COMPLEX POWERS AND NON-COMPACT MANIFOLDS

BERND AMMANN, ROBERT LAUTER, VICTOR NISTOR, AND ANDRÁS VASY

Abstract. We study the complex powers A^z of an elliptic, strictly positive pseudodifferential operator A using an axiomatic method that combines the approaches of Guillemin and Seeley. In particular, we introduce a class of algebras, "extended Weyl algebras," whose definition was inspired by Guillemin's paper [12]. An extended Weyl algebra can be thought of as an algebra of "abstract pseudodifferential operators." Many algebras of pseudodifferential operators are extended Weyl algebras. Several results typical for algebras of pseudodifferential operators (asymptotic completeness, construction of Sobolev spaces, boundedness between apropriate Sobolev spaces, ...) generalize to extended Weyl algebras. Most important, our results may be used to obtain precise estimates at infinity for A^z, when $A > 0$ is elliptic and defined on a non-compact manifold, provided that a suitable ideal of regularizing operators is specified (a submultiplicative Ψ^*–algebra). We shall use these results in a forthcoming paper to study pseudodifferential operators and Sobolev spaces on manifolds with a Lie structure at infinity (a certain class of non-compact manifolds).

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Introduction

Results about the complex powers A^z of strictly positive, elliptic pseudodifferential operators have proved useful to study the asymptotic of eigenvalues of self-adjoint pseudodifferential operators [12], Sobolev spaces, and even certain non-linear differential equations (see [9, 36, 10] and the references therein). The study of complex powers has been inaugurated in Seeley's celebrated paper [38], where he

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proved that if $A > 0$ is an elliptic differential operator of order $m > 0$ on a compact manifold, then A^z is a pseudodifferential operator of order $m\mathrm{Re}\,(z)$. For previous results on complex powers, see [7, 22, 23, 12, 37, 34, 40, 38].

In [12], Guillemin has developed another approach to the construction of complex powers. His approach is axiomatic, in the sense that it works for operators in a "Weyl algebra." Weyl algebras were introduced in the aforementioned paper and are a generalization of algebras of pseudodifferential operators on *compact* manifolds. Thus, in particular, on compact manifolds, an operator of negative order in a Weyl algebra is compact.

In this paper, we generalize Guillemin's approach to include non-compact manifolds in which a certain algebra of regularizing operators is specified. We thus drop the condition that an operator of negative order be compact. This forces us to replace the axioms of a Weyl algebra, with the axioms of an "extended Weyl algebra." In particular, we have to carefully check certain analytic facts that were obvious in the case of a Weyl algebra.

An extended Weyl algebra should be thought of as an abstract algebra of pseudodifferential operators. Many results on pseudodifferential operators on a non-compact manifold generalize to extended Weyl algebras. These algebras provide a natural framework to discuss complex powers, Sobolev spaces, continuity, self-adjointness, resolvents, holomorphic families of operators, ellipticity, and parametrices for several classes of algebras of pseudodifferential operators on non-compact manifolds. To be able to construct complex powers, we also need to assume that our Weyl algebra satisfies two additional properties, namely, that the kernel of the principal symbol map consists of operators of lower order (a familiar property of pseudodifferential operators) and that the given ideal of regularizing operators is a submultiplicative Fréchet algebra that contains the inverse of any L^2-invertible element (*i.e.,* that it is a Ψ^*-algebra in the sense of Gramsch [13, 19, 35]).

An important role in our proofs is played by a theorem of Banach from 1948, [4], which states that inversion is continuous on a Fréchet algebra whose set of invertible elements is a G_δ subset (:= a countable intersection of open subsets). Any Ψ^* algebra has an open set of invertible elements, so inversion is continuous on a Ψ^*-algebra.

One of the main applications of our results will be to study the complex powers of operators on manifolds with a Lie structure at infinity [2]. The manifolds with a Lie structure at infinity were introduced, informally, in [27, 29], and were studied more systematically in [1].

The paper is organized as follows. In the first section we introduce the axioms of an extended Weyl algebra $\mathcal{W}$, following [12], and we include two examples. Anticipating, let us just say now that an extended Weyl algebra is an algebra $\mathcal{W} = \cup \mathcal{W}^\mu$, $\mu \in \mathbb{C}$, of (possibly) unbounded operators with a common domain contained in a given Hilbert space $\mathcal{H}$ and such that $\mathcal{W}^\mu \mathcal{W}^\nu = \mathcal{W}^{\mu+\nu}$ and satisfying several axioms. An example is the algebra of (sums of) classical pseudodifferential operators of complex order on a compact manifold, with $\mathcal{W}^\mu$ the space of operators of order (at most) μ. Certain algebras of pseudodifferential operators on non-compact manifolds also provide us with examples of extended Weyl algebras.

In Section 2 we discuss some of the basic properties of extended Weyl algebras $\mathcal{W}$, including boundedness in the given Hilbert space. We prove in particular that any elliptic, symmetric operator $T \in \mathcal{W}^m$ (that is of order m), $m > 0$, is automatically essentially self-adjoint. Intuitively, this means that extended Weyl algebras model geometric operators on complete manifolds. The setting of operators on cones (see [11, 23, 28], for example), is not directly covered in our approach. The domains of operators of positive order, which turn out to be some generalizations of the usual Sobolev spaces, are studied in Section 3. In Section 4, we discuss holomorphic families of operators, including asymptotic completeness. In Section 5 we proved that an elliptic operator in $\mathcal{W}^\mu$, $\mathrm{Re}\,(\mu) \geq 0$, that is invertible as an unbounded operator on the ambient Hilbert space, will have an inverse in our extended Weyl algebra $\mathcal{W}$, provided that the ideal of regularizing operators is a spectrally invariant Fréchet algebra. We also prove the continuity of the adjoint and the multiplication in this section. In Sections 2 – 4 we do not assume that multiplication $S^m_{1,0} \times \mathcal{W}^{-\infty} \ni (a,T) \to q(a)T$ is continuous (this is Axiom (vii)). Holomorphic families of variable order, called "special holomorphic families," are studied in Section 6. Special holomorphic families are then used in Section 7 to construct complex powers of elliptic operators, by combining the methods of [12, 38] and [40]. In the process, we are led to study the behavior of the resolvent $(T+\imath t)^{-1}$, where $T \in \mathcal{W}^m$, $m > 0$, is elliptic and symmetric, and we prove, in particular, that it is bounded in $\mathcal{W}^{-m}$, for $|t| \geq 1$. This allows us to recover the results of [12, 38].

It is worthwhile mentioning that the setting of [12, 38] is such that their Weyl algebras are complete in the Guillemin topology (*i.e.,* the topology on the set of regularizing operators is given by the operator norms $H^{(s)} \to H^{(s')}$, for any s, s'). This simplifies greatly the proofs. However, certain applications, for example to manifolds with cylindrical ends [30] require us to go beyond that. It would be interesting and important to see to what extent the functional calculus with symbolic functions of [14] extends to the setting of extended Weyl algebras.

Throughout this paper, the notation ":=" means "the left hand side is defined to be equal to the right hand side."

1. Extended Weyl algebras

We introduce now a class of algebras generalizing the class of Weyl algebras introduced in [12]. We follow the spirit of [12] when defining these algebras. In fact, the main difference is that we do not impose compactness conditions on operators of negative order, which in turn forces us to make very explicit analytic assumptions. Then, in order to construct complex powers, we need the ideal of operators of order $-\infty$ to satisfy some strong analytic conditions that will be made precise below (Conditions (σ) and (ψ)).

1.1. **Notation and preliminaries.** We need to recall first the definition of some classes of symbols. Let $V \to M$ be a smooth orthogonal bundle over a manifold M, possibly with corners, endowed with an orthogonal connection ∇. Let

$$\langle v \rangle = (1 + \|v\|^2)^{-1/2}\,, \quad v \in V. \tag{1}$$

We shall denote by $S^m_{1,0}(V)$ the space of type $(1,0)$–symbols of order m on V [16, 42]. Recall that the space $S^m_{1,0}(V)$ is defined as follows. If $V = M \times \mathbb{R}^N$, then $S^m_{1,0}(V)$ consists of the smooth functions $a : V \to \mathbb{C}$ such that

$$\|\partial_x^\alpha \partial_\xi^\beta (\varphi_i(x) a(x,\xi))\| \leq C(\alpha,\beta)\, \langle \xi \rangle^{m-|\beta|}, \tag{2}$$

for any $x \in M$ and $\xi \in \mathbb{R}^N$. The topology is given by the best constants $C(\alpha,\beta)$ in Equation (2).

To define the space $S_{1,0}(V)$ and its topology in general, we first choose a locally finite covering (U_i) of M by open subsets on which V is trivial. Then choose trivializations ψ_i of V on U_i and a smooth partition of unity φ_i subordinate to the cover U_i. Then a smooth complex-valued function a on the total space of the bundle V is an element of $S^m_{1,0}(V)$ if $\varphi_i(x)a(\xi,x)$ is a type $(1,0)$–symbol of order m for all i. This definition is independent of the choice of the covering, trivialization, and partition of unity used. If M is compact, the topology is obtained by considering a finite partition of unity and the best constants in the definitions of symbols. (This topology is thus a quotient topology.)

We shall also need the case of a manifold M of bounded geometry and $V = TM$. Then we specify the families U_i and ψ_i to be given by normal coordinates around each point and requiring

$$\|\partial_x^\alpha \partial_\xi^\beta(\varphi_i(x)a(x,\xi))\| \leq C(\alpha,\beta)\,\langle\xi\rangle^{m-|\beta|}, \tag{3}$$

for any multi-indices $\alpha \in \mathbb{N}^l$ and $\beta \in \mathbb{N}^k$, and any i. Note that the definition of $S^m_{1,0}(V)$ does not depend on the choice of U_i, ψ_i, φ_i.

Since most of our symbols are defined on V, we shall usually omit V from the notation, thus write $S^m_{1,0} := S^m_{1,0}(V)$.

Also, recall that a symbol $a \in S^m_{1,0}$ is called *homogeneous of order (at most)* $\mu = m + it$, $t \in \mathbb{R}$, if there exists $R > 0$ such that

$$a(rv) = r^\mu a(v)\,, \quad \text{for all } \ r \in (0,\infty)\,, \ \text{and } v \in V\,, \ \text{ such that } \ |t|, \|v\| \geq R.$$

Moreover, a symbol $a \in S^m_{1,0}$ is called a *classical symbol of order* $\mu = m + it \in \mathbb{C}$, $t \in \mathbb{R}$, if there exist symbols $a_j \in S^{m-j}_{1,0}$, homogeneous of order $\mu - j$ such that

$$a(v) - \sum_{j=0}^{k-1} a_j(v) \in S^{m-k}_{1,0}.$$

(If this is the case, we write $a \sim \sum_{j=0}^\infty a_j$.) The subspace of classical symbols of order (at most) μ will be denoted $S^\mu_{cl}(V) \subset S^m_{1,0}(V)$, as usual. We shall often write $S^\mu_{cl} := S^\mu_{cl}(V)$.

The following lemma is probably well known (see [16, Lemma18.1.10] for the case $m = 0$).

Lemma 1.1. *Let $m \geq 0$, $k \in \mathbb{R}$, and $S^m_{1,0} := S^m_{1,0}(V)$, as above. Also, denote*

$$Ell^m_\mu := S^m_{1,0} \cap \{|a(v)| \geq \mu^{-1}\langle v\rangle^m \ \text{ if } |v| \geq \mu\}.$$

Then

$$S^k_{1,0}(\mathbb{C}) \times Ell^m_\mu \to S^{mk}_{1,0}\,, \qquad (f,a) \mapsto f \circ a \tag{4}$$

is a well-defined and continuous for any $\mu > 0$. A similar statement holds for $S^k_{1,0}(\mathbb{C})$ replaced with $S^k(\mathbb{R})$ or with $S^k([\epsilon,\infty))$, if one restricts to real valued symbols in $S^m_{1,0}$ or, respectively, to symbols that are $\geq \epsilon$.

Proof. We can assume that V is trivial. Thus it is enough to verify the condition of Equation (2). We shall do this by induction on $|\alpha| + |\beta|$. Let $|\alpha| = |\beta| = 0$. By the definition of $S^k_{1,0}(\mathbb{C})$ we have $|f(a(v))| \leq C_1\langle a(v)\rangle^k$. If $k \geq 0$, then $\langle a(v)\rangle^k \leq C_2\langle v\rangle^{km}$ gives the desired inequality

$$|f(a(v))| \leq C_3\langle v\rangle^{km}. \tag{5}$$

Here the C_i's are constants. For $k < 0$ we use the "ellipticity condition" $|a(v)| \geq \mu^{-1}\langle v\rangle^m$ to conclude (5).

The inductive step follows from the identity

$$X(f \circ a) = (f' \circ a)X(a),$$

for any vector field X on V.

The proof of the last two statements of the lemma follow by restriction. $\square$

If X and Y are two locally convex spaces, we shall denote by $\mathcal{L}(X, Y)$ the space of continuous linear maps $X \to Y$. We shall write $\mathcal{L}(X, X) = \mathcal{L}(X)$. Also, we shall denote by T^* the adjoint of a (possibly unbounded) operator T between Banach spaces.

1.2. **The axioms of an extended Weyl algebra.** We now introduce the conditions defining an "extended Weyl algebra."

