity. In particular this is true for a torus T^m, though as both the sphere and the torus have zero Gromov norm.

A good candidate for replacing φ in dimension 3 is the ratio of simplicial volumes mod $\mathbf{Z}$, namely

$$v(M,N) = \frac{\| M \|}{\| N \|} (mod\ \mathbf{Z}) \in [0,1),$$

which is defined for all M, N with positive Gromov norm.

Notice that for (closed manifolds M) the simplicial volume $\| M \|$ depends only on the fundamental group $\pi_1(M)$ of M. In particular it vanishes for simply connected manifolds, making it less useful in dimensions at least 4.

Remark 4.2. If M^n covers N^n then (see [8]) $v(M,N) = 0$. The converse holds true for surfaces of genus at least 2, from Hurwitz formula.

The norm ratio has been extensively studied for hyperbolic manifolds in dimension 3, where it coincides with the volume ratio, in connection with commensurability problems (see e.g. [15]). In particular the values $v(M^3, N^3)$ accumulate on 1 since the set of volumes of closed hyperbolic 3-manifolds has an accumulation point. The simplicial volume is zero for a Haken 3-manifold iff the manifold is a graph manifold (from a result of T.Soma), and conjecturally the simplicial volume is the sum of (the hyperbolic) volumes of the hyperbolic components of the manifold.

However it seems that this invariant is not appropriate in dimensions higher than 3 (even if one restricts to aspherical manifolds). Here are two arguments in the favour of this claim:

Proposition 4.3.
- *Let us suppose that M^n is a ramified covering of N^n over the complex K^{n-2}. Assume that both the branch locus K^{n-2} and its preimage in M^n can be engulfed in a simply connected codimension one submanifold. Then $v(M^n, N^n) = 0$.*
- *Assume that there is a map $f : M^n \to N^n$ such that the kernel $\ker(f_* : \pi_1(M) \to \pi_1(N))$ is an amenable group. Then $v(M^n, N^n) = 0$.*

Proof. One uses the fact that for any simply connected codimension one submanifold $A^{n-1} \subset M^n$ one has $\| M \| = \| M - A \|$ (see [8], p.10 and 3.5). The second part follows from [8], since under the hypothesis on the kernel of f_* one obtains $\frac{\|M\|}{\|N\|} = deg(f)$. □

Remark 4.4. For $n \geq 4$ A.Sambusetti ([13]) constructed many examples of manifolds M^n and N^n satisfying the second condition, which are therefore far from being a fibration, though as the invariant v vanishes. It seems that there are not such examples in dimension 3. At least for hyperbolic Haken N^3 any M^3 dominating N^3 with amenable kernel must be a covering (from the rigidity result of Soma and Thurston).

Remark 4.5. If there is branched covering $f : M^n \to N^n$ then the branch locus is of codimension 2. This explains why $\varphi(M,N)$ is trivial in high dimensions. A possible extension of φ would have to take into account the (minimal) complexity of the branch locus, (e.g. its minimal Betti number) over all branched coverings. Given N this must be bounded from above (as it is for N a sphere). However it seems that such invariants are far from being explicitly computable.

5. Higher codimension

The higher codimension presents new features with respect to codimension 0. Remark first that the existence of the Hopf fibrations $S^3 \to S^2, S^7 \to S^4, S^{15} \to S^8$, shows that

$$\varphi(S^3, S^2) = \varphi(S^7, S^4) = \varphi(S^{15}, S^8) = 0.$$

Conversely one can state:

Proposition 5.1. *The values of $m > n > 1$ for which $\varphi(S^m, S^n) = 0$ are exactly those arising in the Hopf fibrations i.e. $n \in \{2, 4, 8\}$ and $m = 2n - 1$.*

Proof. The fiber F of the fibration $f : S^m \to S^n$ is a homology $(n-1)$-sphere (hence $m = 2n-1$) and the transgression map $H^{n-1}(F) \to H^n(S^n)$ is an isomorphism, by using Gysin's sequence. In particular, if u denotes a $(n-1)$-form on S^{2n-1} which represents the Poincaré dual of F and v is the volume form on S^n then one obtains $f^*(v) = du$.
One computes the Hopf invariant $H(f)$ as:

$$H(f) = \int_{S^{2n-1}} u \wedge du = \int_{x \in S^n} \left(\int_{f^{-1}(x)} u \right) v.$$

However it is clear that:

$$d(\int_{f^{-1}(x)} u) = 0,$$

hence $\int_{f^{-1}(x)} u$ is a constant function on S^n, hence it is 1. Therefore f has Hopf invariant one and Adams's theorem (see [1]) implies the claim. □

However we can suspend this maps and get examples of pairs with non-trivial φ.

Proposition 5.2. *One has*

$$\varphi(S^4, S^3) = \varphi(S^8, S^5) = \varphi(S^{16}, S^9) = 2.$$

Conversely, if $\varphi(S^m, S^n) = 2$ then the pair (m, n) is one of those from above.

Proof. Fix a Hopf map $f : S^3 \to S^2$, and extend it to $B^4 \to B^3$ by taking the cone and smoothing it at 0. Then glue together two copies of B^4 along the boundary. One gets a smooth map having two critical points. Since S^4 cannot fiber over S^3 it follows that $\varphi(S^4, S^3) \geq 1$. On the other hand if there is a map with precisely one critical point then $int(B^4)$ would fiber over $int(B^3)$ with fiber S^1, which is impossible by homotopical reasons. The other cases are similar. Conversely, if $S^{m-1} \times (0, 1)$ fibers (with compact fibers) over $S^{n-1} \times (0, 1)$, then S^{m-1} fibers over S^{n-1}, hence the claim follows from the previous proposition. □

This might be used to construct other examples with finite φ in the respective dimensions. For instance one finds that $\varphi(\Sigma^8, S^5) = \varphi(\Sigma^{16}, \Sigma^9) = 2$, where Σ^n denotes an exotic n-sphere.

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