PSEUDODIFFERENTIAL OPERATORS ON MANIFOLDS WITH A LIE STRUCTURE AT INFINITY

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Abstract. Several interesting examples of non-compact manifolds M_0 whose geometry at infinity is described by Lie algebras of vector fields $\mathcal{V} \subset \Gamma(M; TM)$ (on a compactification of M_0 to a manifold with corners M) were studied by Melrose and his collaborators for instance in [30, 33, 48]. In [1], the geometry of manifolds described by Lie algebras of vector fields – baptised "manifolds with a Lie structure at infinity" there – was studied from an axiomatic point of view. In this paper, we define and study the algebra $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$, which is an algebra of pseudodifferential operators canonically associated to a manifold M_0 with the Lie structure at infinity $\mathcal{V} \subset \Gamma(M; TM)$. We show that many of the properties of the usual algebra of pseudodifferential operators on a compact manifold extend to $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$. We also consider the algebra $\mathrm{Diff}^*_{\mathcal{V}}(M_0)$ of differential operators on M_0 generated by $\mathcal{V}$ and $\mathcal{C}^\infty(M)$, and show that $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$ is a "microlocalization" of $\mathrm{Diff}^*_{\mathcal{V}}(M_0)$. Finally, we introduce and study semi-classical and "suspended" versions of the algebra $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$. our construction solves a problem posed by Melrose in [30].

Contents

Introduction

It is a fundamental problem to study geometric operators on non-compact manifolds. However, in order to obtain stronger and more precise results, one has to restrict oneself to suitable classes of non-compact manifolds M_0. Let M be a compact manifold with corners such that $M_0 = M \smallsetminus \partial M$. In [30], it was formulated a far reaching program to study the analytic properties of geometric differential

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operators on M_0, provided that the geometry at infinity of M_0 is described by a Lie algebra of vector fields $\mathcal{V} \subset \Gamma(M; TM)$. An important ingredient in that program is to define a suitable pseudodifferential calculus on M_0 adapted in a certain sense to $(M, \mathcal{V})$. This pseudodifferential calculus was called a "microlocalization of $\mathrm{Diff}^*_{\mathcal{V}}(M_0)$" in [30], where $\mathrm{Diff}^*_{\mathcal{V}}(M_0)$ is the algebra of differential operators on M_0 generated by $\mathcal{V}$ and $\mathcal{C}^\infty(M)$. (See Section 2.)

In [33] and several other papers, Melrose and his collaborators have completed this program in several important cases [9, 24, 25, 26, 29, 31, 33, 34, 50]. One of the main points is that the geometric operators on manifolds with a Lie structure at infinity identify with degenerate differential operators on the compactification M. This type of differential operators appear naturally, for example, also in the study of boundary value problems on manifolds with singularities. Numerous important results in this direction were obtained also by Schulze and his collaborators, who typically worked in the framework on the Boutet de Monvel algebras. See [42, 41] and the references therein. Other important cases in which this program was completed can be found in [17, 18, 19, 36, 38]. An important motivation for the construction of these algebras, even before [30], was the study of boundary value problems using singular integral operators and analysis on locally symmetric spaces. See for example [3, 5, 7, 21, 20, 27].

In [1], the geometry of manifolds whose geometry is described by Lie algebras of vector fields $\mathcal{V}$ on a compact manifolds with corners M was studied from an axiomatic point of view, while trying to keep the assumptions on the Lie algebra $\mathcal{V}$ to a minimum. These manifolds were called "manifolds with a Lie structure at infinity" in [1] (we recall their definition in Section 1). In fact, one of the main points of that paper was that a very small set of axioms on $\mathcal{V}$ is actually enough to prove many of the *geometric* properties of M_0 usually needed for analysis. However, we know (for example from the papers mentioned above) that this is not the case for the *analytic* properties of M_0. In that case, one typically imposes additional conditions on $\mathcal{V}$. See [30, 33] for examples.

Nevertheless, the study of the analytic properties of geometric differential operators on M_0 that depend only on the principal symbol should require only few additional assumptions on $\mathcal{V}$, if any. It is one of the purposes of this paper to identify analytic properties of differential operators in $\mathrm{Diff}^*_{\mathcal{V}}(M_0)$ that do not require *any* additional assumptions on $\mathcal{V}$. For example, we give an elementary definition of an algebra $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$ of pseudodifferential operators on M_0 that is canonically associated to the manifold with a Lie structure at infinity M_0 and microlocalizes $\mathrm{Diff}^*_{\mathcal{V}}(M_0)$. The existence of this algebra was predicted by Melrose in [30]. We also show that the algebra $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$ is invariant under the diffeomorphisms of M_0 obtained by exponentiating the vector fields $X \in \mathcal{V}$ and under conjugation with complex powers of the functions that define the faces of the compactification M.

The explicit construction of the algebra $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$ microlocalizing $\mathrm{Diff}^*_{\mathcal{V}}(M_0)$ in the sense of [30] is, roughly, as follows. First, $\mathcal{V}$ defines an extension of TM_0 to a vector bundle $A \to M$ ($M_0 = M \smallsetminus \partial M$). Denote $V_r := \{d(x,y) < r\} \subset M_0^2$ and $(A)_r = \{v \in A, \|v\| < r\}$. Let $r > 0$ be less than the injectivity radius of M_0 and $V_r \ni (x,y) \mapsto (x, \tau(x,y)) \in (A)_r$ be a local inverse of the Riemannian exponential map $TM_0 \ni v \mapsto \exp_x(-v) \in M_0 \times M_0$. Let χ be a smooth function on A with support in $(A)_r$, $\chi = 1$ on $(A)_{r/2}$. For any $a \in S^m_{1,0}(A^*)$, we define

$$(1)\quad \big[a_\chi(D)u\big](x) = (2\pi)^{-n}\int_{M_0}\left(\int_{T^*_x M_0} e^{i\tau(x,y)\cdot\eta}\chi(x,\tau(x,y))a(x,\eta)u(y)\,d\eta\right)dy.$$

The algebra $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$ is then generated linearly by the operators $a_\chi(D)$ and $b_\chi(D)\exp(X_1)\ldots\exp(X_k)$, $a \in S^\infty(A^*)$, $b \in S^\infty(A^*)$, and $X_j \in \mathcal{V}$. (We need to introduce the operators $b_\chi(D)\exp(X_1)\ldots\exp(X_k)$, where $\exp(X_j)$ is the exponential of the vector field X_j, to make our space closed under products.) The definition of these operators of order $-\infty$ is in fact the crucial part of the construction.

In [30] Melrose outlined the construction of a pseudodifferential calculus on manifolds with Lie structure at infinity provided certain manifolds with corners (Blow-ups) can be constructed. In the present paper we fill this important gap by using a deep result of Crainic and Fernandez [6] on the integration of Lie algebroids.

A closely related situation is encountered when one considers a product of a manifold with a Lie structure at infinity M_0 by a Lie group G and operators G invariant on $M_0 \times G$. We obtain in this way an algebra $\Psi^\infty_{1,0,\mathcal{V}}(M_0;G)$ of G–invariant pseudodifferential operators on $M_0 \times G$ with similar properties. The algebra $\Psi^\infty_{1,0,\mathcal{V}}(M_0;G)$ arises in the study of the analytic properties of differential geometric operators on some higher dimensional manifolds with a Lie structure at infinity. When $G = \mathbb{R}^q$, this algebra is essentially one of Melrose's suspended algebras. In fact, it plays the same role as the suspended algebras, namely it typically appears as a quotient of the algebra $\Psi^\infty_{1,0,\mathcal{V}}(M_0')$, for a suitable manifold with a Lie structure at infinity M_0'. These issues will be, however, addressed in a future publication. We have $\Psi^\infty_{1,0,\mathcal{V}}(M_0;G) = \Psi^\infty_{1,0,\mathcal{V}}(M_0)$ when G is reduced to a point. We also introduce a semi-classical variant of the algebra $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$, denoted $\Psi^\infty_{1,0,\mathcal{V}}(M_0[[h]])$, consisting of semi-classical families of operators in $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$.

In a forthcoming paper, we shall prove that the algebras $\Psi^\infty_{1,0,\mathcal{V}}(M_0;G)$ give rise to "extended Weyl algebras," (extended Weyl algebras were introduced in [2] following Guillemin's original definition [11]). The results of [2] then will allow us to show that the complex powers of strictly positive, elliptic operators $P \in \Psi^\infty_{1,0,\mathcal{V}}(M_0;G)$ are again pseudodifferential operators in a slight enlargement of these algebras. Moreover, this will also allow us to to construct a canonical scale of Sobolev spaces on M_0 and to prove the boundedness of our pseudodifferential operators as operators on this scale.

The paper is organized as follows: In Section 1 we recall the definition of manifolds with a Lie structure at infinity and some of their basic properties. In Section 2 we define the space $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$, where (M, M_0, A) is a manifold with a Lie structure at infinity and $\mathcal{V} = \Gamma(M, A)$. In Section 3 we prove that $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$ is an algebra. We also prove in this section several other properties of this algebra. In Section 5 we define the algebras $\Psi^\infty_{1,0,\mathcal{V}}(M_0[[h]])$ and $\Psi^\infty_{1,0,\mathcal{V}}(M_0;G)$, which are generalizations of the algebra $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$. The first of these two algebras consists of the semi-classical (or adiabatic) families of operators in $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$. The second algebra is a subalgebra of the algebra of G–invariant, properly supported pseudodifferential operators on $M_0 \times G$, where G is a Lie group.

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1. Manifolds with Lie structure at infinity

For the convenience of the reader, let us recall the definition of a Riemannian manifold with a Lie structure at infinity and some of its basic properties.

1.1. **Preliminaries.** In the sequel, by a *manifold* we shall always understand a C^∞-manifold *possibly with corners*, whereas a *smooth manifold* is a C^∞-manifold *without corners*. By definition, every point p in a manifold with corners M has a coordinate neighborhood diffeomorphic to $[0,\infty)^k \times \mathbb{R}^{n-k}$ such that the transition functions are smooth up to the boundary. We call p then a point of *boundary depth at most k* and write $\operatorname{depth}(p) \le k$. Points p with $\operatorname{depth}(p) \le k$ but not $\operatorname{depth}(p) \le k-1$ are said to be of *boundary depth k*. (This terminology is in agreement with the terminology for stratified spaces, if we stratify a manifold with corners by its open subfaces, see [23].) The closure of a connected component of points of boundary depth k is called a *face of codimension k*. Faces of codimension 1 are also called *hyperfaces*. For simplicity, we always assume that each hyperface H of a manifold with corners M is an embedded submanifold and has a defining function, that is, that there exists a smooth function $x_H \ge 0$ on M such that

$$H = \{x_H = 0\} \quad \text{and} \quad dx_H \neq 0 \quad \text{on} \quad H.$$

For the basic facts on the analysis on manifolds with corners we refer to the forthcoming book [28]. We denote by ∂M the union of all non-trivial faces of M. Usually, we write M_0 for the interior of M, *i.e.*, $M_0 := M \smallsetminus \partial M$.