Recall that a limit-of-Frechet space or, for short, an *LF–space* $X = \cup_{k\geq 1} X_k$ is a inductive limit $X = \lim\limits_{\to} X_k$, $k = 1, 2, \ldots$, where each X_k is a Fréchet space, $X_k \subset X_{k+1}$ is a closed subspace whose topology coincides with is the subspace topology. We endow X with the inductive limit topology. This implies, in particular, that X_k also carries the subspace topology of X, or equivalently by defintion, that X is a *strict* inductive limit. (See, for example, [33, II.6.1–6.6] for a thorough discussion.) By an *LF-algebra* we shall mean an LF–space $A = \cup_{n\geq 1} A_n$, which is endowed with an algebra structure such that $A_k A_l \subset A_{k+l}$ and the multiplication $A_k \times A_l \to A_{k+l}$ is continuous, or equivalently $A \times A \to A$ is continuous.

In this section we consider a tuple $(\mathcal{W}^{-\infty}, \mathcal{H}, V, \mathcal{P}, q)$ satisfying the following axioms:

(i) $\mathcal{H}$ is a Hilbert space and $\mathcal{W}^{-\infty} = \cup_{k\geq 1} \mathcal{W}_k^{-\infty}$ is an LF–algebra continuously embedded in $\mathcal{L}(\mathcal{H})$ such that $^* : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$, $T \mapsto T^*$, maps $\mathcal{W}^{-\infty}$ continuously onto itself.

(ii) There exists an injective operator $R = R^* \in \mathcal{W}_1^{-\infty}$.

(iii) $V \to M$ is a vector bundle over a compact manifold M, possibly with corners, or $V = TM$ and M is with bounded geometry. Furthermore, q is a map $q : S^\infty_{1,0} \to \mathrm{Hom}(\mathcal{W}^{-\infty}\mathcal{H}, \mathcal{H})$, $q(1) = I$, satisfying:
  (a) $q(a)$ is symmetric for a real; and
  (b) The restriction $q|_{S^{-\infty}} : S^{-\infty} \to \mathcal{W}_1^{-\infty}$ is well defined and continuous.

(iv) $q(S^m_{1,0})\mathcal{W}_k^{-\infty} \subset \mathcal{W}_{k+1}^{-\infty}$.

(v) The map $\mathcal{P} : S^\infty_{1,0} \times S^\infty_{1,0} \to S^\infty_{1,0}$ is bilinear and satisfies:
  (a) $q(a)q(b) - q(\mathcal{P}(a, b)) : S^\infty_{1,0} \times S^\infty_{1,0} \to \mathcal{W}_1^{-\infty}$ is well defined and continuous;
  (b) The induced maps $\mathcal{P} : S^m_{1,0} \times S^n_{1,0} \to S^{m+n}_{1,0}$, $\mathcal{P} : S^\mu_{cl} \times S^\nu_{cl} \to S^{\mu+\nu}_{cl}$, and $\mathcal{P}_0 : S^m_{1,0} \times S^n_{1,0} \to S^{m+n-1}_{1,0}$, $\mathcal{P}_0(a, b) = \mathcal{P}(a, b) - ab$, are well defined and continuous.

(vi) There exists $d > 0$ and a continuous seminorm p on $S^{-d-1}_{1,0}$ such that $q(a)$ extends to a bounded operator on $\mathcal{H}$ with $\|q(a)\| \leq p(a)$, for all $a \in S^{-d-1}_{1,0}$.

(vii) The induced map

$$S^m_{1,0} \times \mathcal{W}_k^{-\infty} \to \mathcal{W}_{k+1}^{-\infty}, \qquad (a, T) \mapsto q(a)T,$$

(which is well defined by axiom (iv)) is continuous.

Note that Axiom (ii) implies that $\mathcal{W}^{-\infty}\mathcal{H}$ is dense in $\mathcal{H}$ because $R\mathcal{H}$ is dense for any $R = R^*$ satisfying Axiom (ii). Hence, $q(a)$ is an operator densely defined in $\mathcal{H}$ and the following axioms make sense.

Comments. Let us observe that the Axiom (iv) implies that if $T \in \mathcal{W}_k^{-\infty}$ and $a \in S_{cl}^{\infty} := S_{cl}^{\infty}(V)$, then $Tq(a) = (q(\overline{a})T^*)^* \in \mathcal{W}_{k+1}^{-\infty}$. In particular, $\mathcal{W}^{-\infty}$ is closed under both left and right multiplication by operators of the form $q(a)$.

Two other conditions, Conditions (σ) and (ψ) will be considered below and will be used to construct complex powers, but they are not part of the definition of an extended Weyl algebra. Before stating these conditions, let us notice that the axioms above give rise to an algebra.

Proposition 1.2. *Let $\mathcal{H}$, $\mathcal{W}^{-\infty} = \cup_{k\geq 1}\mathcal{W}_k^{-\infty}$, and q satisfy the Axioms* (i)–(vii) *above. Define $\mathcal{W}$ to be the algebraic sum of the spaces $\mathcal{W}_k^{\mu} := q(S_{cl}^{\mu}) + \mathcal{W}_k^{-\infty}$, $\mu \in \mathbb{C}$, as above. Then $\mathcal{W}$ is an algebra of (possibly) unbounded operators on $\mathcal{H}$ with dense common domain $\mathcal{W}^{-\infty}\mathcal{H}$ and satisfying $\mathcal{W}_k^{\mu}\mathcal{W}_l^{\nu} \subset \mathcal{W}_{k+l}^{\mu+\nu}$ Similarly, $\mathcal{W}_{1,0}$, the union of the increasing sequence of subspaces $\mathcal{W}_{1,0}^m := q(S_{1,0}^m) + \mathcal{W}^{-\infty}$, $m \in \mathbb{Z}$, is also an algebra and $\mathcal{W} \subset \mathcal{W}_{1,0}$.*

We shall endow the spaces $\mathcal{W}_k^{\mu}$ and $\mathcal{W}_{1,0}^m$ introduced in the statement above with the quotient topology with respect $q(S_{cl}^{\mu}) \oplus \mathcal{W}_k^{-\infty} \to \mathcal{W}_k^{\mu}$ and $q(S_{1,0}^m) \oplus \mathcal{W}^{-\infty} \to \mathcal{W}_{1,0}^m$. Let $\mathcal{W}^j := \cup_{k\geq 1}\mathcal{W}_k^j$. Similarly, we endow $\mathcal{W}_{1,0,k}^m := q(S_{1,0}^m) + \mathcal{W}_k^{-\infty}$ with the quotient topology. We clearly have $\mathcal{W}_{1,0}^m = \cup_k \mathcal{W}_{1,0,k}^m$.

Proof. First let us notice that Axiom (v) gives that

$$q(S_{cl}^{\mu})q(S_{cl}^{\nu}) \subset \mathcal{W}_1^{\mu+\nu}. \tag{6}$$

Hence $\mathcal{W}_k^{\mu}\mathcal{W}_l^{\nu} \subset \mathcal{W}_{k+l}^{\mu+\nu}$, $\mu, \nu \in \mathbb{C} \cup \{-\infty\}$, by Axiom (iv) and the property $\mathcal{W}_k^{-\infty}\mathcal{W}_l^{-\infty} \subset \mathcal{W}_{k+l}^{-\infty}$ of an LF–algebra. □

Definition 1.3. Assume the Axioms (i) – (vii). Then the algebra $\mathcal{W}$ of Proposition 1.2 above will be called *an extended Weyl algebra.* The map $q : S_{1,0}^{\infty} \to \mathrm{Hom}(\mathcal{W}^{-\infty}\mathcal{H}, \mathcal{H})$ will be called the *quantization map.*

Our definition differs from the definition in [12] in form but not in substance.

1.3. **Conditions (σ) and (ψ).** For the construction of complex powers, we shall need our extended Weyl algebra to satisfy two other conditions, Conditions (σ) and (ψ), which we now introduce.

(σ) *The map*

$$S_{cl}^m \oplus \mathcal{W}^{-\infty} \ni (a, T) \mapsto q(a) + T \in \mathcal{W}^m := q(S_{cl}^m) + \mathcal{W}^{-\infty} \subset \mathcal{W} \tag{7}$$

has kernel $\{(a, -q(a)), a \in S_{cl}^{-\infty}\} \subset S_{cl}^{-\infty} \oplus \mathcal{W}^{-\infty}$.

If this is the case, the algebra $(\cup_{m\in\mathbb{Z}}\mathcal{W}^m)/\mathcal{W}^{-\infty}$ will be a topologically filtered algebra, in the sense of [5, 6].

(ψ) $\mathcal{W}^{-\infty} = \mathcal{W}_1^{-\infty}$ (so $\mathcal{W}^{-\infty}$ is actually a Fréchet algebra) *with topology generated by a submultiplicative family of seminorms $\|\cdot\|_n$, $n \geq 0$, such that for any* *operator of the form $I + R$, $R \in \mathcal{W}^{-\infty}$, that is invertible as a bounded operator on $\mathcal{H}$, there is an $R_1 \in \mathcal{W}^{-\infty}$ with $(I+R)^{-1} = I + R_1$.*

Recall that a *Ψ^*-algebra with unit* is a Fréchet algebra with unit $\mathfrak{B}$ that is continuously embedded in $\mathcal{L}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$, such that $\mathfrak{B}^{-1} = \mathcal{L}(\mathcal{H})^{-1} \cap \mathfrak{B}$. (Here $\mathfrak{B}^{-1}$ denotes the group of invertible elements of $\mathfrak{B}$, and the last condition means that $\mathfrak{B}$ is *spectrally invariant* in $\mathcal{L}(\mathcal{H})$.) Ψ^*-algebras were introduced by Gramsch, see [13, 19, 35].

Thus, in Gramsch's terminology, condition (ψ) is equivalent to saying that $\mathcal{W}^{-\infty} + \mathbb{C}I$ is a Ψ^*-algebra whose topology is generated by a multiplicative family of seminorms.

It is useful now to recall an old result of Banach [4], which says that the inversion in a Fréchet algebra is continuous if, and only if, its group of invertible elements is a G_δ-set. In particular, inversion is continuous on a Ψ^*-algebra.

Although (ψ) does not hold in some pseudo-differential algebras that one constructs directly, such as the small b-calculus $\Psi_{\mathrm{b}}^*(M)$, it does hold for their Guillemin completion (see Proposition 5.2 and the example below).

1.4. **Examples.** We now briefly discuss some examples. More details will be provided in [2].

Example 1. *The scattering calculus on $\mathbb{R}^n$:* [8, 29, 31, 39] Let B be the closed unit ball in $\mathbb{R}^n$. The map $\Phi : \mathbb{R}^d \ni x \mapsto x\langle x\rangle^{-1} \in B$ defines a diffeomorphism onto the interior of B (the "radial compactification map").

We shall take $M = B$, $V = B \times \mathbb{R}^d$, $\mathcal{H} = L^2(\mathbb{R}^d)$, and q to be the Weyl quantization,

$$\big[q(a)u\big](x) = (2\pi\imath)^{-d} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} e^{\imath\langle x-y,\xi\rangle} a(\Phi(\frac{x+y}{2}), \xi) u(y)\, dy \right) d\xi.$$

and

$$\mathcal{W}^{-\infty} = \mathcal{W}_1^{-\infty} := q(S^{-\infty}),$$

with the induced topology. Let $\Delta = d^*d$ be the Laplace operator, then we can take $R = e^{-t\Delta}$, $t > 0$. The results of the papers mentioned above (see also [14, 16, 35, 42]) show that this is indeed an extended Weyl algebra.

Pseudodifferential operators on a compact, manifold without boundary form an extended Weyl algebra as well.

Example 2. *Compact manifolds:* Let M be a compact manifold. We take $V = T^*M$, $\mathcal{H} = L^2(M)$, and let $\mathcal{W} = \mathcal{W}_1^{-\infty}$ be the algebra of smoothing operators, $\mathcal{W} = \Psi^{-\infty}(M) \simeq \mathcal{C}^\infty(M \times M)$ acting on $L^2(M)$ by convolution. To define a suitable quantization map q, let us cover M with finitely many coordinate neighborhoods $U_\alpha \simeq \mathbb{R}^d$. Let q_α be the quantization map corresponding to this identification of U_α with $\mathbb{R}^d$ (as in the example above) and let $\sum_\alpha \varphi_\alpha^2$ be a smooth partition of unity subordinated to U_α. Then we define

$$q(a) = \sum_\alpha \varphi_\alpha q_\alpha(a) \varphi_\alpha\,. \tag{8}$$

Note that, this quantization map depends on the choice of the trivializations and of the cut-off functions.

We can again take $R = e^{-t\Delta}$, $t > 0$. It is a classical fact that we obtain in this way an extended Weyl algebra (it also follows, by localization, from the first example).

Example 3. Let M be a smooth compact manifold with boundary and let $x \in \mathcal{C}^\infty(M)$ be a boundary defining function (so $x \geq 0$, ∂M is its zero set, and $dx \neq 0$ at ∂M). Let $V = {}^{\mathrm{b}}T^*M$, the dual bundle of ${}^{\mathrm{b}}TM$, whose smooth sections are the smooth vector fields on M that are tangent to ∂M. The small calculus $\mathcal{W}^m = \Psi_{\mathrm{b}}^m(M)$ is defined in [28] as the algebra of operators whose Schwartz kernel is conormal (of order m) to the lifted diagonal on the blown up space $M_{\mathrm{b}}^2 = [M \times M; \partial M \times \partial M]$ and vanishes to infinite order at the left and right faces, i.e. at the lifts of $\partial M \times M$ and $M \times \partial M$ to M_{b}^2. The elements of $\mathcal{W}^{-\infty}$ are those operators whose Schwartz kernel is in addition smooth. (It is easy to write down a quantization map q as in the previous example.) Moreover, let $\mathcal{H} = L_{\mathrm{b}}^2(M) = x^{1/2}L^2(M)$, be the natural L^2-space with respect to b-densities, defined using V. Then M does not satisfy (ψ): typically the inverses of operators in $I + \mathcal{W}^{-\infty}$, even if they exist in $\mathcal{L}(\mathcal{H})$, have Schwartz kernels that vanish only to finite order at the left and right faces. However, it does satisfy all other axioms, including (σ). Then Proposition 5.2 will show that the Guillemin completion $\overline{\mathcal{W}}^{-\infty}$, i.e. the completion of $\mathcal{W}^{-\infty}$ in the topology of the operator norms $H_{\mathrm{b}}^s(M) \to H_{\mathrm{b}}^{s'}(M)$, satisfies all axioms. This illustrates the power, and limitations, of our results: for a b-metric g, we show that $(\Delta_g + 1)^z$ is in $\overline{\mathcal{W}}^{2z}$. However, it certainly does not lie in $\mathcal{W}^{2z}$, although it does lie in a 'large b-calculus' [22]. Note that in a similar closures of the b-calculus have been studied in detail by Mantlik [24].

Manifolds with bounded geometry can also be included in our framework.

Example 4. Let M be a manifold with bounded geometry and injectivity radius r. By definition of "bounded geometry" $r > 0$. Take $\mathcal{W}^\mu = B\Psi^\mu(M)$ [17, 41], and $\mathcal{W}_k^\mu$ be the operators with support in the set $M_{kr/2}^2$ of pairs of points $(x, y) \in M^2$ at distance $\leq kr/2$. The topology on $\mathcal{W}_k^{-\infty}$ is as defined in [17, 41], which in fact coincides with the restriction of the Guillemin topology (see Proposition 5.2 below) on this subspace.

Our main interest is in manifolds with a Lie structure at infinity, which generalize the first two examples above and refine the algebras $B\Psi^\infty$ of the last example.

Example 5. Let (M, A) ba a manifold with a Lie structure at infinity and $\mathcal{W}^\mu = \Psi^\mu(M, A)$ be the algebras of pseudodifferential operators canonically associated to it, see [1, 2]. Let r be the injectivity radius of M. We let as above $\mathcal{W}_k^\mu$ be the operators with support in the set $M_{kr/2}^2$ introduced in the previous example. This example will be discussed in detail in [2].

Yet another example is obtained by considering the so called "Φ–calculus" of Mazzeo and Melrose, see [26].

2. First consequences of the definitions

We now establish the first consequences of the axioms. One of our goals is to construct Sobolev spaces, in the next section. Only Axioms (i)–(vi) are needed to define Sobolev spaces (Axiom (vii) will not be needed, so in this section and next two sections we do not assume it).

As above, let $\mathcal{W} = \sum \mathcal{W}_k^\mu$, $\mathcal{W}_k^\mu := q(S_{cl}^\mu) + \mathcal{W}_k^{-\infty}$, $\mu \in \mathbb{C}$, and $\mathcal{W}_{1,0} = \sum \mathcal{W}_{1,0}^m$, $\mathcal{W}_{1,0}^m := q(S_{1,0}^m) + \mathcal{W}^{-\infty}$, $m \in \mathbb{Z}$.

As in [12], an element $A \in M_N(\mathcal{W}^\mu)$ is called *elliptic*, if there exists $B \in \mathcal{W}^{-\mu}$ such that $AB - I \in M_N(\mathcal{W}^{-1})$. Note that our assumptions (especially those for

$\mathcal{P}_0$) give

$$AB - I \in M_N(\mathcal{W}^{-1}) \Leftrightarrow BA - I \in M_N(\mathcal{W}^{-1}).$$

Similarly, a symbol $a \in M_N(S^m_{1,0})$ is called *elliptic* if there exists $b \in M_N(S^{-m}_{1,0})$ such that $ab - 1 \in S^{-1}_{1,0}$, as usual.

Lemma 2.1. *Let $T \in \mathcal{W}^m_{1,0}$ and $T^\sharp := T^*|_{\mathcal{W}^{-\infty}\mathcal{H}}$. Then $T^\sharp \in \mathcal{W}^{\overline{m}}_{1,0}$ and $T \mapsto T^\sharp$ defines a conjugate-linear involution on $\mathcal{W}_{1,0}$ satisfying $(\mathcal{W}^\mu_k)^\sharp \subset \mathcal{W}^{\overline{\mu}}_k$. Moreover, $q(\overline{a}) = q(a)^\sharp$ and is hence contained in $q(a)^*$, the adjoint of $q(a)$, for any $a \in S^m_{1,0} = S^m_{1,0}(V)$.*

Proof. Let $a = b + \imath c \in S^m_{1,0} = S^m_{1,0}(V)$ with b and c real valued. Then

$$(q(a)\xi, \eta) = (q(b)\xi, \eta) + \imath(q(c)\xi, \eta) = (\xi, q(b)\eta) + \imath(\xi, q(c)\eta) = (\xi, q(\overline{a})\eta), \tag{9}$$

for any $\xi, \eta \in \mathcal{W}^{-\infty}\mathcal{H}$, since $q(b)$ and $q(c)$ are symmetric by the Axiom (iii). Let $P \in \mathcal{W}_{1,0}$, then $P = q(a) + T$, with $a \in S^m_{1,0}$ and $P \in \mathcal{W}^{-\infty}$. Then Equation (9) implies that $P^*\xi = (q(\overline{a}) + T^*)\xi$, for any $\xi \in \mathcal{W}^{-\infty}\mathcal{H}$. Thus $P^\sharp \in \mathcal{W}_{1,0}$ and $(\mathcal{W}^\mu)^\sharp = \mathcal{W}^{\overline{\mu}}$. Moreover, $(P^\sharp)^\sharp = q(a) + T = P$ and hence $^\sharp$ is an involution of $\mathcal{W}_{1,0}$ and $\mathcal{W}$. Also,

$$((P_1P_2)^\sharp\xi, \eta) = ((P_1P_2)^*\xi, \eta) = (\xi, P_1P_2\eta) = (P_2^\sharp P_1^\sharp \xi, \eta),$$

for any $\xi, \eta \in \mathcal{W}^{-\infty}\mathcal{H}$, and hence $^\sharp$ is anti-linear. It is easy to check that $\lambda T^\sharp = \overline{\lambda} T^\sharp$, and hence $^\sharp$ is conjugate-linear as well. □

This lemma then gives the following.

Proposition 2.2. *(i) There exists a continuous seminorm p' on $S^0_{1,0} := S^0_{1,0}(V)$ such that $\|q(a)\| \le p'(a)$, the norm here being the norm of bounded operators on $\mathcal{H}$.*
(ii) The map $S^0_{1,0} \ni a \mapsto q(a) \in \mathcal{L}(\mathcal{H})$ is well defined and continuous.
(iii) If $a \in S^m_{1,0}$, $m \ge 0$, is elliptic and real valued, then $q(a)$ is essentially self-adjoint.

Proof. For proving (i) and (ii) we will use the symbolic calculus and Hörmander's trick [15, 16]

For $a \in S^0_{1,0}$ we take a constant $M \ge |a| + 1$, $M \in \mathbb{R}$. We can assume a real, for simplicity.

Then we define

$$b_0 = (M^2 - a^2)^{1/2} \in S^0_{1,0} \tag{10}$$

Let $r_0 = M^2 - \mathcal{P}(a, a) - \mathcal{P}(b_0, b_0)$. Then $r_0 \in S^{-1}_{1,0}$, $q(r_0) - (M^2 - q(a)^2 - q(b_0)^2) \in \mathcal{W}^{-\infty}$ and r_0 will depend continuously on a because $\mathcal{P}(a', b') - a'b' \in S^{-1}_{1,0}$, if $a' \in S^0_{1,0}$ and $b' \in S^0_{1,0}$, and the induced map is continuous (Axiom (v)).

Let $b_1 = b_0 + b_1'$, with

$$b_1' = b_0^{-1}\big[r_0 - M^2 - \mathcal{P}(a, a)\big]/2 \in S^{-1}_{1,0}$$

and $r_1 = M^2 - \mathcal{P}(a, a) - \mathcal{P}(b_1, b_1)$. Then $r_1 \in S^{-2}_{1,0}$, $q(r_1) - (M^2 - q(a)^2 - q(b_1)^2) \in \mathcal{W}^{-\infty}$ and r_1 will depend continuously on a. Iterating this construction, we see that we can assume that we have constructed $b_d \in S^0_{1,0}$, $r_d \in S^{-d-1}_{1,0}$, both depending continuously on a, such that $r_d = 0$ if $a = 0$ and

$$R := q(r_d) - (M^2 - q(a)^2 - q(b_d)^2) \in \mathcal{W}^{-\infty}$$

also depends continuously on a.

Now we use Axiom (vi) to conclude that for sufficiently large d the operator $q(a)$ is bounded and $\|q(a)\|$ is continuous in a.

We then obtain

$$\|q(a)\xi\|^2 = (q(a)\xi, q(a)\xi) = M^2\|\xi\|^2 + (R\xi, \xi) - (q(r_d)\xi, \xi) - \|q(b)\xi\|^2, \tag{11}$$

for any $\xi \in \mathcal{W}^{-\infty}\mathcal{H}$, which is dense in $\mathcal{H}$, by Axiom (ii). This gives

$$\|q(a)\|^2 \leq (\|a\|_\infty + 1)^2 + \|R\| + \|q(r_d)\| = C(a),$$

where $C(a)$ is a continuous function of $a \in S^0_{1,0}$, by our discussion above. This proves (ii). The first part, (i), is then a direct consequence of (ii).

To prove (iii), we can assume $m > 0$, otherwise the statement is trivial, because $q(a)$ is bounded in that case. Since $q(a)$ is symmetric by the Axiom (iii), it is enough to check that $(q(a) \pm \imath I)(\mathcal{W}^{-\infty}\mathcal{H})$ are dense [32, Theorem VIII.3 and Corollary]. Using the asymptotic completeness of the space of symbols, we can find $b \in S^{-m}_{1,0}$ such that $ab - 1 \in S^{-1}_{1,0}$ and $(q(a) + \imath I)q(b) - I \in \mathcal{W}^{-\infty}$.

Choose $R = R^* \in \mathcal{W}^{-\infty}$ to be an injective map. Then

$$P = (q(a) - \imath I)(R^*R + q(\overline{b})q(b)),$$

also satisfies $(q(a) + \imath I)P = I + T$, for some $T \in \mathcal{W}^{-\infty}$. Note that

$$(R^*R + q(\overline{b})q(b)\xi, \xi) \geq (R\xi, \xi) > 0$$

if $\xi \in \mathcal{W}^{-\infty}\mathcal{H}$, $\xi \neq 0$, by Lemma 2.1. Now, if $(q(a) + \imath I)(\mathcal{W}^{-\infty})\mathcal{H}$ is not dense, we can find $\eta \in \mathcal{H}$, $\eta \neq 0$, perpendicular to $(q(a) + \imath I)P(\mathcal{W}^{-\infty}\mathcal{H})$.

This gives that $(I + T)^*\eta$ is perpendicular to $\mathcal{W}^{-\infty}\mathcal{H}$. Since the latter is dense in $\mathcal{H}$ by Axiom (ii), we obtain that $(I + T)^*\eta = 0$, and hence $\eta = -T^*\eta \in \mathcal{W}^{-\infty}\mathcal{H}$ is in the domain of all operators in $\mathcal{W}$. But then

$$(\eta, (q(a) + \imath I)P\zeta) = (P^*(q(a) - \imath I)\eta, \zeta) = 0$$

for all $\zeta \in \mathcal{W}^{-\infty}\mathcal{H}$, which implies as above that

$$(R^*R + q(b)^*q(b))(q(a) + \imath I)(q(a) - \imath I)\eta = 0.$$

Now this is a contradiction, because all operators in the above product are injective. (Note that we have used above $(q(a) + \imath I)^*\eta = (q(a) - \imath I)\eta$ and similarly for the other operators above. The main point of the proof was to show that this is indeed possible.)

The proof for the other choice of sign in $(q(a) \pm \imath I)$ is the same, and hence our proof of (iii) is complete. □

For further reference, let us mention here a consequence of being elliptic and invertible in the algebra $\mathcal{W}^\infty$.

Proposition 2.3. *Let $m \geq 0$, $A \in M_N(\mathcal{W}^m)$, and $B \in M_N(\mathcal{W}^{-m})$ be such that $AB = BA = I$ in $M_N(\mathcal{W})$. Then $\overline{A}$, the closure of A, is invertible with inverse $\overline{B}$. In particular, the domain of $\overline{A}$ is $\overline{B}\mathcal{H}$.*

Proof. Use the boundedness of B and the definitions. □

3. Sobolev spaces

We now define some abstract Sobolev space $H^{(s)}$ as in Guillemin's paper. Recall from the previous section that only the Axioms (i)–(vi) are assumed in this and next section.

As in Guillemin's paper, we define the Sobolev space $H^{(s)}$, $s \geq 0$, to be the domain of (the closure of) P, where $P \in \mathcal{W}^s$ is elliptic. This definition is independent of the choice of P, by Axioms (v) and (vi), because if P_1 is another such elliptic operator, then we can find $Q \in \mathcal{W}^0$ and $R \in \mathcal{W}^{-\infty}$ such that $P_1 = QP + R$ and both Q and R are bounded. Let $r \in S^1_{cl}$ be a symbol with

$$r(\xi) = \|\xi\| \qquad \text{for all } \xi \in V \text{ with } \|\xi\| \geq 2 \tag{12}$$

and

$$P_s := \frac{1}{2}\big(I + q(r^{s/2})^2)\big). \tag{13}$$

We endow $H^{(s)}$ with the norm defined by

$$\|f\|^2_{(s)} := \|P_s f\|^2,$$

$f \in \mathcal{H}$ in the domain of (the closure of) P_s. If $s < 0$, we define $P_s := P_{-s}^{-1}$ and $H^{(s)}$ to be the completion of $\mathcal{H}$ in the norm $\xi \to \|\xi\|_{(s)} := \|P_s\xi\|$. Then the inner product of $\mathcal{H}$ extends to a bilinear pairing $H^{(s)} \otimes H^{(-s)} \to \mathbb{C}$ that identifies $H^{(-s)}$ with the dual of $H^{(s)}$ as in [20], for example.