A Lie subalgebra $\mathcal{V} \subseteq \Gamma(M, TM)$ of the Lie algebra of all smooth vector fields on M is said to be *a structural Lie algebra of vector fields* provided it is a finitely generated, projective $\mathcal{C}^\infty(M)$-module and each $V \in \mathcal{V}$ is tangent to all hyperfaces of M. (We shall denote the sections of a vector bundle $V \to X$ by $\Gamma(X, V)$, unless X is understood, in which case we shall write simply $\Gamma(V)$.) By the Serre-Swan theorem [15], there exists a smooth vector bundle $A_\mathcal{V} \to M$ together with a natural map

$$\begin{array}{ccc} \varrho_\mathcal{V} : A_\mathcal{V} & \longrightarrow & TM \\ & \searrow \quad \swarrow & \\ & M & \end{array} \tag{2}$$

such that $\mathcal{V} = \varrho_\Gamma(\Gamma(A_\mathcal{V}))$. The vector bundle $A_\mathcal{V}$ turns out to be a *Lie algebroid* over M. (Recall that a vector bundle $A \to M$, together with a Lie algebra structure on the space $\Gamma(A)$ of smooth sections of A and a bundle map $\varrho : A \to TM$ is called a *Lie algebroid* provided that $\varrho_\Gamma([X,Y]) = [\varrho_\Gamma(X), \varrho_\Gamma(Y)]$ and $[X, fY] = f[X,Y] + (\varrho_\Gamma(X)f)Y$ for any smooth sections X and Y of A and any smooth function f on M. Here, we have written $\varrho_\Gamma : \Gamma(A) \to \Gamma(TM)$ for the map naturally induced by the map $\varrho : A \to TM$. The map ϱ is called the *anchor of* A.)

We thus see that there exists an equivalence between structural Lie algebras of vector fields $\mathcal{V} = \Gamma(A_\mathcal{V})$ and Lie algebroids $\varrho : A \to TM$ such that the induced map $\varrho_\Gamma : \Gamma(M, A) \to \Gamma(M, TM)$ is injective and has range in the Lie algebra $\mathcal{V}_b(M)$ of all vector fields that are tangent to all hyperfaces of M. Since the induced map ϱ_Γ for the Lie algebroid associated to a structural Lie algebra is one-to-one, we will write Xf instead of $\varrho_\Gamma(X)f$ for the action of the sections of a Lie algebroid on functions.

Definition 1.1. A *Lie structure at infinity* on a smooth manifold M_0 is a triple (M_0, M, A), where M is a compact manifold, possibly with corners, and $\mathcal{V} := \Gamma(M, A) \subset \Gamma(M, TM)$ is a structural Lie algebra of vector fields on M with the following properties:

(a) M_0 is diffeomorphic to the interior $M \smallsetminus \partial M$ of M,
(b) the anchor map $\varrho : A \to TM$ restricts to an isomorphism $A|_{M \smallsetminus \partial M} \to TM|_{M \smallsetminus \partial M}$.

Note that for a given manifold M_0 in general there can exist many Lie structures at infinity. Examples of Lie structures at infinity were discussed in [1]. Some interesting and highly non-trivial examples of Lie structures at infinity on $\mathbb{R}^n$ are obtained from the N-body problem [48].

If M_0 is compact without boundary, then it follows from the above definition that $M = M_0$ and $A_\mathcal{V} = TM$, so a Lie structure at infinity on M_0 gives no additional information on M_0. The interesting cases are thus the ones when M_0 is non-compact.

We identify from now on M_0 with $M \smallsetminus \partial M$ and $A|_{M_0}$ with TM_0. Because A and $\mathcal{V}$ determine each other up to isomorphism, we sometimes specify a Lie structure at infinity on M by the pair (M, A).

Elements in the enveloping algebra $\mathrm{Diff}^*_\mathcal{V}(M)$ of $\mathcal{V}$ are called *$\mathcal{V}$-differential operators on M*; by the injectivity of the induced structural map $\varrho_\mathcal{V} : \Gamma(A_\mathcal{V}) \to \Gamma(TM)$, the algebra of $\mathcal{V}$-differential operators can be realized as a subalgebra of all differential operators on M, in particular they act continuously on the space $\mathcal{C}^\infty(M)$. Moreover, the order of differential operators induces a filtration $\mathrm{Diff}^m_\mathcal{V}(M_0)$, $m \in \mathbb{N}_0$, on the algebra $\mathrm{Diff}^*_\mathcal{V}(M_0)$. Since $\mathrm{Diff}^*_\mathcal{V}(M_0)$ is a $\mathcal{C}^\infty(M)$-module, we can introduce $\mathcal{V}$-differential operators acting between sections of smooth vector bundles $E, F \to M$, $E, F \subset M \times \mathbb{C}^N$ by

$$\mathrm{Diff}^*_\mathcal{V}(M_0; E, F) := e_F M_N(\mathrm{Diff}^*_\mathcal{V}(M_0)) e_E \,, \tag{3}$$

where $e_E, e_F \in M_N(\mathcal{C}^\infty(M))$ are the projections onto E and, respectively, F. It follows that $\mathrm{Diff}^*_\mathcal{V}(M_0; E, E) =: \mathrm{Diff}^*_\mathcal{V}(M_0; E)$ is an algebra that is closed under adjoints and contains all geometric operators on M_0 that are associated to a metric on M_0 that comes from a metric on A. (See [1].)

Since any metric on A induces a natural metric on $TM_0 = A|_{M_0}$, we obtain the following definition.

Definition 1.2. A manifold M_0 with a Lie structure at infinity (M, A) and with metric g_0 on TM_0 obtained from the restriction of a metric g on A is called a *Riemannian manifold with a Lie structure at infinity.*

The geometry of a Riemannian manifold (M_0, g_0) with a Lie structure $(M, \mathcal{V})$ at infinity has been studied in [1]. For instance, (M_0, g_0) is necessarily of infinite volume and complete. Moreover, all the covariant derivatives of the Riemannian curvature tensor are bounded. Under additional mild assumptions, we also know that the injectivity radius is bounded from below by a positive constant, *i.e.,* (M_0, g_0) is of bounded geometry. (A *manifold with bounded geometry* is a Riemannian manifold with positive injectivity radius and with bounded covariant derivatives of the curvature tensor, see [43] and references therein).

On a Riemannian manifold M_0 with a uniform structure at infinity (M_0, M, A), the exponential map $\exp_p : T_pM_0 \to M_0$ is well-defined for all $p \in M_0$ and extends to a differentiable map $\exp_p : A_p \to M$ depending smoothly on $p \in M$. A convenient way to introduce the exponential map is via the geodesic spray, as done in [1]. A related phenomenon is that any vector field $X \in \Gamma(A)$ is integrable. The resulting diffeomorphism of M_0 will be denoted ψ_X.

We define the boundary depth of $p \in M$ as the codimension of the open face containing it. The following proposition is a consequence of those results.

Proposition 1.3. *For any X in $\Gamma(M; A)$, $p \in M$, the boundary depth of $\psi_X(p)$ equals to the boundary depth of p.*

Proof. This follows right away from the assumption that all vector fields in $\mathcal{V}$ are tangent to all faces. □

2. Kohn-Nirenberg quantization and pseudodifferential operators

Throughout this section (M_0, M, A) will be a fixed manifold with Lie structure at infinity and $\mathcal{V} := \Gamma(A)$. In particular, $A := A_{\mathcal{V}}$. *We shall also fix a metric g on $A \to M$, which induces a metric g_0 on M_0.* We are going to introduce a pseudodifferential calculus on M_0 that microlocalizes the algebra of $\mathcal{V}$-differential operators $\mathrm{Diff}^*_{\mathcal{V}}(M_0)$ on M given by the Lie structure at infinity.

2.1. **Riemann-Weyl fibration.** Fix now a Riemannian metric g on the bundle A, and let $g_0 = g|_{M_0}$ be its restriction to the interior M_0 of M. We shall use this metric to trivialize all density bundles on M. Denote by $\pi : TM_0 \to M_0$ the natural projection. Define

$$\Phi : TM_0 \longrightarrow M_0 \times M_0, \quad \Phi(v) := (x, \exp_x(-v)),\ x = \pi(v). \tag{4}$$

Recall that for $v \in T_xM$ we have $\exp_x(v) = \gamma_v(1)$ where γ_v is the unique geodesic with $\gamma_v(0) = \pi(v) = x$ and $\gamma_v'(0) = v$. It is known that there is an open neighborhood U of the zero-section M_0 in TM_0 such that $\Phi|_U$ is a diffeomorphism onto an open neighborhood V of the diagonal $\Delta_{M_0} \subseteq M_0 \times M_0$.

To fix notation, let E be a real vector space together with a metric or a vector bundle with a metric. We shall denote by $(E)_r$ the set of all vectors v of E with $|v| < r$.

We shall also assume from now on that r_0, the injectivity radius of (M_0, g_0), is positive. We know that this is true under some additional mild assumptions and we conjectured that the injectivity radius is always positive [1]. Thus, for each $0 < r \leq r_0$, the restriction $\Phi|_{(TM_0)_r}$ is a diffeomorphism onto an open neighborhood V_r of the diagonal Δ_{M_0}. It is for this reason that we need the positive injectivity radius assumption.

We continue, by slight abuse of notation, to write Φ for that restriction. Following [8] we call Φ a *Riemann-Weyl fibration*, however, note that in [8] the Riemann-Weyl fibrations are defined in a slightly different way; the difference will be of no importance for us. For the sake of completeness note that the inverse of Φ is given by

$$M_0 \times M_0 \supseteq V_r \ni (x, y) \longmapsto (x, \tau(x, y)) \in (TM_0)_r\,,$$

where $\tau(x, y) \in T_xM_0$ is the tangent vector at x to the shortest geodesic $\gamma : [0, 1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

2.2. **Symbols and conormal distributions.** Let $\pi : E \to M$ be a smooth vector bundle with orthogonal metric g. Let

$$\langle \xi \rangle := \big(1 + g(\xi, \xi)\big)^{1/2}. \tag{5}$$

We shall denote by $S^m_{1,0}(E)$ the symbols of type $(1, 0)$ in Hörmander's sense [14]. Recall that they are defined, in local coordinates, by the standard estimates

$$|\partial_x^\alpha \partial_\xi^\beta a(\xi)| \leq C_{K,\alpha,\beta} \langle \xi \rangle^{m-|\beta|}, \quad \pi(\xi) \in K,$$

where K is a compact trivializing subset (*i.e.,* $\pi^{-1}(K) \simeq K \times \mathbb{R}^n$) and α and β are multi-indices. (See any book on pseudodifferential operators or the corresponding discussion in [2].)

Let us briefly recall the definition of classical symbols on E. Let $\overline{E}$ be the radial compactification of E in the sense of [32], i.e. the fiber-bundle obtained by including the *sphere at infinity* $\mathbb{S}(E) := (E \setminus \{0\})/\mathbb{R}_+$ to E. Then the map

$$E \ni \xi \longmapsto \langle \xi \rangle^{-1} \xi \in E$$

identifies E with the set $(E)_1$ of vectors of length < 1 in E. We can thus identify $\overline{E}$ with the set $(E)_{\leq 1}$ of vectors of length ≤ 1 in E. We shall use this identification to define the smooth structure on $\overline{E}$. Then $E \ni \xi \mapsto \langle \xi \rangle^{-1} \in [0, \infty)$ extends to a smooth map ϱ on $\overline{E}$, which is in fact a defining function for the boundary face $\mathbb{S}(E)$ of $\overline{E}$.

A *classical symbol of order* $\mu \in \mathbb{C}$ *on* E is a function $a : E \to \mathbb{C}$ such that $\varrho^\mu a$ extends to a smooth function on $\overline{E}$. We write $S^\mu_{cl}(E)$ for the space of classical symbols of order $\mu \in \mathbb{C}$. We define $\sigma^{(\mu)}(a) := (\varrho^\mu a)|_{\mathbb{S}(E)} \in \mathcal{C}^\infty(\mathbb{S}(E))$. For any $a \in S^\mu_{cl}(E)$ there is a unique μ–homogeneous smooth function a_{prin} on $E \smallsetminus 0$ with $\sigma^{(\mu)}(a) = \sigma^{(\mu)}(\eta a_{\mathrm{prin}})$ where $\eta \in \mathcal{C}^\infty(E)$ vanishes in a neighborhood of the zero section and equals 1 outside another larger neighborhood. The function a_{prin} is called the *homogeneous principal part of* a. As we are always working with the same fixed boundary defining function ϱ we can identify smooth functions on $\mathbb{S}(E)$ with μ–homogeneous smooth function via $\sigma^{(\mu)}$. As a consequence of this, $\sigma^{(\mu)}(a)$ is identified with the homogeneous principal part a_{prin}. No confusion can arise as it will be clear from the context whether $\sigma^{(\mu)}(a)$ will denote a function on $\mathbb{S}(E)$ or a μ–homogeneous function on $E \smallsetminus 0$.