Proposition 3.1. *(i) An operator $T \in \mathcal{W}$ defines a continuous map $T : H^{(s)} \to H^{(s')}$ if, and only if, $P_{s'}TP_{-s}$ extends to a bounded operator on $\mathcal{H}$.*

(ii) In particular, any $P \in \mathcal{W}^z$ extends by continuity to a bounded map $H^{(s)} \to H^{(s-r)}$ for $r = \mathrm{Re}\,(z)$.

Proof. We have from definition that $P_s : H^{(s)} \to \mathcal{H}$ is an isometric isomorphism and that $P_{-s} = (P_s)^{-1}$ for all s. (We first replace all operators by their closures, note however that no confusion can arise, in view of Proposition 2.3.) This takes care of (i).

Let $z = r + i\mu$, $r, \mu \in \mathbb{R}$. The second part is proved directly using the symbolic calculus for the four possible signs of s and $s' := s - r$.

If $-s, s' \geq 0$, then $P_{s'}TP_{-s} \in \mathcal{W}^{\imath\mu}$ is a bounded operator.

If $s, s' \geq 0$, the symbolic calculus tells us that we can write $T = T_1P_s + R$, where $T_1 \in \mathcal{W}^{-s'}$ and $R \in \mathcal{W}^{-\infty}$. Then

$$P_{s'}TP_{-s} = P_{s'}T_1P_sP_{-s} + P_{s'}RP_{-s} = P_{s'}T_1 + P_{s'}RP_{-s}$$

is bounded because $P_{s'}T_1 \in \mathcal{W}^{\imath\mu}$ and $P_{s'}RP_{-s} \in \mathcal{W}^{-\infty}$ are bounded.

If $-s, -s' \geq 0$, we proceed similarly by writing $T = P_{-s'}T_1 + R$ with $T_1 \in \mathcal{W}^{s+\imath\mu}$ and $R \in \mathcal{W}^{-\infty}$.

If $s, -s' \geq 0$, we proceed similarly by writing $T = P_{-s'}T_1P_s + R$ with $T_1 \in \mathcal{W}^{\imath\mu}$ and $R \in \mathcal{W}^{-\infty}$. ∎

4. Holomorphic families of operators

Recall from the previous sections that only the Axioms (i)–(vi) are assumed in this section.

Let $\mathcal{O}(\Omega)$ be the space of holomorphic functions on some open subset $\Omega \subset \mathbb{C}$. If X is a Fréchet space, we denote by $\mathcal{O}(\Omega, X)$ the space of holomorphic (or complex differentiable) functions on Ω with values in X.

We shall assume from now on throughout the paper that our algebra $\mathcal{W}$ satisfies condition (σ) of Section 1.

Condition (σ) allows us to define the principal symbol $\sigma^{(s)}(T)$ of an operator $T \in \mathcal{W}^s$ as in the classical case of algebras of pseudodifferential operators. Namely,

$$\sigma^{(s)} : \mathcal{W}^s \to S^s_{cl}/S^{s-1}_{cl} \subset \mathcal{C}^\infty(V \smallsetminus 0)$$

satisfies $\sigma^{(s)}(q(a)) = a + S^{s-1}_{cl}(V)$, which determines it completely. (Let $V \smallsetminus 0$ be the set of non-zero vectors of the vector bundle V. We identify S^s_{cl}/S^{s-1}_{cl} with the space of smooth, homogeneous of order s functions on $V \smallsetminus 0$.) The maps $\sigma^{(s)}$ and q immediately extend to $M_N(\mathcal{W}^s)$. Then A is elliptic if, and only if, $\sigma^{(s)}(A)$ is invertible everywhere on $V \setminus \{0\}$.

We now establish an analogue of the asymptotic completeness for holomorphic families of operators in $\mathcal{W}$. We shall write $M_N(X)$ for $M_N(\mathbb{C}) \otimes X$, for any Fréchet space X.

Proposition 4.1. *(i) Let $a_i \in \mathcal{O}(\Omega, M_N(S^{\mu-i}_{cl}))$, $i = 0, 1, 2, \ldots$, be holomorphic functions on some open subset $\Omega \in \mathbb{C}$. Then there exists a holomorphic function $a \in \mathcal{O}(\Omega, M_N(S^\mu_{cl}))$ such that for all $k = 1, 2, \ldots$*

$$a(z) - \sum_{i=0}^{k-1} a_i(z) \in M_N(S^{\mu-k}_{cl}).$$

(ii) For any $F \in \mathcal{O}(\Omega, M_N(\mathcal{W}^\mu_k))$ there exists $f \in \mathcal{O}(\Omega, M_N(S^\mu_{cl}))$ such that $F(z) - q(f(z)) \in M_N(\mathcal{W}^{-\infty}_k)$ is a holomorphic function.

(iii) Similarly, let $A_i \in \mathcal{O}(\Omega, M_N(\mathcal{W}^{\mu-i}_k))$, $i = 0, 1, 2, \ldots$, be holomorphic functions on some open subset $\Omega \in \mathbb{C}$. Then there exists a holomorphic function $A \in M_N(\mathcal{O}(\Omega, \mathcal{W}^\mu_k))$ such that for all $l = 1, 2, \ldots$

$$A(z) - \sum_{i=0}^{l-1} A_i(z) \in M_N(\mathcal{W}^{\mu-l}_k).$$

If a is as in (i) above, we shall say that a is an *asymptotic sum* of the sequence a_i, $i \geq 0$. Similarly, if A is as in (iii) above, we shall say that A is an *asymptotic sum* of the sequence A_i, $i \geq 0$.

Proof. Assume $\mu = 0$, $N = 1$, and $\Omega = \mathbb{C}$, for simplicity.

Let B be the set of vectors of length at most one in V, which we identify with the radial compactification of V using the map $V \ni \xi \longmapsto \langle \xi \rangle^{-1}\xi \in V$. This identifies S^0_{cl} with $\mathcal{C}^\infty(B)$ and we let

$$p_n(a) = \max_{l=0}^n \|\nabla^l a\|_\infty$$

be the standard seminorms defining the topology on $\mathcal{C}^\infty(B^*)$.

We observe that S^{-n-1}_{cl} is contained in the closure of $S^{-\infty}$ in the topology defined by the seminorm p_n. By applying this to the Taylor coefficients of $a_j(z)$, for any j, we obtain that there exist holomorphic functions $r_j \in \mathcal{O}(\Omega, S^{-\infty})$ such that $p_j(a_{j+1}(z) - r_{j+1}(z)) \leq 2^{-j}$ for any $|z| \leq j$. This proves that the series $\sum_j (a_j(z) - r_j(z))$ is uniformly convergent on the compact subsets of $\mathbb{C}$. Let $a(z)$ be its sum. Then $a \in \mathcal{O}(\Omega, S^0_{cl})$. This proves (i).

To prove (ii), proceed as follows. Let χ be a smooth function on V, $\chi = 1$ outside the unit ball and $\chi = 0$ on the set of vectors of length $\leq 1/2$. Define $a_0(z) = \chi\sigma^{(0)}(A(z))$, which is a holomorphic function with values in S^0_{cl}. We define $a_n(z)$, $n \geq 1$, by induction by

$$a_n(z) = q\Big[\chi\sigma^{(-n)}\big(A(z) - \sum_{j=0}^{n-1} q(a_j(z))\big)\Big] .$$

Then use (i) to construct $a \in \mathcal{O}(\Omega, S^0_{cl})$ such that $a(z) - \sum_{i=0}^{k-1} a_i(z) \in S^{-k}_{cl}$. Then a satisfies the requirements of (ii).

We now turn to (iii). First, using (ii), we find $a_i \in \mathcal{O}(\Omega, S^{-i}_{cl})$ such that $A_i(z) - q(a_i(z)) \in \mathcal{W}^{-\infty}$. Then we use (i) for the sequence a_i to find $a \in \mathcal{O}(\Omega, S^0_{cl})$ such that $a(z) - \sum_{i=0}^{k-1} a_i(z) \in S^{-k}_{cl}$. We can let then $A = q(a)$. □

We shall write

$$\mathcal{O}(\Omega, \mathcal{W}^{\mu}) := \cup\mathcal{O}(\Omega, \mathcal{W}^{\mu}_k).$$

Standard reasonings then give the following result.

Corollary 4.2. *(i) Let $A \in \mathcal{O}(\Omega, \mathcal{W}^{\mu})$ be a holomorphic, pointwise elliptic family. Then there exists $B \in \mathcal{O}(\Omega, \mathcal{W}^{-\mu})$ such that $AB - I, BA - I \in \mathcal{O}(\Omega, \mathcal{W}^{-\infty})$.*

(ii) There exists $b \in \mathcal{O}(\mathbb{C}, S^0_{cl})$, $b(0) = 1$, such that $q(r^z)q(r^{-z}b(z)) - I \in \mathcal{W}^{-\infty}$ and, similarly, $q(r^{-z}b(z))q(r^z) - I \in \mathcal{W}^{-\infty}$ with r defined via (12).

Proof. (i) follows right away from Proposition 4.1 using standard arguments [16, 42]. Namely, let $a \in \mathcal{O}(\Omega, S^{\mu}_{cl})$ be such that $A(z) - q(a(z)) \in \mathcal{W}^{-\infty}$ for all $z \in \Omega$. Choose $\chi \in \mathcal{C}^{\infty}(V)$ to be equal to zero in a neighborhood of 0 containing all zeroes of $a(z)$, for all z, and equal to 1 outside some larger neighborhood of $0 \in V$. Let $b(z) := \chi a(z)^{-1}$, which is defined to be zero where $a(z) = 0$. Let $B_0(z) = q(b(z)) \in \mathcal{O}(\Omega, \mathcal{W}^{-\mu})$. Then $AB_0 = I + R_1(z)$, where $R_1 \in \mathcal{O}(\Omega, \mathcal{W}^{-1})$. We can find $R_{1n} \in \mathcal{O}(\Omega, \mathcal{W}^{-n}_1)$ such that $R_{1n}(z) - R_1(z)^n \in \mathcal{W}^{-\infty}$. Define B then to be an asymptotic sum of the sequence $(-1)^n B_0 R_{1n}$ (see Proposition 4.1 (iii)).

We now apply (i) to the holomorphic, elliptic family $A(z) = q(r^z)q(r^{-z})$ to deduce that there exists $a \in \mathcal{O}(\Omega, S^0_{cl})$ such that $q(r^z)q(r^{-z})q(a(z)) - I \in \mathcal{W}^{-\infty}$. Then $q(r^{-z})q(a(z)) = q(r^{-z}b(z)) + r(z)$, where $b \in \mathcal{O}(\Omega, S^0_{cl})$ and $r \in \mathcal{O}(\Omega, \mathcal{W}^{-\infty})$, by Axiom (v). □

5. Spectral invariance

In this section we prove the continuity of the multiplication $\mathcal{W}^{\mu}_k \times \mathcal{W}^{\nu}_l \to \mathcal{W}^{\mu+\nu}_{k+l}$, we introduce the Guillemin completion of an extended Weyl algebra, which, in fact, is again an extended Weyl algebra satisfying also Condition (ψ). Then we establish the spectral invariance of extended Weyl algebras satisfying conditions (σ) and (ψ).

From now on and throughout the rest of the paper, $\mathcal{W}$ will denote an extended Weyl algebra, i.e. we assume Axioms (i)–(vii).

As before let $r \in S^1_{cl}$ be a symbol satisfying (12). We now establish the partial continuity of the multiplication on an extended Weyl algebra.

Proposition 5.1. *The map $\mathcal{P} : S^{\mu}_{cl} \times S^{\nu}_{cl} \to S^{\mu+\nu}_{cl}$ and the multiplication maps*

$$\mathcal{W}^{\mu}_k \times \mathcal{W}^{-\infty}_l \to \mathcal{W}^{-\infty}_{k+l}, \quad \mathcal{W}^{-\infty}_k \times \mathcal{W}^{\mu}_l \to \mathcal{W}^{-\infty}_{k+l}, \quad \mathcal{W}^{\mu}_k \times \mathcal{W}^{\nu}_l \to \mathcal{W}^{\mu+\nu}_{k+l},$$

where $\mu, \nu \in \mathbb{C}$ and $k, l = 1, 2, \ldots$, are well defined and continuous.

Proof. Axiom (i) tells us that $\mathcal{W}^{-\infty} = \cup_{k\geq 1}\mathcal{W}_k^{-\infty}$ is an LF–algebra, and hence the multiplication map

$$\mathcal{W}_k^{-\infty} \times \mathcal{W}_l^{-\infty} \to \mathcal{W}_{k+l}^{-\infty}$$

is continuous. Axiom (vii) shows that the multiplication map

$$S_{cl}^{\mu} \times \mathcal{W}_k^{-\infty} \ni (a,T) \to q(a)T \in \mathcal{W}_{k+1}^{-\infty}$$

is continuous. Hence the induced map

$$(S_{cl}^{\mu} \oplus \mathcal{W}_k^{-\infty}) \times \mathcal{W}_l^{-\infty} \to \mathcal{W}_{k+l}^{-\infty}$$

is also continuous (note that we used $k + l \geq k + 1$). The continuity of the first two maps is then a consequence of the definition of $\mathcal{W}_k^{\mu} := q(S_{cl}^{\mu}) + \mathcal{W}_k^{-\infty}$. For the continuity of the right multiplication, we also use the continuity of $T \mapsto T^*$ on $\mathcal{W}_k^{-\infty}$, see Axiom (i).

Let $\mathcal{R}(a,b) = q(a)q(b) - q(\mathcal{P}(a,b)) \in \mathcal{W}_1^{-\infty}$. Axiom (v) and the definition of the topology on $\mathcal{W}_1^{\mu+\nu} := q(S_{cl}^{\mu+\nu}) + \mathcal{W}_1^{-\infty}$ shows that the maps

$$S_{cl}^{\mu} \times S_{cl}^{\nu} \ni (a,b) \to \big(\mathcal{P}(a,b), \mathcal{R}(a,b)\big) \in S_{cl}^{\mu+\nu} \oplus \mathcal{W}_1^{-\infty} \quad \text{and}$$
$$S_{cl}^{\mu+\nu} \oplus \mathcal{W}_1^{-\infty} \ni (a,T) \to q(a) + T \in \mathcal{W}_1^{\mu+\nu}$$

are well defined and continuous. Their composition is the product $q(a)q(b)$, which hence depends continuously on a and b in the above spaces of symbols. Then the same arguments as above, together with the continuities already proved, show that the multiplication

$$\mathcal{W}_k^{\mu} \times \mathcal{W}_l^{\nu} \to \mathcal{W}_{k+l}^{\mu+\nu}$$

is well defined and continuous. □

Denote by $\|\cdot\|_n$ the operator norms $H^{(-n)} \to H^{(n)}$. Let $\overline{\mathcal{W}}^{-\infty}$ be the closure of $\mathcal{W}^{-\infty}$ in the topology defined by the family of norms $(\|\cdot\|_n)$, $n \in \mathbb{Z}_+$. Also, let

$$\overline{\mathcal{W}} := \mathcal{W} + \overline{\mathcal{W}}^{-\infty}.$$

We shall call $\overline{\mathcal{W}}$ and $\overline{\mathcal{W}}^{-\infty}$ the *Guillemin completions* of $\mathcal{W}$ and $\mathcal{W}^{-\infty}$, respectively. We shall also write $\overline{\mathcal{W}}^{s} := \mathcal{W}^{s} + \overline{\mathcal{W}}^{-\infty}$.

Proposition 5.2. *Let $\mathcal{W}$ be an extended Weyl algebra. Then the Guillemin completion $\overline{\mathcal{W}}$ is an extended Weyl algebra that satisfies (ψ) with respect to the family $(\|\cdot\|_n)$, $n \in \mathbb{Z}_+$, of operator norms $H^{(-n)} \to H^{(n)}$.*

(It is not assumed that $\mathcal{W}$ satisfies (ψ), nor that it is a Fréchet algebra. The conclusion is that $\overline{\mathcal{W}}$ has these properties. Also, in applications, one will have to check that $\overline{\mathcal{W}}$ satisfies Condition (σ) in order to be able to construct complex powers. However, this condition is automatically satisfied in practice for algebras of pseudodifferential operators.)

Proof. We shall check that $\overline{\mathcal{W}}$ satisfies all the axioms defining an extended Weyl algebra. To start with, the axioms (i), (ii), and (vi) are automatically satisfied.

Recall that $\mathcal{L}(X,Y)$ denotes the space of continuous linear maps between two locally convex vector spaces X and Y. If X and Y are Banach spaces, we denote by $\|\cdot\|_{\mathcal{L}(X,Y)}$ the canonical norm on $\mathcal{L}(X,Y)$. In particular,

$$\|\cdot\|_n = \|\cdot\|_{\mathcal{L}(H^{(-n)},H^{(n)})}$$

and $\|T\|_n = \|P_n T P_n\|_0$, where $P_s := \frac{1}{2}\Big(I + q(r^{s/2})^2)\Big)$ is as introduced in Equation (13). In particular, all semi-norms $\|\cdot\|_n$ are continuous on $\mathcal{W}_k^{-\infty}$, and hence the inclusion map

$$\mathcal{W}_k^{-\infty} \hookrightarrow \overline{\mathcal{W}}^{-\infty} \tag{14}$$

is continuous, for any k. The continuity of this map is enough to conclude that Axioms (iii), (v), and (vii) are satisfied.

Let us notice that $\|T\|_{\mathcal{L}(H^{(s)}, H^{(s')})} \leq \|T\|_n$, whenever $s \geq -n$ and $s' \leq n$. This gives

$$\|T_1 T_2\|_n \leq \|T_1\|_{\mathcal{L}(H^{(0)}, H^{(n)})} \|T_2\|_{\mathcal{L}(H^{(-n)}, H^{(0)})} \leq \|T_1\|_n \|T_2\|_n.$$

Thus the family of seminorms $\|\cdot\|_n$ is submultiplicative.

The only thing that is left to prove is that the algebra $\mathbb{C} + \overline{\mathcal{W}}^{-\infty}$ is spectrally invariant in $\mathcal{L}(\mathcal{H})$ (that is, that $\mathbb{C} + \overline{\mathcal{W}}^{-\infty} \cap \mathcal{L}(\mathcal{H})^{-1} = (\mathbb{C} + \overline{\mathcal{W}}^{-\infty})^{-1}$), but this follows from Proposition A.1 due to the definition of the Sobolev spaces.. $\square$

It is worth pointing out that in a different axiomatic setting spectral invariance of the closure of an algebra of bounded operators with respect to the family $\|\cdot\|_n$ of seminorms has been studied before by Mantlik [25].

We shall need the following lemma.

Lemma 5.3. *If $\mathcal{W}$ is an extended Weyl algebra satisfying (ψ) and $w \in \mathbb{C}$, then we can find $Q \in \mathcal{W}^w$ such that $Q^{-1} \in \mathcal{W}^{-w}$.*

Proof. Let $b \in \mathcal{O}(\mathbb{C}, S^0_{cl})$, $b(0) = 0$, such that $R(z) := q(r^z) q(r^{-z} b(z)) - I \in \mathcal{W}^{-\infty}$, as in Corollary 4.2(ii). Then $R \in \mathcal{O}(\mathbb{C}, \mathcal{W}^{-\infty})$ is holomorphic and $R(0) = 0$. Hence $I + R(z) = q(r^z) q(r^{-z} b(z))$ is invertible on $\mathcal{H}$ for $|z| < \epsilon$, for ϵ small. Then $q(r^z)^{-1} = q(r^{-z} b(z))(I + R(z))^{-1} \in \mathcal{W}^{-z}$, for $|z| < \epsilon$ small.

Chose now $k \in \mathbb{N}$ large enough so that $|w| < k\epsilon$. Then we can take $Q = q(r^{w/k})^k$. $\square$

We then obtain

Theorem 5.4. *Let $\mathcal{W}$ be an extended Weyl algebra satisfying (ψ). Also, let $P \in M_N(\mathcal{W}^z)$, $\mathrm{Re}\,(z) \geq 0$. We replace P with its closure, if necessary. Assume that P is elliptic and invertible as a (possibly) unbounded operator on $\mathcal{H}$, then P^{-1} is (the closure of an element) in $M_N(\mathcal{W}^{-z})$.*

Proof. Below, we shall replace all operators in $\mathcal{W}$ by their closures, when necessary. We shall assume $N = 1$, for simplicity.

Write $z = a + bi$, $a, b \in \mathbb{R}$, and let $Q \in \mathcal{W}^{-z}$ be such that $Q^{-1} \in \mathcal{W}^z$, which is possible by Lemma 5.3. Then $P_1 := PQ \in \mathcal{W}^0$ is elliptic, injective, and bounded. The ellipticity of Q^{-1} guarantees the range of (the closure of) Q is $H^{(a)}$ and the ellipticity of P guarantees that the domain of P is also $H^{(a)}$. Thus $P_1 = PQ : \mathcal{H} \to \mathcal{H}$ is also surjective, and hence it is invertible (on $\mathcal{H} = H^{(0)}$). We can hence replace P by P_1 and hence assume that it is bounded.

Let then $P \in \mathcal{W}^0$ be invertible. At this point the proof can be completed using [19], but we prefer to give a self-contained proof. We want to prove that $P^{-1} \in \mathcal{W}^0$. In any case, there exists a sequence of bounded operators $Q_n \in \mathcal{W}^0$ such that $Q_n \to P^{-1}$ in norm. (This is seen as follows, the functional calculus applied to the bounded, invertible operator PP^* shows that there exists a sequence of polynomials f_n such that $f_n(PP^*) \to (PP^*)^{-1}$. Then we can take $Q_n := P^* f_n(PP^*)$).

Let Q be a parametrix for P. That is, let $Q \in \mathcal{W}^0$ be such that $R := PQ - I \in \mathcal{W}^{-\infty}$. (We use here the ellipticity of P.) Then

$$P^{-1} = Q - P^{-1}R = \lim(Q - Q_n R),$$

in the norm of bounded operators (the only one used in this proof). Then $Q' := Q - Q_n R$ is also a parametrix of P and is moreover invertible if we choose n large enough. If we write $PQ' = I + R'$, then $I + R'$ is invertible as a bounded operator on $\mathcal{H}$, and hence $(I+R')^{-1} - I \in \mathcal{W}^{-\infty}$, by axiom (ψ). Thus $P^{-1} = Q'(I+R')^{-1} \in \mathcal{W}^0$. The proof is now complete. □

6. Special holomorphic families

Let $\mathcal{W}$ be an extended Weyl algebra, and let Ω be an open subset of $\mathbb{C}$.

We shall need the following result.

Proposition 6.1. *We have the following continuity properties*

(i) The map $z \mapsto r^z \in S^m_{1,0} := S^m_{1,0}(V)$ is holomorphic for $\mathrm{Re}\,(z) < m$.

(ii) The functions $z \mapsto q(r^z)T \in \mathcal{W}^{-\infty}_{k+1}$, $z \mapsto Tq(r^z) \in \mathcal{W}^{-\infty}_{k+1}$, and $z \mapsto q(r^z)\xi \in \mathcal{H}$ are holomorphic in $z \in \mathbb{C}$ for any fixed $T \in \mathcal{W}^{-\infty}_k$ and any fixed $\xi \in \mathcal{W}^{-\infty}\mathcal{H}$.

Proof. Recall that $r \in S^1_{cl}$ is a fixed symbol satisfying $r \geq 1$ and $r = \|v\|$ if $\|v\| \geq 2$. Fix $m \in \mathbb{R}$. To prove that the map

$$z \mapsto r^z \in S^m_{1,0}$$

is holomorphic on $\{\mathrm{Re}\,(z) < m\}$, we shall check that the complex derivative

$$\partial_z r^z = \lim_{w \to 0} \frac{r^{z+w} - r^z}{w} = r^z \log r \in S^m_{1,0}$$

exists for $\mathrm{Re}\,(z) < m$. By Proposition 5.1, it is enough to prove this for $m = 0$.

Let $S^0_{1,0}([1,\infty))$ be the space of order zero symbols on $[1,\infty)$, endowed with the usual semi-norms $p_n(a) := \sup_{x,\ k \leq n} |(x\partial_x)^k a|$. We shall first check that the function

$$\{\mathrm{Re}\,(z) < 0\} \ni z \mapsto x^z \in S^0([0,\infty))$$

is holomorphic. Now $x^k \partial_x^k x^z = P(z) x^z$, with P a polynomial. Taylor's formula then gives that

$$|P(z+w)x^{z+w} - P(z)x^z - w\partial_z(P(z)x^z)| \leq C|w|^2 x^{-\epsilon}(\log x)^2 \leq CC_\epsilon |w|^2 \tag{15}$$

for $\mathrm{Re}\,(z) < -2\epsilon$ and $|w| < \epsilon$, where C depends only on the polynomial P and $C_\epsilon = \sup_{x \geq 1} x^{-\epsilon}(\log x)^2 < \infty$. Since $x^k \partial_x$ is a linear operator, this is enough to prove that

$$p_n\Big(\frac{x^{z+w} - x^z}{w} - x^z \log r\Big) \to 0, \quad \text{as } |w| \to 0,$$

for any n. Consequently, $z \to x^z \in S^0_{1,0}([1,\infty))$ is holomorphic for $\mathrm{Re}\,(z) < 0$.

(i) then follows from the continuity of the map

$$S^m_{1,0}([1,\infty)) \ni a \to a \circ r \in S^m_{1,0} := S^m_{1,0}(V),$$

(take $\epsilon = 1$ in Lemma 1.1).

The holomorphy on $\mathbb{C}$ of the maps $z \mapsto q(r^z)T \in \mathcal{W}^{-\infty}_{k+1}$ and $T \mapsto aq(r^z) \in \mathcal{W}^{-\infty}_{k+1}$ then follows from Axioms (i) and (vii). In turn, this then gives right away that the map $z \mapsto q(r^z)T\xi$, for $T \in \mathcal{W}^{-\infty}$ and $\xi \in \mathcal{H}$, is holomorphic, and hence that $z \mapsto q(r^z)\xi$ is holomorphic on $\mathbb{C}$ for $\xi \in \mathcal{W}^{-\infty}\mathcal{H}$. □

We now introduce special holomorphic families.

Definition 6.2. A family of operators $A(z) \in M_N(\mathcal{W}^{mz+d})$, $m, d \in \mathbb{R}$, is called a *special holomorphic family of order* $mz + d$ *on* Ω if there exist $k \in \mathbb{N}$, $b \in \mathcal{O}(\Omega, M_N(S^d_{cl}))$, and $R \in \mathcal{O}(\Omega, M_N(\mathcal{W}^{-\infty}_k))$ such that

$$A(z) = q(r^{mz}b(z)) + R(z). \tag{16}$$

A *special holomorphic family* is a special holomorphic family of order $mz + d$ in a neighborhood of 0, for some $m, d \in \mathbb{R}$.