As in the classical theory of pseudodifferential operators in addition to symbols we need the space of inverse Fourier transforms of symbols, i.e. of conormal distributions. Let $X \subset Y$ be an embedded submanifold of a manifold with corners Y. On a small neighborhood V of X in Y we define a structure of a vector bundle over X, such that X is the zero section of V. As a bundle V is isomorphic to the normal bundle of X in Y. Then we define the space of *distributions on* Y *that are conormal of order* m *to* X, denoted $I^m(Y, X)$, to be the space of distributions on M that are smooth on $Y \smallsetminus X$ and, that are, in a tubular neighborhood $V \to X$ of X in Y, the inverse Fourier transforms of elements in $S^m(V^*)$ along the fibers of $V \to X$. (For simplicity, we have ignored the density factors, see [13, 14] for details).

In particular, for $E = A^*$, where $A \to M$ is the vector bundle (Lie algebroid) appearing in the triple (M_0, M, A) (=Lie structure at infinity), we obtain the following. First, recall that we have fixed a metric g on A. Then the inverse of the

Fourier transform $\mathcal{F}^{-1}_{\text{fiber}}$, along the fibers of A^* gives a map

$$\mathcal{F}^{-1}_{\text{fiber}} : S^m_{1,0}(A^*) \longrightarrow \mathcal{C}^{-\infty}(A) := \mathcal{C}^\infty_c(A)' , \quad \langle \mathcal{F}^{-1}_{\text{fiber}} a, \varphi \rangle := \langle a, \mathcal{F}^{-1}_{\text{fiber}} \varphi \rangle, \tag{6}$$

where $\pi : A \to M$ is the canonical projection, $a \in S^m_{1,0}(A^*)$, φ is a smooth, compactly supported function, and

$$\mathcal{F}^{-1}_{\text{fiber}}(\varphi)(\xi) := (2\pi)^{-n} \int_{\pi(\zeta)=\pi(\xi)} e^{\imath\langle \xi, \zeta\rangle} \varphi(\zeta)\, d\zeta.$$

Then $I^m(A, M)$ is the image of $S^m_{1,0}(A^*)$ through the above map. We shall call this space the space of distributions on A conormal to M, following the standard conventions. The spaces $I^m(TM_0, M_0)$ and $I^m(M_0^2, \Delta_{M_0})$ are defined similarly. For more details on conormal distributions (on manifolds with corners) we refer to [13, 14, 45] and the forthcoming book [28].

We shall also use the spaces $I^m(TM_0, M_0)$ and $I^m(M_0^2, \Delta_{M_0})$, which are defined similarly.

The main use of spaces of conormal distributions is in relation to pseudodifferential operators. For example, since we have

$$I^m(M_0^2, \Delta_{M_0}) \subseteq \mathcal{C}^{-\infty}(M_0^2) := \mathcal{C}^\infty_c(M_0^2)' ,$$

we can associate to a distribution in $K \in I^m(M_0^2, \Delta_{M_0})$ a continuous linear map

$$T_K : \mathcal{C}^\infty_c(M_0) \to \mathcal{C}^{-\infty}(M_0) := \mathcal{C}^\infty_c(M_0)', \tag{7}$$

by the Schwartz-kernel theorem. Then a well known result of Hörmander [13, 14] states that T_K is a pseudodifferential operator on M_0 and that all pseudodifferential operators on M_0 are obtained in this way.

Recall now that $(A)_r$ denotes the set of vectors of norm $< r$ of the vector bundle A. We agree to write $I^m_{(r)}(A, M)$ for all $k \in I^m(A, M)$ with $\operatorname{supp} k \subseteq (A)_r$. The space $I^m_{(r)}(TM_0, M_0)$ is defined in an analogous way. Then restriction defines a map

$$\mathcal{R} : I^m_{(r)}(A, M) \longrightarrow I^m_{(r)}(TM_0, M_0). \tag{8}$$

Recall that r_0 denotes the injectivity radius of M_0 and that we assume $r_0 > 0$. Similarly, the Riemann–Weyl fibration Φ of Equation (4) defines, for any $0 < r \le r_0$, a map

$$\Phi_* : I^m_{(r)}(TM_0, M_0) \to I^m(M_0^2, M_0). \tag{9}$$

We shall write $I^m_c(\,\cdot\,, \cdot\,; \cdot\,)$ for all conormal distributions in $I^m(\,\cdot\,, \cdot\,; \cdot\,)$ that are compactly supported. We shall also use the following subscripts

- “cl” to designate the distributions that are “classical,” in the sense that they correspond to classical pseudodifferential operators.
- “c” to denote distributions that have compact support.
- “pr” to indicate operators that are properly supported or distributions that give rise to such operators.

Occasionally, we shall use the double subscripts “cl,pr” and “cl,c”. Note that “c” implies “pr”.

2.3. **Kohn-Nirenberg quantization.** For notational simplicity, we use the metric g_0 on M_0 (obtained from the metric on A) to trivialize the half-density bundle $\Omega^{1/2}(M_0)$. In particular, we identify $\mathcal{C}_c^\infty(M_0, \Omega^{1/2})$ with $\mathcal{C}_c^\infty(M_0)$. Let $0 < r \leq r_0$ be arbitrary. Each smooth function χ, with $\chi = 1$ close to $M \subseteq A$ and support contained in the set $(A)_r$, induces a map $q_{\Phi,\chi} : S^m_{1,0}(A^*) \longrightarrow \mathcal{C}^{-\infty}(M_0^2)$,

$$q_{\Phi,\chi}(a) := \Phi_* \left(\mathcal{R}\left(\chi \mathcal{F}^{-1}_{\text{fiber}}(a)\right)\right), \tag{10}$$

such that

$$q_{\Phi,\chi}(S^m_{1,0}(A^*)) \subset I^m(M_0^2, \Delta_{M_0}).$$

Following Melrose, we call the map $q_{\Phi,\chi}$ the *Kohn-Nirenberg quantization map.* It will play an important role in what follows.

Lemma 2.1. *Let $0 < r \leq r_0$. If χ_j are smooth functions with support $(A)_r$ and $\chi_j = 1$ in some neighborhoods of $M \subseteq A$, then the difference $q_{\Phi,\chi_1}(a) - q_{\Phi,\chi_2}(a)$ has a smooth Schwartz kernel. Moreover, the map $S^m_{1,0}(A^*) \to \mathcal{C}^\infty(A)$ that maps $a \in S^m_{1,0}(A^*)$ to the Scwartz kernel of $q_{\Phi,\chi_1}(a) - q_{\Phi,\chi_2}(a)$ is continuous, where the right hand side is endowed with the topology of uniform C^∞-convergence on compact subsets.*

Proof. Since the singular support of $q_{\Phi,\chi_j}(a)$ is contained in the diagonal Δ_{M_0} and $\chi_1 - \chi_2$ vanishes there, we have $\operatorname{sing\,supp}(q_{\Phi,\chi_1}(a) - q_{\Phi,\chi_2}(a)) = \emptyset$.

To prove the continuity of the map $S^m_{1,0}(A^*) \ni a \mapsto q_{\Phi,\chi_1}(a) - q_{\Phi,\chi_2}(a) \in \mathcal{C}^\infty(A)$, it is enough, using a partition of unity, to assume that $A \to M$ is a trivial bundle. Then our result follows from the standard estimates for oscilatory integrals (*i.e.,* by formally writing $|v|^2 \int e^{\imath\langle v,\xi\rangle} a(\xi) d\xi = \int (D_\xi^2 e^{\imath\langle v,\xi\rangle}) a(\xi) d\xi$ and then integrating by parts, see [14, 46, 47] for example). □

We now verify that the quantization map $q_{\Phi,\chi}$ (Equation (10)) gives rise to pseudodifferential operators.

Lemma 2.2. *Let $r \leq r_0$ be arbitrary. For each $a \in S^m_{1,0}(A^*)$ and each $\chi \in \mathcal{C}_c^\infty((A)_r)$ with $\chi = 1$ close to $M \subseteq A$, the distribution $q_{\Phi,\chi}(a)$ is the Schwartz-kernel of a pseudodifferential operator $a_\chi(D)$ on M_0, which is properly supported if $r < \infty$. If $a \in S^\mu_{cl}(A^*)$, then $a_\chi(D)$ is a classical pseudodifferential operator satisfying*

$$\sigma^{(\mu)}(a_\chi(D)) = \sigma^{(\mu)}(a) \in \mathcal{C}^\infty(T^*M_0 \smallsetminus 0). \tag{11}$$

On the left hand side of this equation, $\sigma^{(\mu)}$ denotes the usual principal symbol.

Proof. Denote also by χ the "multiplication by χ" map $\chi : I^m(TM_0, M_0) \to I^m_{(r)}(TM_0, M_0)$. Then

$$a_\chi(D) = T \circ \Phi_* \circ \mathcal{R} \circ \chi \circ \mathcal{F}^{-1}_{\text{fiber}}(a) := T_{\Phi_*(\mathcal{R}(\chi \mathcal{F}^{-1}_{\text{fiber}}(a)))} = T \circ q_{\Phi,\chi}(a) \tag{12}$$

where T is defined in Equation (7). Hence $a_\chi(D)$ is a pseudodifferential operator by the Hörmander's result mentioned above [13, 14] (stating that the distribution conormal to the diagonal are exactly the Schwartz kernels of pseudifferential operators. Since $\chi\mathcal{R}(a)$ is properly supported, so will be the operator $a_\chi(D)$).

For (11) we use the principal symbol map for conormal distributions [13, 14], and the fact that the restriction of the anchor $A \to TM$ to the interior $A|_{M_0}$ is the identity. (This also follows from the Equation (13) below.) This proves our lemma. □

Let us now make the formula for the induced operator $a_\chi(D) : \mathcal{C}_c^\infty(M_0) \to \mathcal{C}_c^\infty(M_0)$ more explicit. Neglecting the density factors in the formula, for $u \in \mathcal{C}_c^\infty(M_0)$ we obtain

$$a_\chi(D)u(x) = \int_{M_0} (2\pi)^{-n} \int_{T_x^* M_0} e^{i\tau(x,y)\cdot\eta} \chi(x, \tau(x,y)) a(x,\eta) u(y)\, d\eta\, dy\,.$$

Specializing to the case of Euclidean space $M_0 = \mathbb{R}^n$ with the standard metric we have $\tau(x,y) = x - y$, and hence

$$a_\chi(D)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\eta} \chi(x, x-y) a(x,\eta) u(y)\, d\eta\, dy\,, \tag{13}$$

i.e., the well-known formula for the Kohn-Nirenberg-quantization on $\mathbb{R}^n$, if $\chi = 1$. Lemma 2.1 allows us up to an operator with smooth kernel to assume that this is indeed the case.

We have the following simple corollary.

Corollary 2.3. *The map $\sigma_{tot} : S_{1,0}^m(A^*) \to \Psi^m(M_0)/\Psi^{-\infty}(M_0)$,*

$$\sigma_{tot}(a) := a_\chi(D) + \Psi^{-\infty}(M_0)$$

is independent of the choice of the function $\chi \in \mathcal{C}_c^\infty((A)_r)$ used to define $a_\chi(D)$ in Lemma 2.2.