Let us make now the simple observation that Lemma 2.1 shows that $A(\overline{z})^\sharp$ is a special holomorphic family whenever $A(z)$ is a special holomorphic family.

A special holomorphic family can be described in several ways. Recall that $\mathcal{O}(\Omega, \mathcal{W}^w) = \cup_{k\geq 1}\mathcal{O}(\Omega, \mathcal{W}^w_k)$, for any $w \in \mathbb{C} \cup \{\infty\}$.

Proposition 6.3. *(i) A function $A(z) \in M_N(\mathcal{W}^{mz+d})$ is a special holomorphic family of order $mz+d$ on a neighborhood of 0 if, and only if, there exist holomorphic functions $B_1(z) \in M_N(\mathcal{W}^d)$ and $R_1(z) \in M_N(\mathcal{W}^{-\infty})$, defined in a possibly smaller neighborhood of 0, such that $A(z) = q(r^{mz}I_N)B_1(z) + R_1(z)$.*

(ii) Let $T \in \mathcal{O}(\Omega, M_N(\mathcal{W}^{-\infty}))$ and $A(z)$ be a special holomorphic family on Ω. Then $z \mapsto A(z)T(z)$ and $z \mapsto T(z)A(z)$ are in $\mathcal{O}(\Omega, M_N(\mathcal{W}^{-\infty}))$.

Proof. Let $N = 1$ and $d = 0$, for simplicity. Assume $A(z) = q(r^{mz}a(z)) + R(z)$ is a holomorphic family, as in the Definition 6.2. Let $b(z)$ be as in Corollary 4.2, namely,

$$q(r^{mz})q(r^{-mz}b(z)) - I \in \mathcal{W}^{-\infty}.$$

We define $B_1(z) = q(r^{-mz}b(z))q(r^{mz}a(z)) \in \mathcal{W}^0_2$. Then $B_1(z)$ and $R_1(z) := q(r^{mz})B(z) - A(z) \in \mathcal{W}^{-\infty}$ are the desired holomorphic families.

Conversely, assume that $A(z) = q(r^{mz})B_1(z) + R_1(z)$, with B_1 and R_1 as in the statement of the proposition. By Proposition 4.1, there exists $c \in \mathcal{O}(\Omega, S^0_{cl})$ such that $B_1(z) - q(c(z)) \in \mathcal{W}^{-\infty}$. Let

$$a(z) = r^{-mz}\mathcal{P}(r^{mz}, c(z)).$$

Then $a \in \mathcal{O}(\Omega, S^0_{cl})$ and $q(r^{mz})q(c(z)) - q(r^{mz}a(z)) \in \mathcal{W}^{-\infty}_1$ by Axiom (v) and Proposition 6.1. Moreover,

$$\begin{aligned} &R(z) := q(r^{mz})B_1(z) + R_1(z) - q(r^{mz}a(z)) \\ &= \big(q(r^{mz})B_1(z) - q(r^{mz})q(c(z))\big) + \big(q(r^{mz})q(c(z)) - q(r^{mz}a(z))\big) + R_1(z) \in \mathcal{W}^{-\infty}_k. \end{aligned}$$

We continue to assume $N = 1$ and $d = 0$. (ii) follows from the descriptions of a special holomorphic family in (i) and Proposition 6.1(ii) as follows. Let us write $A(z) = q(r^{mz}a(z)) + R(z)$ for some $a \in \mathcal{O}(\Omega, S^0_{cl})$ and $R(z) \in \mathcal{O}(\Omega, \mathcal{W}^{-\infty}_l)$. Choose k such that $T \in \mathcal{O}(\Omega, \mathcal{W}^{-\infty}_k)$.

Then by 6.1(i), $r^{mz}a(z) \in S^N_{1,0}$ is a holomorphic family for $\mathrm{Re}\,(z) < N/m$. The continuity of the multiplication map $S^N_{1,0} \times \mathcal{W}^{-\infty}_k \ni (a, T) \mapsto q(a)T \in \mathcal{W}^{-\infty}_{k+1}$ then proves that $q(r^{mz}a(z))T(z)$ is a holomorphic function in $\mathcal{O}(\Omega, \mathcal{W}^{-\infty}_{k+1})$. Since $\mathcal{W}^{-\infty}$ is an LF–algebra, $R(z)T(z)$ is also in $\mathcal{O}(\Omega, \mathcal{W}^{-\infty}_{k+l})$. This proves (ii) for $A(z)T(z)$. The proof for $T(z)A(z)$ is the same. $\square$

Let us observe that by taking adjoints (and by replacing z with $\overline{z}$, any family of the form $A(z) = B_1(z)q(r^{mz}I_N) + R_1(z)$, with $B_1(z) \in M_N(\mathcal{W}^d)$ and $R_1(z) \in M_N(\mathcal{W}^{-\infty})$, is also a special holomorphic family of order $mz + d$.

The following consequence of the above Proposition will be useful later on.

Corollary 6.4. *Let $A(z) \in \mathcal{W}^{mz}$ define a special holomorphic family. Then $\|A(z)\|$ is continuous on the set $\Omega \cap \{\mathrm{Re}\,(mz) < 0\}$ and is bounded on the compact subsets of $\Omega \cap \{\mathrm{Re}\,(mz) \leq 0\}$.*

Proof. Let us write $A(z) = q(a(z)r^{mz}) + R(z)$ with $a \in \mathcal{O}(\Omega, S^0_{cl})$ and $R \in \mathcal{O}(\Omega, \mathcal{W}^{-\infty})$, by Proposition 6.3. Then $a(z)r^{mz}$ defines a continuous function $\{\mathrm{Re}\,(mz) < 0\} \to S^0_{1,0}$, and hence $\|q(a(z)r^{mz})\|$ is continuous on $\Omega \cap \{\mathrm{Re}\,(mz) < 0\}$ by Proposition 2.2(i). For the second part of the statement, let us observe that the map $z \mapsto r^z \in S^0_{1,0}$ is bounded on $\{\mathrm{Re}\,(mz) \leq 0\}$ and the map $z \mapsto a(z) \in S^0_{cl}$ is continuous on Ω. The result then follows from Proposition 2.2(i). □

The asymptotic completeness of Proposition 4.1 extends to special holomorphic families right away, giving the following technical result, which will however be crucial in constructing complex powers.

Corollary 6.5. *Let $A_i \in M_N(\mathcal{W}^{mz+d-i})$ be special holomorphic families of order $mz + d - i$, $i \in \mathbb{Z}_+$, defined on some open subset $\Omega \in \mathbb{C}$. Then there exists a special holomorphic family $A(z)$ of order $mz + d$ on Ω such that $A(z) - \sum_{i=0}^{k-1} A_i(z) \in M_N(\mathcal{W}^{mz+d-k})$.*

Proof. The proof is the same for all values of N, so we shall assume that $N = 1$, for simplicity. Let us write $A_i(z) = q(r^{mz}a_i(z)) + R_i(z)$, with $a_i \in \mathcal{O}(\Omega, S^{d-i}_{cl})$. We can choose by Proposition 4.1 $a \in \mathcal{O}(\Omega, S^d_{cl})$ to be an asymptotic sum of the families a_i, $i = 0, 1, 2, \ldots$. Then we can take $A(z) = q(r^{mz}a(z))$. □

We conclude this section with a result that will allow us to construct resolvents.

Proposition 6.6. *Assume that the extended Weyl algebra satisfies the Conditions (σ) and (ψ), and let $A(z)$ be a special holomorphic family defined in an open neighbhorhood of 0 in $\mathbb{C}$ such that $A(0) = I$. Then there exists a special holomorphic family $A_1(z)$, defined in a possibly smaller neighborhood of 0 such that*

$$A(z)A_1(-z) = A_1(-z)A(z) = I.$$

Proof. Let $A(z) = q(r^{mz})B(z) + R(z) \in M_N(\mathcal{W}^{mz+d})$. We shall assume $N = m = 1$ and $d = 0$, for simplicity. We can assume $b(0) = 1$ and $R(0) = 0$. Then $C(z) := q(r^{-z})A(z)$ is a holomorphic function with values in $\mathcal{W}^0$ (defined in a small neighborhood of 0). Moreover, $C(z)$ is elliptic. By the holomorphic asymptotic completeness (Proposition 4.1), we can find a holomorphic function $C_1(z) \in \mathcal{W}^0$ such that

$$R(z) := C_1(z)C(z) - I \in \mathcal{W}^{-\infty}.$$

By Proposition 5.1, R is also holomorphic in a possibly smaller neighborhood of 0. We can also assume that $C_1(0) = I$, and hence $R(0) = 0$. By Axiom (ψ), $(I + R(z))^{-1} = (I + R_1(z))$, for some holomorphic function $R_1(z)$ that is again defined in a neighborhood of 0. (This uses the fact that in a Ψ^*-algebra the set of invertible elements is open and hence inversion is continuous, by Banach's theorem mentioned also above, see [4].) let

$$A_1(z) = (I + R_1(-z))C_1(-z)q(r^z).$$

Then $A_1(-z)A(z) = I$. By Proposition 6.3(i), $A_1(z)$ is a special holomorphic family. □

7. Complex powers of elliptic operators

We now turn to the construction of complex powers of an elliptic, strictly positive operator $A \in M_N(\mathcal{W}^m)$, $m \geq 0$, so that $A^m \in M_N(\mathcal{W}^{mz})$. We start by using Guillemin's method of constructing complex powers in our slightly modified setting of extended Weyl algebras. Then we adapt some arguments from [40].

We want to prove the following two theorems, analogues of Theorems 5.1 and 5.2 in [12]. For the ring of pseudodifferential operators, these theorems are due to Seeley [38]. See also [40].

Let $\mathcal{W}$ be an extended Weyl algebra and let $r \in S^1_{cl}$ be as before, namely $r \geq 1$, $r(\xi) = \|\xi\|$, for all $\xi \in V$, such that $\|\xi\| \geq 2$. We shall assume that our extended Weyl algebra satisfies condition (σ).

7.1. **Symbolic complex powers.**

We first establish the following preliminary result, which is, in a certain sense, a result about the symbolic part of complex powers.

Proposition 7.1. *Recall that $\mathcal{W}$ is an extended Weyl algebra satisfying Condition (σ). Let $m \geq 0$, $A \in M_N(\mathcal{W}^m)$ be elliptic, and suppose that there exists $a \in \mathcal{O}(\mathbb{C}, M_N(S^0_{cl}))$, such that*

$$a(z+w) = a(z)a(w) \quad \textit{and} \quad \sigma^{(m)}(A) = a(1)r^m. \tag{17}$$

Then there exists a special holomorphic family $A(z) \in M_N(\mathcal{W}^{mz})$ such that

$$\begin{gathered} A(z)A(w) - A(z+w) \in \mathcal{W}^{-\infty}, \quad \sigma^{(mz)}(A(z)) = a(z)r^{mz}, \quad \textit{and} \\ A(1) - A \in M_N(\mathcal{W}^{-\infty}), \quad z, w \in \mathbb{C}. \end{gathered} \tag{18}$$

Moreover, this family is unique modulo smoothing operators.

Note that, by replacing $A(z)$ with $A(z) + I_N - A(0)$, we can assume that $A(0) = I_N := \mathrm{id}_{\mathcal{H}^N}$ in the Proposition above.

Proof. The proof is done as in Guillemin's paper. We assume that the reader is familiar with the details of that paper. Also, in the proof, we shall assume $N = 1$, for simplicity of notation. (The extension to the case $N > 1$ is non-trivial, however. To treat that case, we proceed as in [7].)

First we recall that the second holomorphic cohomology group of $\mathbb{C}$ with values in S^μ_{cl} satisfies $H^2_{hol}(\mathbb{C}, S^\mu_{cl}) = 0$, [12, Theorem 5.3]. For the benefit of the reader, we now briefly recall the main steps of the proof.

The main idea is to approximate $A(z)$ by a sequence $A_n(z)$ such that

$$A_n(z)A_n(z') - A_n(z+z') \in \mathcal{W}^{m(z+z')-n-1} \quad \text{and} \quad A_n(1) - A \in \mathcal{W}^{m-n-1}.$$

To start with, one can take $A_0(z) = q(a(z)r^{mz})$ (this works even for $m = 0$). Then one can solve $A_{n+1}(z) = A_n(z) + q(r^{mz}b(z))$, where $b(z) \in S^{-n-1}_{cl}$, using the vanishing of $H^2_{hol}(\mathbb{C}, S^\mu_{cl})$. Moreover, at each step the solution is unique, because two solutions differ by a linear map (modulo lower order terms). This linear map is uniquely determined at by the condition $A_n(1) - A \in \mathcal{W}^{m-n-1}$. The holomorphic asymptotic completeness (Corollary 6.5) then gives the result. ☐

Theorem 7.2. *Recall that $\mathcal{W}$ is an extended Weyl algebra satisfying Conditions (σ) and (ψ). Let $m \geq 0$, $A \in M_N(\mathcal{W}^m)$, and $a(z) \in M_N(S^0_{cl}) = M_N(S^0_{cl}(V))$,*

$z \in \mathbb{C}$, satisfy (18) *of the above Proposition. Then we can find a special holomorphic family $A_z \in M_N(\mathcal{W}^{mz})$ such that*

$$A_zA_w = A_{z+w}, \quad \sigma^{(mz)}(A_z) = a(z)r^{mz}, \quad \textit{and} \quad A_1 - A \in M_N(\mathcal{W}^{-\infty}), \quad . \tag{19}$$

$z, w \in \mathbb{C}$.

To prove Theorem 7.2, we again proceed as in [12]. This is contained in the following lemma.

Let us fix some notation before proceeding with the proof. We begin by choosing $A(z)$ as in Proposition 7.1. We may assume also that $A(z) = I_N$. Then we write $A(z)A(w) = A(z+w) + F(z,w)$, with $F : \mathbb{C}^2 \to \mathcal{W}^{-\infty}$ a holomorphic function. Let

$$P := [\partial_z A(z)]|_{z=0} \quad \text{and} \quad \mathrm{Q}(z) := [\partial_w F(z,w)]|_{w=0}.$$

Then

$$P \in q(S^{\epsilon}_{1,0}) + \mathcal{W}^{-\infty}, \ \ \epsilon > 0, \quad \text{and} \quad \partial_z A(z) = A(z)P - Q(z). \tag{20}$$

Lemma 7.3. *Let $P := [\partial_z A(z)]|_{z=0} \in q(S^{\epsilon}_{1,0}) + \mathcal{W}^{-\infty}$, $\epsilon > 0$, be as above. Then there exists a unique special holomorphic family of operators $A(z) \in \mathcal{W}^{mz}$, defined on $\mathbb{C}$, such that $A_0 = I$,*

$$\partial_z A_z = A_z P \quad \textit{and} \quad A_z - A(z) \in \mathcal{W}^{-\infty}.$$

This family then also satisfies $A_zA_w = A_{z+w}$ and $A_zP = PA_z$, for any $z, w \in \mathbb{C}$.

Proof. Let $A_1(z)$ be a special holomorphic family, defined in a small neighborhood of 0 in $\mathbb{C}$ such that $A_1(-z) = A(z)^{-1}$, for all z close to 0. The existence of this family is given by Proposition 6.6 above. Then the function $z \mapsto Q(z)A(z^{-1}) = Q(z)A_1(-z) \in \mathcal{W}^{-\infty}$ is holomorphic, by Proposition 6.3(ii).

Let $\| \cdot \|_n$ be the sequence of Banach algebra norms on $\mathcal{W}^{-\infty}$ defining the topology on $\mathcal{W}^{-\infty}$, as in Condition (ψ). Denote by X_n be the completion of $\mathcal{W}^{-\infty}$ in the norm $\| \cdot \|_n$, and let $R(z) \in X_n$, with z in a small neighborhood of 0 in $\mathbb{C}$ satisfy the *holomorphic* differential equation

$$\partial_z R(z) = (I + R(z))Q(z)A(z)^{-1}. \tag{21}$$

with initial condition $R(0) = 0$. Because this equation is linear, a solution will exist on the domain Ω of $Q(z)A(z)^{-1}$, by the existence theorem for ordinary differential equations in Banach spaces. This solution does not depend on choice of the norm $\| \cdot \|_n$, and hence $R \in \mathcal{W}^{-\infty}$. Then $A_z := (I + R(z))A(z)$ satisfies the differential equation $A_z' = A_zP$, for $z \in \Omega$, as operators on $\mathcal{W}^{-\infty}\mathcal{H}$. This solution is unique.

Similarly, the solution of the differential equation $A(z)' = A(z)P$ with initial condition $A(0) = B$ is unique and is given by $A(z) = BA_z$. This can be used to prove that

$$A_zA_{z'} = A_{z+z'}, \tag{22}$$

by differentiating with respect to z'. (Here $|z|, |z'| < \epsilon$, for some $\epsilon > 0$ small enough such that the ball of radius 2ϵ is contained in Ω.) We define then $A_z = (A_{z/n})^n$, where $n > |z|/\epsilon$.

By differentiating Equation (22) with respect to z, we obtain the relation $A_zP = PA_z$, for any $z \in \mathbb{C}$. $\square$

7.2. **Semiclassical estimates.** We have now proved Theorem 7.2. Before proceeding to construct the complex powers, we need to establish some asymptotic formulas, in the spirit of the estimates in [40] for parameter dependent pseudodifferential operators. Recall that we assume in this section that $\mathcal{W}$ is an extended Weyl algebra satisfying Condition (σ).

Let $\mathcal{P}_0(a,b) := q(a)q(b) - q(ab)$. Below we use crucially that

$$\mathcal{P}_0 : S^m_{1,0} \times S^{m'}_{1,0} \to \mathcal{W}^{m+m'-1} \tag{23}$$

is continuous (Axiom v). We fix an elliptic real-valued symbol $c \in S^m_{1,0}$, $m > 0$. (We note however that our arguments remain valid if $\arg c$ is bounded away from α, for some $\alpha \in [0, 2\pi]$.) We shall need the following lemma, which is a particular case of [14, Lemma 2]. We include a proof, for completeness.

Lemma 7.4. *The family $t \mapsto (c+it)^{-1}$, $|t| \geq 1$, is bounded in $S^{-m}_{1,0}$ and, moreover, $(c+it)^{-1} \to 0$ in $S^{-m+\epsilon}_{1,0}$, for any $\epsilon > 0$.*

Proof. First, using Lemma 1.1, we reduce to the case $m = 1$ and $c = x \in S^1_{1,0}(\mathbb{R})$.

Then we notice that

$$|(x + \imath t)|^{-1} \leq C\langle x\rangle^{-a}|t|^{a-1} \tag{24}$$

for any $t \geq 1$, $0 \leq a \leq 1$, and $x \in \mathbb{R}$, where C is a constant independent of t, x, or a. Since $\partial_x^k(x + \imath t)^{-1} = (-1)^k k!(x + \imath t)^{-k-1}$, we obtain that $|x^{1+k}\partial_x^k(x + \imath t)^{-1}|$ is bounded, by using (24) for $a = 1$.

To check the last part, we can assume $\epsilon < 1$. Using (24) for $a = 1 - \epsilon$, we obtain

$$|x^{1-\epsilon+k}\partial_x^k(x + \imath t)^{-1}| \leq Ck!|t|^{-\epsilon}$$

for $|t| \geq 1$. This completes the proof. □

The following is an immediate consequence of the above lemma.

Corollary 7.5. *The family $q(c+it)q((c+it)^{-1}) - I$ is bounded in $\mathcal{W}^{-1}$ for $|t| \geq 1$. The proof that it converges to 0 in $\mathcal{W}^{-1+\epsilon}$, for any $\epsilon > 0$, is completely similar.*

Proof. First, $q((c + it)^{-1})$ is bounded in $\mathcal{W}^{-m}$, by lemma 7.4. Since $q(c) \in \mathcal{W}^m$, we obtain from Proposition 2.2(ii) that

$$q(c)q((c + it)^{-1}) - q(c(c + it)^{-1})$$

is bounded in $\mathcal{W}^{-1}$ by (23). Thus,

$$\begin{aligned}
&q(c + it)q((c + it)^{-1}) - I \\
&\quad= \left[q(c)q((c + it)^{-1}) - q(c(c + it)^{-1})\right] + \left[q(c(c + it)^{-1}) + itq((c + it)^{-1})\right] - I \\
&\quad= \left[q(c)q((c + it)^{-1}) - q(c(c + it)^{-1})\right] + q((c + it)(c + it)^{-1}) - q(1) \\
&\quad= q(c)q((c + it)^{-1}) - q(c(c + it)^{-1})
\end{aligned}$$

is bounded in $\mathcal{W}^{-1}$ and converges to 0 in $\mathcal{W}^{-1+\epsilon}$, for any $\epsilon > 0$. □

We are now ready to prove one of our main results on resolvents in the framework of extended Weyl algebras. Recall that in this section we are assuming $\mathcal{W}$ to satisfy condition (σ).

Theorem 7.6. *Assume $\mathcal{W}$ satisfies Condition (ψ). Let $T = q(c) + R$, with $R \in \mathcal{W}^{-\infty}$ and $c \in S^m_{1,0}$ real valued, elliptic, as above. Then*

$$(T + \imath t)^{-1} = q((c + \imath t)^{-1})(I + R_1(t)),$$

where $R_1(t)$ is bounded in $\mathcal{W}^{-1}$ and converges to 0 in $\mathcal{W}^0$, as $|t| \to \infty$.

In particular, $(T + \imath t)^{-1}$ is bounded in $\mathcal{W}^{-m}$ for $|t| \geq 1$.

Proof. Let us write $T = q(c)+R$, with $R \in \mathcal{W}^{-\infty}$ and $c \in S^m_{1,0}$ real valued, elliptic, as above. Then

$$R_2(t) := q((c + \imath t)^{-1})(T + \imath t) - I = q((c + \imath t)^{-1})q(c + \imath t) + q((c + \imath t)^{-1})R - I \to 0$$

in $\mathcal{W}^{-1+\epsilon}$, for any $\epsilon > 0$. In particular, if $\epsilon = 1$, we obtain that $\|R_2(t)\| \to 0$ as $|t| \to \infty$, by Proposition 2.2(ii). Hence $I+R_2(t)$ will be invertible on $\mathcal{H}$ for $|t|$ large. Assumption (ψ) together with Theorem 5.4 show that the set of invertible elements of the Fréchet algebra $\mathcal{W}^0$ is open. Hence, by Banach's theorem mentioned also earlier, inversion is continuous on $\mathcal{W}^0$. Using the spectral invariance of $\mathcal{W}^0$ (again by Theorem 5.4) and the continuity of the inversion for the Fréchet algebra $\mathcal{W}^0$, we obtain that $(I + R_2(t))^{-1} = I + R_1(t)$, with $R_1(t) \in \mathcal{W}^0$ and $R_1(t) \to 0$ in $\mathcal{W}^0$. This completes the proof. □

Corollary 7.7. *Let $T \in \mathcal{W}^m$, $m > 0$, $T^\sharp = T$, elliptic. Then there exists a parametrix family $t \mapsto G_t$, such that G_t is bounded in $\mathcal{W}^{-m}$, converges to 0 in $\mathcal{W}^{-m+\epsilon}$, for any $\epsilon > 0$, and $(T + \imath t)G_t - I$ is bounded in $\mathcal{W}^{-\infty}$.*

Proof. Let $E_1(t) = I - q(c + it)q((c + it)^{-1}) - R$, so $E_1(t)$ is bounded in $\mathcal{W}^{-1}$. Then the Neumann series $\sum_j E_1(t)^j$ can be summed asymptotically, uniformly in t, to get a bounded family $E(t) \in \mathcal{W}^{-1}$. Thus $F_N(t) = E(t) - \sum_{j \leq N-1} E_1(t)^j$ is uniformly bounded in $\mathcal{W}^{-N}$, so

$$\begin{aligned}&(I - E_1(t))(I + E(t)) - I\\ &\quad = (I - E_1(t))(I + \sum_{j\leq N-1} E_1(t)^j + F_N(t)) = E_1(t)^N + (I - E_1(t))F_N(t)\end{aligned}$$

is bounded in $\mathcal{W}^{-N}$ as well. Let $G_t = q((c + it)^{-1})(I + E(t))$. Then

$$(q(c) + it)G_t - I = (I - E_1(t))(I + E(t)) - I$$

is bounded in $\mathcal{W}^{-N}$ for all N, hence in $\mathcal{W}^{-\infty}$.

The convergence to 0 is proved similarly. □

Of course, a similar calculation gives a left parametrix, and shows that these may be taken equal. The parametrix identity then yields the following.

Proposition 7.8. *There exists a bounded family of operators $P_t \in \mathcal{W}^{-m}$ such that $(q(c) + it)^{-1} - P_t$ is uniformly bounded as a map between any Sobolev spaces, in fact*

$$\|(q(c) + it)^{-1} - P_t\|_{\mathcal{L}(H^{(l)},H^{(l')})} \leq C|t|^{-1}.$$

Proof. We know that $(q(c)+it)G_t - I = E_t$ and $G_t(q(c)+it) - I = F_t$ are bounded in $\mathcal{W}^{-\infty}$. Thus,

$$\begin{aligned}(q(c) + it)^{-1} &= (G_t(q(c) + it) - F_t)(q(c) + it)^{-1} = G_t - F_t(q(c) + it)^{-1}\\ &= G_t - F_t(q(c) + it)^{-1}((q(c) + it)G_t - E_t)\\ &= G_t - F_tG_t + F_t(q(c) + it)^{-1}E_t.\end{aligned}$$

The first two terms are bounded in $\mathcal{W}^{-m}$, respectively, $\mathcal{W}^{-\infty}$. Moreover,

$$\|(q(c)+it)^{-1}\|_{\mathcal{L}(L^2,L^2)} \leq |t|^{-1},$$

in particular bounded, and $E_t \in \mathcal{L}(H^{(l)},\mathcal{H})$, $F_t \in \mathcal{L}(\mathcal{H},H^{(l')})$ are bounded, so the last term has the desired properties. □

7.3. **Complex powers.** We shall now use this theorem to construct complex powers of positive definite operators in $\mathcal{W}$.

Recall that a (possibly unbounded) operator on a Hilbert space $\mathcal{H}$ is *positive* if $(A\xi,\xi) \geq 0$ for any ξ in the domain of A, and is called *positive definite* if $(A\xi,\xi) \geq \epsilon(\xi,\xi)$, for some $\epsilon > 0$, independent of ξ in the domain of A. If $A \in M_N(\mathcal{W}^m)$ is elliptic and positive, then we can define the complex powers A^z by the spectral theorem, as above. To prove that A^z are pseudodifferential operators, we assume below that A is also invertible (and hence positive definite). (Recall that every positive $A \in \mathcal{W}$ is automatically essentially self-adjoint by Proposition 2.2.)

We are now ready to state and prove the main result of this section.