Proof. This follows right away from the previous lemma, Lemma 2.2. □

A first glance at the explicit formula (13) might lead one to believe that the space of operators $a_\chi(D)$, $a \in S_{1,0}^\infty(A^*)$ does not depend on the choice of the Lie structure at infinity for M_0. However, since the symbol $a \in S_{1,0}^m(T^*M_0)$ is the restriction of a smooth function on A^*, the Lie structure at infinity determines how $a(p,\eta)$ stabilizes when p runs to infinity. As explained in [1], there is no canonical Lie structure at infinity for flat Euclidean space. So, (13) yields several possible pseudodifferential calculi on manifold $\mathbb{R}^n$, including the so called "scattering calculus" (also called the "SG-calculus" by some authors), see [4, 10, 33, 38, 40, 44].

In particular, we see that not all pseudodifferential operators in $\Psi^m(M_0)$ are of the form $a_\chi(D)$ for some symbol $a \in S_{1,0}^m(A^*)$, not even if we assume that they are properly supported, because they do not have the correct behavior at infinity. On the other hand, the set $q_{\Phi,\chi}(S_{1,0}^\infty(A^*))$ of all operators of the form $a_\chi(D)$ with $a \in S_{1,0}^*(A^*)$ is not closed under composition. In order to make it closed under composition, we are going to include more (but not all) operators of order $-\infty$ in our calculus.

Since any $X \in \Gamma(A)$ is by definition tangent to all boundary faces of M and M is compact, X generates a global flow $\Psi_X : \mathbb{R} \times M \to M$. Evaluationg at $t = 1$ yields a diffeomorphism denote by

$$\psi_X := \Psi_X(1, \cdot) : M \to M. \tag{14}$$

One should not confuse the flow Ψ_X with the geodesic exponential map, which also plays a prominent role here.

Recall that we have fixed a manifold M_0, a Lie structure at infinity (M, A) on M_0, and a metric g on A. Also, we assumed that the injectivity radius r_0 of the induced metric g_0 on M_0 is positive.

We continue to assume that the injectivity radius r_0 of our fixed manifold with a Lie structure at infinity (M, M_0, A) is strictly positive.

Definition 2.4. Fix $0 < r < r_0$ and $\chi \in \mathcal{C}_c^\infty((A)_r)$ such that $\chi = 1$ in a neighborhood of $M \subseteq A$. For $m \in \mathbb{R}$, the space $\Psi^m_{1,0,\mathcal{V}}(M_0)$ of *pseudodifferential operators generated by the Lie structure at infinity* (M, A) is the linear space of operators $\mathcal{C}_c^\infty(M_0) \to \mathcal{C}_c^\infty(M_0)$ generated by $a_\chi(D)$, $a \in S^m_{1,0}(A^*)$, and $b_\chi(D)\psi_{X_1}\ldots\psi_{X_k}$, $b \in S^{-\infty}(A^*)$ and $X_j \in \Gamma(A)$, $\forall j$.

Similarly, the space $\Psi^m_{cl,\mathcal{V}}(M_0)$ of *classical pseudodifferential operators generated by the Lie structure at infinity* (M, A) is obtained by using classical symbols a in the construction above.

It is implicit in the above definition that the spaces $\Psi^{-\infty}_{1,0,\mathcal{V}}(M_0)$ and $\Psi^{-\infty}_{cl,\mathcal{V}}(M_0)$ are the same. They will typically be denoted by $\Psi^{-\infty}_{\mathcal{V}}(M_0)$. As usual, we shall denote

$$\Psi^{\infty}_{1,0,\mathcal{V}}(M_0) := \cup_{m\in\mathbb{Z}}\Psi^m_{1,0,\mathcal{V}}(M_0) \quad \text{and} \quad \Psi^{\infty}_{cl,\mathcal{V}}(M_0) := \cup_{m\in\mathbb{Z}}\Psi^m_{cl,\mathcal{V}}(M_0).$$

At first sight, the above definition depends on the choice of the metric g on A. However, we shall soon prove that this is not the case.

As for the usual algebras of pseudodifferential operators, we have the following basic property of the principal symbol.

Proposition 2.5. *The principal symbol establishes isomorphisms*

$$\sigma^{(m)} : \Psi^m_{1,0,\mathcal{V}}(M_0)/\Psi^{m-1}_{1,0,\mathcal{V}}(M_0) \to S^m_{1,0}(A^*)/S^{m-1}_{1,0}(A^*) \tag{15}$$

and

$$\sigma^{(m)} : \Psi^m_{cl,\mathcal{V}}(M_0)/\Psi^{m-1}_{cl,\mathcal{V}}(M_0) \to S^m_{cl}(A^*)/S^{m-1}_{cl}(A^*). \tag{16}$$

Proof. This follows from the classical case of the spaces $\Psi^m(M_0)$ using also Equation (11) of Lemma 2.2. □

3. The product

We continue to denote by $(M, M_0, \mathcal{V})$, $\mathcal{V} = \Gamma(A)$, a fixed manifold with a Lie structure at infinity and with positive injectivity radius. In this section we want to show that the spaces $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$ are the images of algebras of pseudodifferntial operators on groupoids (Theorem 3.3). This theorem will have several implications in the following section, in particular it will follow directly that $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$ is closed under multiplication (see Proposition 4.3).

At first, we will use a Proposition that directly follows a theorem by Crainic and Fernandes [6]

Proposition 3.1 (Cranic–Fernandes)**.** *Any Lie algebroid arising from a Lie structure at infinity is actually the Lie algebroid to a differentiable groupoid (*i.e., *it is* integrable*).*

This proposition should be thought of as an analog of Lie's third theorem. Lie's third theorem for Lie algebras states that every finite dimensional Lie algebra is the Lie algebra of a Lie group. However, the analog of Lie's theorem for Lie algebroids does not hold: there are Lie algebroid which are not Lie algebroids to a Lie groupoid [22]. A slightly weaker form of this result, which is however enough for the proof of Melrose's conjecture was obtained in [35].

Remark 3.2. We suspect that any proof of the fact that $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$ is closed under multiplication is equivalent to the integrability of A. In fact, Melrose has implicitely noticed this in [30] for particular M_0, by showing that the kernels of the pseudodifferential operators on M_0 that he constructed naturally live on a modified product space $M^2_{\mathcal{V}}$. In his case $M^2_{\mathcal{V}}$ was a blow-up of the product $M \times M$, and hence was a *larger* compactification of the product $M_0 \times M_0$. The kernels of his operators naturally extended to conormal distributions on this larger product $M^2_{\mathcal{V}}$. The product and adjoint were defined in terms of suitable maps between $M^2_{\mathcal{V}}$ and some fibered product spaces $M^3_{\mathcal{V}}$, which are suitable blow-ups of M^3 and hence larger compactifications of M^3_0. This in principle leads to a solution the problem of microlocalizing $\mathcal{V}$ stated in the introduction whenever one can define the spaces $M^2_{\mathcal{V}}$ and $M^3_{\mathcal{V}}$. Moreover, by requiring these spaces to be compact, Melrose obtained a larger and, in many aspects, a more suitable algebra of pseudodifferential operators then it is provided by the groupoid construction (we will recall the groupoid construction below). However, the problem is that by requiring the spaces $M^2_{\mathcal{V}}$ to be compact, multiplication cannot be defined everywhere (think of the problem of defining the multiplication on the *two-point* compactification of $\mathbb{R}$).

For the convenience of the reader we will give a short summary of the theory of pseudodifferential operators on groupoids. Details and proofs of the statements will be omitted but references will be included.

A *groupoid* is a small category all of whose morphisms are invertible. Let $\mathcal{G}$ denote the set of morphisms and M denote the set of objects of a given groupoid. Then each $g \in \mathcal{G}$ will have a domain $d(g) \in M$ and a range $r(g) \in M$ such that the product $g_1 g_2$ is defined precisely when $d(g_1) = r(g_2)$. Moreover, it follows that the multiplication (or composition) is associative and every element in $\mathcal{G}$ has an inverse. We shall identify the set of objects M with their identity morphisms, so $M \subset \mathcal{G}$. One can think then of a groupoid as being a group, except that the multiplication is only partially defined. By abuse of notation, we shall use the same notation for the groupoid and its set of morphisms ($\mathcal{G}$ in this case).

A *differentiable groupoid* is a groupoid $\mathcal{G}$ such that the space of arrows $\mathcal{G}$ and the space of units M are manifolds with corners, all its structural maps (multiplication, inverse, domain, range) are differentiable, the domain and range maps (*i.e.,* d and r) are submersions. (The definition of a submersion of manifolds with corners is such that the submanifolds $d^{-1}(x)$ and $r^{-1}(x)$ have no corners, for any $x \in M$). Note that M is then an embedded submanifold of $\mathcal{G}$.

The *d–vertical tangent space to $\mathcal{G}$*, denoted $T_{vert}\mathcal{G}$, is the union of the tangent spaces to the fibers of $d : \mathcal{G} \to M$, that is

$$T_{vert}\mathcal{G} := \cup_{x \in M} T\mathcal{G}_x = \ker d_*, \tag{17}$$

the union being a disjoint union. The *Lie algebroid of* $\mathcal{G}$, denoted $A(\mathcal{G})$ is defined to be the restriction of the d–vertical tangent space to the set of units M, that is, $A(\mathcal{G}) = \cup T_x\mathcal{G}_x$, $x \in M$. This shows that the space of sections of $A(\mathcal{G})$ identifies canonically with the space of sections of the d-vertical tangent bundle (= d-vertical vector fields) that are *right invariant* with respect to the action of $\mathcal{G}$. This also implies a canonical isomorphism between the vertical tangent bundle and the lifting of $A(\mathcal{G})$ via the range map r to $\mathcal{G}$:

$$r^* A(\mathcal{G}) \simeq T_{vert}\mathcal{G}. \tag{18}$$

The structure of Lie algebroid on $A(\mathcal{G})$ is induced by the Lie brackets on the spaces $\Gamma(T\mathcal{G}_x)$, $\mathcal{G}_x := d^{-1}(x)$. This is possible since the Lie bracket of two right invariant vector fields is again right invariant. The anchor map in this case is given by the differential of r, $r_* : A(\mathcal{G}) \to TM$.

If $\mathcal{G}$ is a differentiable groupoid with units M, then there is associated to it a pseudodifferential calculus (or algebra of pseudodifferential operators) $\Psi^{\infty}_{1,0}(\mathcal{G})$, whose operators of order m form a linear space denoted $\Psi^{m}_{1,0}(\mathcal{G})$, $m \in \mathbb{R}$, such that $\Psi^{m}_{1,0}(\mathcal{G})\Psi^{m'}_{1,0}(\mathcal{G}) \subset \Psi^{m+m'}_{1,0}(\mathcal{G})$. This calculus is defined as follows: $\Psi^{m}_{1,0}(\mathcal{G})$ consists of smooth families of pseudodifferential operators (P_x), $x \in M$, which are right invariant with respect to multiplication by elements of $\mathcal{G}$ and are "uniformly supported." To define what uniformly supported means, let us observe that the right invariance of the operators P_x implies that their distribution kernels K_{P_x} descend to a distribution $k_P \in I^m(\mathcal{G}, M)$. Then the family $P = (P_x)$ is called *uniformly supported* if, by definition, k_P has compact support. If P is uniformly supported, then each P_x is properly supported. The right invariance condition means, for $P = (P_x) \in \Psi^{\infty}_{1,0}(\mathcal{G})$, that right multiplication $\mathcal{G}_x \ni g' \mapsto g'g \in \mathcal{G}_y$ maps P_y to P_x, whenever $d(g) = y$ and $r(g) = x$. By definition, the map

$$\Psi^{\infty}_{1,0}(\mathcal{G}) \ni P = (P_x) \mapsto e_z(P) := P_z \in \Psi^{\infty}_{1,0}(\mathcal{G}_z) \tag{19}$$

is an algebra morphism for any $z \in M$.

If we require that the operators P_x be classical of order $\mu \in \mathbb{C}$, we obtain spaces $\Psi^{\mu}(\mathcal{G})$ having similar properties. These spaces were considered in [36].

Recall that the groupoid $\mathcal{G}$ is called *d-connected* if $\mathcal{G}_x := d^{-1}(x)$ is a connected set, for any $x \in M$. If there exists a differentiable groupoid $\mathcal{G}$ whose Lie algebroid $A(\mathcal{G})$ is $A(\mathcal{G}) \simeq A$, then there exists also a d-connected differentiable groupoid with this property. In that case, the interior M_0 of M is easily seen to be an open, invariant set, hence the vector representation π_{M_0} is well-defined on $\Psi^{\infty}_{cl}(\mathcal{G})$ and allows us to associate to a pseudodifferential operator P on $\mathcal{G}$ a pseudodifferential operator $\pi_{M_0}(P) : \mathcal{C}^{\infty}_{c}(M_0) \to \mathcal{C}^{\infty}_{c}(M_0)$ [19], see Equation (20) below. (Quite often one has that, up to a canonical identification, $\mathcal{G}_x = M_0$ and $\pi_{M_0}(P) = P_x$, for $x \in M_0$.)

The morphism π_{M_0} is defined as follows. If $\varphi \in \mathcal{C}^{\infty}_{c}(M_0)$, $\varphi \circ r$ is a smooth function on $\mathcal{G}$, and we can let the family (P_x) act along each of $\mathcal{G}_x$ to obtain the function $P(\varphi \circ r)$ on $\mathcal{G}$ defined by $P(\varphi \circ r)|_{\mathcal{G}_x} = P_x(\varphi \circ r|_{\mathcal{G}_x})$. The fact that P_x is a smooth family guarantees that $P(\varphi \circ r)$ is again smooth. Then it is not difficult to check that $P(\varphi \circ r) = \varphi_0 \circ r$, for some function $\varphi_0 \in \mathcal{C}^{\infty}_{c}(M_0)$. We shall then let

$$\pi_{M_0}(P)\varphi = \varphi_0. \tag{20}$$

The fact that P is uniformly supported guarantees that φ_0 will also have compact support in M_0. A different but related description of π_{M_0} will be obtained in the proof of the next theorem.

All results and constructions above remain true for classical pseudodifferential operators. This gives the algebra $\Psi^{\infty}_{cl}(\mathcal{G})$ consisting of families $P = (P_x)$ of *classical* pseudodifferential operators satisfying all the previous conditions.

Theorem 3.3. *Let M_0 be a manifold with a Lie structure at infinity, (M, A), as above. Also, let $\mathcal{G}$ be a d-connected groupoid with units M and with $A(\mathcal{G}) \simeq A$. Then $\Psi^{m}_{1,0,\mathcal{V}}(M_0) = \pi_{M_0}(\Psi^{m}_{1,0}(\mathcal{G}))$ and $\Psi^{m}_{cl,\mathcal{V}}(M_0) = \pi_{M_0}(\Psi^{m}_{cl}(\mathcal{G}))$.*

Proof. We shall consider only the first equality. The case of classical operators can be treated in exactly the same way.

We refer to [19] or [36] for the concepts and results on groupoids and algebras of pseudodifferential operators on groupoids not explained below or before the statement of this theorem.

Here is, briefly, the idea of the proof. Let $P = (P_x) \in \Psi^m_{1,0}(\mathcal{G})$. Then the Schwartz kernels of the operators P_x, a smooth family of conormal distributions in $I^m(\mathcal{G}_x^2, \mathcal{G}_x)$, descends, by right invariance, to a distribution $k_P \in I^m_c(\mathcal{G}, M)$ (*i.e.,* to a compactly supported distribution on $\mathcal{G}$, conormal to M) called the *convolution kernel of* P. The map $P \to k_P$ is an isomorphism [36] with inverse $T : I^m_c(\mathcal{G}, M) \to \Psi^m_{1,0}(\mathcal{G})$.

Fix a metric on $A \to M$. The resulting exponential map (reviewed below) then gives rise, for $r > 0$ small enough, to an open embedding

$$\alpha : (A)_r \to \mathcal{G}, \tag{21}$$

which is a diffeomorphism onto its image. This diffeomorphism then gives rise to an embedding

$$\alpha_* : I^m((A)_r, M) \to I^m(\mathcal{G}, M) \tag{22}$$

such that for each χ as above

$$\pi_{M_0}\big(\alpha_*(\chi\mathcal{F}^{-1}_{\text{fiber}}(a))\big) = a_\chi(D) \in \Psi^m(M_0). \tag{23}$$

This will allow to show that the range of $\pi_{M_0}(\mathfrak{A})$ contains the linear span of all operators P of the form $P = a_\chi(D)$, $a \in S^m_{cl}(A^*)$, $m \in \mathbb{Z}$, and that conversely, every operator P of this form is in the range of π_{M_0}. This reduces the problem to verifying that

$$\pi_{M_0}(\Psi^{-\infty}(\mathcal{G})) = \Psi^{-\infty}_{\mathcal{V}}(M_0). \tag{24}$$

Using a partition of unity, this in turn will be reduced to Equation (23). Now let us give complete details.

Let $\mathcal{G}_x^x := d^{-1}(x) \cap r^{-1}(x)$, which is a group for any $x \in M_0$, by the axioms of a groupoid. Then $\mathcal{G}_x^x \simeq \mathcal{G}_y^y$ whenever there exists $g \in \mathcal{G}$ with $d(g) = x$ and $r(g) = y$ (conjugate by g). We shall assume M connected, for simplicity. Our above informal description of the proof can be conveniently formalized and visualized using the following diagram whose morphisms are defined below:

$$\begin{tikzcd}
S^m_{1,0}(A) \arrow[r, "\mathcal{F}^{-1}_{\text{fiber}}"] & I^m(A, M) \arrow[r, "\chi"] & I^m_{(r)}(A, M) \arrow[r, "\mathcal{R}"] \arrow[d, "\alpha_*"'] & I^m_{(r)}(TM_0, M_0) \arrow[d, "\Phi_*"] \\
& \Psi^m_{1,0}(\mathcal{G}) \arrow[r, "\simeq"] \arrow[d, "e_x"'] & I^m_c(\mathcal{G}, M) \arrow[r, "l_*"] \arrow[d, "\mu_1^*"'] & I^m(M_0^2, M_0) \arrow[d, equal] \\
& \Psi^m_{1,0}(\mathcal{G}_x)^\Gamma \arrow[r, "\simeq"] & I^m_{pr}(\mathcal{G}_x^2, \mathcal{G}_x)^\Gamma \arrow[r, "\tilde{r}_*"] \arrow[d, "\simeq"'] & I^m(M_0^2, M_0) \arrow[d, "\simeq"] \\
& & \Psi^m_{pr}(\mathcal{G}_x)^\Gamma \arrow[r, "r_*"] & \Psi^m_{cl}(M_0)
\end{tikzcd}$$

We now define the morphisms appearing in the above diagram in such a way that it will turn out to be a commutative diagram. Some of the maps above are defined only if $x \in M_0$. (Also, recall that the index "pr" means "properly supported.")

To begin with, the maps $\mathcal{F}^{-1}_{\text{fiber}}$, χ, $\mathcal{R}$, Φ_*, and e_x have already been defined.

The four isomorphisms not named are the T morphisms defined in various places earlier. Namely, the top isomorphism is from [36] and all the other isomorphisms are the canonical identifications between pseudodifferential operators and distributions on product spaces that are conormal to the diagonal (via the Schwartz kernels). In fact, the top isomorphism T is completely determined by the requirement that the left-most square (containing e_x) be commutative.

We let $\mu_1(g', g) = g'g^{-1}$ and μ_1^* be the map induced at the level of kernels by μ_1 by pull-back (which is seen to be defined in this case because μ_1 is a submersion and its range is transverse to M).

It is only slightly more difficult task to define r_*. We shall have to make use of a minimum of groupoid theory here. Let $x \in M_0$ be arbitrary for a moment. Then

$$r : T_x\mathcal{G}_x = A(\mathcal{G})_x \to T_x M_0 = A(\mathcal{G})_x$$

is an isomorphism by one of the axioms defining the Lie structure at infinity (namely $A|_{M_0} = TM_0$). Since the Lie algebra of $\mathcal{G}_x^x$ is isomorphic to the kernel of the anchor map $\varrho : A(\mathcal{G})_x \to T_x M$, we see that $\mathcal{G}_x^x$ is a discrete group if, and only if, $x \in M_0$. Fix now $x \in M_0$. Then $r : \mathcal{G}_x \to M_0$ is surjective. Hence $r : \mathcal{G}_x \to M_0$ is a covering map with group $\Gamma := \mathcal{G}_x^x$. In particular, Γ acts freely on $\mathcal{G}_x$, $\mathcal{G}_x/\Gamma = M_0$, and $\mathcal{C}^\infty(\mathcal{G}_x)^\Gamma = \mathcal{C}^\infty(M_0)$. Since P is Γ invariant and properly supported, the map $P_x : \mathcal{C}^\infty(\mathcal{G}_x) \to \mathcal{C}^\infty(\mathcal{G}_x)$, descends to a map $\mathcal{C}^\infty(M_0) \to \mathcal{C}^\infty(M_0)$, which is by definition $r_*(P)$. More precisely, if φ is a smooth function on M_0, then it lifts to $\varphi \circ r$, which is a Γ-invariant function on $\mathcal{G}_x$. Hence $P(\varphi \circ r)$ is defined (because P is properly supported) and is in turn Γ invariant. Thus there exists a function $\varphi_0 \in \mathcal{C}^\infty(M_0)$ such that $P(\varphi \circ r) = \varphi_0 \circ r$. The operator $r_*(P)$ is then given by $r_*(P)\varphi := \varphi_0$. This process is in fact very similar to the one defining the vector representation π_{M_0}, and what we have achieved here is to prove that

$$\pi_{M_0}(P) = r_*(e_x(P)). \tag{25}$$

We also obtain that

$$\pi_{M_0} \circ T = r_* \circ T \circ \mu_1^*, \tag{26}$$

by the commutativity of the left-most diagram.

The commutativity of the bottom square completely determines the morphism $\tilde{r}_*$. However, we shall also need an explicit form of this map. This can be obtained as follows. Recall that $\mathcal{G}_x$ is a covering of M_0 with group $\Gamma := \mathcal{G}_x^x$. This allows us to identify $I^m_{pr}(\mathcal{G}_x^2, \mathcal{G}_x)^\Gamma$ with $I^m_{pr}((\mathcal{G}_x^2)^\Gamma, \mathcal{G}_x^\Gamma)$. The map $\tau : (\mathcal{G}_x^2)/\Gamma \to M_0^2$ is also a covering map. This allows us to identify a distribution with small support in $(\mathcal{G}_x^2)/\Gamma$ with a distribution with support in a small subset of M_0^2. These identifications then extend by summation along the fibers of $\tau : (\mathcal{G}_x^2)/\Gamma \to M_0^2$ to define a distribution $\tau_*(u) \in \mathcal{D}'(M_0^2)$, for any distribution u on $(\mathcal{G}_x^2)/\Gamma$ whose support is such that it intersects γK only for finitely many components of $\tau^{-1}(U)$, for any locally trivializing open set $U \subset M_0$. The morphism $\tilde{r}_*$ identifies then with τ_*. Also, observe for later use that

$$\tau(g', g) = (r(g'), r(g)) = (r(g'g^{-1}), d(g'g^{-1})) = (r(\mu_1(g, g')), d(\mu_1(g, g'))).$$

Next, we must have $l_* := \tilde{r}_* \circ \mu_1^*$, by the commutativity requirement. For this morphism we have a similar description, but simpler, $l_*(u)$ is obtained by first restricting a distribution u to $d^{-1}(M_0) = r^{-1}(M_0)$ and then by applying to this restriction the map $(d, r) : d^{-1}(M_0) \to M_0^2$.