Theorem 7.9. *Let $\mathcal{W}$ be an extended Weyl algebra satisfying Conditions (σ) and (ψ). Let $m \geq 0$ and $A \in M_N(\mathcal{W}^m)$ be positive definite and elliptic. Assume $\sigma^{(m)}(A) > 0$. Then $A^z \in M_N(\mathcal{W}^{mz})$, for all $z \in \mathbb{C}$, and defines a special holomorphic family such $\sigma^{(mz)}(A^z) = \sigma^{(m)}(A)^z$.*

In particular, $A^z = q(a(z)r^{mz}) + R(z)$, with $a(z) \in \mathcal{O}(\mathbb{C}, S^0_{cl})$ and $R(z) \in \mathcal{O}(\mathbb{C},\mathcal{W}^{-\infty})$.

Proof. The last part is nothing but the definition of a special holomorphic family (Definition 6.2 and Proposition 6.3). We shall assume $N = 1$, for simplicity. Recall that $T^\sharp := T^*|_{\mathcal{W}^{-\infty}\mathcal{H}}$ (Lemma 2.1).

The positivity of A shows that $\sigma^{(m)}(\xi) > 0$ if $\xi \neq 0$. We can then define the powers $\sigma^{(m)}(A)^z$. Then we apply Proposition 7.1 to $a(z) = \sigma^{(m)}(A)^z$, obtaining the family $A(z)$. By replacing $A(z)$ with $1/2(A(z) + A(\overline{z})^\sharp)$, if necessary, we can also assume $A(\overline{z}) = A(z)^\sharp$.

Then we construct A_z as in the proof of Lemma 7.3. The family $A^\sharp_{\overline{z}}$ will satisfy the same differential equation as A_z (namely $\partial_z A_z = A_z P = P A_z$, using the notation of Lemma 7.3). By the uniqueness of the solutions of this equation with $A_0 = I$, we obtain

$$A_{\overline{z}} = A^\sharp_z. \tag{25}$$

The resulting family $A_z \in \mathcal{W}^{mz}$ is then a special holomorphic family satisfying

$$A_0 = I, \quad A_1 - A \in \mathcal{W}^{-\infty}, \quad \sigma^{(mz)}(A_z) = a(z), \quad \text{and} \quad A_{z+w} = A_z A_w, \quad z,w \in \mathbb{C}.$$

It satisfies the additional condition $A^*_{\imath s} = A_{-\imath s}$, for $s \in \mathbb{R}$, by Equation (25). Since the function $A_z\xi$ is holomorphic for $\xi \in \mathcal{W}^{-\infty}\mathcal{H}$, we obtain that $A_{\imath s}$, $s \in \mathbb{R}$, is a strongly continuous one-parameter group of unitaries.

Let us prove now that $A_z = (A_1)^z$. By Stone's theorem [32], we have that $A_{\imath s} = e^{\imath s T}$, where T is a self-adjoint operator and $\partial_s A_{\imath s}\xi = A_{\imath s} T\xi$, for any ξ in the domain of T (this is the actual definition of the domain of T). In particular, the domain of T contains $\mathcal{W}^{-\infty}\mathcal{H}$ (because $A_{is}\xi$ is differentiable for $\xi \in \mathcal{W}^{-\infty}\mathcal{H}$) and $T = P$ on $\mathcal{W}^{-\infty}\mathcal{H}$, by the Cauchy-Riemann equations.

We claim that $\sigma(T) = \sigma(P) \subset [-M,\infty)$. Indeed, we know from Corollary 6.4 that the function $z \mapsto \|A_z\|$ is bounded on the compact subsets of $\mathrm{Re}\,(z) \leq 0$. Let

$\|A_z\| \leq M$, if $|z| \leq 1$ and $\mathrm{Re}\,(z) \leq 0$. Then $\|A_z\| \leq M^{|z|+1}$ for all $\mathrm{Re}\,(z) \leq 0$. Let

$$R(\lambda) := \int_0^\infty e^{\lambda t} A_{-t} dt, \tag{26}$$

which is convergent if $\mathrm{Re}\,(\lambda) < -M$ and satisfies $(P-\lambda)R(\lambda) = R(\lambda)(P-\lambda) = I$.

We now check that $A_{\imath s+t}\xi = e^{(\imath s+t)T}\xi$, for any $s,t \in \mathbb{R}$ and any $\xi \in \mathcal{W}^{-\infty}\mathcal{H}$. We have already checked that this is true for $t = 0$. Fix s and ξ. Because $\sigma(T) \subset [-M,\infty)$, $e^{(\imath s+t)T}$ is bounded for $t \leq 0$, and hence the function

$$F(t) := A_{\imath s+t}\xi - e^{(\imath s+t)T}\xi$$

is defined and continuous for $t \leq 0$. By the same argument, F is differentiable when $t < 0$. Its derivative is $F'(t) = 0$, for $t < 0$, because $T\xi = P\xi$, $\xi \in \mathcal{W}^{-\infty}\mathcal{H}$. This proves that $F(t) = 0$ for all $t \leq 0$. Therefore $A_{\imath s+t}\xi = e^{(\imath s+t)T}\xi$, for any $s,t \in \mathbb{R}$, $t \leq 0$, and any $\xi \in \mathcal{W}^{-\infty}\mathcal{H}$. For $t > 0$, we have

$$A_{\imath s+t}\xi = (A_{-\imath s-t})^{-1}\xi = (e^{-(\imath s+t)T})^{-1}\xi = e^{(\imath s+t)T}\xi,$$

for any $\xi \in \mathcal{W}^{-\infty}\mathcal{H}$.

Finally,

$$A_z = e^{\imath zT} = (e^{\imath T})^z = (A_1)^z.$$

We now use the results of the previous subsection. The resolvent identity gives for $\mathrm{Re}\,(z) < -1$ that

$$A^z = (A_1)^z + (A^z - (A_1)^z) = A_z + \frac{1}{2\pi i}\int_\gamma \lambda^z(\lambda I - A)^{-1}(A_1 - A)(\lambda I - A_1)^{-1} d\lambda.$$

We can now use then Theorem 7.6 to conclude that the right hand side of the above integral converges in $\mathcal{W}^{-\infty}$.

The full conclusion follows using $A^{z+k} = A^k A^z$ with $\mathrm{Re}\, z < -1$. □

This yields the following results (due to Seeley [38], Schrohe [35], and Kordyukov [18]).

Corollary 7.10. *Let $A \in \Psi^m(M)$ (respectively, $A \in \Psi^m_{sc}(\mathbb{R}^n)$, or $A \in B\Psi^m(M)$) (see Examples 1-3). Assume $A > 0$, $m \geq 0$, and A elliptic if $m > 0$. Then $A^z \in \Psi^m(M)$ (respectively, $A \in \Psi^z_{sc}(\mathbb{R}^n)$, or $A \in B\Psi^z(M)$).*

Appendix A. Semi-ideals and holomorphic functional calculus

For the convenience of the reader, we include a special case of a construction in [19] that was used in Section 3 to define an algebra that is closed under holomorphic functional calculus containing a given Weyl algebra.

Recall that a subspace $\mathcal{J} \subseteq \mathcal{B}$ of a unital algebra $\mathcal{B}$ is said to be a *semi-ideal* provided we have $xby \in \mathcal{J}$ for all $x,y \in \mathcal{J}$ and all $b \in \mathcal{B}$. Suppose in addition that $\mathcal{B}$ is a unital C^*-algebra. Then we have $f(x) \in \mathcal{J}$ for all f that are holomorphic in a neighborhood of the spectrum $\sigma_{\mathcal{B}}(x)$ of $x \in \mathcal{J}$ and satisfy $f(0) = 0$, i.e. the semi-ideal $\mathcal{J}$ is closed under holomorphic functional calculus in $\mathcal{B}$.

Let now $T : \mathcal{H} \supseteq \mathcal{D}(T) \to \mathcal{H}$ be a closed operator and $\mathfrak{A} \subset \mathcal{L}(\mathcal{H})$ be a closed, symmetric subalgebra (*i.e.,* a C^*–subalgebra). We are going to associate a family of semi-ideals $\mathcal{J}_k := \mathcal{J}_k(T)$, $k \geq 0$, to T. Let $\mathcal{J}_0 := \mathfrak{A} \subset \mathcal{L}(\mathcal{H})$ be a C^*-subalgebra, and let $\mathcal{J}_1$ be the space of all $A \in \mathcal{J}_0$ such that

(a) $A(\mathcal{H}) \subseteq \mathcal{D}(T)$ and $\omega^\ell_T(A) := TA \in \mathcal{J}_0$.
(b) There exists $\omega^r_T(A) \in \mathcal{J}_0$ with $\omega^r_T(A)f = ATf$ for all $f \in \mathcal{D}(T)$.

(c) $A(\mathcal{H}) \subseteq \mathcal{D}(T)$, and there exists $\omega_T^{\ell,r}(A) \in \mathcal{J}_0$ such that $\omega_T^{\ell,r}(A)f = TATf$ for all $f \in \mathcal{D}(T)$.

Moreover, let $\mathcal{J}_{k+1}$ be the space of all $A \in \mathcal{J}_k$ with $\omega_T^{\ell}(A), \omega_T^{r}(A), \omega_T^{\ell,r}(A) \in \mathcal{J}_k$. The spaces $\mathcal{J}_k$ are naturally endowed with the following norms: for $A \in \mathcal{J}_0$, let $p_0(A) = \|A\|_{\mathcal{B}(\mathcal{H})}$, and for $A \in \mathcal{J}_{k+1}$ let

$$p_{k+1}(A) := p_k(A) + p_k(\omega_T^{\ell}(A)) + p_k(\omega_T^{r}(A)) + p_k(\omega_T^{\ell,r}(A)) .$$

The projective limit $\mathcal{J}_\infty := \bigcap_{k\geq 0} \mathcal{J}_k$ is given the topology induced by the system of norms $(p_k)_{k\geq 0}$. The main properties of this construction are summarized in the following Proposition; the straightforward proof can be found in [19, 21].

Proposition A.1.

(a) *For $k \geq 0$, $(\mathcal{J}_k, p_k)$ is a unital Banach algebra.*
(b) *$(\mathcal{J}_\infty, (p_k)_{k\geq 0})$ is a submultiplicative Fréchet algebra.*
(c) *The canonical inclusion $\mathcal{J}_k \hookrightarrow \mathcal{J}_0$ is continuous for all $k \geq 0 \cup \{\infty\}$.*
(d) *For $k \geq 0 \cup \{\infty\}$, $\mathcal{J}_k$ is a semi-ideal in the C^*-algebra $\mathfrak{A}$; in particular, $\mathcal{J}_k$ is closed under holomorphic functional calculus in $\mathcal{B}(\mathcal{H})$. Moreover, the canonical map*

$$\mathcal{J}_k \times \mathfrak{A} \times \mathcal{J}_k \to \mathcal{J}_k$$

is jointly continuous.

Remark A.2. Note that it is not clear, and in general not true that the spaces $\mathcal{J}_k$ are symmetric subspaces of $\mathcal{B}(\mathcal{H})$. However, we easily obtain this property by considering the spaces

$$\mathcal{J}_k^* := \{A \in \mathcal{J}_k : A^* \in \mathcal{J}_k\}\,, k \geq 0 \cup \{\infty\} .$$

Note that Proposition A.1 remains true for these smaller spaces.

Remark A.3. Roughly speaking, $A \in \mathfrak{A}$ is in $\mathcal{J}_k$ if for all $0 \leq N, N' \leq k$, the operators $T^N A T^{N'}$ extend to bounded operators on $\mathcal{H}$ that belong to the given C^*-subalgebra $\mathfrak{A} \subseteq \mathcal{B}(\mathcal{H})$.

Furthermore, note that naturally associated to $T : \mathcal{H} \supseteq \mathcal{D}(T) \to \mathcal{H}$ there is a discrete scale of Hilbert spaces, namely

$$\begin{aligned} \mathcal{H}_0(T) &:= \mathcal{H} \\ \mathcal{H}_1(T) &:= \mathcal{D}(T) \\ \mathcal{H}_k(T) &:= \{f \in \mathcal{H}_{k-1}(T) : Tf \in \mathcal{H}_{k-1}(T)\}\,, k \geq 2 . \end{aligned}$$

The spaces $\mathcal{H}_k(T)$ are endowed with the iterated graph norms with respect to T, i.e. $q(f) := \|f\|_{\mathcal{H}}$ and $q_k(f) := q_{k-1}(f) + q_k(Tf)$, $f \in \mathcal{H}_k(T)$, $k \geq 1$. The Hilbert space $\mathcal{H}_k(T)$ is said to be the *Sobolev space of order k associated to T*. The intersection $\mathcal{H}_\infty(T) := \bigcap_{k=0}^{\infty} \mathcal{H}_k(T)$ is endowed with the system of norms $(q_k)_{k\geq 0}$.

Proposition A.4. *For $k \geq 0 \cup \{\infty\}$, the canonical bilinear map*

$$\mathcal{J}_k \times \mathcal{H}_0(T) \longrightarrow \mathcal{H}_k(T) : (A, f) \longmapsto Af$$

is well-defined and continuous.

Proof. This follows immediately by induction from the definitions. ∎

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Universität Hamburg, Fachbereich 11 – Mathematik, Bundesstrasse 55, D - 20146 Hamburg, Germany, http://www.berndammann.de/publications,
E-mail address: `ammann@math.uni-hamburg.de`

Universität Mainz, Fachbereich 17 – Mathematik, D - 55099 Mainz, Germany
E-mail address: `lauter@mathematik.uni-mainz.de`

Pennsylvania State University, Math. Dept., University Park, PA 16802
E-mail address: `nistor@math.psu.edu`

M.I.T., Department of Mathematics, Cambridge, MA 02139
E-mail address: `andras@math.mit.edu`