To define α_*, choose first a metric on A. This metric then lifts via $r : \mathcal{G} \to M$ to $T_{vert}\mathcal{G} \simeq r^*A(\mathcal{G})$, by Equations (17) and (18). The induced metric give rise using the (geodesic) exponential mapto maps

$$A_x \simeq A(\mathcal{G})_x = T_x\mathcal{G}_x \to \mathcal{G}_x, \quad x \in M.$$

This gives rise to a map $(A)_r$ into $\mathcal{G}$, which, by the Inverse mapping theorem, is seen to be a diffeomorphism onto its image. It, moreover, sends the zero section of A to the units of $\mathcal{G}$. Then α_* is the resulting map at the level of conormal distributions. (Note that $\mathcal{G}_x$ is the isometric covering of a complete manifold, and hence it is complete.)

We have completed the definition of all morphisms in the above diagram. To prove that it is commutative, it will be enough to check that

$$l_* \circ \alpha_* = \Phi_* \circ \mathcal{R}.$$

This however follows from the above description of the map l_*, since (d, r) is injective on $\alpha((A)_r)$ and $r : \mathcal{G}_y \to M_0$ is an isometric covering, thus preserving the exponential maps.

The commutativity of the above diagram finally shows that

$$\begin{aligned}(27) \quad a_\chi(D) := T \circ q_{\Phi,\chi}(a) = T \circ \Phi_* \circ \mathcal{R} \circ \chi \circ \mathcal{F}^{-1}_{\text{fiber}}(a) \\ = \pi_{M_0} \circ T \circ \alpha_* \circ \chi \circ \mathcal{F}^{-1}_{\text{fiber}}(a) = \pi_{M_0}(Q),\end{aligned}$$

where $Q = T \circ \alpha_* \circ \chi \circ \mathcal{F}^{-1}_{\text{fiber}}(a)$ and $a \in S^m_{1,0}(A^*)$. Thus every operator of the form $a_\chi(D)$ is in the range of π_{M_0}.

Let us notice for the rest of our argument that the definition of the vector representation can be extended by the same formula to arbitrary right invariant families of operators $P = (P_x)$, $P_x : \mathcal{C}^\infty(\mathcal{G}_x) \to \mathcal{C}^\infty(\mathcal{G}_x)$, such that the induced operator $P : \mathcal{C}^\infty_{\text{c}}(\mathcal{G}) \to \cup\mathcal{C}^\infty(\mathcal{G}_x)$ has range in $\mathcal{C}^\infty_{\text{c}}(\mathcal{G})$. We shal use this in the following case. Let $X \in \mathcal{V}$. Then X defines by exponentiation a diffeomorphism of M, see Equation (14). Let $\tilde{X}$ be its lift to a d-vertical vector field on $\mathcal{G}$ (*i.e.*, on each $\mathcal{G}_x$ we obtain a vector field, and this family of vector fields is right invariant). Recall that X can be integrated to a global flow on M (this follows from the fact that M is compact). A result from [16, Appendix] (see also [35]) then shows that $\tilde{X}$ can be integrated to a global flow. Let us denote by $\tilde{\psi}_X$ the *family* of diffeomorphisms of each $\mathcal{G}_x$ obtained in this way. It follows then from the definition that

$$(28) \qquad \pi_{M_0}(\tilde{\psi}_X) = \psi_X.$$

The Equations (27) and (28) then give

$$(29) \qquad \pi_{M_0}(Q\tilde{\psi}_{X_1} \dots \tilde{\psi}_{X_n}) = a_\chi(D)\psi_{X_1} \dots \psi_{X_n} \in \Psi^{-\infty}_{1,0,\mathcal{V}}(M_0),$$

for any $a \in S^{-\infty}(A^*)$ and $Q = T \circ \alpha_* \circ \chi \circ \mathcal{F}^{-1}_{\text{fiber}}(a)$. Also $Q\tilde{\psi}_{X_1} \dots \tilde{\psi}_{X_n} \in \Psi^{-\infty}(\mathcal{G})$, since the product of a regularizing operator with the operator induced by a diffeomorphism is regularizing. We have thus proved that $\pi_{M_0}(\Psi^m_{1,0}(\mathcal{G}) \supset \Psi^m_{1,0,\mathcal{V}}(M_0)$. Let us now prove the oposite inclusion, that is that $\pi_{M_0}(\Psi^m_{1,0}(\mathcal{G}) \subset \Psi^m_{1,0,\mathcal{V}}(M_0)$.

Let $Q \in \Psi^m_{1,0}(\mathcal{G})$ be arbitrary and let $b = T^{-1}(Q)$. Let χ_0 be a smooth function on $\mathcal{G}$ that is equal to 1 in a neighborhood of M in $\mathcal{G}$ and with support in $\alpha((A)_r)$ and such that $\chi = 1$ on the support of $\chi_0 \circ \alpha$. Then $b_0 := \chi_0 b$ is in the range of $\alpha_* \circ \chi \circ \mathcal{F}^{-1}_{\text{fiber}}$, because any distribution $u \in I^m_{(r)}(A, M)$ is in the range of $\mathcal{F}^{-1}_{\text{fiber}}$, if $r < \infty$. Then the difference $b - b_0$ is smooth. Because $\mathcal{G}$ is d-connected, we can use a similar construction as the one for b_0 and a partition of unity argument to obtain that

$$T(b - b_0) = \sum_{j=1}^{l} T(b_j)\tilde{\psi}_{X_{j1}} \dots \tilde{\psi}_{X_{jn}} \tag{30}$$

for some distributions $b_j \in \chi I^{-\infty}_{(r)}(A, M)$ and vector fields $X_{jk} \in \mathcal{V}$. (This is easily understood in the case of groups, when it amounts to the possibility of covering any given compact set by finitely many translations of a given open neighborhood of the identity. The argument in general is the same as the argument used to define the basic coordinate neighborhoods on $\mathcal{G}$ in [35]. The basic coordinate neighborhoods on $\mathcal{G}$ were used in that paper to define the smooth structure on the groupoid $\mathcal{G}$.) Let a_j be such that $b_j = \alpha_* \circ \chi \circ \mathcal{F}^{-1}_{\text{fiber}}(a_j)$, for $a_0 \in S^m_{1,0}(A^*)$ and $a_j \in S^{-\infty}_{1,0}(A^*)$, if $j > 0$. Then Equations (27) and (29) show that

$$\pi_{M_0}(Q) = a_0(D) + \sum_{j=1}^{l} a_j(D)\psi_{X_{j1}} \dots \psi_{X_{jn}} \in \Psi^m_{1,0,\mathcal{V}}(M_0). \tag{31}$$

We have thus proved that $\pi_{M_0}(\Psi^m_{1,0}(\mathcal{G})) = \Psi^m_{1,0,\mathcal{V}}(M_0)$, as desired. This completes our proof. $\square$

The existence of a d-connected groupoid integrating A was proved in [35] for the Lie algebras arising in [30] and in [6] in general, building on the ideas in [35].

4. Properties of $\Psi^\infty_{1,0,\mathcal{V}}(M_0)$

Theorem 3.3 has several consequences similar to the results in [24, 33, 31, 34, 39, 42].

First we obtain the following boundedness result.

Corollary 4.1. *Each operator $P \in \Psi^m_{1,0,\mathcal{V}}(M_0)$, $P : \mathcal{C}^\infty_c(M_0) \to \mathcal{C}^\infty(M_0)$, extends to a continuous linear operator $P : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$. Moreover, any $P \in \Psi^0_{1,0,\mathcal{V}}(M_0)$ is bounded on $L^2(M_0)$.*

Proof. The first part is a direct consequence of the fact that any $P \in \Psi^m_{1,0,\mathcal{V}}(M_0)$ is properly supported.

It was proved in [19] that $\pi_{M_0}(P_1)$ is bounded. (In fact, the proof given there was for classical operators, but it extends right away to type $(1, 0)$ operators.) This proves the second part. $\square$

We also obtain that the algebras $\Psi^m_{1,0,\mathcal{V}}(M_0)$ and $\Psi^m_{1,0,\mathcal{V}}(M_0)$ are independent of the choices made to define them.

Corollary 4.2. *The spaces $\Psi^m_{1,0,\mathcal{V}}(M_0)$ and $\Psi^m_{1,0,\mathcal{V}}(M_0)$ are independent of the choice of the metric on A and the function χ used to define it, but depend, in general, on the Lie structure at infinity (M, A) on M_0.*

Proof. The space $\Psi^m_{1,0}(\mathcal{G})$ does not depend on the metric on A or on the function χ and neither does the vector representation π_{M_0}. Then use Theorem 3.3. The proof is the same for classical operators. $\square$

An important consequence is that $\Psi^m_{1,0,\mathcal{V}}(M_0)$ and $\Psi^m_{1,0,\mathcal{V}}(M_0)$ are filtered algebras, as it is the case of the usual algebra of pseudodifferential operators on a compact manifold.

Proposition 4.3. *Using the above notation, we have that*

$$\Psi^m_{1,0,\mathcal{V}}(M_0)\Psi^{m'}_{1,0,\mathcal{V}}(M_\mathcal{V}) \subseteq \Psi^{m+m'}_{1,0,\mathcal{V}}(M_0) \quad \textit{and} \quad \Psi^m_{cl,\mathcal{V}}(M_0)\Psi^{m'}_{cl,\mathcal{V}}(M_\mathcal{V}) \subseteq \Psi^{m+m'}_{cl,\mathcal{V}}(M_0)\,,$$

for all $m, m' \in \mathbb{C} \cup \{-\infty\}$.

Proof. Use Theorem 3.3 and the fact that π_{M_0} preserves the product. □

Part (i) of the following result is an analog of a standard result about the b-calculus [31], whereas the second formula is the independence of diffeomorphisms of the algebras $\Psi^\infty_{cl,\mathcal{V}}(M_0)$, in the framework of manifolds with a Lie structure at infinity. Recall that if $X \in \Gamma(A)$, we have denoted by $\psi_X := \Psi_X(1, \cdot) : M \to M$ the diffeomorphism defined by integrating X (and specializing at $t = 1$).

Proposition 4.4. *(i) Let x be a defining function of some hyperface of M. Then $x^s\Psi^m_{1,0,\mathcal{V}}(M_0)x^{-s} = \Psi^m_{1,0,\mathcal{V}}(M_0)$ and $x^s\Psi^m_{cl,\mathcal{V}}(M_0)x^{-s} = \Psi^m_{cl,\mathcal{V}}(M_0)$ for any $s \in \mathbb{C}$.*
(ii) Similarly,

$$\psi_X\Psi^m_{1,0,\mathcal{V}}(M_0)\psi_X^{-1} = \Psi^m_{1,0,\mathcal{V}}(M_0) \quad \textit{and} \quad \psi_X\Psi^m_{cl,\mathcal{V}}(M_0)\psi_X^{-1} = \Psi^m_{cl,\mathcal{V}}(M_0),$$

for any $X \in \Gamma(A)$.

Proof. We have that $x^s\Psi^m_{cl}(\mathcal{G})x^{-s} = \Psi^m_{cl}(\mathcal{G})$, for any $s \in \mathbb{C}$, by [19]. A similar result for type $(1,0)$ operators is proved in the same way as in [19]. This proves (a) because $\pi_{M_0}(x^sPx^{-s}) = x^s\pi_{M_0}(P)x^{-s}$.

Similarly, using the notations of Theorem 3.3, we have $\tilde\psi_X\Psi^m_{cl}(\mathcal{G})\tilde\psi_X^{-1} = \Psi^m_{cl}(\mathcal{G})$, for any $X \in \Gamma(A) = \mathcal{V}$. By the diffeomorphism invariance of the space of pseudodifferential operators, $\tilde\psi_XP\tilde\psi_X^{-1}$ defines a right invariant family of pseudodifferential operators on $\mathcal{G}$ for any such right invariant family $P = (P_x)$, as in the proof of Theorem 3.3. To check that the family $P_1 := \psi_XP\psi_X^{-1}$ has a compactly supported convolution kernel, denote by $(\mathcal{G})_a = \{g, dist(g, d(g)) \le a\}$. Then observe that $\operatorname{supp}(\tilde\psi_XP\tilde\psi_X^{-1}) \subset \mathcal{G}_{d+2\|X\|}$ whenever $\operatorname{supp}(P) \subset (\mathcal{G})_d$. Then use Equation (28) to conclude the result.

The proof for type $(1,0)$ operators is the same. □

Let us notice that the same proof gives (ii) above for any diffeomorphism of M_0 that extends to an automorphism of (M, A). Recall that an autormorphism of the Lie algebroid (M, A) is a morphism of vector bundles (φ, ψ), $\varphi : M \to M$, $\psi : A \to A$, such that φ and ψ are diffeomorphisms and we have the following compatibility with the anchor map ϱ:

$$\varrho \circ \psi = \varphi_* \circ \varrho.$$

For any $X \in \Gamma(A)$, denote by $a_X : A^* \to \mathbb{C}$ the function defined by $a_X(\xi) = \xi(X)$. Then there exists a unique Poisson structure on A^* such that $\{a_X, a_Y\} = a_{[X,Y]}$. It is related to the Poisson structure $\{\,\cdot\,,\cdot\,\}^{T^*M}$ on T^*M via the formula

$$\{f_1 \circ \varrho^*, f_2 \circ \varrho^*\}^{T^*M} = \{f_1, f_2\} \circ \varrho^*,$$

where $\varrho^* : T^*M \to A^*$ denotes the dual to the anchor map ϱ. In particular, in $\{\,\cdot\,,\cdot\,\}$ and $\{\,\cdot\,,\cdot\,\}^{T^*M}$ coincide on M_0.

Proposition 4.5. *We have that*

$$\sigma^{(m+m'-1)}([P,Q]) = \{\sigma^{(m)}(P), \sigma^{(m')}(Q)\}$$

for any $P \in \Psi^m_{1,0}(M_0)$ and any $Q \in \Psi^{m'}_{1,0}(M_0)$, where $\{\,\cdot\,,\,\cdot\,\}$ is the usual Poisson bracket on A^*.

Proof. The Poisson structure on T^*M_0 is induced from the Poisson structure on A^*. In turn, the Poisson structure on T^*M_0 determine the Poisson structure on T^*M_0, because the later is dense in A^*. The desired result then follows from the similar result that is known for pseudodifferential operators on M_0 and the Poisson bracket on T^*M_0. □

We conclude with the following result, which is independent of the previous considerations, but sheds some light on them. The invariant differential operators on $\mathcal{G}$ are generated by d–vertical invariant vector fields on $\mathcal{G}$, that is by $\Gamma(A(\mathcal{G}))$. We have by definition that $\pi_{M_0} = \varrho : \Gamma(M; A(\mathcal{G})) \to \Gamma(M; TM)$, and hence π_{M_0} maps the algebra of invariant differential operators onto $\mathcal{G}$ to $\mathrm{Diff}^*_{\mathcal{V}}(M_0)$. In particular, the proof of Theorem 3.3 (more precisely Equation (27)) can be used to prove the following result, which we will however prove also without making appeal to Theorem 3.3.

Proposition 4.6. *Let $X \in \Gamma(A)$ and denote by $a_X(\xi) = \xi(X)$ the associated linear function on A^*. Then $a_X \in S^1(A^*)$ and $a_X(D) = -\imath X$. Moreover,*

$$\{a_\chi(D),\, a = \text{ polynomial in each fiber }\} = \mathrm{Diff}^*_{\mathcal{V}}(M_0).$$

Proof. We continue to use a fixed metric on A to trivialize any density bundle. Let $u = \mathcal{F}^{-1}_{\text{fiber}}(a)$, where $a \in S^m_{cl}(A^*)$ polynomial in each fiber. By the Fourier inversion formula (and integration by parts), u is supported on M, which is the same thing as saying that u is a distribution the form $\langle u, f\rangle = \int_M P_0 f(x) d\mathrm{vol}(x)$, with P_0 a differential operator acting along the fibers of $A \to M$ and $f \in \mathcal{C}^\infty_c(A)$. It then follows from the definition of $a_\chi(D)$, from the formula above for $u = \mathcal{F}^{-1}_{\text{fiber}}(a)$, and from the fact that $\chi = 1$ in a neighborhood of the support of u that

$$a_\chi(D)f(x) = [P_0 f(\exp_x(-v))]|_{v=0}, \quad v \in T_x M_0. \tag{32}$$

Let $X_1, X_2, \ldots, X_m \in \Gamma(A)$ and

$$a = a_{X_1} a_{X_2} \ldots a_{X_m} \in S^m(A^*). \tag{33}$$

Then the differential operator P_0 above is given by the formula

$$P_0 f(x) = \int_{A^*_x} a(\xi)\mathcal{F}^{-1} f(\xi),$$

with the inverse Fourier transform $\mathcal{F}^{-1}$ being defined along the fiber A_x. Hence

$$P_0 = \imath^m X_1 X_2 \ldots X_m,$$

with each X_j being identified with the family of constant coefficients differential operators along the fibers of $A \to M$ that acts along A_x as the derivation in the direction of $X_j(x)$.

For any $X \in A$, we shall denote by ψ_{tX} the one parameter subgroup of diffeomorphisms of M generated by X. (Note that ψ_{tX} is defined for any t because M is compact and X is tangent to all faces of M.) We thus obtain an action of ψ_{tX}

on functions by $[\psi_{tX}(f)](x) = f(\exp(tX)x)$. Then the differential operator P_0 is associated to a as in Equation (33) is given by

$$P_0(f \circ \exp)|_M = \imath^m \big[\partial_1 \partial_2 \ldots \partial_m \psi_{t_1 X_1 + t_2 X_2 + \ldots + t_m X_m} f\big]|_{t_1 = \ldots = t_m = 0}. \tag{34}$$

Then Equations (32) and (34) give

$$a_\chi(D) f = \imath^m \big[\partial_1 \ldots \partial_m \exp(-t_1 X_1 - \ldots - t_m X_m) f\big]|_{t_1 = \ldots = t_m = 0}, \tag{35}$$

In particular, $a_X(D) = -\imath X$, for any $X \in \Gamma(A)$.
This proves that

$$a_\chi(D) \in \mathrm{Diff}^*_{\mathcal{V}}(M_0), \tag{36}$$

by Campbell-Hausdorff formula [12, 37], which states that $a_\chi(D)$ is generated by $X_1, X_2, \ldots, X_n$ (and their Lie brackets), and hence that it is generated by $\mathcal{V}$, which was assumed to be a Lie algebra.

Let us prove now that any differential operator $P \in \mathrm{Diff}^*_{\mathcal{V}}(M_0)$ is of the form $a_\chi(D)$, for some polynomial symbol a on A^*. This is true if P has degree zero. Indeed, assume P is the multiplication by $f \in \mathcal{C}^\infty(M)$. Lift f to an order zero symbol on A^*, by letting this extension to be constant in each fiber. Then $P = f(D)$. We shall prove our statement by induction on the degree m of P. By linearity, we can reduce to the case $P = \imath^{-m} X_1 \ldots X_m$, where $X_1, \ldots, X_m \in \Gamma(A)$. Let $a = a_{X_1} \ldots a_{X_m}$. Then

$$\sigma_m(a_\chi(D))(\xi) = a(\xi) = X_1(\xi) \ldots X_m(\xi) = \sigma_m(P),$$

and hence $Q := a_\chi(D) - \imath^{-m} X_1 \ldots X_m \in \mathrm{Diff}^{m-1}_{\mathcal{V}}(M_0)$. By the induction hypothesis, $Q = b_\chi(D)$ for some polynomial symbol of order at most $m-1$ on A^*. This completes the proof. □

From this we obtain the following corollary.

Corollary 4.7. *Let* $\mathrm{Diff}(M_0)$ *be the algebra of all differential operators on* M_0*. Then*

$$\Psi^\infty_{1,0,\mathcal{V}}(M_0) \cap \mathrm{Diff}(M_0) = \mathrm{Diff}^*_{\mathcal{V}}(M_0).$$

Proof. We know from the above proposition that

$$\Psi^\infty_{1,0,\mathcal{V}}(M_0) \cap \mathrm{Diff}(M_0) \supset \mathrm{Diff}^*_{\mathcal{V}}(M_0).$$

Conversely, assume $P \in \Psi^m_{1,0,\mathcal{V}}(M_0) \cap \mathrm{Diff}(M_0)$. We shall prove by induction on m that $P \in \mathrm{Diff}^m_{\mathcal{V}}(M_0)$. If $m = 0$ then P is the multiplication with a smooth function f on M_0. But then $f = \sigma^{(0)}(P) \in S^0(A^*)$ is constant along the fibers of $A^* \to M$, and hence $f \in \mathcal{C}^\infty(M) \subset \mathrm{Diff}^*_{\mathcal{V}}(M_0)$. Assume now that the statement is proved for P of order $< m$. We shall prove it then for P of order m. Then $a := \sigma^{(m)}(P)$ is a polynomial symbol in $S^m(A^*)$. Thus $a_\chi(D) \in \mathrm{Diff}^m_{\mathcal{V}}(M_0)$, by Proposition 4.6. But then $\sigma^{(m)}(P - a_\chi(D)) = 0$, by Lemma 2.2, and hence $P - a_\chi(D) \in \Psi^{m-1}_{1,0,\mathcal{V}} \cap \mathrm{Diff}(M_0)$. By the induction hypothesis $P - a_\chi(D) \in \mathrm{Diff}^{m-1}_{\mathcal{V}}(M_0)$. This completes the proof. □

5. Group actions and semi-classical limits

One of the most convenient features of manifolds with a Lie structure at infinity is that questions on the analysis on these manifolds often reduce to questions on the analysis on simpler manifolds. These simpler manifolds are manifolds of the same dimension but endowed with certain non-trivial group actions. Harmonic analysis techniques then allow us to ultimately reduce our questions to analysis on lower dimensional manifolds with a Lie structure at infinity. In this section, we discuss the algebras $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0, G)$ that generalize the algebras $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0)$ when group actions are considered. These algebras are necessary for the reductions mentioned above. Then we discuss a semi-classical version of the algebra $\Psi^{\infty}_{1,0,\mathcal{V}}(M_0)$.

5.1. **Group actions.** We shall consider the following setting. Let M_0 be a manifold with a Lie structure at infinity (M, A), and $\mathcal{V} = \Gamma(A)$, as above. Also, let G be a Lie group with Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$. We shall denote by $\mathfrak{g}_M$ the bundle $M \times \mathfrak{g} \to M$. Then

$$\mathcal{V}_G := \mathcal{V} \oplus \mathcal{C}^{\infty}(M, \mathfrak{g}) \simeq \Gamma(A \oplus \mathfrak{g}_M) \tag{37}$$

has the structure of a Lie algebra with respect to the bracket $[\,\cdot\,,\,\cdot\,]$ which is defined such that on $\mathcal{C}^{\infty}(M, \mathfrak{g})$ it coincides with the pointwise bracket, on $\mathcal{V}$ it coincides with the original bracket, and, for any $X \in \mathcal{V}$, $f \in \mathcal{C}^{\infty}(M)$, and $Y \in \mathfrak{g}$, we have

$$[X, f \otimes Y] := X(f) \otimes Y \,.$$

(Here $f \otimes Y$ denotes the function $\xi : M \to \mathfrak{g}$ defined by $\xi(m) = f(m)Y \in \mathfrak{g}$.)

The main goal of this subsection is to indicate how the results of the Section (2) extend to $\mathcal{V}_G$, after we replace A with $A \oplus \mathfrak{g}_M$, M_0 with $M_0 \times G$, and M with $M \times G$. The resulting constructions and definitions will yield objects that are invariant with respect to the action of G on itself by *right* translations.

We now proceed by analogy with the construction of the operators $a_\chi(D)$ in Subsection 2.3. First, we identify a section of $\mathcal{V}_G := \mathcal{V} \oplus \mathcal{C}^{\infty}(M, \mathfrak{g}) \simeq \Gamma(A \oplus \mathfrak{g}_M)$ with a right G-invariant vector field on $M_0 \times G$. At the level of vector bundles, this corresponds to the map

$$p : T(M_0 \times G) = TM_0 \times TG \to TM_0 \times \mathfrak{g}, \tag{38}$$

where the map $TG \to \mathfrak{g}$ is defined using the trivialization of TG by *right* invariant vector fields. Let $p_1 : M \times G \to M$ be the projection onto the first component and p_1^*A be the lift of A to $M \times G$ via p_1.

The map p defined in the Equation (38) can then be used to define the lift

$$p^*(u) \in I^m(p_1^*A \oplus TG, M \otimes G), \tag{39}$$

for any distribution $u \in I^m(A \oplus \mathfrak{g}_M, M)$. In particular, $p^*(u)$ will be a right G-invariant distribution. Then we define $\mathcal{R}$ to be the restriction of distributions from $p_1^*A \oplus TG$ to distributions on $TM_0 \times TG = T(M_0 \times G)$.

We endow $M_0 \times G$ with the metric obtained from a metric on A and a right invariant metric on G. This allows us to define the exponential map, thus obtaining, as in Section 2, a differentiable map

$$\Phi : (TM_0 \times TG)_r = (T(M_0 \times G))_r \to (M_0 \times G)^2 \tag{40}$$

that is a diffeomorphism onto an open neighborhood of the diagonal, provided that $r < r_0$, where r_0 is the injectivity radius of $M_0 \times G$. We shall denote as before by

$$\Phi_* : I_c^m((TM_0 \times TG)_r, M_0 \times G) \to I_c^m((M_0 \times G)^2, M_0 \times G)$$

the induced map on conormal distributions.

The inverse Fourier transform will give a map

$$\mathcal{F}^{-1}_{\text{fiber}} : S^m_{1,0}(A^* \oplus \mathfrak{g}_M^*) \longrightarrow I^m(A \oplus \mathfrak{g}_M, M), \tag{41}$$

defined by the same formula as before (Equation (6)). Finally, we shall also need a smooth function χ on $A \oplus \mathfrak{g}_M$ that is equal to 1 in a neighborhood of the zero section and has support inside $(A \oplus \mathfrak{g}_M)_r$.

We can then define the quantization map in the G-equivariant case by

$$q_{\Phi,\chi,G} := \Phi_* \circ \mathcal{R} \circ p^* \circ \chi \circ \mathcal{F}^{-1}_{\text{fiber}} : S^m_{1,0}(A^* \oplus \mathfrak{g}_M^*) \longrightarrow \mathcal{I}^m((M_0 \times G)^2, M_0 \times G). \tag{42}$$

(The main difference with the definition in Equation (10) is that we included the map p^*, which is the lift of distributions in $I^m(A \oplus \mathfrak{g}_M, M)$ to G-invariant distributions in $I^m(p_1^* A \oplus TG, M_0 \otimes G)$, see Equation (39).) Then

$$a_\chi(D) = T \circ q_{\Phi,\chi,G}, \tag{43}$$

as before.

With the definition of the quantization map, all the results of Section 2 remain valid, with the appropriate modifications. In particular, we obtain the following definition of the algebra of G-equivariant pseudodifferential operators associated to (A, M, G).

Definition 5.1. For $m \in \mathbb{R}$, the space $\Psi^m_{1,0,\mathcal{V}}(M_0, G)$ of *G-equivariant pseudodifferential operators generated by the Lie structure at infinity* (M, A) is the linear space of operators $\mathcal{C}_c^\infty(M_0 \times G) \to \mathcal{C}_c^\infty(M_0 \times G)$ generated by $a_\chi(D)$, $a \in S^m_{1,0}(A^* \oplus \mathfrak{g}_M^*)$, and $b_\chi(D)\psi_{X_1} \ldots \psi_{X_k}$, $b \in S^{-\infty}(A^* \oplus \mathfrak{g}_M^*)$ and $X_j \in \Gamma(A \oplus \mathfrak{g}_M)$.

The space $\Psi^m_{cl,\mathcal{V}}(M_0, G)$ of *classical G-equivariant pseudodifferential operators generated by the Lie structure at infinity* (M, A) is defined similarly, but using classical symbols a.

With this definition, all the results on the algebras $\Psi^m_{1,0,\mathcal{V}}(M_0)$ and $\Psi^m_{cl,\mathcal{V}}(M_0)$ extend right away to the spaces $\Psi^m_{1,0,\mathcal{V}}(M_0, G)$ and $\Psi^m_{cl,\mathcal{V}}(M_0, G)$. In particular, these spaces are algebras, are independent of the choice of the metric on A used to define them, and have the usual symbolic properties of the algebras of pseudodifferential operators.

The only thing that maybe needs more explanations is with what we replace π_{M_0} in the G-equivariant case, because in the G-equivariant case we no longer use the vector representation. Let $\mathcal{G}$ be a groupoid integrating $\mathcal{G}$. Then $\mathcal{G} \times G$ integrates $A \oplus \mathfrak{g}_M$. If $P = (P_x) \in \Psi^m_{1,0}(\mathcal{G} \times G)$, then we consider $\pi_0(P)$ to be the operator induced by P_x on $(\mathcal{G}_x/\mathcal{G}_x^x) \times G$, $x \in M_0$, the later space being a quotient of $(\mathcal{G} \times G)_x$. We shall use then π_0 instead of π_{M_0} in the G-equivariant case. (By the proof of Theorem 3.3, $\pi_0 = \pi_{M_0}$, if G is reduced to a point.)

The main reason for considering the algebras $\Psi^m_{1,0,\mathcal{V}}(M_0, G)$ and their classical counterparts is the following. Let (M_0, M, A) be a manifold with a Lie structure at infinity. Let $N_0 \subset M$ be a submanifold such that $T_x N_0 = \varrho(A_x)$ for any $x \in N_0$. Moreover, assume that N_0 is completely contained in an open face $F \subset M$ such that $N := \overline{N_0}$ is a submanifold with corners of F and $N_0 = N \smallsetminus \partial N$. Then the restriction $A|_{N_0}$ is such that the Lie bracket on $\mathcal{V} = \Gamma(A)$ descends to a Lie bracket on $\Gamma(A|_{N_0})$. (This is due to the fact that the space I of functions vanishing on N is invariant for derivations in $\mathcal{V}$. Then $I\mathcal{V}$ is an ideal of $\mathcal{V}$, and hence $\mathcal{V}/I\mathcal{V} \simeq \Gamma(A|_N)$ is naturally a Lie algebra.)

Assume now that there exists a Lie group G and a vector bundle $A_1 \to N$ such that $A|_N \simeq A_1 \oplus \mathfrak{g}_N$ and $\mathcal{V}_1 := \mathcal{V}|_N \simeq \Gamma(A_1)$. Then $\mathcal{V}_1$ is a Lie algebra and (N_0, N, A_1) is also a manifold with a Lie structure at infinity. In many cases (certainly for many of the most interesting examples) one obtains for any Lie group H a natural morphism

$$R_N : \Psi^m_{1,0,\mathcal{V}}(M_0; H) \to \Psi^m_{1,0,\mathcal{V}_1}(N_0; G \times H). \tag{44}$$

For example, the morphisms considered in [19] are of the form (8). However, we do not know exactly what are the conditions under which the morphism R_N above is defined.

Let $\mathfrak{h} = \operatorname{Lie} H$ and $\mathfrak{h}_N = M \times \operatorname{Lie} H$. Then, at the level of kernels the morphism defined by Equation (44) corresponds to the restriction maps

$$r_N : I^m(A^* \oplus \mathfrak{h}^*_N, M) \to I^*(A^*|_N \oplus \mathfrak{h}^*_N, N) \simeq I^*(A^*_1|_N \oplus \mathfrak{g}_N \oplus \mathfrak{h}^*_N, N)$$

in the sense that $R_N(a_\chi(D)) = (r_N(a))(D)$.

5.2. **Semi-classical limits.** We now define the algebra $\Psi^m_{1,0,\mathcal{V}}(M_0[[h]])$, an element of which will be, roughly speaking, a semi-classical family of operators (T_t), $T_t \in \Psi^m_{1,0,\mathcal{V}}(M_0)$ $t \in (0,1]$. See [49] for some applications of semi-classical analysis.

Definition 5.2. For $m \in \mathbb{R}$, the space $\Psi^m_{1,0,\mathcal{V}}(M_0[[h]])$ of *pseudodifferential operators generated by the Lie structure at infinity* (M, A) is the linear space of families of operators $T_t : \mathcal{C}^\infty_c(M_0 \times G) \to \mathcal{C}^\infty_c(M_0 \times G)$, $t \in (0,1]$, generated by

$$a_\chi(t, tD)\,, \quad a \in S^m_{1,0}([0,1] \times A^* \oplus \mathfrak{g}^*_M m),$$

and

$$b_\chi(t, tD)\psi_{tX_1(t)} \dots \psi_{tX_k(t)}, \quad b \in S^{-\infty}([0,1] \times A^* \oplus \mathfrak{g}^*_M), \quad X_j \in \Gamma([0,1] \times A \oplus \mathfrak{g}_M).$$

The space $\Psi^m_{cl,\mathcal{V}}(M_0[[h]])$ of *semi-classical families of pseudodifferential operators generated by the Lie structure at infinity* (M, A) is defined similarly, but using classical symbols a.

Thus we consider families of operators (T_t), $T_t \in \Psi^m_{1,0,\mathcal{V}}(M_0)$, defined in terms of data a, b, X_k, that extends smoothly to $t = 0$, with the interesting additional feature that the cotangent variable is rescaled as $t \to 0$.

Again, all the results on the algebras $\Psi^m_{1,0,\mathcal{V}}(M_0)$ and $\Psi^m_{cl,\mathcal{V}}(M_0)$ extend right away to the spaces $\Psi^m_{1,0,\mathcal{V}}(M_0[[h]])$ and $\Psi^m_{cl,\mathcal{V}}(M_0[[h]])$, except maybe Proposition 4.6 and its corollary, Corollary 4.7, that need to be properly reformulated.

Another variant of the above constructions is to consider families of manifolds with a Lie structure at infinity. The necessary changes are obvious though, and we will not discuss them here.

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